COUNTING FRIEZES IN TYPE $E_6$

MICHAEL CUNTZ AND PIERRE-GUY PLAMONDON

Friezes (also called “friezes of type A” and “SL$_2$-friezes”) were introduced by H. Coxeter in [4] and later studied by H. Coxeter and J. Conway in [2, 3]. It was observed by P. Caldero that the theory of cluster algebras of S. Fomin and A. Zelevinsky [5] allows for a far-reaching generalization of the original notion of frieze; this generalization was first studied in [1]. Since then, many generalizations and variations on the notion of friezes have been introduced, as can be seen in the survey paper [7].

It is known that, for a given non-Dynkin type, there are infinitely many friezes of that type. However, it follows from [2, 3] that friezes of Dynkin type $A$ come in a finite number (given by Catalan numbers), and it was proved in [6] that friezes of Dynkin type $B, D$ and $G$ also come in a finite (explicit) number (the result for type $D_4$ was also proved in [8]). It was conjectured in [9, 8, 6] that the number of friezes in type $E_6$ is 868; in [6] the precise number of friezes in any Dynkin type is also conjectured.

In this Appendix, we settle the case of type $E_6$, and obtain the result for type $F_4$ as a corollary.

Theorem 1. The number of friezes of type $E_6$ is exactly 868.

Since Dynkin type $F_4$ is a folding of type $E_6$, it follows from the work of [6] that we then have:

Corollary 2. The number of friezes of type $F_4$ is exactly 112.

Our proof relies on a reduction to 2-friezes (whose definition we recall below); our strategy is to show that the entries in a 2-frieze of height 3 are bounded.

We have attempted to apply the methods used in this Appendix to types $E_7$ and $E_8$, without success.

1. 2-FRIEZES

We shall not be using the definition of a frieze from [1], but rather a slightly different notion, that of a 2-frieze as defined in [8].

Definition 3. A 2-frieze of height $h$ is an array of positive integers $(a_{i,j})$, where

- $i \in \mathbb{Z}$ and $j \in \{0, 1, \ldots, h + 1\}$;
- for all $i \in \mathbb{Z}$, we have that $a_{i,0} = a_{i,h+1} = 1$;
- for all $i \in \mathbb{Z}$ and all $j \in \{1, \ldots, h\}$, we have that $a_{i,j} = a_{i-1,j}a_{i+1,j} - a_{i,j-1}a_{i,j+1}$.

The reason we are interested in 2-friezes is the following result.

Theorem 4 ([8]). Any frieze of type $E_6$ determines a unique 2-frieze of height 3.

Our strategy to prove Theorem 1 is thus to prove that the number of 2-friezes of height 3 is finite. We will do this by showing that there is a bound on the possible values appearing in a 2-frieze of height 3.

2. TWO CHOICES OF INITIAL VARIABLES FOR 2-FRIEZES OF HEIGHT 3

If we fix the entries of the first two rows of a 2-frieze of height 3 to be $s, t, u, v, w$ and $x$, then we get the following expressions for all its entries:

The second author is supported by the French ANR grant SC3A (ANR-15-CE40-0004-01) and by a PEPS “Jeune chercheuse, jeune chercheur” grant.
where

\[ X = \frac{svx + sw + tw + tvx + tw^2 + twx}{stuvw}, \]

\[ Y = \frac{stvx + sw^2 + stwx + suvwx + sw^2 + t^2vw + t^2vx + t^2w^2 + twx + tww}{stuwvx}, \]

and

\[ Z = \frac{stw + suv + sw + t^2w + tuw + tvx}{tuvwx}. \]

If, instead, we fix the six first entries of the leftmost non-trivial column, then we get:

\[ A = \frac{svx - stw + suw - svx + t - swx - tw - x + 1}{stuvwx - stvw^2 - stv^2x + stuv - su^2wx + suvx + suvw - sv - t^2vwx + t^2w^2 + twvx + tvx - 2tw - wx + 1}. \]
\[
B = \begin{pmatrix}
stuvwx - stuw - sv^2x + suw + su - sv - t^2vx + t^2w + tw + tv - tw - t - u + 1 \\
stuwwx - stuw^2 - stvx + stuw - sv^2vx + suwx + suw + sv - stuvwx + t^2w^2 + tuwx + tvx - 2tw - ux + 1 \\
suvwx - suw^2 - sv - 2x + suv - tuvw + tw - tw + uw - ux + vx - v - w + 1 \\
stuwvx - stuw^2 + stvx + stuw - sv^2vx + suwx + suw - sv - t^2vwx + t^2w^2 + tuwx + tvx - 2tw - ux + 1.
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
stuvwx - stuw - sv^2x + suw + su - sv - t^2vx + t^2w + tw + tv - tw - t - u + 1 \\
stuwwx - stuw^2 - stvx + stuw - sv^2vx + suwx + suw + sv - stuvwx + t^2w^2 + tuwx + tvx - 2tw - ux + 1 \\
suvwx - suw^2 - sv - 2x + suv - tuvw + tw - tw + uw - ux + vx - v - w + 1 \\
stuwvx - stuw^2 + stvx + stuw - sv^2vx + suwx + suw - sv - t^2vwx + t^2w^2 + tuwx + tvx - 2tw - ux + 1.
\end{pmatrix}
\]

3. FINITUDE OF THE NUMBER OF 2-FRIEZES OF HEIGHT 3

We use the second choice of initial variables of the previous section. We can assume, without loss of generality, that the greatest entry in the first and third columns is \( u \). Then

\[
u \geq suw - sv - tw + 1
\]

\[
\geq suw - su - wu + 1 \quad \text{(since } u \geq t, v) = \frac{(s-1)(w-1)}{u} - u + 1.
\]

Therefore

\[
1 \geq (s-1)(w-1) - 1 + \frac{1}{u} > (s-1)(w-1) - 1.
\]

Hence \((s-1)(w-1) < 2\). This implies that \( s = 1 \) or \( w = 1 \) or \( s = w = 2 \).

3.1 The case \( s = 1 \) or \( w = 1 \). If \( s = 1 \) or \( w = 1 \), then the associated frieze of type \( E_6 \) contains a frieze of type \( D_5 \), and it is known [6] that there are only 187 of these. Thus there is a finite number of cases where \( s = 1 \) or \( w = 1 \).

3.2 The case \( s = w = 2 \). If \( s = w = 2 \), then consider the following inequalities:

\[
u \geq suw - sv - tw + 1
\]

\[
= 4u - 2t - 2v + 1
\]

and \( u \geq v \), which implies that \( 3u \geq 4u - 2t + 1 \), so

\[
u \leq 2t - 1.
\]

But together with \( u \geq tvx - tw - ux + 1 = tvx - 2t - ux + 1 \) this yields

\[
(2t - 1)(x + 1) \geq tvx - 2t + 1,
\]

and hence

\[
4t \geq tvx + 2 - 2tx + x = x + 2 + (v - 2)tx > (v - 2)tx.
\]

Thus we obtain \((v - 2)x < 4\) which gives \( v \leq 6 \). For symmetry reasons, the same argument produces \( t \leq 6 \). But then \( u \leq 35 \) since \( tv - u \geq 1 \). Hence we have reduced the problem to a finite number of cases. In fact, an easy computation shows that the only solution is

\[
(s, t, u, v, w, x) = (2, 4, 5, 4, 2, 1)
\]

in which case the (transposed) 2-frieze is:

\[
\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 4 & 5 & 4 & 2 & 1 & 1 & 2 & 4 \\
1 & 2 & 6 & 11 & 6 & 2 & 1 & 1 & 2 & 6 \\
1 & 2 & 4 & 5 & 4 & 2 & 1 & 1 & 2 & 4 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Alternatively, assuming that there is no entry 1 in the first and third column, we have \( x \geq 2 \), so \((v - 2)x < 4\) implies \( v \leq 3 \). Then \( t \leq 3 \) and \( u \leq 8 \) which produces less cases and has no solution.

Thus the number of 2-friezes of height 3 is finite, and we know a bound on the values appearing in such a 2-frieze. A computer check then allows to show that there are only 868 such 2-friezes. Moreover, by [8], the number of friezes of type \( E_6 \) is at most the number of 2-friezes of height 3. Since we know from [9, 8, 6] that this number is at least 868, we have thus proved that the number is exactly 868. This finishes the proof of Theorem 1.
References


Michael Cuntz
Institut f"{u}r Algebra, Zahlentheorie und Diskrete Mathematik, Fakult"{a}t f"{u}r Mathematik und Physik, Gottfried Wilhelm Leibniz Universit"{a}t Hannover, Welfengarten 1, D-30167 Hannover, Germany
E-mail address: cuntz@math.uni-hannover.de

Pierre-Guy Plamondon
Laboratoire de Mathématiques d’Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France
E-mail address: pierre-guy.plamondon@math.u-psud.fr