$(X_1, \dots, X_n)$ iid with pdf $f(x) = \begin{cases} \frac{1}{\theta} & \text{if } 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$

parameter $\theta > 0$.

① Each X_i has the same distribution (i.d. = identically distributed)

$$E(X_i) = \int_0^\theta \frac{1}{\theta} x \, dx = \frac{1}{\theta} \left[\frac{x^2}{2}\right]_0^\theta = \frac{\theta}{2}$$

$$Var(X_i) = E(X_i^2) - (EX_i)^2 = \int_0^\theta \frac{1}{\theta} x^2 dx - \left(\frac{\theta}{2}\right)^2 = \frac{\theta^2}{12}$$

② We can express θ in function of $E(X_i)$, the 1st moment of X_i:
$\theta = 2E(X_i)$. The method of moments replaces the theoretical moment $E(X_i)$ with the sample moment $\overline{X} = \frac{1}{n} \sum_{i=1}^n X_i$.

The method of moments estimate of θ is $\boxed{\hat{\theta} = 2\overline{X}}$

③ • $Bias(\hat{\theta}) = E(\hat{\theta}) - \theta$

$$E(\hat{\theta}) = 2E(\overline{X}) = 2E\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \underset{\text{linearity of the expectation}}{=} \frac{2}{n} \sum_{i=1}^n E(X_i)$$

$$\underset{X_i \text{ same i.d.}}{=} \frac{2}{n} \times n \frac{\theta}{2} = \theta$$

$\Leftrightarrow Bias(\hat{\theta}) = 0$, $\hat{\theta}$ is unbiased.

• $$Var(\hat{\theta}) = Var(2\overline{X}) = 4\, Var\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{4}{n^2} Var\left(\sum_{i=1}^n X_i\right)$$

$$\underset{X_i \text{ same independent}}{=} \frac{4}{n^2} \sum_{i=1}^n Var(X_i) \underset{\text{i.d.}}{=} \frac{4}{n^2} \times n \frac{\theta^2}{12} = \frac{1}{3} \frac{\theta^2}{n}$$

$$s.e.(\hat{\theta}) = \frac{\theta}{\sqrt{3n}}.$$

• We use the bias-variance decomposition of the MSE.
Since $\hat{\theta}$ is unbiased:

$$MSE(\hat{\theta}) = Bias^2(\hat{\theta}) + Var(\hat{\theta}) = Var(\hat{\theta}) = \frac{\theta^2}{3n}.$$

$MSE \longrightarrow 0$ as $n \longrightarrow +\infty$

$\Rightarrow \hat{\theta}$ is a consistent estimator of θ