Questions to practice before the third test

Let $X_1, X_2, \dots, X_{16}$ be an i.i.d. sample of size $n = 16$ generated by a one-parameter probability distribution P_θ. θ is the parameter of interest. Notation : $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

1. Find a 95% confidence interval for θ in the following models (*justify* the formula) :

   (a) $P_\theta = \text{Bernoulli}(\theta)$, $\theta \in [0,1]$; $\widehat{\theta} = \overline{X}$; observed value of $\overline{X} = 0.72$.

   (b) $P_\theta = \text{Poisson}(\theta)$, $\theta = \mathbb{E}(X_i) > 0$; $\widehat{\theta} = \overline{X}$; observed value of $\overline{X} = 2.5$

   (c) $P_\theta = \text{Uniform}(0,\theta)$, $\theta > 0$; $\widehat{\theta} = 2\overline{X}$; observed value of $\overline{X} = 1.25$

   (d) $P_\theta = \text{Normal } \mathcal{N}(\theta, 1)$, $\theta \in \mathbb{R}$; $\widehat{\theta} = \overline{X}$; observed value of $\overline{X} = 2.5$.

   (e) $P_\theta = \text{Exponential}(\theta)$, $\theta = \mathbb{E}(X_i) > 0$; $\widehat{\theta} = \overline{X}$; observed value of $\overline{X} = 2.5$.

   (f) $P_\theta = \text{Normal } \mathcal{N}(0, \theta)$, $\theta > 0$; $\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$; observed value of $\sum_{i=1}^{n} X_i^2 = 27.1$.

   (g) $P_\theta = \text{Normal } \mathcal{N}(\theta, \sigma^2)$, $\theta \in \mathbb{R}, \sigma^2 > 0$; $\widehat{\theta} = \overline{X}$; observed value of $\overline{X} = 2.5$, observed value of $\sum_{i=1}^{n} X_i^2 = 9.7$.

2. Give the (pivotal) test statistic for testing $H_0 : \theta = 2$ versus $H_1 : \theta > 2$ in the models of Question 1.(b)(d)(e)(g).

   Compute the p-value of the tests.

   What do you change if $H_1 : \theta < 2$ or $H_1 : \theta \neq 2$?

3. Suppose that $P_\theta = \mathcal{N}(\theta, 1)$. Consider testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. Let the rejection region be $\mathcal{R} = \{\overline{X} > c\}$

   - Show that the critical value c of the test of level 5% is $c = 0.41$.
   - Show that the type II error of the test of level 5% is 0.9%.

4. Suppose that P_θ is the Bernoulli distribution.

   Consider the test that rejects $H_0 : \theta = 0.5$ in favor of $H_1 : \theta = 0.6$ if $\{(\overline{X} - 1/2) > 1/10\}$.

   What is the (approximate) level of this test ? (*Answer : 21%*)

   What is the type II error value ? (*Answer : 50%*)