INDICES AND GENERIC BASES FOR CLUSTER ALGEBRAS

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In their paper [9] published in 2002 , S. Fomin and A. Zelevinsky laid the foundations of the theory of cluster algebras. One of their aims was to give a combinatorial approach to the study of Kashiwara/Lusztig's canonical bases in quantum groups. For this reason, the search for suitable bases of cluster algebras is an important problem in the theory. The aim of this extended abstract is to put forward a method, using the additive categorification of cluster algebras by means of 2-Calabi–Yau triangulated categories (see [1], [13]), which will hopefully lead to the construction of such bases. This method is based on that of C. Geiss, B. Leclerc and J. Schröer [11], who obtained "generic" bases for a large class of cluster algebras. Previously, such bases were constructed explicitly by G. Dupont [7], see also the work of M. Ding, J. Xiao and F. Xu [6], for cluster algebras associated with affine quivers.

1. Setting : 2-Calabi–Yau triangulated categories

We will work over the field $\mathbb{C}$ of complex numbers. Let $\mathcal{C}$ be a triangulated $\mathbb{C}$-category with suspension functor Σ. We will make the following assumptions :

- $\mathcal{C}$ is Hom-finite (*i.e.* all morphism spaces are finite-dimensional);
- $\mathcal{C}$ is 2-Calabi–Yau (*i.e.* for any objects X and Y of $\mathcal{C}$, there exists a bifunctorial isomorphism $\operatorname{Hom}_{\mathcal{C}}(X,Y) \cong D\operatorname{Hom}_{\mathcal{C}}(Y,\Sigma^2 X)$, where $D = \operatorname{Hom}_{\mathbb{C}}(?,\mathbb{C})$ is the standard duality);
- $\mathcal{C}$ admits a cluster-tilting object T (*i.e.* for any object X of $\mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(T,\Sigma X)$ vanishes iff X is in $\operatorname{add} T$) which is non-degenarate (*i.e.* if T' is obtained from T by iterated mutations, then the Gabriel quiver of $\operatorname{End}_{\mathcal{C}}(T')$ has no oriented cycles of length ≤ 2).

Example 1.1. *C. Amiot's generalized cluster category* [1] *associated to a quiver with potential* [5] *satisfies the above assumptions, provided that the quiver with potential is Jacobi-finite.*

Example 1.2. *Let Q be a finite quiver without oriented cycles. To any element w of the Weyl group associated to Q, A. Buan, O. Iyama, I. Reiten and J. Scott* [3] *(see also* [10]*) associated a category $\underline{\mathcal{C}}_w$ which satisfies the above assumptions. To any reduced expression $\mathbf{i}$ of w, they associated a cluster-tilting object $V_{\mathbf{i}}$ of $\underline{\mathcal{C}}_w$.*

2. Generic basis of C. Geiss, B. Leclerc and J. Schröer

For any finite-dimensional $\mathbb{C}$-algebra A with Gabriel quiver Q, denote by $\operatorname{rep}_{\mathbf{d}}(A)$ the variety of finite-dimensional representations of A of dimension vector $\mathbf{d}$; it is a closed subvariety of $\bigoplus_{\alpha:i\to j} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{d_i},\mathbb{C}^{d_j})$. Let $\operatorname{rep}(A)$ be the union of all the $\operatorname{rep}_{\mathbf{d}}(A)$. For any irreducible component $\mathcal{Z}$ of $\operatorname{rep}(A)$, we let $\underline{\dim}\,\mathcal{Z} = \mathbf{d}$ if $\mathcal{Z}$ is contained in $\operatorname{rep}_{\mathbf{d}}(A)$.

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A significant result in the search for a good basis of cluster algebras is the following theorem of C. Geiss, B. Leclerc and J. Schröer, which requires the definition [11, Section 7.1] of strongly reduced components of $\operatorname{rep}(A)$.

Theorem 2.1 (Theorem 5 of [11])**.** *Let $\underline{\mathcal{C}}_w$ be as in the setting of Example 1.2, with a cluster-tilting object $V_{\mathbf{i}}$. Let $A = \operatorname{End}_{\underline{\mathcal{C}}_w}(V_{\mathbf{i}})$, and let Q be the Gabriel quiver of A. Then a basis of the cluster algebra $\mathcal{A}_Q$ is given by the set*

$$\left\{ \mathbf{x}^{\mathbf{e}} \psi_{\mathcal{Z}} \,\middle|\, \mathcal{Z} \text{ is strongly reduced}, \mathbf{e} \in \mathbb{N}^{Q_0}, \mathbf{e} \text{ and } \underline{\dim}\, \mathcal{Z} \text{ have disjoint support} \right\},$$

where $\psi_{\mathcal{Z}}$ is the generic value taken by Y. Palu's cluster character [13] *inside $\mathcal{Z}$.*

Remarks 2.2.
1. *The generic basis obtained is dual to the semicanonical basis of G. Lusztig* [12].
2. *The authors of* [11] *conjecture that the set defined in Theorem 2.1 is a basis of the associated cluster algebra, even outside of the setting of Example 1.2.*

3. Results

Let $\mathcal{C}$ be a category which satisfies the assumptions of section 1. Let T be a cluster-tilting object of $\mathcal{C}$, let $A = \operatorname{End}_{\mathcal{C}}(T)$ and let Q be the Gabriel quiver of A. Denote by $K_0(\operatorname{proj} A)$ the Grothendieck group of the category of finite-dimensional projective A-modules ; its elements are called *indices*, as in [13].

Theorem 3.1. *There exists a canonical map $I : K_0(\operatorname{proj} A) \to \mathcal{A}_Q^+$, where $\mathcal{A}_Q^+$ is the upper cluster algebra of* [2]*. In the setting of Example 1.2, the image of I is the generic basis of* [11].

Remarks 3.2.
1. *Let T_1 and T_0 be objects of $\operatorname{add} T$; then $P_i = \operatorname{Hom}_{\mathcal{C}}(T, T_i)$ is in $\operatorname{add} A$, for $i = 1, 2$. In that case, I sends the index $[P_0] - [P_1]$ to the generic value taken by Y. Palu's cluster character on the cones of morphisms in $\operatorname{Hom}_{\mathcal{C}}(T_1, T_0)$.*
2. *The canonical decomposition of H. Derksen and J. Fei* [4] *of a morphism in $\operatorname{Hom}_A(P_1, P_0)$ yields a factorization of $I([P_0] - [P_1])$.*

Theorem 3.3. *The map I commutes with mutations.*

Remark 3.4. *This theorem links results of* [11] *to a conjecture of V. Fock and A. Goncharov* [8, Conjecture 4.1]. *Together with the results of G. Lusztig and of C. Geiss, B. Leclerc and J. Schröer, the theorem allows to prove this conjecture in the setting of Example 1.2.*

Theorem 3.5. *There is a commutative diagram*

$$\begin{array}{ccc} \{\textit{strongly reduced comp. of } \operatorname{rep}(A)\} & \hookrightarrow & K_0(\operatorname{proj} A) \\ & \searrow^{\psi} & \downarrow{\scriptstyle I} \\ & & \mathcal{A}_Q^+, \end{array}$$

where ψ is the map described in Theorem 2.1.

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