Questions to practice before Thurday's evaluation test

Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample generated by a one-parameter probability distribution P_θ. θ is the parameter of interest. Notation : $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$.

1. Find the bias, standard error and MSE of the estimator $\widehat{\theta}$ in the following cases : (*hint : for each parametric model, first calculates or finds in wikipedia or in a probability book the theoretical moments of the probability distribution)*

   - $P_\theta = \text{Bernoulli}(\theta)$, $\theta \in [0,1]$; $\widehat{\theta} = \overline{X}$.
   - $P_\theta = \text{Poisson}(\theta)$, $\theta > 0$; $\widehat{\theta} = \overline{X}$.
   - $P_\theta = \text{Uniform}(0,\theta)$, $\theta > 0$; $\mathit{pdf}(x) = \frac{1}{\theta}$ if $0 \le x \le \theta$, $\mathit{pdf}(x) = 0$ otherwise.
   - $P_\theta = \text{Normal } \mathcal{N}(\theta,1)$, $\theta \in \mathbb{R}$; $\widehat{\theta} = \overline{X}$.
   - $P_\theta = \text{Exponential}(\theta)$, $\theta > 0$; $\mathit{pdf}(x) = \frac{1}{\theta}\exp(-x/\theta)$ if $x \ge 0$, $\mathit{pdf}(x) = 0$ otherwise ; $\widehat{\theta} = \overline{X}$.
   - $P_\theta = \text{Normal } \mathcal{N}(0,\theta)$, $\theta > 0$; $\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

2. Find the method of moments estimator of θ in the following cases :

   - $P_\theta(X_i = 1) = \theta = 1 - P_\theta(X_i = -1)$, $\theta \in [0,1]$.
   - $P_\theta = \text{Geometric}(\theta)$, $\theta > 0$; $P_\theta(X_i = k) = \theta(1-\theta)^{k-1}, \ldots, k = 1, 2, 3, \ldots$
   - $P_\theta = \text{Normal } \mathcal{N}(\mu, \sigma^2)$, $\mu \in \mathbb{R}$,$\sigma^2 > 0$, $\theta = \sigma^2 = \text{Var}(X_i)$.
   - $P_\theta = \text{Gamma}(\alpha,\beta)$, $\theta = (\alpha,\beta)$, $\mathit{pdf}(x) = \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$, $x > 0$, $\mathbb{E}(X_i) = \alpha\beta$, $\text{Var}(X_i) = \alpha\beta^2$.

3. Let $X \sim \text{Exponential}(\theta)$. Find the cdf F of X and its quantile of order q, $q \in [0,1]$, as a function of θ.