COUNTING FRIEZES IN TYPE E_6

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Friezes (also called "friezes of type A" and "SL_2-friezes") were introduced by H. Coxeter in [4] and later studied by H. Coxeter and J. Conway in [2, 3]. It was observed by P. Caldero that the theory of cluster algebras of S. Fomin and A. Zelevinsky [5] allows for a far-reaching generalization of the original notion of frieze; this generalization was first studied in [1]. Since then, many generalizations and variations on the notion of friezes have been introduced, as can be seen in the survey paper [7].

It is known that, for a given non-Dynkin type, there are infinitely many friezes of that type. However, it follows from [2, 3] that friezes of Dynkin type A come in a finite number (given by Catalan numbers), and it was proved in [6] that friezes of Dynkin type B, D and G also come in a finite (explicit) number (the result for type D_4 was also proved in [8]). It was conjectured in [9, 8, 6] that the number of friezes in type E_6 is 868; in [6] the precise number of friezes in any Dynkin type is also conjectured.

In this Appendix, we settle the case of type E_6, and obtain the result for type F_4 as a corollary.

Theorem 1. *The number of friezes of type E_6 is exactly* 868*.*

Since Dynkin type F_4 is a folding of type E_6, it follows from the work of [6] that we then have:

Corollary 2. *The number of friezes of type F_4 is exactly* 112*.*

Our proof relies on a reduction to 2-friezes (whose definition we recall below); our strategy is to show that the entries in a 2-frieze of height 3 are bounded.

We have attempted to apply the methods used in this Appendix to types E_7 and E_8, without success.

1. 2-friezes

We shall not be using the definition of a frieze from [1], but rather a slightly different notion, that of a 2-frieze as defined in [8].

Definition 3. *A* 2-frieze of height h *is an array of positive integers $(a_{i,j})$, where*

- *$i \in \mathbb{Z}$ and $j \in \{0, 1, \dots, h+1\}$;*
- *for all $i \in \mathbb{Z}$, we have that $a_{i,0} = a_{i,h+1} = 1$;*
- *for all $i \in \mathbb{Z}$ and all $j \in \{1, \dots, h\}$, we have that $a_{i,j} = a_{i-1,j}a_{i+1,j} - a_{i,j-1}a_{i,j+1}$.*

The reason we are interested in 2-friezes is the following result.

Theorem 4 ([8])**.** *Any frieze of type E_6 determines a unique 2-frieze of height* 3*.*

Our strategy to prove Theorem 1 is thus to prove that the number of 2-friezes of height 3 is finite. We will do this by showing that there is a bound on the possible values appearing in a 2-frieze of height 3.

2. Two choices of initial variables for 2-friezes of height 3

If we fix the entries of the first two rows of a 2-frieze of height 3 to be s, t, u, v, w and x, then we get the following expressions for all its entries:

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$$\begin{array}{ccccc}
1 & s & t & u & 1 \\
1 & v & w & x & 1 \\
1 & \frac{v+w}{s} & \frac{vx+w}{t} & \frac{w+x}{u} & 1 \\
1 & \frac{svx+sw+tv+tw}{stv} & X & \frac{tw+tx+uvx+uw}{tux} & 1 \\
1 & \frac{suvx+suw+tuw+tvw+tvx}{tuvw} & Y & \frac{stw+suvx+suw+tvx+twx}{stwx} & 1 \\
1 & \frac{sux+tw+tx+uw}{uwx} & Z & \frac{suv+sw+tv+tw}{svw} & 1 \\
1 & \frac{t+u}{x} & \frac{su+t}{w} & \frac{s+t}{v} & 1 \\
1 & u & t & s & 1 \\
1 & x & w & v & 1
\end{array}$$

where

$$X = \frac{suvx + suw + tvw + tvx + tw^2 + twx}{stuw},$$

$$Y = \frac{stuvx + stw^2 + stwx + suvwx + suw^2 + t^2vw + t^2vx + t^2w^2 + t^2wx + tuvw + tuw^2}{stuvwx}, \quad \text{and}$$

$$Z = \frac{stw + suvx + suw + t^2w + tuw + tvx}{tvwx}.$$

If, instead, we fix the six first entries of the leftmost non-trivial column, then we get:

$$\begin{array}{ccccc}
1 & s & \frac{sux-s-tx+t}{tvx-tw-ux+1} & B & 1 \\
1 & t & su-t & \frac{suw-sv+tv-tw-u+1}{tvx-tw-ux+1} & 1 \\
1 & u & tv-u & suw-sv-tw+1 & 1 \\
1 & v & uw-v & tvx-tw-ux+1 & 1 \\
1 & w & vx-w & \frac{tvx-tw+uw-ux-v+1}{suw-sv-tw+1} & 1 \\
1 & x & \frac{svw-sw+w-x}{suw-sv-tw+1} & C & 1 \\
1 & \frac{sux-s-tx+1}{suw-sv-tw+1} & A & \frac{svx-sw-x+1}{tvx-tw-ux+1} & 1 \\
1 & B & \frac{sux-s-tx+t}{tvx-tw-ux+1} & s & 1 \\
1 & \frac{suw-sv+tv-tw-u+1}{tvx-tw-ux+1} & su-t & t & 1
\end{array}$$

where

$$A = \frac{stvx - stw + suwx - sux - svx + s - twx + tw + x - 1}{stuvwx - stuw^2 - stv^2x + stvw - su^2wx + suvx + suw - sv - t^2vwx + t^2w^2 + tuwx + tvx - 2tw - ux + 1},$$

$$B = \frac{stuvx - stuw - sr^2x + suw + su - sv - t^2vx + t^2w + tux + tv - tw - t - u + 1}{stuvwx - stuw^2 - stv^2x + stvw - su^2wx + suvx + suw - sv - t^2vwx + t^2w^2 + tuwx + tvx - 2tw - ux + 1},$$
$$C = \frac{suvwx - suw^2 - sv - 2x + svw - tvwx + tvx + tw^2 - tw + uw - ux + vx - v - w + 1}{stuvwx - stuw^2 - stv^2x + stvw - su^2wx + suvx + suw - sv - t^2vwx + t^2w^2 + tuwx + tvx - 2tw - ux + 1}.$$

3. Finiteness of the number of 2-friezes of height 3

We use the second choice of initial variables of the previous section. We can assume, without loss of generality, that the greatest entry in the first and third columns is u. Then

$$\begin{aligned} u &\geq suw - sv - tw + 1 \\ &\geq suw - su - wu + 1 \text{ (since } u \geq t, v) \\ &= u(s-1)(w-1) - u + 1. \end{aligned}$$

Therefore

$$\begin{aligned} 1 &\geq (s-1)(w-1) - 1 + \frac{1}{u} \\ &> (s-1)(w-1) - 1. \end{aligned}$$

Hence $(s-1)(w-1) < 2$. This implies that $s = 1$ or $w = 1$ or $s = w = 2$.

3.1. **The case $s = 1$ or $w = 1$.** If $s = 1$ or $w = 1$, then the associated frieze of type E_6 contains a frieze of type D_5, and it is known [6] that there are only 187 of these. Thus there is a finite number of cases where $s = 1$ or $w = 1$.

3.2. **The case $s = w = 2$.** If $s = w = 2$, then consider the following inequalities:

$$\begin{aligned} u &\geq suw - sv - tw + 1 \\ &= 4u - 2t - 2v + 1 \end{aligned}$$

and $u \geq v$, which implies that $3u \geq 4u - 2t + 1$, so

$$u \leq 2t - 1.$$

But together with $u \geq tvx - tw - ux + 1 = tvx - 2t - ux + 1$ this yields

$$(2t-1)(x+1) \geq tvx - 2t + 1,$$

and hence

$$4t \geq tvx + 2 - 2tx + x = x + 2 + (v-2)tx > (v-2)tx.$$

Thus we obtain $(v-2)x < 4$ which gives $v \leq 6$. For symmetry reasons, the same argument produces $t \leq 6$. But then $u \leq 35$ since $tv - u \geq 1$. Hence we have reduced the problem to a finite number of cases. In fact, an easy computation shows that the only solution is

$$(s, t, u, v, w, x) = (2, 4, 5, 4, 2, 1)$$

in which case the (transposed) 2-frieze is:

1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	4	5	4	2	1	1	2	4	5	4	2	1
1	2	6	11	6	2	1	1	2	6	11	6	2	1
1	2	4	5	4	2	1	1	2	4	5	4	2	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1

Alternatively, assuming that there is no entry 1 in the first and third column, we have $x \geq 2$, so $(v-2)x < 4$ implies $v \leq 3$. Then $t \leq 3$ and $u \leq 8$ which produces less cases and has no solution.

Thus the number of 2-friezes of height 3 is finite, and we know a bound on the values appearing in such a 2-frieze. A computer check then allows to show that there are only 868 such 2-friezes. Moreover, by [8], the number of friezes of type E_6 is at most the number of 2-friezes of height 3. Since we know from [9, 8, 6] that this number is at least 868, we have thus proved that the number is exactly 868. This finishes the proof of Theorem 1.

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