LARGE DEVIATIONS FOR THREE DIMENSIONAL SUPERCRITICAL PERCOLATION

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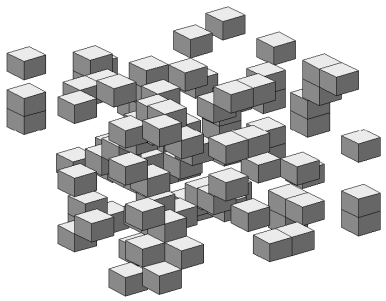
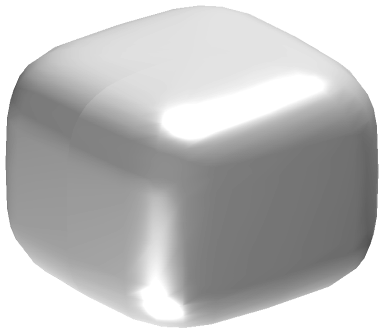
Abstract. We consider Bernoulli bond percolation on the three dimensional cubic lattice in the supercritical regime. We prove a large deviation principle for the rescaled configuration, from which a picture of the Wulff crystal of the model emerges.

Résumé (Grandes déviations pour la percolation supercritique en dimension trois). Nous considérons la percolation Bernoulli sur les arêtes du réseau cubique de dimension trois dans le régime supercritique. Nous prouvons un principe de grande déviation pour la configuration renormalisée, duquel émerge une image du cristal de Wulff du modèle.





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CHAPTER 1

INTRODUCTION

The ground–breaking monograph of Dobrushin, Kotecký and Shlosman [31] has initiated in the past years an intense research activity around the study of phase separation for the two dimensional Ising model (see [21,46,47,48,49,58,59,66,67,68]). The so–called Wulff construction in this context is now fairly well understood. The next challenge is to analyze phase separation and coexistence in higher dimensions. The aim of this work is to propose a way to do that for the three dimensional Bernoulli percolation model.

We consider Bernoulli bond percolation on the three dimensional lattice $\mathbb{Z}^3$. Edges between nearest neighbours are independently open with probability p and closed with probability $1-p$. It is known that this model has a phase transition at a value p_c strictly between 0 and 1: for $p<p_c$ the open clusters are finite and for $p>p_c$ there exists a unique infinite open cluster C_∞ (see [42]). We focus here on the supercritical regime where $p>p_c$. Aside from the infinite cluster, the configuration contains finite clusters of arbitrary large sizes. We wish to understand the geometry of these large clusters. The presence at a particular location of a large finite cluster is an event of low probability: for Bernoulli percolation in dimension d, for $p>p_c$, there exist two positive constants c_1, c_2 such that

$$\forall n\in\mathbb{N}\quad \exp(-c_1 n^{(d-1)/d})\leq P(n\leq \operatorname{card} C(0)<\infty)\leq \exp(-c_2 n^{(d-1)/d})\,,$$

where $C(0)$ is the open cluster containing the origin. This is a result from Kesten and Zhang [50] (It was proved under the assumption that p is strictly larger than the limit of the slab critical points. Grimmett and Marstrand proved that this limit coincides with p_c, see [43] or the second edition of [42]). An historic account of the successive refinements of this type of bounds is given in [3]. This estimate is based on the fact that the occurrence of a large finite cluster is due to a surface effect. Indeed at the frontier of a large open cluster C, there is a set $\partial_{\mathrm{e}} C$ of closed edges, called the edge boundary of C, whose macroscopic components look like a large surface separating the sites of the cluster from the outside world.

Alexander, Chayes and Chayes have obtained much more precise results in the two dimensional case [3] (which were further refined by Alexander [2]). Let us sum up the main points of their work. In dimension two, a component of the edge boundary is a closed curve of the plane. The most likely curves to realize the event

$\{\, n \leq \operatorname{card} C(0) < \infty \,\}$ are close to a very specific deterministic curve, namely the boundary of the so–called Wulff crystal of the model. More precisely, it is possible to define an angle dependent surface tension $\sigma(\nu)$ for the model, which characterizes the exponential decay of the probability of having a long flat interface of closed edges orthogonal to the direction ν. Consider next the variational problem of minimizing the σ–surface energy of a closed curve γ (that is the linear integral of $\sigma(\nu)$ along γ, where ν is the normal to γ) under the constraint that the curve γ encloses an area at least one. The unique solution of this problem is the boundary of a suitable dilation of the Wulff crystal of σ

$$\mathcal{W}_\sigma \;=\; \{\, x \in \mathbb{R}^2 : x \cdot \nu \leq \sigma(\nu) \text{ for all unit vectors } \nu \,\}\,.$$

Besides, the probability that the cluster $C(0)$ has inside the macroscopic components of $\partial_{\mathrm{e}} C(0)$ a density larger than the typical density $\theta(p)$ of the infinite cluster is of order $\exp(-\operatorname{const}\operatorname{card} C(0))$, because this event requires a volume effect. Hence, up to surface order large deviation events, the area enclosed by the macroscopic components of $\partial_{\mathrm{e}} C(0)$ and $\theta^{-1}\operatorname{card} C(0)$ are comparable. Finally,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \ln P(n \leq \operatorname{card} C(0) < \infty) \;=\; -(\theta \operatorname{area} \partial\mathcal{W}_\sigma)^{-1/2} \int_{\partial\mathcal{W}_\sigma} \sigma(\nu_{\mathcal{W}_\sigma}(x))\, dx\,.$$

Furthermore the solutions of the previous variational problem are stable with respect to the Hausdorff distance between curves, that is, up to translations, any minimizing sequence converges towards the boundary of a suitable dilated Wulff crystal. As a consequence, conditionally on the event $\{\, n \leq \operatorname{card} C(0) < \infty \,\}$, with probability tending rapidly to 1 as n goes to ∞ (say faster than any inverse power of n), the Hausdorff distance between the rescaled curve $n^{-1/2}$(outer component of $\partial_{\mathrm{e}} C(0)$) and the boundary of the dilated Wulff crystal $(\theta \operatorname{area} \partial\mathcal{W}_\sigma)^{-1/2}\mathcal{W}_\sigma$ is less than any fixed positive real number. Alexander, Chayes and Chayes prove also a single droplet Theorem, which is close in spirit to the Wulff construction Theorem of Dobrushin, Kotecký and Shlosman [31] for the two dimensional Ising model: for $\lambda < (\operatorname{diam} \mathcal{W}_\sigma)^{-2} \operatorname{area} \partial\mathcal{W}_\sigma$, for any positive η, conditionally on the event

$$\{\, \operatorname{card} C_\infty \cap [-L/2, L/2]^2 \leq (1-\lambda)\theta L^2 \,\}\,,$$

(C_∞ is the infinite cluster), with probability tending rapidly to one as L goes to ∞, there is inside $[-L/2, L/2]^2$ a finite open cluster of cardinality approximatively $\theta\lambda L^2$ whose edge boundary is at a Hausdorff distance less than ηL from $\lambda^{1/2} L\, (\operatorname{area} \partial\mathcal{W}_\sigma)^{-1/2} \partial\mathcal{W}_\sigma$.

Our original motivation was to prove similar results in the three dimensional case. As noted in [3,31], a new formulation of the results themselves is required in higher dimension: indeed the Hausdorff distance between the boundaries is not adequate any more, because very thin and long filaments create insignificant surface energy while increasing dramatically the Hausdorff distance. However the main obstacle so far seems to have been the extension of the skeleton coarse graining technique [49] which relies on the possibility of approximating in a suitable way a polygonal line on the lattice by a coarser one and on a combinatorial bound for the number of polygonal lines. The structure of two dimensional surfaces is so rich compared to the one of curves that it seems hard to find a similar combinatorial argument in higher dimension (see the introduction of [49]). Therefore a new strategy is needed. A natural way is to leave the discrete setting and to try to work from the start into the continuum. The combinatorial argument should then be replaced by a compactness property. Thus we must embed our objects into a continuous topological space in which the level sets of the surface energy are compact. If the volume happened to be continuous, we would have in addition existence of solutions for our variational problems, a highly desirable feature. This picture has already a taste of large deviations theory: the surface energy should be a good rate function. Subsequently, why not seek for a large deviation principle (in this yet undefined ideal space) governed by the surface energy? The results concerning the Wulff crystal would then be natural consequences of the large deviation principle; the law of the random objects under a volume constraint would concentrate exponentially fast around the ones minimizing the surface energy with respect to this volume constraint. Indeed, whenever a large deviation principle holds, the random objects solve automatically the variational problems associated with the rate function. Thus it is very reasonable to think that the probabilistic results on the Wulff crystal (at least those dealing with rough estimates) might be included in a general large deviations setting. Large deviations theory itself does not provide the required probabilistic tools, yet it suggests efficient abstract guidelines to attack the problem.

Although there exists a substantial literature devoted to the study of stochastic geometry, we found no existing result on large deviations for general random sets. Thus we started by proving the simplest such result, namely the analog of the Cramér Theorem for random sets [17]. We then tested the feasibility of some aspects of the large deviations approach to the Wulff crystal in the case of two dimensional Bernoulli percolation [18]. There we prove large deviation principles for the finite cluster shape in the Hausdorff and L^1 metric, but with the help of the skeleton coarse graining technique, instead of working from the start in the continuum.

To achieve this appealing programme, we should first find the ideal continuous space to work in. It is clear that this space must contain the smooth surfaces Γ and that the surface energy $\mathcal{I}(\Gamma)$ of a smooth surface Γ has to be

$$\mathcal{I}(\Gamma) \;=\; \int_\Gamma \tau(\nu_\Gamma(x))\, d\mathcal{H}^2(x)$$

where $\nu_\Gamma(x)$ is the normal vector to Γ at x, $\tau(\nu)$ is the surface tension of the model in the direction ν and $\mathcal{H}^2$ is the two dimensional Hausdorff measure in $\mathbb{R}^3$. The question of extending surface energy functionals like the above one to more general non smooth surfaces and to define nice topologies on surfaces has been a long sought quest. The literature on the subject is very rich and it is still an active area of research (see [4,14,20] and the references therein). It is of course linked to problems about the existence and the regularity of minimal surfaces, like the Plateau problem (find the surface of least area among those bounded by a given curve). As far as we understand, the most satisfactory setting for having nice compactness properties is the theory of currents and the tools of Geometric Measure Theory (see the monumental book of Federer [35]). Besides, a general proof of the Wulff isoperimetric Theorem has been achieved by Taylor [69,70,71] in this framework. This Theorem (originally due to Wulff [75], followed by Dinghas [30]) states that, up to Lebesgue negligible sets, the Wulff crystal

$$\mathcal{W}_\tau = \{ x \in \mathbb{R}^3 : x \cdot w \leq \tau(w) \text{ for all unit vectors } w \}$$

is the only solution to the variational problem

$$\text{minimize } \mathcal{I}(E) = \int_{\partial E} \tau(\nu_E(x))\, d\mathcal{H}^2(x) \quad \text{with vol}\, E \geq \text{vol}\, \mathcal{W}_\tau\,.$$

Notice here that Dobrushin, Kotecký and Shlosman [31] already mentioned the setting of Geometric Measure Theory for generalizing their Theorem in higher dimension. Fortunately enough, the objects involved in Taylor's proof of the Wulff isoperimetric Theorem are a restricted class of currents which can be identified with the Caccioppoli sets, also called the sets of locally finite perimeter. This is a different geometric theory which was historically initiated by Caccioppoli [15,16] and subsequently developed by De Giorgi [24,25,26,27]. The goal of Caccioppoli was to build a general theory of integration for differential forms and to extend the classical Gauss–Green Theorem to sets whose boundary is not C^1. Independently, De Giorgi was seeking to generalize some isoperimetric problems, starting with the Gauss–Green Theorem. This theory lead among other things to the solution of the general Plateau problem in arbitrary dimension (see the book of Giusti [41] and the references therein). It is much more accessible to analysts than the general theory of currents. For instance the Wulff isoperimetric Theorem has been reworked and slightly generalized in this context by Fonseca and Müller [36,37,39]. Thus we choose to work within the theory of the Caccioppoli sets. This framework is the most natural one to analyze geometric interfaces between stable coexisting phases. As a matter of fact, the phenomenological theory of phase transitions is developed within this setting (see for instance [38] for a recent paper and some entries in the corresponding literature).

The fundamental idea of employing this geometric setting to analyze models of statistical mechanics has appeared for the first time in the context of the Ising model with Kac potentials, where Alberti, Bellettini, Cassandro, Presutti [5,7] used the BV setting (the equivalent functional formulation of the sets of finite perimeter) to study surface order large deviations. The subsequent work of Benois, Bodineau, Buttà, Presutti [9] contains the idea of transforming the configuration on the mesoscopic level in order to get estimates on the probability of the local presence of an interface (see the explanation after the central lemma in chapter 8). Finally, Benois, Bodineau and Presutti [10] provided a relevant exponential tightness argument, which shows that the random law of the system concentrates very fast near functions of bounded variations. Although we were not inspired by these works, the historical credit for introducing the appropriate geometric setting for studying microscopic statistical models should be given to these works on the Ising model with Kac potentials: the global philosophy of their approach is indeed quite robust and is similar to ours in several essential respects. However, these works lose the microscopic structure of the model: when the range of interactions is finite, the estimates are not precise and the correct surface tension factor is recovered only in the mean–field limit where the range of interactions tends to infinity and everything becomes isotropic.

After extensive discussions with Dmitry Ioffe, we believe that two obstacles hindered the completion of the full program in the Kac model (that is, deriving the macroscopic effects induced by a truly microscopic model). The first one is that the afore–mentioned works did not have an operational approach to define the surface tension on fixed finite interaction scales. The second one is that in Ising type models, it is necessary to control the dependence between events in distant regions (or equivalently the influence of boundary conditions). By working in Bernoulli percolation, the second obstacle is immediately lifted. And we succeeded here in dealing with the first obstacle. Namely, we are able to implement the coarse–graining results of Pisztora [60] in order to recover the exact direction–dependent surface tension factor in the probabilistic estimates for the local presence of an interface. It should be stressed that the true hard part of the probabilistic work is contained in the results of Pisztora.

Recently, the second obstacle has been overcome and the case of the nearest–neighbour Ising model has been successfully handled in two parallel works [13,19]. While [13] is limited to sufficiently low temperatures, the domain of temperatures covered by [19] is conjectured to be the whole segment of phase coexistence.

We would like also to mention some possible future directions of research. In the specific percolation model, one would like to analyze the geometry of the Wulff crystal and to understand whether it has facets or not throughout the supercritical regime (see [56] for the corresponding question in the Ising model). The topological stability of the Wulff crystal is also linked with the delicate questions of the sharp large deviations and the random fluctuations of the interfaces. Besides the Wulff crystal, there are numerous issues of the phenomenological theory of phase coexistence which one should try to analyze from a microscopic point of view (see the book of Visintin [72] and the references therein). Moreover it is possible to refine our large deviation principles, depending on the amount of information required. Caccioppoli partitions correspond to a very convenient level of precision to get the single droplet Theorem. Another very interesting topic is of course the dynamics. It is naturally expected that several reasonable choices of microscopic dynamics induce the macroscopic dynamics associated to the so–called motion by mean curvature. In his last years, De Giorgi became interested in the problem of the geometric evolution of Caccioppoli sets and he introduced the concept of minimal barriers. Another available tool is the method of the auxiliary function and the use of viscosity solutions (see [8] and the references therein).

We next describe more precisely the setting and our results. The chosen continuous space is the space of the Borel subsets of $\mathbb{R}^3$ endowed with the L^1 or L^1_{loc} topology. We first extract from the Bernoulli percolation model a direction dependent surface tension τ. For a unit vector ν, let A be a unit square orthogonal to ν, let $\mathrm{cyl}\, A$ be the cylinder $A+\mathbb{R}\nu$, then $\tau(\nu)$ is equal to the limit, as n goes to ∞, of the quantity

$$-\frac{1}{n^2}\ln P\begin{pmatrix}\text{inside } n\mathrm{cyl}\, A \text{ there exists a finite set of closed edges } E \text{ which cuts}\\ n\mathrm{cyl}\, A \text{ in at least two unbounded components and the edges of } E \text{ at}\\ \text{distance less than 5 from } \partial n\mathrm{cyl}\, A \text{ are at distance less than 5 from } nA\end{pmatrix}$$

This function τ satisfies an important functional inequality, namely the weak triangle inequality: for any triangle (ABC) in $\mathbb{R}^3$, if ν_A, ν_B, ν_C are the exterior normal unit vectors to the sides $[BC]$, $[AC]$, $[AB]$ in the plane containing (ABC), then

$$|BC|_2\tau(\nu_A) \,\leq\, |AC|_2\tau(\nu_B)+|AB|_2\tau(\nu_C)\,.$$

By a result of Aizenman, Chayes, Chayes, Fröhlich and Russo [1], τ is positive in the supercritical regime $p>p_c$. Furthermore it is continuous, invariant under the isometries which leave $\mathbb{Z}^3$ invariant. The Wulff crystal $\mathcal{W}_\tau$ associated to τ is convex, closed, bounded and contains the origin 0 in its interior. The surface energy of a Borel subset A of $\mathbb{R}^3$ is then defined as

$$\mathcal{I}(A)\,=\,\sup\,\Big\{\,\int_A \mathrm{div} f(x)\,dx : f\in C^1_0(\mathbb{R}^3,\mathcal{W}_\tau)\,\Big\}$$

where $C^1_0(\mathbb{R}^3,\mathcal{W}_\tau)$ is the set of the C^1 vector functions defined on $\mathbb{R}^3$ with values in $\mathcal{W}_\tau$ having compact support and div is the usual divergence operator. The surface energy $\mathcal{I}(A)$ is infinite unless A is a set of finite perimeter in the sense of Caccioppoli and De Giorgi. It is very natural that we end up with a quantity like the above one. Indeed Bernoulli percolation is a model for the microscopic flow of particles in a random medium and the surface energy $\mathcal{I}(A)$ is a quantitative measure of the induced macroscopic flow from A to its complement.

We shall first prove a large deviation principle for the rescaled cluster $C(0)$ containing the origin. For r positive, we denote by $\mathcal{V}(C(0),r)$ the set of the points of $\mathbb{R}^3$

whose distance to $C(0)$ is strictly less than r. If $f(n)$ is a function from $\mathbb{N}$ to $\mathbb{N}$ such that $n/f(n)^2$ goes to ∞ as n goes to ∞ and $f(n) \geq \kappa \ln n$ for all n, where κ is a sufficiently large constant, then, conditionally on the event $\{\operatorname{card} C(0) < \infty\}$, the sequence of random Borel sets $(n^{-1}\mathcal{V}(C(0), f(n)))_{n\in\mathbb{N}}$ satisfies a weak large deviation principle in $(\mathcal{B}(\mathbb{R}^3), L^1)$ with speed n^2 and rate function $\mathcal{I}$. Unfortunately, the functional $\mathcal{I}$ is not a good rate function when considered on $(\mathcal{B}(\mathbb{R}^3), L^1)$ (because $\mathbb{R}^3$ is not bounded) and we are not able to prove the full large deviation principle. However we propose an enhanced large deviation upper bound which is enough for our applications. Alternatively, we may work with the L^1_{loc} topology; the functional $\mathcal{I}$ is a good rate function on the space $(\mathcal{B}(\mathbb{R}^3), L^1_{\mathrm{loc}})$ and we get a large deviation principle with this weaker topology. Under the additional hypothesis that $f(n)/\ln n$ goes to ∞ as n goes to ∞, we prove that up to large deviations of order n^2, $\operatorname{card} C(0)$ and $\theta \operatorname{vol} \mathcal{V}(C(0), f(n))$ are of the same order (θ is the density of the infinite cluster). These results provide a description of the asymptotic shapes of the large finite clusters. Together with the Wulff isoperimetric Theorem, they imply that, for the three dimensional Bernoulli bond percolation model on the cubic lattice, for $p > p_c$,

$$\lim_{n\to\infty} \frac{1}{n^2} \ln P\big(n^3 \leq \operatorname{card} C(0) < \infty\big) \;=\; -\frac{\mathcal{I}(\mathcal{W}_\tau)}{(\theta \operatorname{vol} \mathcal{W}_\tau)^{2/3}}\,.$$

If $f(n)$ is a function from $\mathbb{N}$ to $\mathbb{N}$ such that both $n/f(n)^2$ and $f(n)/\ln n$ go to ∞ as n goes to ∞, the law of $n^{-1}\mathcal{V}(C(0), f(n))$ conditioned on the event $\{n^3 \leq \operatorname{card} C(0) < \infty\}$ concentrates exponentially fast around a shape which is a suitable dilation of the Wulff crystal of τ.

The key ingredients of the proof are the results of Pisztora [60] specialized to the case of percolation, which describe precisely the typical structure of the configuration on the intermediate mesoscopic scale $f(n)$. These results seem to have been designed intentionally to perform the type of coarse graining we need. Our philosophy is to do the coarse graining around the clusters rather than around the interfaces. The central lemma for the large deviation upper bound gives a uniform probabilistic estimate for the local presence of a collection of open clusters which create a small flat interface in the L^1 topology. The estimate is uniform with respect to the localization, the size and the direction of the interface. Hence the central lemma provides a link between the discrete microscopic structure of the model and the continuous macroscopic effects.

Our final aim is to prove a single droplet Theorem similar to [3]. Let $f(n)$ be a function from $\mathbb{N}$ to $\mathbb{N}$ such that $n/f(n)^2$ goes to ∞ as n goes to ∞ and $f(n) \geq \kappa \ln n$ for all n, where κ is a sufficiently large constant. We consider the collection of random sets

$$n^{-1}\mathcal{V}(\mathcal{C}, f(n)) = \Big\{\, n^{-1}\mathcal{V}(C, f(n)),\ C \text{ open cluster of diameter strictly larger than } f(n) \Big\}$$

which lives in the space $\mathrm{BC}(\mathbb{R}^3)$ of the countable collections of non negligible Borel subsets of $\mathbb{R}^3$. Starting with the topology L^1_{loc} on the space of the Borel subsets of $\mathbb{R}^3$, we build a metric D_{loc} on $\mathrm{BC}(\mathbb{R}^3)$. We prove that the sequence of random collections of Borel sets $(n^{-1}\mathcal{V}(\mathcal{C}, f(n)))_{n\in\mathbb{N}}$ satisfies a large deviation principle in $(\mathrm{BC}(\mathbb{R}^3), D_{\mathrm{loc}})$ with speed n^2 and rate function $\mathcal{I}$ defined by

$$\mathcal{I}(\mathcal{A}) \;=\; \frac{1}{2}\sum_{A\in\mathcal{A}} \mathcal{I}(A)$$

if the sets of the collection $\mathcal{A}$ form a partition of $\mathbb{R}^3$, and $\mathcal{I}(\mathcal{A}) = \infty$ otherwise. The collections of sets which form a partition of $\mathbb{R}^3$ and whose elements are Caccioppoli sets have been introduced by Congedo and Tamanini [22,23,53] under the name of Caccioppoli partitions to deal with image segmentation problems. It is rather natural to encounter questions common to percolation and image segmentation. Indeed our rate function $\mathcal{I}$ on the space of the collections of sets is a particular case of the famous Mumford–Shah functional [57].

Although the previous large deviation principle in the metric D_{loc} is enough to get a version of the single droplet Theorem inside a bounded set, in order to get a result which is closer in spirit to the single droplet Theorem of [3], that is involving the infinite cluster C_∞, we need to work with a stronger metric D. Roughly speaking, two collections of sets are at D distance less than δ if there exists a correspondence between the sets of the collections having volume larger than δ such that two corresponding sets are at L^1 distance less than δ. Let U be a bounded Borel subset of $\mathbb{R}^3$. We consider the random collection of open clusters $\mathcal{C}(nU, f(n))$ which is

$$\Big\{\, C \text{ finite open cluster of diameter strictly larger than } f(n) \text{ intersecting } nU \Big\}$$

and we define the random Borel collection $\mathcal{VC}(nU, f(n))$ to be

$$\mathcal{VC}(nU, f(n)) \;=\; \Big\{\, \mathcal{V}(C, f(n)),\, C \in \mathcal{C}(nU, f(n)) \Big\} \cup \Big\{\, \mathbb{R}^3 \setminus \bigcup_{C\in\mathcal{C}(nU,f(n))} \mathcal{V}(C, f(n)) \Big\}\,.$$

We prove that the sequence of random collections of Borel sets $n^{-1}\mathcal{VC}(nU, f(n))$ satisfies a weak large deviation principle in $(\mathrm{BC}(\mathbb{R}^3), D)$ with speed n^2 and rate function $\mathcal{I}$. Finally we obtain an enhanced large deviation upper bound similar in spirit to the enhanced upper bound for one cluster.

We state finally our single droplet Theorem. Let Λ be a cubic box in $\mathbb{R}^3$ such that $\mathrm{vol}\,\Lambda = 1$ and let λ be small enough so that some translate of the dilated Wulff crystal $(\lambda/\mathrm{vol}\,\mathcal{W}_\tau)^{1/3}\mathcal{W}_\tau$ is included in Λ. Then

$$\lim_{n\to\infty} \frac{1}{n^2} \ln P\Big(\mathrm{card}\, C_\infty \cap n\Lambda \le (1-\lambda)\theta n^3\Big) \;=\; -(\lambda/\mathrm{vol}\,\mathcal{W}_\tau)^{2/3}\, \mathcal{I}(\mathcal{W}_\tau)\,.$$

Let $f(n)$ be a function from $\mathbb{N}$ to $\mathbb{N}$ such that both $n/f(n)^2$ and $f(n)/\ln n$ go to ∞ as n goes to ∞. Conditionally on the above event, with probability going exponentially fast to 1 as n goes to ∞, there exists one finite open cluster C_0 and x in $\mathbb{R}^3$ such that $x+(\lambda/\mathrm{vol}\,\mathcal{W}_\tau)^{1/3}\mathcal{W}_\tau$ is included in Λ, $n^{-1}\mathcal{V}(C_0,f(n))$ is close to $x+(\lambda/\mathrm{vol}\,\mathcal{W}_\tau)^{1/3}\mathcal{W}_\tau$, $\Lambda\setminus n^{-1}\mathcal{V}(C_\infty,f(n))$ is close to $\Lambda\setminus(x+(\lambda/\mathrm{vol}\,\mathcal{W}_\tau)^{1/3}\mathcal{W}_\tau)$ and the total volume of the $f(n)$–neighbourhoods of the other clusters having diameter strictly larger than $f(n)$ and intersecting $n\Lambda$ is small.

The proof of the large deviation upper bound follows the same line as for the case of one cluster and it relies on the same central lemma. Here again compactness plays a crucial role; we rely on a compactness result for Caccioppoli partitions proved by Congedo and Tamanini [23]. The proof of the lower bound involves a delicate approximation of Caccioppoli partitions through polyhedral partitions, based on a result of Quentin de Gromard [61,62].

The large deviation principles are stated in chapter 2, together with their application to the Wulff crystal. Chapter 3 is an informal sketch of the proofs for the single cluster case. The notation and the model are introduced in chapter 4. In chapter 5 we build the surface tension τ and we study its main properties. In chapter 6 we sum up the important facts about the theory of Caccioppoli sets and the Wulff Isoperimetric Theorem and we build the surface energy $\mathcal{I}$ associated to the surface tension of our model. In chapter 7 we state the results of Pisztora (specialized to three dimensional Bernoulli percolation) as well as some general coarse–graining Lemmata. The central Lemma which is essential for proving the large deviation upper bound is the object of chapter 8. Chapter 9 is the proof of the large deviation principle for a single cluster. In chapter 10 we define a metric on the space of the collections of sets. We build then the surface energy for Caccioppoli partitions in chapter 11 and we state a Wulff Theorem for Caccioppoli partitions. Chapter 12 is the proof of the large deviation principles for the whole configuration. We include in the appendix the basic large deviations terminology which we use to state our results.

Acknowledgements. I thank Geoffrey Grimmett for suggesting me the problem and for his helpful advice.

CHAPTER 2

THE LARGE DEVIATION PRINCIPLES

This chapter contains the statements of the main results, that is, the large deviation principles and their application to the Wulff crystal. We consider first the case of one single cluster and then the case of the whole configuration. The proofs of the large deviation principles are deferred in the remaining chapters. The results concerning the Wulff crystal are mere consequences of the large deviation principles and the Wulff isoperimetric Theorem, hence we proved them here.

2.1. The finite cluster shape

We start by describing the topological setting of our first large deviation principle. We denote by $\mathcal{B}(\mathbb{R}^3)$ the Borel subsets of $\mathbb{R}^3$ and by $\mathcal{L}^3$ the three dimensional Lebesgue measure. When dealing with topological questions on the space $\mathcal{B}(\mathbb{R}^3)$, we identify Borel sets whose symmetric difference is Lebesgue negligible, so that the map $(A_1, A_2) \in \mathcal{B}(\mathbb{R}^3) \mapsto \mathcal{L}^3(A_1 \Delta A_2)$ is a metric on $\mathcal{B}(\mathbb{R}^3)$, which we call the L^1 metric.

We consider the Bernoulli bond percolation model on $\mathbb{Z}^3$ with parameter p, where p is a fixed value in the supercritical regime $p_c < p < 1$ (see chapter 4 for more precisions). In chapter 5, we extract from this model a direction dependent surface tension τ, which is a map from the unit sphere S^2 of $\mathbb{R}^3$ to $\mathbb{R}^+$. This function τ is positive (because $p > p_c$), continuous, invariant under the isometries which leave $\mathbb{Z}^3$ invariant, and satisfies the weak triangle inequality. We denote by $\mathcal{W}_\tau$ the Wulff set associated to the surface tension τ, called also the crystal of τ,

$$\mathcal{W}_\tau \,=\, \{\, x \in \mathbb{R}^3 : x \cdot w \leq \tau(w) \text{ for all } w \text{ in } S^2 \,\}\,.$$

Since τ is continuous and bounded away from 0, its crystal $\mathcal{W}_\tau$ is convex, closed, bounded and contains the origin 0 in its interior [37, Proposition 3.5]. The surface energy $\mathcal{I}(A)$ of a Borel set A is defined as

$$\mathcal{I}(A) \,=\, \sup \,\Big\{ \int_A \mathrm{div} f(x)\, d\mathcal{L}^3(x) : f \in C_0^1(\mathbb{R}^3, \mathcal{W}_\tau) \Big\}$$

where $C_0^1(\mathbb{R}^3, \mathcal{W}_\tau)$ is the set of the C^1 vector functions defined on $\mathbb{R}^3$ with values in $\mathcal{W}_\tau$ having compact support and div is the usual divergence operator, defined for a

C^1 vector function f with scalar components (f_1, f_2, f_3) as

$$\operatorname{div} f \;=\; \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\,.$$

As a consequence of the classical Gauss–Green Theorem, for a set A having a smooth boundary ∂A, the surface energy $\mathcal{I}(A)$ is the surface integral of τ on the boundary of A, that is

$$\mathcal{I}(A) \;=\; \int_{\partial A} \tau(\nu_A(x))\, d\mathcal{H}^2(x)$$

where $\nu_A(x)$ is the exterior normal vector to ∂A at x and $\mathcal{H}^2$ is the two dimensional Hausdorff measure in $\mathbb{R}^3$.

We denote by P the Bernoulli product measure on the set of edges between nearest neighbour sites of $\mathbb{Z}^3$ corresponding to the parameter p. Let $C(0)$ be the open cluster containing the origin. We set for l in $\mathbb{N}$

$$\mathcal{V}_\infty(C(0), l) \;=\; \{\, x \in \mathbb{R}^3 : \exists\, y \in C(0) \quad |x-y|_\infty < l \,\}$$

where $|\quad|_\infty$ is the standard supremum norm. We denote by $\widehat{P}$ the measure P conditioned on the event that $C(0)$ is finite i.e.

$$\widehat{P}(\cdot) \;=\; P(\cdot / \operatorname{card} C(0) < \infty)\,.$$

Let $\widehat{\mathcal{I}}$ be the map from $\mathcal{B}(\mathbb{R}^3)$ to $\mathbb{R}^+ \cup \{\infty\}$ defined by

$$\forall\, A \in \mathcal{B}(\mathbb{R}^3) \qquad \widehat{\mathcal{I}}(A) \;=\; \begin{cases} \mathcal{I}(A) & \text{if} \quad \mathcal{L}^3(A) < \infty \\ \infty & \text{if} \quad \mathcal{L}^3(A) = \infty \end{cases}$$

Theorem 2.1. *(The finite cluster shape, topology L^1)*
There exists a constant $\kappa = \kappa(p)$ depending on p such that if $f(n)$ is a function from $\mathbb{N}$ to $\mathbb{N}$ such that $n/f(n)^2$ goes to ∞ as n goes to ∞ and $f(n) \geq \kappa \ln n$ for all n, then, under $\widehat{P}$, the sequence of random Borel sets $(n^{-1}\mathcal{V}_\infty(C(0), f(n)))_{n\in\mathbb{N}}$ satisfies a weak large deviation principle in $(\mathcal{B}(\mathbb{R}^3), L^1)$ with speed n^2 and rate function $\widehat{\mathcal{I}}$: for any open subset $\mathcal{O}$ of $(\mathcal{B}(\mathbb{R}^3), L^1)$,

$$-\inf\{\, \widehat{\mathcal{I}}(A) : A \in \mathcal{O} \,\} \;\leq\; \liminf_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\big(n^{-1}\mathcal{V}_\infty(C(0), f(n)) \in \mathcal{O}\big)\,,$$

and, for any compact subset $\mathcal{K}$ of $(\mathcal{B}(\mathbb{R}^3), L^1)$,

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\big(n^{-1}\mathcal{V}_\infty(C(0), f(n)) \in \mathcal{K}\big) \;\leq\; -\inf\{\, \widehat{\mathcal{I}}(A) : A \in \mathcal{K} \,\}\,.$$

Remark. Although the random sets $\mathcal{V}_\infty(C(0), f(n))$, $n \in \mathbb{N}$, are connected, the definition of the rate function involves no connectedness condition, because we work up to Lebesgue negligible sets.

This first large deviation principle is weak because the upper bound holds only for compact sets. The usual way to go from a weak large deviation principle to a full large deviation principle (where the upper bound holds for any closed set) is to prove an exponential tightness result. We couldn't do that and we don't know whether the full large deviation principle holds. However we are able to get an enhanced upper bound.

Theorem 2.2. *(Enhanced upper bound for the finite cluster shape, topology L^1) Let $f(n)$ be a function satisfying the same hypothesis as in Theorem* 2.1. *For any closed subset $\mathcal{F}$ of $(\mathcal{B}(\mathbb{R}^3), L^1)$,*

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}(n^{-1}\mathcal{V}_\infty(C(0), f(n)) \in \mathcal{F}) \ \leq\ -\sup_{\delta>0}\ \inf \left\{ \widehat{\mathcal{I}}(A) : A \in \mathcal{V}_{L^1}(\mathcal{F},\delta) \right\}$$

where $\mathcal{V}_{L^1}(\mathcal{F},\delta) = \left\{ E \in \mathcal{B}(\mathbb{R}^3) : \exists\, A \in \mathcal{F}\ \ \mathcal{L}^3(A\Delta E) < \delta \right\}$ is the L^1 δ–neighbourhood of $\mathcal{F}$.

Was the rate function $\widehat{\mathcal{I}}$ a good rate function, the upper bound of Theorem 2.2 would imply the full large deviation principle. Unfortunately, the level sets of $\widehat{\mathcal{I}}$ are not compact and there do exist closed subsets $\mathcal{F}$ of $(\mathcal{B}(\mathbb{R}^3), L^1)$ such that

$$\sup_{\delta>0}\ \inf \left\{ \widehat{\mathcal{I}}(A) : A \in \mathcal{V}_{L^1}(\mathcal{F},\delta) \right\} \ <\ \inf \left\{ \widehat{\mathcal{I}}(A) : A \in \mathcal{F} \right\}.$$

We illustrate this is the simpler situation where τ is independent of the direction and $\mathcal{I}$ becomes the classical perimeter $\mathcal{P}$, which for a smooth set is the $\mathcal{H}^2$ measure of the boundary. Consider the family $\mathcal{F}$ consisting of the sets $A(n)$, $n \geq 1$, defined by

$$A(n) \ =\ B\big((0,0,n+1),1\big) \cup \left\{ (x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1/n^2,\ |x_3| \leq n \right\}.$$

The set $\mathcal{F}$ is closed with respect to the topology L^1. For each n, the perimeter of $A(n)$ is $\mathcal{H}^2(\partial A(n)) = 4\pi + 2\pi(2 + 1/n^2)$. Thus $\inf \left\{ \mathcal{P}(A) : A \in \mathcal{F} \right\} = 8\pi$. Yet for any δ positive, for n strictly larger than $2\pi/\delta$, the ball $B\big((0,0,n+1),1\big)$ belongs to $\mathcal{V}_{L^1}(\mathcal{F},\delta)$ whence $\inf \left\{ \mathcal{P}(A) : A \in \mathcal{V}_{L^1}(\mathcal{F},\delta) \right\} \leq 4\pi$ and

$$\sup_{\delta>0}\ \inf \left\{ \mathcal{P}(A) : A \in \mathcal{V}_{L^1}(\mathcal{F},\delta) \right\} \ \leq\ 4\pi \ <\ 8\pi = \inf \left\{ \mathcal{P}(A) : A \in \mathcal{F} \right\}.$$

The reason of this defect is that the topology L^1 on $\mathcal{B}(\mathbb{R}^3)$ is too strong and lacks compactness. It might therefore be more convenient to work with a weaker topology, namely the topology L^1_{loc}, which is defined next. For K a compact subset of $\mathbb{R}^3$, we denote by $\mathcal{L}^3_K$ the map from $\mathcal{B}(\mathbb{R}^3)$ to $\mathbb{R}^+$ which to a Borel set E associates the Lebesgue measure of $E \cap K$, i.e, $\mathcal{L}^3_K(E) = \mathcal{L}^3(E \cap K)$. We say that a sequence $(E_n)_{n\in\mathbb{N}}$ of Borel sets converges in L^1_{loc} towards a Borel set E if for any compact set K

$$\lim_{n\to\infty} \mathcal{L}^3_K(E_n \Delta E) \,=\, 0\,.$$

This notion of convergence corresponds to the topology on $\mathcal{B}(\mathbb{R}^3)$ generated by the following system of neighbourhoods: a fundamental system of neighbourhoods of a Borel set E is

$$V(E,K,\varepsilon) \,=\, \big\{\, F\in\mathcal{B}(\mathbb{R}^3) : \mathcal{L}^3_K(E\Delta F) < \varepsilon \,\big\}\,, \quad K \text{ compact}\,,\ \varepsilon>0\,.$$

We may construct a metric d_{loc} on $\mathcal{B}(\mathbb{R}^3)$ corresponding to the topology L^1_{loc} by using an increasing sequence of compact sets filling the whole space $\mathbb{R}^3$. Let us consider for instance the balls $(B(0,j), j\in\mathbb{N})$ and define $\psi(A)=\sum_{j\geq 1} j^{-5}\mathcal{L}^3(B(0,j)\cap A)$ for a Borel set A. The set function ψ is a finite measure on $\mathcal{B}(\mathbb{R}^3)$ which has the same null sets as the Lebesgue measure $\mathcal{L}^3$. We define next a metric d_{loc} by:

$$\forall\, A_1,A_2\in\mathcal{B}(\mathbb{R}^3) \qquad d_{\mathrm{loc}}(A_1,A_2) \,=\, \psi(A_1\Delta A_2)\,.$$

The topology generated by d_{loc} on $\mathcal{B}(\mathbb{R}^3)$ is the topology L^1_{loc}.

Theorem 2.3. *(The finite cluster shape, topology L^1_{loc})*
Let $f(n)$ be a function satisfying the same hypothesis as in Theorem 2.1. Under $\widehat{P}$, the sequence of random Borel sets $(n^{-1}\mathcal{V}_\infty(C(0),f(n)))_{n\in\mathbb{N}}$ satisfies a large deviation principle in $(\mathcal{B}(\mathbb{R}^3),L^1_{\mathrm{loc}})$ with speed n^2 and rate function $\widehat{\mathcal{I}}$: for any Borel set $\mathcal{E}$ included in $\mathcal{B}(\mathbb{R}^3)$,

$$\begin{aligned} -\inf\{\,\widehat{\mathcal{I}}(A) : A\in\operatorname{int}\mathcal{E}\,\} &\leq \liminf_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\in\mathcal{E}\big)\\ &\leq \limsup_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\in\mathcal{E}\big) \,\leq\, -\inf\{\,\widehat{\mathcal{I}}(A) : A\in\operatorname{clo}\mathcal{E}\,\}\end{aligned}$$

(where $\operatorname{int}\mathcal{E}$ and $\operatorname{clo}\mathcal{E}$ are the interior and the closure of $\mathcal{E}$ in $\mathcal{B}(\mathbb{R}^3)$ with respect to the topology L^1_{loc}).

If the full large deviation principle would hold in the topology L^1, an application of the contraction principle with the identity map from $(\mathcal{B}(\mathbb{R}^3),L^1)$ to $(\mathcal{B}(\mathbb{R}^3),L^1_{\mathrm{loc}})$ would yield immediately the large deviation principle in the topology L^1_{loc}. Yet the enhanced upper bound of Theorem 2.2 is enough to get the large deviation upper bound in the L^1_{loc} topology. More precisely, we have the following implications:

$$\begin{gathered} L^1 \text{ lower bound } \;\Rightarrow\; L^1_{\mathrm{loc}} \text{ lower bound}\,,\\ L^1 \text{ enhanced upper bound } \;\Rightarrow\; L^1_{\mathrm{loc}} \text{ upper bound } \;\Rightarrow\; L^1 \text{ weak upper bound}\,.\end{gathered}$$

We will prove the large deviation lower bound for the topology L^1. For the clarity of the exposition, we will prove first the large deviation upper bound for the topology

L^1_{loc} (at this stage the proof of Theorems 2.1, 2.3 will be completed) and we will finish with the proof of the enhanced upper bound of Theorem 2.2. The proof of the enhanced upper bound is more involved because the analysis is not confined to a bounded set: the cluster $C(0)$ might have a very large diameter, due to the presence of thin and long filaments having a small surface energy. Let us check the first implication of the second line, the other ones being more classical. Let $\mathcal{E}$ be a closed subset of $(\mathcal{B}(\mathbb{R}^3), L^1_{\text{loc}})$. We must show that

$$-\sup_{\delta>0}\ \inf\left\{\,\widehat{\mathcal{I}}(A) : A\in\mathcal{V}_{L^1}(\mathcal{E},\delta)\,\right\}\ \le\ -\inf\left\{\,\widehat{\mathcal{I}}(A) : A\in\mathcal{E}\,\right\}.$$

If the left–hand side is equal to $-\infty$, the inequality is certainly true. Otherwise, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of Borel sets such that $-\widehat{\mathcal{I}}(A_n)$ converges to the left–hand quantity as n goes to ∞ and A_n belongs to $\mathcal{V}_{L^1}(\mathcal{E},1/n)$ for all n in $\mathbb{N}$. In particular, we have

$$\lim_{n\to\infty}\ \inf\left\{\,d_{\text{loc}}(A_n,E) : E\in\mathcal{E}\,\right\}\ =\ 0\,.$$

Since the level sets of $\mathcal{I}$ on $(\mathcal{B}(\mathbb{R}^3), L^1_{\text{loc}})$ are compact, the sequence $(A_n)_{n\in\mathbb{N}}$ admits a subsequence converging towards a Borel set A' in L^1_{loc}. Necessarily, A' belongs to the L^1_{loc} closure of $\mathcal{E}$, which is $\mathcal{E}$ itself. Because $\widehat{\mathcal{I}}$ is lower semicontinuous, we obtain

$$\inf\left\{\,\widehat{\mathcal{I}}(A) : A\in\mathcal{E}\,\right\}\ \le\ \widehat{\mathcal{I}}(A')\ \le\ \lim_{n\to\infty}\widehat{\mathcal{I}}(A_n)\ =\ \sup_{\delta>0}\ \inf\left\{\,\widehat{\mathcal{I}}(A) : A\in\mathcal{V}_{L^1}(\mathcal{E},\delta)\,\right\}.$$

The next result links the cardinality of $C(0)$ to the volume of $\mathcal{V}_\infty(C(0),f(n))$.

Proposition 2.4. *Let $\theta = P(\operatorname{card} C(0)=\infty)$ be the density of the infinite cluster. Let $f(n)$ be a function from $\mathbb{N}$ to $\mathbb{N}$ such that both $n/f(n)^2$ and $f(n)/\ln n$ go to ∞ as n goes to ∞. Under $\widehat{P}$, the two sequences of positive random variables $(n^{-3}\operatorname{card} C(0))_{n\in\mathbb{N}}$ and $(\theta n^{-3}\mathcal{L}^3(\mathcal{V}_\infty(C(0),f(n))))_{n\in\mathbb{N}}$ are exponentially contiguous, i.e., for any positive δ,*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\Big(|\operatorname{card} C(0)-\theta\mathcal{L}^3(\mathcal{V}_\infty(C(0),f(n)))|\ge\delta n^3\Big)\ =\ -\infty\,.$$

Remark. The hypothesis on the function $f(n)$ is stronger than for the large deviation principles. Indeed it is required here that $f(n)/\ln n$ converges to ∞ as n goes to ∞.

We show now how a picture of the Wulff crystal for three dimensional Bernoulli percolation emerges with the help of the large deviation principle and Proposition 2.4.

Theorem 2.5. *(Asymptotics of the finite cluster shape)*
In the three dimensional Bernoulli bond percolation model on the cubic lattice, for $p > p_c$,

$$\lim_{n\to\infty} \frac{1}{n^2} \ln P\big(n^3 \le \operatorname{card} C(0) < \infty\big) \;=\; -\frac{\mathcal{I}(\mathcal{W}_\tau)}{(\theta\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}}\,.$$

If $f(n)$ *is a function from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that both* $n/f(n)^2$ *and* $f(n)/\ln n$ *go to* ∞ *as* n *goes to* ∞, *the law of* $n^{-1}\mathcal{V}_\infty(C(0), f(n))$ *conditioned on the event* $\{\, n^3 \le \operatorname{card} C(0) < \infty \,\}$ *concentrates exponentially fast around a shape which is a suitable dilation of the Wulff crystal of* τ. *More precisely, for any positive* δ, *if* P_n *is the conditional probability given by* $P_n(\cdot) = P(\cdot | n^3 \le \operatorname{card} C(0) < \infty)$, *then*

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln P_n\Big(\inf_{x\in\mathbb{R}^3} \mathcal{L}^3\big((x + (\theta\mathcal{L}^3(\mathcal{W}_\tau))^{-1/3}\mathcal{W}_\tau)\Delta n^{-1}\mathcal{V}_\infty(C(0), f(n))\big) \;\ge\; \delta\Big) < 0.$$

Proof. Proposition 2.4 shows that, under $\widehat{P}$, at the level of the large deviations of order n^2, the event $\{\, n^3 \le \operatorname{card} C(0) \,\}$ is equivalent to the event

$$\{\, \mathcal{L}^3(n^{-1}\mathcal{V}_\infty(C(0), f(n))) \ge 1/\theta \,\}\,.$$

Let $\mathcal{E}$ be the set $\{\, E \in \mathcal{B}(\mathbb{R}^3) : \mathcal{L}^3(E) \ge 1/\theta \,\}$. Notice that $\mathcal{E}$ is closed for the topology L^1. The interior of $\mathcal{E}$ is the set $\{\, E : \mathcal{L}^3(E) > 1/\theta \,\}$. The variational results related to the Wulff crystal (see chapter 6) yield

$$\inf\{\, \widehat{\mathcal{I}}(E) : E \in \operatorname{int}\mathcal{E} \,\} \;=\; \inf\{\, \widehat{\mathcal{I}}(E) : \mathcal{L}^3(E) > 1/\theta \,\} \;=\; \frac{\mathcal{I}(\mathcal{W}_\tau)}{(\theta\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}}\,,$$

and for any positive δ,

$$\begin{aligned}\inf\,\big\{\, \widehat{\mathcal{I}}(E) : E \in \mathcal{V}_{L^1}(\mathcal{E},\delta) \,\big\} &\ge \inf\,\big\{\, \widehat{\mathcal{I}}(E) : \mathcal{L}^3(E) > 1/\theta - \delta \,\big\}\\ &= \left(\frac{(1/\theta - \delta)}{\mathcal{L}^3(\mathcal{W}_\tau)}\right)^{2/3} \mathcal{I}(\mathcal{W}_\tau)\,.\end{aligned}$$

Letting δ go to zero, we see that

$$\sup_{\delta>0}\, \inf\,\big\{\, \widehat{\mathcal{I}}(E) : E \in \mathcal{V}_{L^1}(\mathcal{E},\delta) \,\big\} \;=\; \frac{\mathcal{I}(\mathcal{W}_\tau)}{(\theta\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}} \;=\; \inf\,\big\{\, \widehat{\mathcal{I}}(E) : E \in \operatorname{int}\mathcal{E} \,\big\}\,.$$

The weak large deviation principle of Theorem 2.1 and the enhanced upper bound of Theorem 2.2 yield that the limit $\lim_{n\to\infty} n^{-2} \ln P\big(n^3 \le \operatorname{card} C(0) < \infty\big)$ exists and is equal to the above quantity.

We turn now to the proof of the second claim of the Theorem. By the Wulff isoperimetric Theorem (see chapter 6), the Borel sets E minimizing the surface energy $\widehat{\mathcal{I}}(E)$ under the volume constraint $\mathcal{L}^3(E) \ge 1/\theta$ are translates of a suitable dilation of the Wulff crystal; more precisely, the solutions of this variational problem are the elements of

$$\mathcal{S}^* \;=\; \big\{\, x + (\theta\mathcal{L}^3(\mathcal{W}_\tau))^{-1/3}\mathcal{W}_\tau : x \in \mathbb{R}^3 \,\big\}\,.$$

We write

$$\limsup_{n\to\infty}\; \frac{1}{n^2}\ln P_n\Big(\inf_{x\in\mathbb{R}^3}\mathcal{L}^3\big((x+(\theta\mathcal{L}^3(\mathcal{W}_\tau))^{-1/3}\mathcal{W}_\tau)\;\Delta\; n^{-1}\mathcal{V}_\infty(C(0),f(n))\big)\;\geq\;\delta\Big)\;=\;$$
$$\limsup_{n\to\infty}\;\frac{1}{n^2}\ln\widehat{P}\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\in\mathcal{E}\setminus\mathcal{V}_{L^1}(\mathcal{S}^*,\delta)\big)\;+\;(\theta\mathcal{L}^3(\mathcal{W}_\tau))^{-2/3}\mathcal{I}(\mathcal{W}_\tau)\,.$$

Applying the enhanced upper bound of Theorem 2.2, for γ such that $0<\gamma<\delta/4$,

$$\limsup_{n\to\infty}\;\frac{1}{n^2}\ln\widehat{P}\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\in\mathcal{E}\setminus\mathcal{V}_{L^1}(\mathcal{S}^*,\delta)\big)\;\leq$$
$$-\,\inf\,\big\{\,\widehat{\mathcal{I}}(E) : E\in\mathcal{V}_{L^1}\big(\mathcal{E}\setminus\mathcal{V}_{L^1}(\mathcal{S}^*,\delta),\gamma\big)\,\big\}\,.$$

Yet the set $\mathcal{V}_{L^1}\big(\mathcal{E}\setminus\mathcal{V}_{L^1}(\mathcal{S}^*,\delta),\gamma\big)$ is included in

$$\big\{\,E\in\mathcal{B}(\mathbb{R}^3) : \mathcal{L}^3(E)>1/\theta-\gamma,\,\forall\,A\in\mathcal{S}^*\quad\mathcal{L}^3(E\Delta A)\geq\delta-\gamma\,\big\}\,.$$

Let $\lambda=\big((1/\theta-\gamma)/\mathcal{L}^3(\mathcal{W}_\tau)\big)^{1/3}$. If E belongs to the preceding set, then for any x in $\mathbb{R}^3$,

$$\begin{aligned}\mathcal{L}^3\big(\lambda^{-1}E\Delta(x+\mathcal{W}_\tau)\big)\;&=\;\lambda^{-3}\mathcal{L}^3\big(E\Delta(\lambda x+\lambda\mathcal{W}_\tau)\big)\\ &\geq\;\lambda^{-3}\mathcal{L}^3\big(E\Delta(\lambda x+(\theta\mathcal{L}^3(\mathcal{W}_\tau))^{-1/3}\mathcal{W}_\tau)\big)\;-\;\lambda^{-3}\mathcal{L}^3\big((\theta\mathcal{L}^3(\mathcal{W}_\tau))^{-1/3}\mathcal{W}_\tau\Delta\lambda\mathcal{W}_\tau\big)\\ &\geq\;\lambda^{-3}(\delta-2\gamma)\,.\end{aligned}$$

Therefore the preceding set is further included inside the set

$$\big\{\,E\in\mathcal{B}(\mathbb{R}^3),\,\mathcal{L}^3(\lambda^{-1}E)\geq\mathcal{L}^3(\mathcal{W}_\tau),\,\forall\,x\in\mathbb{R}^3\;\;\mathcal{L}^3(\lambda^{-1}E\Delta(x+\mathcal{W}_\tau))\geq\lambda^{-3}(\delta-2\gamma)\,\big\}\,.$$

We apply now the stability result associated to the Wulff isoperimetric Theorem (see chapter 6): there exists a non–decreasing function $s:\mathbb{R}^+\setminus\{0\}\mapsto\mathbb{R}^+\setminus\{0\}$ such that the infimum of $\widehat{\mathcal{I}}$ on the above set is larger than $\lambda^2\mathcal{I}(\mathcal{W}_\tau)+\lambda^2 s(\lambda^{-3}(\delta-2\gamma))$. It follows that

$$\inf\,\big\{\,\widehat{\mathcal{I}}(E) : E\in\mathcal{V}_{L^1}\big(\mathcal{E}\setminus\mathcal{V}_{L^1}(\mathcal{S}^*,\delta),\gamma\big)\,\big\}\;\geq\;\lambda^2\mathcal{I}(\mathcal{W}_\tau)+\lambda^2 s(\lambda^{-3}(\delta-2\gamma))\,.$$

For $\gamma<\delta/4$, since s is non–decreasing, $s(\lambda^{-3}(\delta-2\gamma))\geq s(\delta\theta\mathcal{L}^3(\mathcal{W}_\tau)/2)$, whence, coming back to the initial inequality,

$$\limsup_{n\to\infty}\;\frac{1}{n^2}\ln P_n\Big(\inf_{x\in\mathbb{R}^3}\mathcal{L}^3\big((x+(\theta\mathcal{L}^3(\mathcal{W}_\tau))^{-1/3}\mathcal{W}_\tau)\;\Delta\; n^{-1}\mathcal{V}_\infty(C(0),f(n))\big)\;\geq\;\delta\Big)\;\leq$$
$$-\lambda^2\mathcal{I}(\mathcal{W}_\tau)-\lambda^2 s(\delta\theta\mathcal{L}^3(\mathcal{W}_\tau)/2)+(\theta\mathcal{L}^3(\mathcal{W}_\tau))^{-2/3}\mathcal{I}(\mathcal{W}_\tau)\,.$$

We conclude by letting γ go to 0: the quantity on the right–hand side converges to $-(\theta\mathcal{L}^3(\mathcal{W}_\tau))^{-2/3}s(\delta\theta\mathcal{L}^3(\mathcal{W}_\tau)/2)$ which is negative. □

2.2. The continuum limit

We describe now the topological setting of our second type of large deviation principles. We denote by $\mathrm{BC}(\mathbb{R}^3)$ the set of all finite or countable collections of non negligible Borel subsets of $\mathbb{R}^3$. An arrangement of an element $\mathcal{A}$ of $\mathrm{BC}(\mathbb{R}^3)$ is a sequence $(A(i), i\in\mathbb{N})$ of sets in $\mathcal{A}\cup\{\varnothing\}$ such that each set of the collection $\mathcal{A}$ appears exactly once in the sequence $(A(i), i\in\mathbb{N})$ and the empty set $\varnothing$ appears countably many times in the sequence. The metric D on $\mathrm{BC}(\mathbb{R}^3)$ is the following one:

$$\forall\, \mathcal{A}_1,\mathcal{A}_2\in\mathrm{BC}(\mathbb{R}^3)\quad D(\mathcal{A}_1,\mathcal{A}_2)=$$
$$\inf\Big\{\sup_{i\in\mathbb{N}}\mathcal{L}^3(A_1(i)\Delta A_2(i)) : (A_j(i), i\in\mathbb{N}) \text{ arrangement of } \mathcal{A}_j,\ j=1,2\Big\}.$$

The surface energy $\mathcal{I}(\mathcal{A})$ of a collection of Borel sets $\mathcal{A}$ is defined as

$$\mathcal{I}(\mathcal{A})\;=\;\frac{1}{2}\sum_{A\in\mathcal{A}}\mathcal{I}(A)$$

if the sets of the collection form a Borel partition of $\mathbb{R}^3$, and $\mathcal{I}(\mathcal{A})=\infty$ otherwise. We consider the random collection $\mathcal{C}$ of the open clusters of the percolation configuration,

$$\mathcal{C}\;=\;\{\,C : C \text{ open cluster of the configuration}\,\}\,.$$

Remark. The collection $\mathcal{C}$ is a partition of $\mathbb{Z}^3$ into connected sets.

For U a Borel subset of $\mathbb{R}^3$ and l in $\mathbb{N}$, we define $\mathcal{C}(U,l)$ as the collection of the finite clusters of $\mathcal{C}$ intersecting U and having diameter strictly larger than l, that is,

$$\mathcal{C}(U,l)\;=\;\{\,C\in\mathcal{C},\, l<\mathrm{diam}_\infty C<\infty,\, C\cap U\neq\varnothing\,\}\,,$$

and we define the random Borel collection $\mathcal{V}_\infty\mathcal{C}(U,l)$ to be

$$\mathcal{V}_\infty\mathcal{C}(U,l)\;=\;\big\{\,\mathcal{V}_\infty(C,l) : C\in\mathcal{C}(U,l)\,\big\}\cup\big\{\,\mathbb{R}^3\setminus\bigcup_{C\in\mathcal{C}(U,l)}\mathcal{V}_\infty(C,l)\big\}\,.$$

Theorem 2.6. *(Continuum limit around a bounded set, metric D)*
There exists a constant $\kappa=\kappa(p)$ depending on p such that if $f(n)$ is a function from $\mathbb{N}$ to $\mathbb{N}$ such that $n/f(n)^2$ goes to ∞ as n goes to ∞ and $f(n)\geq\kappa\ln n$ for all n, then, for U a bounded Borel subset of $\mathbb{R}^3$, the sequence of random Borel collections of sets $(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU,f(n)))_{n\in\mathbb{N}}$ satisfies a weak large deviation principle in $(BC(\mathbb{R}^3),D)$ with speed n^2 and rate function $\mathcal{I}$: for any open subset $\mathbb{O}$ of $(BC(\mathbb{R}^3),D)$,

$$-\inf\{\,\mathcal{I}(\mathcal{A}):\mathcal{A}\in\mathbb{O}\}\;\leq\;\liminf_{n\to\infty}\frac{1}{n^2}\ln P\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU,f(n))\in\mathbb{O}\big)\,,$$

and, for any compact subset $\mathbb{K}$ *of* $(BC(\mathbb{R}^3), D)$,

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln P\big(n^{-1}\mathcal{V}_\infty \mathcal{C}(nU, f(n)) \in \mathbb{K}\big) \;\leq\; -\inf\{\, \mathcal{I}(\mathcal{A}) : \mathcal{A} \in \mathbb{K} \}\,.$$

As previously, a way to get a full large deviation principle is to weaken the topology. We define next the metric D_{loc} on $\mathrm{BC}(\mathbb{R}^3)$. This metric is discussed in chapter 10. For K a compact set and for $\mathcal{A}_1$, $\mathcal{A}_2$ in $\mathrm{BC}(\mathbb{R}^3)$ we define

$$\begin{aligned} &D_K(\mathcal{A}_1, \mathcal{A}_2) = \\ &\qquad \inf \Big\{ \sup_{i\in\mathbb{N}} \mathcal{L}^3_K(A_1(i)\Delta A_2(i)) : (A_j(i), i \in \mathbb{N}) \text{ arrangement of } \mathcal{A}_j,\, j = 1, 2 \Big\}\,. \end{aligned}$$

We construct a metric D_{loc} on $\mathrm{BC}(\mathbb{R}^3)$ by using an increasing sequence of compact sets filling $\mathbb{R}^3$, for instance the closed balls $(B(0,j), j \in \mathbb{N})$, and setting

$$\forall\, \mathcal{A}_1, \mathcal{A}_2 \in \mathrm{BC}(\mathbb{R}^3) \quad D_{\mathrm{loc}}(\mathcal{A}_1, \mathcal{A}_2) \;=\; \sum_{j\geq 1} j^{-5} D_{B(0,j)}(\mathcal{A}_1, \mathcal{A}_2)\,.$$

A sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of Borel collections of subsets of $\mathbb{R}^3$ converges with respect to D_{loc} towards a Borel collection $\mathcal{A}$ if and only if for any compact set K

$$\lim_{n\to\infty} D_K(\mathcal{A}_n, \mathcal{A}) \;=\; 0\,.$$

This notion of convergence corresponds to the topology on $\mathrm{BC}(\mathbb{R}^3)$ generated by the following system of neighbourhoods: a fundamental system of neighbourhoods of a Borel collection $\mathcal{A}$ is

$$V(\mathcal{A}, K, \varepsilon) \;=\; \big\{\, \mathcal{A}' \in \mathrm{BC}(\mathbb{R}^3) : D_K(\mathcal{A}', \mathcal{A}) < \varepsilon \,\big\}\,, \quad K \text{ compact}\,,\ \varepsilon > 0\,.$$

For $\mathcal{A}$ in $\mathrm{BC}(\mathbb{R}^3)$ and l in $\mathbb{N}$, we define the collection $\mathcal{V}_\infty(\mathcal{A}, l)$ of the l–neighbourhoods of the elements of $\mathcal{A}$ having diameter strictly larger than l, i.e.,

$$\mathcal{V}_\infty(\mathcal{A}, l) \;=\; \{\, \mathcal{V}_\infty(A, l) : A \in \mathcal{A},\, \mathrm{diam}_\infty A > l \}\,.$$

Theorem 2.7. *(Continuum limit on* $\mathbb{R}^3$*, metric* D_{loc}*)*
There exists a constant $\kappa = \kappa(p)$ *depending on* p *such that if* $f(n)$ *is a function from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that* $n/f(n)^2$ *goes to* ∞ *as* n *goes to* ∞ *and* $f(n) \geq \kappa \ln n$ *for all* n*, then the sequence of random Borel collections of sets* $(n^{-1}\mathcal{V}_\infty(\mathcal{C}, f(n)))_{n\in\mathbb{N}}$ *satisfies a large deviation principle in* $(BC(\mathbb{R}^3), D_{\mathrm{loc}})$ *with speed* n^2 *and rate function* $\mathcal{I}$*: for any Borel subset* $\mathbb{E}$ *of* $BC(\mathbb{R}^3)$,

$$\begin{aligned} -\inf\{\, \mathcal{I}(\mathcal{A}) : \mathcal{A} \in \operatorname{int}\mathbb{E} \} \;&\leq\; \liminf_{n\to\infty} \frac{1}{n^2} \ln P\big(n^{-1}\mathcal{V}_\infty(\mathcal{C}, f(n)) \in \mathbb{E}\big) \\ &\leq\; \limsup_{n\to\infty} \frac{1}{n^2} \ln P\big(n^{-1}\mathcal{V}_\infty(\mathcal{C}, f(n)) \in \mathbb{E}\big) \;\leq\; -\inf\{\, \mathcal{I}(\mathcal{A}) : \mathcal{A} \in \operatorname{clo}\mathbb{E} \} \end{aligned}$$

(where $\operatorname{int}\mathbb{E}$ *and* $\operatorname{clo}\mathbb{E}$ *are the interior and the closure of* $\mathbb{E}$ *in* $BC(\mathbb{R}^3)$ *with respect to the metric* D_{loc}*).*

Remark. We stress that the large deviation principles of Theorems 2.6 and 2.7 involve different random objects, namely $\mathcal{V}_\infty\mathcal{C}(nU,f(n))$ and $\mathcal{V}_\infty(\mathcal{C},f(n))$.

We show next that the sequence of the collections $(n^{-1}\mathcal{V}_\infty(\mathcal{C},f(n)))_{n\in\mathbb{N}}$ concentrates exponentially fast near Borel partitions.

Proposition 2.8. *(Concentration near Borel partitions)*
There exists a constant $\kappa=\kappa(p)$ *depending on* p *such that if* $f(n)$ *is a function from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that* $n/f(n)^2$ *goes to* ∞ *as* n *goes to* ∞ *and* $f(n)\geq\kappa\ln n$ *for all* n*, then the sequence of random Borel collections of sets* $(n^{-1}\mathcal{V}_\infty(\mathcal{C},f(n)))_{n\in\mathbb{N}}$ *concentrates near Borel partitions in the following sense: for any compact set* K *and for any positive* δ*,*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\Bigg(\mathcal{L}^3_{nK}\Big(\bigcup_{\substack{A_1,A_2\in\mathcal{V}_\infty(\mathcal{C},f(n))\\ A_1\neq A_2}}A_1\cap A_2\Big)>\delta n^3\Bigg)\;=\;-\infty\,,$$

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\Big(\mathcal{L}^3\Big(nK\quad\setminus\bigcup_{A\in\mathcal{V}_\infty(\mathcal{C},f(n))}A\Big)>\delta n^3\Big)\;=\;-\infty\,.$$

Under the same conditions, for any bounded Borel set U *of* $\mathbb{R}^3$ *and any positive* δ*,*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\Bigg(\mathcal{L}^3\Big(\bigcup_{\substack{A_1,A_2\in\mathcal{V}_\infty\mathcal{C}(nU,f(n))\\ A_1\neq A_2}}A_1\cap A_2\Big)>\delta n^3\Bigg)\;=\;-\infty\,.$$

As a consequence of the previous results, we can deduce some information on the intersection of the infinite cluster C_∞ with a box $n\Lambda$ from the large deviation principle on $\mathcal{C}(n\Lambda,f(n))$.

Corollary 2.9. *There exists a constant* $\kappa=\kappa(p)$ *depending on* p *such that if* $f(n)$ *is a function from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that* $n/f(n)^2$ *goes to* ∞ *as* n *goes to* ∞ *and* $f(n)\geq\kappa\ln n$ *for all* n*, then for any cubic box* Λ*, for any positive* δ*,*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\Big(\mathcal{L}^3_{n\Lambda}\Big(\mathcal{V}_\infty(C_\infty,f(n))\Delta\big(\mathbb{R}^3\setminus\bigcup_{C\in\mathcal{C}(n\Lambda,f(n))}\mathcal{V}_\infty(C,f(n))\big)\Big)\geq\delta n^3\Big)=-\infty.$$

Proof. We compute

$$\begin{aligned}
&\mathcal{L}^3_{n\Lambda}\Big(\mathcal{V}_\infty(C_\infty,f(n))\Delta\big(\mathbb{R}^3\setminus\bigcup_{C\in\mathcal{C}(n\Lambda,f(n))}\mathcal{V}_\infty(C,f(n))\big)\Big)\\
&\leq\mathcal{L}^3_{n\Lambda}\Big(\bigcup_{\substack{A_1,A_2\in\mathcal{V}_\infty(\mathcal{C},f(n))\\ A_1\neq A_2}}A_1\cap A_2\Big)\;+\;\mathcal{L}^3\Big(n\Lambda\setminus\bigcup_{C\in\mathcal{C}}\mathcal{V}_\infty(C,f(n))\Big)\;+\;n^2f(n)\mathcal{H}^2(\partial\Lambda)\,.
\end{aligned}$$

The result follows from Proposition 2.8. □

The next result links the cardinalities of the clusters intersecting a fixed cubic box and the volumes of their $f(n)$–neighbourhoods.

Proposition 2.10. *Let $f(n)$ be a function from $\mathbb{N}$ to $\mathbb{N}$ such that both $n/f(n)^2$ and $f(n)/\ln n$ go to ∞ as n goes to ∞. For any cubic box Λ, any positive δ,*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\Big(\sup_{C\in\mathcal{C}}\big|n^{-3}\mathrm{card}\,(\Lambda\cap n^{-1}C)-\theta\mathcal{L}^3(\Lambda\cap n^{-1}\mathcal{V}_\infty(C,f(n)))\big|\geq\delta\Big)\;=\;-\infty\,.$$

Although the previous results are enough to get a version of the single droplet Theorem inside a bounded set, in order to get a result which is closer in spirit to the single droplet Theorem of [3] (i.e., which involves the infinite cluster C_∞), we need the following enhanced large deviations upper bound, whose proof is more delicate than the previous upper bounds. Indeed, the analysis is not confined to a bounded set and we have to track all the open clusters intersecting a fixed bounded set; these clusters might have a large diameter, because of the presence of thin and long filaments having a small surface energy.

Theorem 2.11. *(Enhanced upper bound around a bounded set, metric D)*
There exists a constant $\kappa=\kappa(p)$ depending on p such that if $f(n)$ is a function from $\mathbb{N}$ to $\mathbb{N}$ such that $n/f(n)^2$ goes to ∞ as n goes to ∞ and $f(n)\geq\kappa\ln n$ for all n, then, for U a bounded Borel subset of $\mathbb{R}^3$, the sequence of random Borel collections of sets $(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU,f(n)))_{n\in\mathbb{N}}$ satisfies the following large deviations enhanced upper bound: for any closed subset $\mathbb{F}$ of $(BC(\mathbb{R}^3),D)$,

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU,f(n))\in\mathbb{F})\;\leq\;-\sup_{\delta>0}\,\inf\,\big\{\,\mathcal{I}(\mathcal{A}):\mathcal{A}\in\mathcal{V}_D(\mathbb{F},\delta)\,\big\}$$

where $\mathcal{V}_D(\mathbb{F},\delta)=\big\{\mathcal{E}\in BC(\mathbb{R}^3):\exists\,\mathcal{A}\in\mathbb{F}\ \ D(\mathcal{A},\mathcal{E})<\delta\big\}$ is the δ–neighbourhood of $\mathbb{F}$ in $(BC(\mathbb{R}^3),D)$.

We end with a formulation of the single droplet Theorem. We recall that C_∞ denotes the unique infinite open cluster of the configuration.

Theorem 2.12. *(Single droplet Theorem)*
Let Λ be a cubic box in $\mathbb{R}^3$ such that $\mathcal{L}^3(\Lambda)=1$. Let λ in $]0,1[$ be such that some translate of the dilated Wulff crystal $(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau$ is included in the interior of Λ. Then

$$\lim_{n\to\infty}\frac{1}{n^2}\ln P\Big(\mathrm{card}\,C_\infty\cap n\Lambda\leq(1-\lambda)\theta n^3\Big)\;=\;-(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}\,\mathcal{I}(\mathcal{W}_\tau)\,.$$

Let $\Lambda(\lambda)\;=\;\big\{\,x\in\Lambda\,:\,x+(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau\;\subset\;\mathrm{clo}\,\Lambda\,\big\}$. Let $f(n)$ be a function from $\mathbb{N}$ to $\mathbb{N}$ such that both $n/f(n)^2$ and $f(n)/\ln n$ go to ∞ as n goes to ∞. For δ

positive, let $\mathcal{E}_n(\Lambda,\lambda,\delta)$ *be the event: there exist* x *in* $\Lambda(\lambda)$ *and one open cluster* C_0 *such that*

$$\mathcal{L}^3\big((n^{-1}\mathcal{V}_\infty(C_\infty,f(n))\cap\Lambda)\,\Delta\,(\Lambda\setminus\big(x+(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau))\big)<\delta\,,$$
$$\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C_0,f(n))\,\Delta\,(x+(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau)\big)\;<\;\delta\,,$$
$$\mathcal{L}^3\Big(\bigcup_{C\in\mathcal{C}(n\Lambda,f(n))\setminus\{C_0\}}n^{-1}\mathcal{V}_\infty(C,f(n))\Big)<\delta\,.$$

Then there exist two positive constants b,c *depending on* p,λ,δ *such that*

$$P\big(\mathcal{E}_n(\Lambda,\lambda,\delta)\,|\,\mathrm{card}\,C_\infty\cap n\Lambda\le(1-\lambda)\theta n^3\big)\;\ge\;1-b\exp(-cn^2)\,.$$

Remark. We consider the case where $\mathcal{L}^3(\Lambda)=1$ only to simplify the statement. Otherwise we should replace λ and $1-\lambda$ by $\lambda\mathcal{L}^3(\Lambda)$ and $(1-\lambda)\mathcal{L}^3(\Lambda)$ respectively in the above statement.

Proof. Let us define

$$\mathbb{E}(\lambda)\;=\;\Big\{\,\mathcal{A}\in\mathrm{BC}(\mathbb{R}^3):\forall\,A\in\mathcal{A}\quad\text{either }\mathcal{L}^3(A)<\infty\text{ or }\mathcal{L}^3\big(\Lambda\cap A\big)\le 1-\lambda\,\Big\}\,.$$

Notice that $\mathbb{E}(\lambda)$ is closed in $\mathrm{BC}(\mathbb{R}^3)$ for the metric D. For x in $\mathbb{R}^3$, we set

$$\mathcal{A}_{\mathcal{W}_\tau}(x,\lambda)\;=\;\Big\{\,x+(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau,\,\mathbb{R}^3\setminus\big(x+(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau\big)\,\Big\}\,.$$

Let δ be positive. On one hand, by Corollary 2.9, Proposition 2.10 and the lower bound of Theorem 2.6,

$$\begin{aligned}
&-\inf\{\,\mathcal{I}(\mathcal{A}):\mathcal{A}\in\ \mathrm{int}\,\mathbb{E}(\lambda+\delta)\,\}\\
&\qquad\le\liminf_{n\to\infty}\ \frac{1}{n^2}\ln P\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(n\Lambda,f(n))\in\mathbb{E}(\lambda+\delta)\big)\\
&\qquad\qquad\le\liminf_{n\to\infty}\ \frac{1}{n^2}\ln P\big(\mathcal{L}^3(n^{-1}\mathcal{V}_\infty(C_\infty,f(n))\cap\Lambda)\le 1-\lambda-\delta/2\big)\\
&\qquad\qquad\qquad\le\liminf_{n\to\infty}\ \frac{1}{n^2}\ln P\big(\mathrm{card}\,C_\infty\cap n\Lambda\le(1-\lambda)\theta n^3\big)\,.
\end{aligned}$$

On the other hand, by Corollary 2.9, Proposition 2.10 and the enhanced upper bound of Theorem 2.11,

$$\begin{aligned}
&\limsup_{n\to\infty}\ \frac{1}{n^2}\ln P\big(\mathrm{card}\,C_\infty\cap n\Lambda\le(1-\lambda)\theta n^3\big)\\
&\qquad\le\limsup_{n\to\infty}\ \frac{1}{n^2}\ln P\big(\mathcal{L}^3(n^{-1}\mathcal{V}_\infty(C_\infty,f(n))\cap\Lambda)\le 1-\lambda+\delta/2\big)\\
&\qquad\qquad\le\limsup_{n\to\infty}\ \frac{1}{n^2}\ln P\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(n\Lambda,f(n))\in\mathbb{E}(\lambda-\delta)\big)\\
&\qquad\qquad\qquad\le\ -\inf\ \big\{\,\mathcal{I}(\mathcal{A}):\mathcal{A}\in\mathcal{V}_D(\mathbb{E}(\lambda-\delta),\delta)\,\big\}\,.
\end{aligned}$$

We examine first the large deviation lower bound. Let x be such that the translated and dilated Wulff crystal $x+(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau$ is included in the interior of Λ. We claim that $\mathcal{A}_{\mathcal{W}_\tau}(x,\lambda+2\delta)$ belongs to the interior of $\mathbb{E}(\lambda+\delta)$ for δ sufficiently small. Indeed, let δ positive be such that $x+((\lambda+2\delta)/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau$ is still included in $\operatorname{int}\Lambda$. Let $\mathcal{A}$ be an element of $\mathrm{BC}(\mathbb{R}^3)$ such that $D(\mathcal{A}_{\mathcal{W}_\tau}(x,\lambda+2\delta),\mathcal{A})<\delta$. Let A be a set of $\mathcal{A}$. Three cases can occur.

- $\mathcal{L}^3(A)<\delta$, then $\mathcal{L}^3(A)<\infty$.
- $\mathcal{L}^3\big(A\Delta(x+((\lambda+2\delta)/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau)\big)<\delta$, then $\mathcal{L}^3(A)<\infty$.
- $\mathcal{L}^3\big(A\Delta(\mathbb{R}^3\setminus(x+((\lambda+2\delta)/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau))\big)<\delta$, then

$$\mathcal{L}^3(A\cap\Lambda)\ \leq\ \mathcal{L}^3\big(\Lambda\setminus(x+((\lambda+2\delta)/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau)\big)+\delta\ \leq\ 1-\lambda-2\delta+\delta=1-\lambda-\delta\,.$$

Thus $\{\,\mathcal{A}\in\mathrm{BC}(\mathbb{R}^3):D(\mathcal{A}_{\mathcal{W}_\tau}(x,\lambda+2\delta),\mathcal{A})<\delta\,\}$ is included in $\mathbb{E}(\lambda+\delta)$. Therefore $\mathcal{A}_{\mathcal{W}_\tau}(x,\lambda+2\delta)$ belongs to the interior of $\mathbb{E}(\lambda+\delta)$ and

$$\inf\{\,\mathcal{I}(\mathcal{A}):\mathcal{A}\in\operatorname{int}\mathbb{E}(\lambda+\delta)\,\}\ \leq\ ((\lambda+2\delta)/\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}\mathcal{I}(\mathcal{W}_\tau)\,.$$

We examine now the large deviation upper bound. Let δ be positive and let $\mathcal{A}$ belong to $\mathcal{V}_D(\mathbb{E}(\lambda-\delta),\delta)$. Suppose that $\mathcal{I}(\mathcal{A})$ is finite. Then $\mathcal{A}$ contains exactly one set A_∞ such that $\mathcal{L}^3(A_\infty)=\infty$ (see chapter 11). By the definition of $\mathcal{V}_D$, there exists an element $\mathcal{A}'$ in $\mathbb{E}(\lambda-\delta)$ such that $D(\mathcal{A},\mathcal{A}')<\delta$. Hence there exists a set A' in $\mathcal{A}'$ such that $\mathcal{L}^3(A_\infty\Delta A')<\delta$; necessarily $\mathcal{L}^3(A')=\infty$ so that $\mathcal{L}^3(A'\cap\Lambda)\leq 1-\lambda+\delta$ and $\lambda-2\delta\leq\mathcal{L}^3(\mathbb{R}^3\setminus A_\infty)<\infty$. The Wulff isoperimetric Theorem (see chapter 6) implies

$$\mathcal{I}(\mathcal{A})\ \geq\ \mathcal{I}(\mathbb{R}^3\setminus A_\infty)\ \geq\ ((\lambda-2\delta)/\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}\mathcal{I}(\mathcal{W}_\tau)\,.$$

Letting δ go to 0, we see that the large deviation upper and lower bounds coincide.

We turn now to the proof of the second claim of the Theorem. We are going to prove that the law of the random collection of sets $n^{-1}\mathcal{V}_\infty\mathcal{C}(n\Lambda,f(n))$ conditioned on the event $\{\,\mathrm{card}\,C_\infty\cap n\Lambda\leq(1-\lambda)\theta n^3\,\}$ concentrates exponentially fast in the metric D on $\mathrm{BC}(\mathbb{R}^3)$ around an element $\mathcal{A}_{\mathcal{W}_\tau}(x,\lambda)$ for some x in $\Lambda(\lambda)$. More precisely, let $\mathcal{E}'_n(\Lambda,\lambda,\delta)$ be the intersection of the events:

- $\displaystyle \mathcal{L}^3_{n\Lambda}\Big(\bigcup_{\substack{A_1,A_2\in\mathcal{V}_\infty(\mathcal{C},f(n))\\ A_1\neq A_2}} A_1\cap A_2\Big)<\delta n^3\,,\qquad \mathcal{L}^3\Big(n\Lambda\ \setminus\bigcup_{A\in\mathcal{V}_\infty(\mathcal{C},f(n))}A\Big)<\delta n^3\,.$
- $\displaystyle \mathcal{L}^3\Big(\bigcup_{\substack{A_1,A_2\in\mathcal{V}_\infty\mathcal{C}(n\Lambda,f(n))\\ A_1\neq A_2}} A_1\cap A_2\Big)<\delta n^3\,.$
- $\displaystyle \mathcal{L}^3_{n\Lambda}\Big(\mathcal{V}_\infty(C_\infty,f(n))\Delta\big(\mathbb{R}^3\ \setminus\bigcup_{C\in\mathcal{C}(n\Lambda,f(n))}\mathcal{V}_\infty(C,f(n))\big)\Big)<\delta n^3.$
- there exists x in $\Lambda(\lambda)$ such that

$$D\Big(\mathcal{A}_{\mathcal{W}_\tau}(x,\lambda),n^{-1}\mathcal{V}_\infty\mathcal{C}(n\Lambda,f(n))\Big)\ <\ \delta\,.$$

Elementary computations show that, for any positive δ, the event $\mathcal{E}'_n(\Lambda,\lambda,\delta)$ is included in the event $\mathcal{E}_n(\Lambda,\lambda,4\delta)$. Hence we need only to estimate the conditional probability of $\mathcal{E}'_n(\Lambda,\lambda,\delta)$. The estimates for the first two points in the definition of the event $\mathcal{E}'_n(\Lambda,\lambda,\delta)$ are obtained in Proposition 2.8. Corollary 2.9 gives the estimate for the third point. Hence we need only to estimate the probability of the fourth point. By the Wulff isoperimetric Theorem for Caccioppoli partitions (see chapter 11), the Borel collections $\mathcal{A}$ minimizing the surface energy $\mathcal{I}(\mathcal{A})$ in $\mathbb{E}(\lambda)$ are exactly the elements of

$$\mathcal{S}^* \;=\; \big\{\, \mathcal{A}_{\mathcal{W}_\tau}(x,\lambda) : x\in\Lambda(\lambda)\,\big\}\,.$$

Using the previous result and Proposition 2.10, we have, for any δ,δ' positive,

$$\limsup_{n\to\infty}\,\frac{1}{n^2}\ln P\Big(\inf_{x\in\Lambda(\lambda)} D\Big(\mathcal{A}_{\mathcal{W}_\tau}(x,\lambda), n^{-1}\mathcal{V}_\infty\mathcal{C}(n\Lambda,f(n))\Big)\;\geq\;\delta \;\Big|\; \operatorname{card} C_\infty\cap n\Lambda\leq(1-\lambda)\theta n^3\Big)\;\leq$$
$$\limsup_{n\to\infty}\,\frac{1}{n^2}\ln P\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(n\Lambda,f(n))\in\mathbb{E}(\lambda-\delta')\setminus\mathcal{V}_D(\mathcal{S}^*,\delta)\big)+(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}\mathcal{I}(\mathcal{W}_\tau)\,.$$

Applying the enhanced upper bound of Theorem 2.11, for any positive γ,

$$\limsup_{n\to\infty}\,\frac{1}{n^2}\ln P\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(n\Lambda,f(n))\in\mathbb{E}(\lambda-\delta')\setminus\mathcal{V}_D(\mathcal{S}^*,\delta)\big)\;\leq$$
$$-\inf\,\big\{\,\mathcal{I}(\mathcal{A}):\mathcal{A}\in\mathcal{V}_D\big(\mathbb{E}(\lambda-\delta')\setminus\mathcal{V}_D(\mathcal{S}^*,\delta),\gamma\big)\,\big\}\,.$$

Yet the set $\mathcal{V}_D\big(\mathbb{E}(\lambda-\delta')\setminus\mathcal{V}_D(\mathcal{S}^*,\delta),\gamma\big)$ is included in

$$\big\{\,\mathcal{A}\in\mathbb{E}(\lambda-\delta'-\gamma):\forall\,\mathcal{A}'\in\mathcal{S}^*\quad D(\mathcal{A},\mathcal{A}')\geq\delta-\gamma\,\big\}\,.$$

We apply now the stability result associated to the Wulff Theorem for Caccioppoli partitions (see the end of chapter 11): for δ',γ small enough, there exists a positive η such that the infimum of $\mathcal{I}$ on the above set is larger than $(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}\mathcal{I}(\mathcal{W}_\tau)+\eta$. The desired conclusion then follows from the initial inequality. $\square$

CHAPTER 3

SKETCH OF THE PROOFS

This chapter is devoted to a sketch of the proofs of the large deviation principles. To ease the reading, we stay here at an informal level. The details, the precise proofs and the bibliographic references are to be found in the subsequent chapters. The order of the presentation is here more natural than logical, hence it does not match exactly the organization of the main text.

3.1. Surface tension

The first main problem is to extract a surface tension from the percolation model. We proceed as follows. Given a unit vector ν, let A be a unit square in $\mathbb{R}^3$ with normal vector ν and let $\operatorname{cyl} A$ be the cylinder with basis A. The surface tension $\tau(\nu)$ in the direction of ν is defined as

$$\tau(\nu) =$$
$$\lim_{n\to\infty} -\frac{1}{n^2}\ln P\begin{pmatrix}\text{in } n\operatorname{cyl} A \text{ there exists a finite set of closed edges } E \text{ such that}\\ \bullet\ E \text{ cuts } n\operatorname{cyl} A \text{ in at least two unbounded components}\\ \bullet \text{ the edges of } E \text{ at distance less than 5 from the boundary}\\ \quad\text{of } n\operatorname{cyl} A \text{ are at distance less than 5 from } nA\end{pmatrix}$$

A subadditive argument yields the existence of this limit: given a large square, we tile it into smaller squares and, supposing that the corresponding event for each small square occurs, we glue together the associated set of closed edges in order to obtain a set of edges realizing the above event for the large square. We then apply the FKG inequality to get an almost subadditive inequality. The condition on the localization of the set of closed edges near the boundary of the cylinder $n\operatorname{cyl} A$ is crucial to perform the glueing without deteriorating the probabilistic estimates. It is furthermore possible to localize the set of closed edges E: we obtain the same limiting value $\tau(\nu)$ if we impose that the edges of E are at distance less than $\phi(n)$ from the plane containing nA, where $\phi(n)$ is any function going to ∞ with n.

The function τ is positive, continuous, invariant under the isometries which leave $\mathbb{Z}^3$ invariant, and it satisfies the weak triangle inequality: for any triangle (ABC) in $\mathbb{R}^3$, if ν_A, ν_B, ν_C are the exterior normal unit vectors to the sides $[BC]$, $[AC]$, $[AB]$ in the plane containing (ABC), then

$$|BC|_2\tau(\nu_A) \le |AC|_2\tau(\nu_B) + |AB|_2\tau(\nu_C)\,.$$

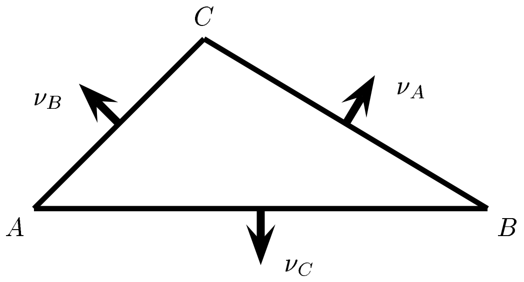

Figure 1: a triangle

We denote by $\mathcal{W}_\tau$ the Wulff set associated to the surface tension τ, called also the crystal of τ, defined by

$$\mathcal{W}_\tau = \left\{ x \in \mathbb{R}^3 : x \cdot w \leq \tau(w) \text{ for any unit vector } w \right\}.$$

Because of the properties of τ, the crystal $\mathcal{W}_\tau$ is convex, closed, bounded and contains the origin 0 in its interior.

In the next step, we obtain probabilistic estimates for the presence of a separating set of closed edges but without constraint on the localization of its boundary. We give first a lower bound: for O a planar set in $\mathbb{R}^3$ with normal vector $\operatorname{nor} O$ and $\phi(n)$ a function going to ∞ with n,

$$\liminf_{n\to\infty} \frac{1}{n^2} \ln P \begin{pmatrix} \text{there exists a set of closed edges in } n\operatorname{cyl} O \\ \text{at distance less than } \phi(n) \text{ from } nO \text{ cutting} \\ n\operatorname{cyl} O \text{ in at least 2 unbounded components} \end{pmatrix} \geq -\mathcal{H}^2(O)\,\tau(\operatorname{nor} O).$$

Secondly we have an upper bound: there exists a positive constant c such that, for any positive ρ, η with $\eta \leq \rho$, any unit vector w, any disc D in $\mathbb{R}^3$ with radius ρ and normal vector w,

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln P \begin{pmatrix} \text{there exists a set of closed edges in } n\operatorname{cyl} D \\ \text{at distance less than } n\eta \text{ from } nD \text{ cutting} \\ n\operatorname{cyl} D \text{ in at least 2 unbounded components} \end{pmatrix}$$
$$\leq -\tau(w)\pi\rho^2(1 - c\sqrt{\eta/\rho}).$$

3.2. Surface energy

To the direction dependent surface tension τ, we associate a surface energy $\mathcal{I}$. The surface energy $\mathcal{I}(A)$ of a Borel set A of $\mathbb{R}^3$ is defined as

$$\mathcal{I}(A) = \sup \left\{ \int_A \operatorname{div} f(x)\, d\mathcal{L}^3(x) : f \in C_0^1(\mathbb{R}^3, \mathcal{W}_\tau) \right\}$$

where $C_0^1(\mathbb{R}^3, \mathcal{W}_\tau)$ is the set of the C^1 vector functions defined on $\mathbb{R}^3$ with values in $\mathcal{W}_\tau$ having compact support and div is the usual divergence operator. The surface energy $\mathcal{I}(A)$ is finite if and only if A is a set of finite perimeter in the sense of Caccioppoli and De Giorgi and in this case it coincides with the surface integral of τ along the reduced boundary of the set A, that is,

$$\mathcal{I}(A) = \int_{\partial^* A} \tau(\nu_A(x))\, d\mathcal{H}^2(x)\,.$$

The reduced boundary $\partial^* A$ of a set of finite perimeter A is the set of the points of the boundary of A where A admits an exterior normal vector $\nu_A(x)$ in a measure theoretic sense. In case A is a smooth set, the reduced boundary $\partial^* A$ coincides with the topological boundary ∂A and $\nu_A(x)$ is the usual exterior normal vector to A at x.

3.3. Proof of the lower bound for a single cluster

Let $\mathcal{O}$ be an open subset of $(\mathcal{B}(\mathbb{R}^3), L^1)$. We have to prove that for any A in $\mathcal{O}$,

$$\liminf_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\big(n^{-1}\mathcal{V}_\infty(C(0), f(n)) \in \mathcal{O}\big) \ \geq\ -\widehat{\mathcal{I}}(A)\,.$$

Because $\mathcal{O}$ is open, there exists a positive δ such that $\{\, E \in \mathcal{B}(\mathbb{R}^3) : \mathcal{L}^3(E\Delta A) < \delta \,\}$ is included in $\mathcal{O}$. Hence it is enough to prove that

$$\liminf_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0), f(n))\Delta A\big) < \delta\Big) \ \geq\ -\widehat{\mathcal{I}}(A)\,.$$

We need only to consider the case where A has finite volume and surface energy (otherwise $\widehat{\mathcal{I}}(A) = \infty$). We prove in chapter 6 that a set of finite perimeter can be approximated in the sense of both volume and surface energy by a sequence of polyhedral sets. This result, proved in the case of isotropic surface energy (τ constant) by De Giorgi, links our definition of the surface energy with the original definition of the perimeter proposed by Caccioppoli, as the infimum of the limits of the surface energies of polyhedral approximations. In case A has finite volume, we can also impose that the approximating polyhedral sets are bounded and connected. Therefore we need only to consider the situation where the set A is polyhedral, its interior is connected and contains the origin. Let $F_1, \dots, F_r$ be the faces of the boundary of A (they are planar polygonal sets), let $\phi(n)$ be a function such that $\phi(n)$ and $n/\phi(n)$ go to ∞ with n, let O, U be open connected sets such that U contains the origin and $U \subset O \subset A$. Whenever $A \setminus U$ has small volume and n is sufficiently large, the event $\{\, \mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0), f(n))\Delta A\big) < \delta \,\}$ contains the intersection of the events

$$\left\{\begin{matrix}\text{there exists a set of closed edges inside } n\operatorname{cyl} F_i \text{ at distance less than} \\ \phi(n) \text{ from } nF_i \text{ cutting } n\operatorname{cyl} F_i \text{ in at least 2 unbounded components}\end{matrix}\right\},\quad 1 \leq i \leq r,$$

$$\left\{\begin{array}{l}\text{the edges at distance less than } \phi(n) \text{ from the plane spanned by } nF_i\\ \text{and at distance less than 5 from the boundary of } n\text{cyl } F_i \text{ are closed}\end{array}\right\},\ 1\leq i\leq r,$$

$$\left\{\begin{array}{l}\text{for the configuration restricted to } nO, \text{ the set } nU \text{ is included in}\\ \text{the } f(n) \text{ neighbourhood of the open cluster containing the origin}\end{array}\right\}.$$

Indeed, the occurrence of the last event of the above list implies that nU is included in $\mathcal{V}_\infty(C(0),f(n))$, while the occurrence of the other events precludes the existence of an open path starting in nO and exiting from $\mathcal{V}_\infty(nA,\phi(n))$, so that $\mathcal{V}_\infty(nA,f(n)+\phi(n))$ contains $\mathcal{V}_\infty(C(0),f(n))$. Since $\phi(n)/n$ goes to 0 and the set O is at positive distance from the boundary of A, for n large, the last event of the list is independent from the other ones; moreover its probability goes to 1 as n goes to ∞ because we are in the supercritical regime where, for n large, the open connected domain nO is likely to be invaded by a huge cluster of density θ (this is a consequence of Pisztora results). The other events deal with the presence of sets of closed edges, hence they are decreasing. The number of edges involved in the second type of events is of order $n\phi(n)$, hence the corresponding probabilities are ruled out at the level of surface order large deviations. We apply the FKG inequality together with our surface tension lower estimate to get

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\Delta A\big)<\delta\Big) &\geq -\sum_{1\leq i\leq r}\mathcal{H}^2(F_i)\tau(\mathrm{nor}\,F_i)\\ &= -\widehat{\mathcal{I}}(A)\,.\end{aligned}$$

3.4. Proof of the weak upper bound for a single cluster

A summary of the basic large deviations techniques and terminology employed here is given in the appendix. We first prove that the sequence of random sets $(n^{-1}\mathcal{V}_\infty(C(0),f(n)))_{n\in\mathbb{N}}$ is $\widehat{\mathcal{I}}$–tight, namely that there exists a positive constant c such that

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\Big(\inf\Big\{\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\Delta A\big):A\in\mathcal{B}(\mathbb{R}^3),\,\widehat{\mathcal{I}}(A)\leq\lambda\Big\}\geq\delta\Big)\leq -c\lambda.$$

For that purpose, we build a random set $\mathrm{fifa}\,C(0)$ having the following properties:

• the sequences of sets $(n^{-1}\mathcal{V}_\infty(C(0),f(n)))_{n\in\mathbb{N}}$ and $(n^{-1}\mathrm{fifa}\,C(0))_{n\in\mathbb{N}}$ are exponentially contiguous for the topology L^1, that is, for any positive δ,

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\Delta n^{-1}\mathrm{fifa}\,C(0)\big)\geq\delta\Big) = -\infty\,.$$

• the law of $n^{-1}\mathrm{fifa}\,C(0)$ concentrates exponentially fast near the sets having finite volume and perimeter, that is, there exists a positive constant c such that,

$$\forall\,M>0\qquad\limsup_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\big(\mathcal{P}(n^{-1}\mathrm{fifa}\,C(0))\geq M\big)\leq -cM\,.$$

The set $\operatorname{fifa} C(0)$ is obtained through a deterministic smoothing procedure applied to the open cluster $C(0)$. We first fatten slightly $C(0)$ (as was done in [50]) and then we fill the microscopic holes to get $\operatorname{fifa} C(0)$.

We next need a local estimate, like the following one:

$$\forall\, A\in\mathcal{B}(\mathbb{R}^3)\,,\quad \widehat{\mathcal{I}}(A)<\infty\,,\quad \forall\,\varepsilon>0\quad \exists\,\delta(A,\varepsilon)>0$$
$$\limsup_{n\to\infty}\ \frac{1}{n^2}\ln\widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\Delta A\big)<\delta(A,\varepsilon)\Big)\ \le\ -\widehat{\mathcal{I}}(A)(1-\varepsilon)\,.$$

Indeed, although the rate function $\widehat{\mathcal{I}}$ is not good, the $\widehat{\mathcal{I}}$–tightness together with the local estimate yield the weak large deviations upper bound. To get the local estimate for a set A having finite volume and perimeter is much more delicate than to get the $\widehat{\mathcal{I}}$–tightness. Indeed we should then recover the surface energy $\mathcal{I}(A)$ in the probabilistic estimate. At the boundary of the open cluster $C(0)$, there is a set of microscopic clusters (of diameter smaller than $f(n)$) which in the limit $n\to\infty$ has a positive density $1-\theta$ with respect to the Lebesgue measure. These small clusters do not create any surface energy, because the surface tension deals with mesoscopic connected sets of closed edges. We must only track these latter sets at the boundary of $C(0)$ to obtain our estimate. Starting from $C(0)$ and the random configuration around it, we define a random set $\operatorname{aglu} C(0)$ by agglutinating the clusters of diameter smaller than $f(n)$ at a distance less than $3f(n)$ from $C(0)$ and putting a unit cube at each vertex belonging to one of these clusters. A key point is that the connected components of closed edges at the boundary of $\operatorname{aglu} C(0)$ have a mesoscopic size and do create a surface energy. Furthermore the sequences of sets $(n^{-1}\mathcal{V}_\infty(C(0),f(n)))_{n\in\mathbb{N}}$ and $(n^{-1}\operatorname{aglu} C(0))_{n\in\mathbb{N}}$ are exponentially contiguous for the topology L^1, so that we need only to prove that

$$\forall\, A\in\mathcal{B}(\mathbb{R}^3),\quad \mathcal{P}(A)<\infty,\quad \mathcal{L}^3(A)<\infty,\qquad \forall\,\varepsilon>0\quad \exists\,\delta(A,\varepsilon)>0$$
$$\limsup_{n\to\infty}\ \frac{1}{n^2}\ln\widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\operatorname{aglu} C(0)\Delta A\big)<\delta(A,\varepsilon)\Big)\ \le\ -\mathcal{I}(A)+\varepsilon\,.$$

At this stage, we use an approximation result which is the counterpart of the polyhedral approximation used in the proof of the lower bound. Let A be a set having finite perimeter. For x in $\mathbb{R}^3$, r positive and ν a unit vector, we denote by $B_-(x,r,\nu)$ the half–ball which is the intersection of the Euclidean ball $B(x,r)$ and the half–space $\{\,y:(y-x)\cdot\nu\le 0\,\}$. Let δ',ε be positive. There exists a finite collection of disjoint balls $B(x_i,r_i)$, $i\in I$, such that: for any i in I, x_i belongs to the reduced boundary ∂^*A of A, r_i belongs to $]0,1[$ and

$$\mathcal{L}^3\big((A\cap B(x_i,r_i))\Delta B_-(x_i,r_i,\nu_A(x_i))\big)\ \le\ \delta'\,r_i^3\,,$$
$$\Big|\mathcal{I}(A)-\sum_{i\in I}\pi r_i^2\tau(\nu_A(x_i))\Big|\ \le\ \varepsilon\,.$$

Here $\nu_A(x)$ is the measure theoretic exterior normal to A at x. This approximation result relies on the description of the reduced boundary and on the Vitali covering Theorem. Now the event $\{\, \mathcal{L}^3(n^{-1}\text{aglu}\, C(0)\Delta A) < \delta \,\}$ is included in the intersection

$$\bigcap_{i\in I} \Big\{ \mathcal{L}^3\big((\text{aglu}\, C(0)\cap B(nx_i,nr_i))\Delta B_-(nx_i,nr_i,\nu_A(x_i))\big) \,\leq\, \delta n^3 + \delta' r_i^3 n^3 \Big\}\,.$$

For i in I, let $\mathcal{C}(i)$ be the collection of the open clusters of the configuration restricted to $\text{aglu}\, C(0)\cap B(nx_i,nr_i)$. Because of the definition of $\text{aglu}\, C(0)$, the clusters of $\mathcal{C}(i)$ are open clusters of the configuration restricted to $B(nx_i,nr_i)$. For n large enough and δ sufficiently small, the above intersection of events is further included in

$$\bigcap_{i\in I} \Big\{ \mathcal{L}^3\big((\textstyle\bigcup_{C\in\mathcal{C}(i)}\mathcal{V}_\infty(C,1/2))\Delta B_-(nx_i,nr_i,\nu_A(x_i))\big) \,\leq\, 3\delta' r_i^3 n^3 \Big\}\,.$$

For x in $\mathbb{R}^3$, w a unit vector, n in $\mathbb{N}$ and r,δ_0 positive, we define $\text{sep}\,(n,x,r,w,\delta_0)$ to be the event: there exists a collection $\mathcal{C}$ of open clusters in the configuration restricted to the ball $B(nx,nr)$ such that $\mathcal{L}^3\big((\bigcup_{C\in\mathcal{C}}\mathcal{V}_\infty(C,1/2))\Delta B_-(nx,nr,w)\big) \,\leq\, \delta_0 r^3 n^3$. The balls $B(x_i,r_i)$ being disjoint, the events $\text{sep}\,(n,x_i,r_i,\nu_A(x_i),3\delta')$ are independent, thus

$$\widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\text{aglu}\, C(0)\Delta A\big) < \delta\Big) \,\leq\, (1-\theta)^{-1}\prod_{i\in I} P\big(\text{sep}\,(n,x_i,r_i,\nu_A(x_i),3\delta')\big)\,.$$

The central Lemma gives an upper bound on $P(\text{sep}\,(n,x,r,w,3\delta'))$, which is uniform with respect to x,w,r, as follows: given a positive ε, we may find δ_0 positive such that for any x in $\mathbb{R}^3$, any unit vector w and r in $]0,1[$,

$$\limsup_{n\to\infty}\,\frac{1}{n^2}\ln P\big(\text{sep}\,(n,x,r,w,\delta_0)\big) \,\leq\, -\tau(w)\pi r^2(1-\varepsilon)\,.$$

Let us now build everything in order. We start with a positive ε. Let δ_0 positive be associated to ε by the central Lemma. We choose a positive δ' smaller than $\delta_0/3$, we apply the approximation result to find the family of balls $B(x_i,r_i)$, $i\in I$, associated to δ' and ε, then we choose δ smaller than the minimum $\min\{\,\delta' r_i^3 : i\in I\,\}$ and we end up with

$$\ln\widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\text{aglu}\, C(0)\Delta A\big) < \delta\Big) \,\leq\, -\ln(1-\theta)+\sum_{i\in I}\ln P\big(\text{sep}\,(n,x_i,r_i,\nu_A(x_i),\delta_0)\big)\,.$$

Dividing by n^2 and taking the supremum limit, we have then

$$\begin{aligned}\limsup_{n\to\infty}\,\frac{1}{n^2}\ln\widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\text{aglu}\, C(0)\Delta A\big) < \delta\Big) &\leq -\sum_{i\in I}\tau(\nu_A(x_i))\pi r_i^2(1-\varepsilon)\\ &\leq -\mathcal{I}(A)(1-\varepsilon)+\varepsilon.\end{aligned}$$

3.5. The central Lemma

Let us recall the result of the central Lemma. For x in $\mathbb{R}^3$, a unit vector w and r,δ positive, we define $\operatorname{sep}(n,x,r,w,\delta)$ to be the event: there exists a collection $\mathcal{C}$ of open clusters in the configuration restricted to the ball $B(nx,nr)$ such that $\mathcal{L}^3\big((\bigcup_{C\in\mathcal{C}}\mathcal{V}_\infty(C,1/2))\Delta B_-(nx,nr,w)\big)\ \leq\ \delta r^3n^3$. The central Lemma states that

$$\lim_{\delta\to 0}\ \sup_{\substack{x\in\mathbb{R}^3,r\in]0,1[,\\ w \text{ unit vector}}}\ \limsup_{n\to\infty}\ \ (n^2\pi r^2\tau(w))^{-1}\ln P\big(\operatorname{sep}(n,x,r,w,\delta)\big)\ \leq\ -1\,.$$

The main difficulty is to relate the event $\operatorname{sep}(n,x,r,w,\delta)$ to an event involved in the definition of the surface tension. Indeed the former event is at first sight quite different from the latter. Yet suppose that the event $\operatorname{sep}(n,x,r,w,\delta)$ occurs and let $\mathcal{C}$ be a collection of open clusters in $B(nx,nr)$ realizing it. Then at the boundary of the union of all the clusters of $\mathcal{C}$, there exists a large set of closed edges which almost cuts the ball $B(nx,nr)$ along the plane $\{\,y:(y-nx)\cdot\nu=0\,\}$. Indeed all the open paths joining $B_-(nx,nr,w)$ and $B_+(nx,nr,w)$ inside $B(nx,nr)$ must either meet $\mathcal{C}\setminus B_-(nx,nr,w)$ or $\mathcal{C}\cap B_+(nx,nr,w)$ (such a path is either entirely in $\mathcal{C}$ or entirely in the complement of $\mathcal{C}$) and these paths are therefore located in a set of volume less than δr^3n^3. Our strategy is to perform some surgery to destroy all "large" paths to get a configuration realizing a typical event related to the definition of the surface tension. The price we pay while doing the surgery has to be negligible compared to the surface effect we wish to capture.

The surgery. Let ρ be strictly smaller than r, but close to r, let η be positive and small and let D be the disc centered at x with radius ρ and normal vector w. To a configuration realizing the event $\operatorname{sep}(n,x,r,w,\delta)$ we associate two random collections of paths $\mathbb{P}_-$ and $\mathbb{P}_+$ as follows. Let $\mathbb{P}_-$ be the collection of all the open paths disjoint from $\mathcal{C}$ connecting the planes $\{\,y:(y-nx)\cdot w=0\,\}$ and $\{\,y:(y-nx)\cdot w=-\eta n\,\}$ in the configuration restricted to the intersection of $\operatorname{cyl}(nD)$ and $\{\,y:-\eta n\leq(y-nx)\cdot w\leq\eta n\,\}$. Let $\mathbb{P}_+$ be the collection of all the open paths inside $\mathcal{C}$ connecting the planes $\{\,y:(y-nx)\cdot w=0\,\}$ and $\{\,y:(y-nx)\cdot w=+\eta n\,\}$ in the configuration restricted to $\operatorname{cyl}(nD)\cap\{\,y:-\eta n\leq(y-nx)\cdot w\leq\eta n\,\}$. The paths of $\mathbb{P}_-\cup\mathbb{P}_+$ belong to $\big(\bigcup_{C\in\mathcal{C}}\mathcal{V}_\infty(C,1/2)\big)\Delta B_-(nx,nr,w)$, hence for n large enough the cardinality of the vertices visited by these paths is smaller than $2\delta r^3n^3$. Moreover each path in $\mathbb{P}_-\cup\mathbb{P}_+$ has a diameter at least ηn. Let l be an intermediate mesoscopic scale, that is l is large compared to $\ln n$ and small compared to $\sqrt{n}$. We consider the partition of $\mathbb{Z}^3$ in cubic boxes of side length l. By the result of Pisztora, inside such a box, with probability of order $1-\exp(-\mathrm{const}\,l)$, there exists a unique crossing open cluster, this cluster has cardinality larger than $\theta l^3/2$ and all other open clusters of the box have a diameter less than $l/3$. We mark as good boxes the boxes where these events occur. Whenever a path of $\mathbb{P}_-\cup\mathbb{P}_+$ visits a good box, it has to include the crossing cluster of the good box (more precisely, whenever it visits a vertex of the good box at distance $l/3$ from the boundary of the box). The number of good boxes visited by the paths of $\mathbb{P}_- \cup \mathbb{P}_+$ has to be less than

$$\frac{\text{cardinality of the set of vertices visited by } \mathbb{P}_- \cup \mathbb{P}_+}{\text{minimum cardinality of a crossing cluster in a good box}} \leq \frac{2\delta r^3 n^3}{\theta l^3/2} = \frac{4}{\theta}\delta r^3 n^3 l^{-3}.$$

For $l/\ln n$ large enough, Pisztora's estimate yields

$$P\big(\text{there are more than } i \text{ bad boxes inside } B(nx,nr)\big) \leq \text{const}\,\exp(-\text{const}\,li)\,.$$

We need also to take into account the boxes which intersect the boundary of $\operatorname{cyl}(nD)$. The number of such boxes is (deterministically) bounded by $100\eta n^2/l^2$. Let a be a large constant. With probability of order $1-\exp(-\text{const}\,an^2)$, the number of boxes (close to the boundary of $\operatorname{cyl}(nD)$, or bad, or good) visited by the paths of $\mathbb{P}_- \cup \mathbb{P}_+$ is less than

$$100\eta n^2/l^2 + an^2/l + (4/\theta)\delta r^3 n^3 l^{-3}\,.$$

We partition the boxes according to the position of their centers with respect to the slabs

$$\big\{\, y \in l\mathbb{Z}^3 : (y-nx)\cdot w \in [li, l(i+1)[\,\big\}\,, \quad i \in \mathbb{Z}\,.$$

Since $\operatorname{cyl}(nD) \cap \big\{\, y : -n\eta \leq (y-nx)\cdot w \leq 0\,\big\}$ contains more than $\eta n/(2l)$ such slabs, certainly there exists a negative index I_- larger than $-\eta n/l$ such that the number of boxes visited by $\mathbb{P}_-$ whose center belongs to the I_-–th slab is less than

$$\frac{\text{number of boxes}}{\text{number of slabs}} \leq \big(100\frac{\eta n^2}{l^2} + \frac{an^2}{l} + \frac{4\delta r^3 n^3}{\theta l^3}\big)/\big(\frac{\eta n}{2l}\big) = 200\frac{n}{l} + 2\frac{an}{\eta} + \frac{8\delta r^3 n^2}{l^2\theta\eta}\,.$$

Similarly, there exists a positive index I_+ smaller than $\eta n/l$ such that the number of boxes visited by $\mathbb{P}_+$ whose center belongs to the I_+–th slab is less than the above quantity. The total number of possibilities for I_-, I_+ and the boxes inside the slabs is less than

$$(\text{number of slabs})^2 \times (\text{number of choices for the boxes in one slab})^2 \leq$$
$$\Big(\frac{\eta n}{l}\Big)^2 \Big(200\frac{n}{l} + 2\frac{an}{\eta} + 8\frac{\delta r^3 n^2}{\theta\eta l^2}\Big)^2 \exp\Big(\Big(400\frac{n}{l} + 4\frac{an}{\eta} + 16\frac{\delta r^3 n^2}{\theta\eta l^2}\Big)\ln(2\pi r^2 n^2 l^{-2})\Big)\,.$$

Given a fixed choice of I_-, I_+ and the boxes inside the corresponding slabs, we transform the configuration ω into a new configuration $\Phi(\omega)$ by closing all the edges belonging to the boundary of the boxes. The number of such edges is less than

$$6l^2 \times \text{number of boxes} \leq 6l^2\Big(400\frac{n}{l} + 4\frac{an}{\eta} + 16\frac{\delta r^3 n^2}{\theta\eta l^2}\Big)\,.$$

Hence the extra factor in the probability estimate caused by this surgery is bounded by

$$\exp -\Big(\Big(2400nl + 24\frac{anl^2}{\eta} + 96\frac{\delta r^3 n^2}{\theta\eta}\Big)\ln\frac{1-p}{2}\Big)\,.$$

In the new configuration $\Phi(\omega)$ restricted to

$$\mathrm{cyl}\,(nD)\cap\{\,y : -n\eta \le (y-nx)\cdot w \le n\eta\,\}\,,$$

there exists no more paths connecting the planes $\{\,y : (y-nx)\cdot w = -\eta n\,\}$ and $\{\,y : (y-nx)\cdot w = +\eta n\,\}$. Indeed any such path of the configuration ω has to visit one of the boxes of one of the slabs I_- or I_+: if the path is disjoint from all the clusters of the collection $\mathcal{C}$, then it meets one of the boxes of the I_-–th slab, if the path is included in a cluster of the collection $\mathcal{C}$, then it meets one of the boxes of the I_+–th slab. Therefore there is in $\phi(\omega)$ a separating set of closed edges in $n\mathrm{cyl}\,D$ at distance less than $n\eta$ from nD cutting $n\mathrm{cyl}\,D$ in at least 2 unbounded components. Combining the previous considerations, we get

$$\begin{aligned}
&P\big(\mathrm{sep}\,(n,x,r,w,\delta)\big) \;\le\; \exp(-\mathrm{const}\,an^2) +\\
&\Big(\frac{\eta n}{l}\Big)^2\Big(200\frac{n}{l}+2\frac{an}{\eta}+8\frac{\delta r^3n^2}{\theta\eta l^2}\Big)^2\times\\
&\exp\Big(\Big(400\frac{n}{l}+4\frac{an}{\eta}+16\frac{\delta r^3n^2}{\theta\eta l^2}\Big)\ln(2\pi r^2n^2l^{-2})\\
&\qquad\qquad -\Big(2400nl+24\frac{anl^2}{\eta}+96\frac{\delta r^3n^2}{\theta\eta}\Big)\ln\frac{1-p}{2}\Big)\\
&\qquad\qquad \times P\begin{pmatrix}\text{there exists a set of closed edges in } n\mathrm{cyl}\,D\\ \text{at distance less than } n\eta \text{ from } nD \text{ cutting}\\ n\mathrm{cyl}\,D \text{ in at least 2 unbounded components}\end{pmatrix}.
\end{aligned}$$

We work now in the regime where $l/\ln n$ and $\sqrt{n}/l$ go to ∞ and we apply our surface tension upper bound estimate:

$$\begin{aligned}
\limsup_{n\to\infty}\quad &(n^2\pi r^2\tau(w))^{-1}\ln P\big(\mathrm{sep}\,(n,x,r,w,\delta)\big) \;\le\\
&-\min\Big\{\,\mathrm{const}\,\frac{a}{r^2},\ \frac{\rho^2}{r^2}(1-\mathrm{const}\sqrt{\eta/\rho}) - \mathrm{const}\,\frac{\delta r}{\eta}\,\Big\}\,.
\end{aligned}$$

We choose now $\eta = \sqrt{\delta}r/3$, $\rho = r\sqrt{1-\delta}$ and we let a go to ∞ in the preceding inequality:

$$\limsup_{n\to\infty}\quad (n^2\pi r^2\tau(w))^{-1}\ln P\big(\mathrm{sep}\,(n,x,r,w,\delta)\big) \;\le\; -1+\delta+\mathrm{const}\,\delta^{1/4}\,.$$

This inequality is uniform with respect to x in $\mathbb{R}^3$, r in $]0,1[$, δ in $]0,\theta/2[$, w a unit vector.

3.6. The enhanced upper bound

We define the surface energy $\mathcal{I}(A,O)$ of a Borel set A of $\mathbb{R}^3$ in an open set O as

$$\mathcal{I}(A,O) \;=\; \sup\;\Big\{\;\int_A \mathrm{div} f(x)\, d\mathcal{L}^3(x) : f \in C_0^1(O,\mathcal{W}_\tau)\,\Big\}$$

where $C_0^1(O,\mathcal{W}_\tau)$ is the set of the C^1 vector functions defined on $\mathbb{R}^3$ with values in $\mathcal{W}_\tau$ having compact support included in O and div is the usual divergence operator.

Let $\mathcal{F}$ be a closed subset of $(\mathcal{B}(\mathbb{R}^3),L^1)$. Let a,δ be positive with $\delta \le 1 \le a$. We write

$$\begin{aligned}&\widehat{P}\big(n^{-1}\mathcal{V}_\infty(C(0),f(n)) \in \mathcal{F}\big) \;\le\\ &\quad \widehat{P}\big(\mathcal{P}(\text{fifa}\, C(0)) \ge an^2\big) \;+\; \widehat{P}\Big(\mathcal{L}^3\big(\mathcal{V}_\infty(C(0),f(n))\Delta \text{fifa}\, C(0)\big) \ge \delta n^3\Big) \;+\; \widehat{P}(\mathcal{E})\,,\end{aligned}$$

where $\mathcal{E}$ is the event

$$\begin{aligned}\Big\{\, &n^{-1}\mathcal{V}_\infty(C(0),f(n)) \in \mathcal{F},\, \mathcal{P}(\text{fifa}\, C(0)) < an^2,\\ &\qquad\qquad \mathcal{L}^3\big(\mathcal{V}_\infty(C(0),f(n))\Delta\text{fifa}\, C(0)\big) \,<\, \delta n^3\,\Big\}\,.\end{aligned}$$

The only new estimate we need is for $\widehat{P}(\mathcal{E})$. Whenever $\mathcal{E}$ occurs, the set $n^{-1}\text{fifa}\, C(0)$ is a bounded set belonging to $\mathcal{V}_{L^1}(\mathcal{F},\delta)$ having perimeter less than a. By the isoperimetric inequality, its volume is less than $c_{\mathrm{iso}}a^{3/2}$, where c_{iso} is the isoperimetric constant of $\mathbb{R}^3$. Therefore there exist at most $8c_{\mathrm{iso}}a^{3/2}/\delta$ balls B centered on $\mathbb{Z}^3$ of radius one such that $\mathcal{L}^3\big(B\cap n^{-1}\text{fifa}\, C(0)\big) \ge \delta$. The bound on the perimeter of $n^{-1}\text{fifa}\, C(0)$ and the isoperimetric inequality relative to the balls imply that most of the volume of $n^{-1}\text{fifa}\, C(0)$ is concentrated in these balls: indeed the remaining volume of $n^{-1}\text{fifa}\, C(0)$ outside of these balls is less than $8a\delta^{1/3}b_{\mathrm{iso}}^{2/3}$, where b_{iso} is the isoperimetric constant for the balls of $\mathbb{R}^3$. Moreover the balls intersecting $n^{-1}\text{fifa}\, C(0)$ are at distance less than an from the origin, hence the number of possible integer centers is bounded by a polynomial function of n. By merging some balls if necessary (that is replacing some balls which are close together by a larger ball), we obtain a finite collection of balls $B(y_i,r_i)$, $1\le i\le m$, such that: the centers belong to $\mathbb{Z}^3\cap B(0,an^2+1)$, the radii r_i are positive integers smaller than $\exp((8c_{\mathrm{iso}}a^{3/2}/\delta)\ln 3)$, the balls $B(y_i,r_i+1)$, $1\le i\le m$, are pairwise disjoint and

$$\mathcal{L}^3\big(n^{-1}\text{fifa}\, C(0)\setminus B(y_1,r_1)\setminus\cdots\setminus B(y_m,r_m)\big) \le 8a\delta^{1/3}b_{\mathrm{iso}}^{2/3}\,.$$

We next look at the configuration inside each ball $B(ny_i,n(r_i+1))$. The preceding large deviations results show that for a fixed ball $B(y,r)$ centered at y in $\mathbb{Z}^3$ with radius r the law of $n^{-1}\big(\mathcal{V}_\infty(C(0),f(n))\cap B(ny,nr)-ny\big)$ satisfies a LDP in $\big(\mathcal{B}(B(0,r)),L^1\big)$ with rate function $\mathcal{I}$. Because of the invariance of the model under integer translations, the corresponding large deviation upper bound is in fact uniform over y in $\mathbb{Z}^3$. The m balls $B(y_i, r_i+1)$, $1 \leq i \leq m$, being disjoint, we get also a uniform large deviation upper bound for what happens simultaneously inside these balls, in the following form: for any closed subset $\mathcal{F}^m$ of $\mathcal{B}(B(0,r_1+1)) \times \cdots \times \mathcal{B}(B(0,r_m+1))$,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n^2} \ln \sup_{y_1,\dots,y_m} P\Big(\big(n^{-1}\mathcal{V}_\infty(C(0),f(n)) \cap B(y_1,r_1+1) - y_1, \dots, \\ n^{-1}\mathcal{V}_\infty(C(0),f(n)) \cap B(y_m,r_m+1) - y_m\big) \in \mathcal{F}^m \Big) \leq \\ -\inf\Big\{ \mathcal{I}(F_1, \operatorname{int} B(0,r_1+1)) + \cdots + \mathcal{I}(F_m, \operatorname{int} B(0,r_m+1)) : (F_1,\dots,F_m) \in \mathcal{F}^m \Big\}\end{aligned}$$

where the supremum ahead of P is taken over the points $y_1, \dots, y_m$ in $\mathbb{Z}^3$ such that the balls $B(y_1,r_1+1), \dots, B(y_m, r_m+1)$ are pairwise disjoint. Since in our case the number of possibilities for the balls $B(y_i, r_i)$, $1 \leq i \leq m$, is bounded by a polynomial function of n, we can decompose $\mathcal{E}$ according to the possible collections of balls and use this large deviation estimate to get

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}(\mathcal{E}) \leq -\inf\Big\{ \mathcal{I}(F_1, \operatorname{int} B(0,r_1+1)) + \cdots + \mathcal{I}(F_m, \operatorname{int} B(0,r_m+1)) \Big\}$$

where the infimum is taken over $m, r_1, \dots, r_m$ in a bounded set of integers (depending on a, δ) and over $(F_1, \dots, F_m)$ in the closure of the set

$$\begin{aligned}\Big\{ \big(E \cap B(y_1,r_1+1) - y_1, \dots, E \cap B(y_m,r_m+1) - y_m\big) : E \in \mathcal{F}, \\ y_1,\dots,y_m \text{ in } \mathbb{Z}^3 \text{ such that the balls } B(y_1,r_1+1),\dots,B(y_m,r_m+1) \text{ are disjoint}, \\ \mathcal{L}^3\big(E \setminus B(y_1,r_1) \setminus \cdots \setminus B(y_m,r_m)\big) \leq \delta + 8a\delta^{1/3} b_{\mathrm{iso}}^{2/3} \Big\}.\end{aligned}$$

Let $\eta = \delta + 8a\delta^{1/3} b_{\mathrm{iso}}^{2/3}$. Whenever $(F_1, \dots, F_m)$ belongs to the closure of the above set, we are able to reconstruct from $F_1, \dots, F_m$ a set F in $\mathcal{V}_{L^1}(\mathcal{F}, 4\eta)$ such that

$$\widehat{\mathcal{I}}(F) \leq \mathcal{I}(F_1, \operatorname{int} B(0,r_1+1)) + \cdots + \mathcal{I}(F_m, \operatorname{int} B(0,r_m+1)) + 2\eta ||\tau||_\infty .$$

The set F is obtained as follows: by definition, there exist E in $\mathcal{F}$ and $y_1, \dots, y_m$ in $\mathbb{Z}^3$ such that the balls $B(y_1,r_1+1), \dots, B(y_m,r_m+1)$ are disjoint and

$$\mathcal{L}^3\big(E \setminus B(y_1,r_1) \setminus \cdots \setminus B(y_m,r_m)\big) \leq \eta ,$$
$$\mathcal{L}^3\big((E \cap B(y_1,r_1+1)) \Delta (y_1+F_1)\big) + \cdots + \mathcal{L}^3\big((E \cap B(y_m,r_m+1)) \Delta (y_m+F_m)\big) < \eta .$$

This implies that

$$\mathcal{L}^3\big(F_1 \cap B(0,r_1+1) \setminus B(0,r_1)\big) + \cdots + \mathcal{L}^3\big(F_m \cap B(0,r_m+1) \setminus B(0,r_m)\big) \leq 2\eta$$

and therefore there exists t in $]0,1[$ such that

$$\mathcal{H}^2\big(F_1 \cap \partial B(0, r_1 + t)\big) \;+\; \cdots \;+\; \mathcal{H}^2\big(F_m \cap \partial B(0, r_m + t)\big) \;\leq\; 2\eta\,.$$

The set F defined by

$$F \;=\; \big(y_1 + (F_1 \cap B(0, r_1 + t))\big) \cup \cdots \cup \big(y_m + (F_m \cap B(0, r_m + t))\big)$$

answers the problem. The difficulty here is to choose t as above, to avoid that too much surface energy is created on the sphere $\partial B(0, r_i + t)$ while cutting F_i at the radius $r_i + t$; this is the reason why we work with the balls $B(y_i, r_i + 1)$ instead of $B(y_i, r_i)$.
We have now

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\big(\mathcal{E}\big) \;\leq\; -\inf\big\{\, \widehat{\mathcal{I}}(F) : F \in \mathcal{V}_{L^1}(\mathcal{F}, 4\eta)\,\big\} + 2\eta||\tau||_\infty\,.$$

Coming back to the initial inequality,

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\big(n^{-1}\mathcal{V}_\infty(C(0), f(n)) \in \mathcal{F}\big) \;\leq$$
$$-\min\big(\mathrm{const}\, a,\, \inf\big\{\, \widehat{\mathcal{I}}(F) : F \in \mathcal{V}_{L^1}(\mathcal{F}, 4\eta)\,\big\} - 2\eta||\tau||_\infty\big)\,.$$

Sending first δ to 0 (so that η goes to 0) and then a to ∞, we get the enhanced upper bound.

3.7. The case of the whole configuration

We first define a metric D which is adequate for working with collections of sets. Roughly speaking, two collections of sets are at distance less than δ if there exists a correspondence between the sets of the collections having volume larger than δ such that two corresponding sets are at L^1 distance less than δ. Starting with the surface tension τ, we build a surface energy functional on the set of the Caccioppoli partitions. We prove that it is a rate function on the space of the collections of sets. Furthermore, its level sets are compact for the weaker metric D_{loc}, which is the local version of the metric D. The proofs of the LDP for the whole configuration rely on the same probabilistic tools than for one single cluster. The additional problems are related to topological and geometric issues. Among those, the approximation of a Caccioppoli partition by a polyhedral partition is quite delicate. Indeed this question cannot be solved easily with the help of the corresponding approximation result for one single Caccioppoli set, because there is the additional global constraint that the approximating collection has to be a partition. Thus we make appeal to a stronger approximation result due to Quentin de Gromard. Apart from that point, the main lines of the proofs are quite similar in spirit to the case of one single cluster and we use extensively the constructions and the results proved in the first part. Let us sum up briefly the proof of the LDP upper bound. Starting with the random collection $\mathcal{C}$ of all the open clusters of the configuration, we build the three random collections

$$\mathcal{V}_\infty(\mathcal{C}, f(n)) = \left\{ \mathcal{V}_\infty(C, f(n)),\, C \in \mathcal{C},\, \mathrm{diam}_\infty C > f(n) \right\},$$
$$\mathrm{fifa}\,\mathcal{C} = \left\{ \mathrm{fifa}\,C,\, C \in \mathcal{C} \right\}, \qquad \mathrm{aglu}\,\mathcal{C} = \left\{ \mathrm{aglu}\,C,\, C \in \mathcal{C} \right\}.$$

We prove that these three collections are exponentially contiguous. We work with the collection $n^{-1}\mathrm{fifa}\,\mathcal{C}$ to prove that the law of $n^{-1}\mathcal{V}_\infty(\mathcal{C}, f(n))$ concentrates exponentially fast around the collections which are Caccioppoli partitions of $\mathbb{R}^3$ having a finite surface energy. We work with the collection $n^{-1}\mathrm{aglu}\,\mathcal{C}$ to estimate the probability of being close to a given Caccioppoli partition of finite perimeter. We finally prove an enhanced large deviations upper bound for the collection of the open clusters intersecting a fixed bounded Borel set of $\mathbb{R}^3$. The spirit of the proof is analogous to the enhanced upper bound for one cluster, yet it is more involved because we deal with collections of sets.

CHAPTER 4

THE MODEL

We start with the notation and the basic definitions. The cardinality of a finite set A is denoted by $\operatorname{card} A$. The symmetric difference between two sets A_1, A_2 is denoted by $A_1 \Delta A_2$. For $\mathcal{A}$ a collection of sets, we denote the union of the sets of $\mathcal{A}$ by $\operatorname{cup} \mathcal{A}$, i.e.,

$$\operatorname{cup} \mathcal{A} \,=\, \bigcup\nolimits_{A \in \mathcal{A}} A\,,$$

and by $\operatorname{overlap} \mathcal{A}$ the union of the intersections of all pairs of distinct sets of $\mathcal{A}$, i.e.,

$$\operatorname{overlap} \mathcal{A} \,=\, \operatorname{cup} \left\{\, A_1 \cap A_2 : A_1, A_2 \in \mathcal{A},\ A_1 \neq A_2 \,\right\}.$$

If f is a function with values in $\mathbb{R}$, we denote by $||f||_\infty$ the supremum of the values of $|f|$.

Constants. A lot of constants appear in the statements of the results and the proofs. We write in parenthesis the particular parameters on which the constants depend. We do not keep a precise track of the constants: hence b, c stand for generic constants and their values differ from place to place.

Topology. Let E be a subset of $\mathbb{R}^3$. We denote its interior by $\operatorname{int} E$, its closure by $\operatorname{clo} E$, its boundary by ∂E. Whenever A is a subset of $\mathbb{R}^3$ of linear dimension 2, that is A spans an hyperplane of $\mathbb{R}^3$, we denote this hyperplane by $\operatorname{hyp} A$ and we use the induced two dimensional topology of $\operatorname{hyp} A$ to define ∂A, $\operatorname{int} A$, $\operatorname{clo} A$. The collection of all the Borel subsets of a set E of $\mathbb{R}^3$ is denoted by $\mathcal{B}(E)$.

Metric. The standard norms of a vector $x = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ are

$$|x|_1 = |x_1| + |x_2| + |x_3|\,, \quad |x|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2}\,, \quad |x|_\infty = \max(|x_1|, |x_2|, |x_3|)\,.$$

Of course $|x|_\infty \leq |x|_2 \leq \sqrt{3}|x|_\infty$ for any x in $\mathbb{R}^3$. The usual scalar product between two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ of $\mathbb{R}^3$ is $x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$. Until the end of the paragraph, the character $*$ stands either for 2 or for ∞. We denote by d_* the metric associated to the norm $|\ \ |_*$, i.e., $d_*(x, y) = |x - y|_*$ for any x, y in $\mathbb{R}^3$. The d_* distance between two subsets E_1 and E_2 of $\mathbb{R}^3$ is

$$d_*(E_1, E_2) \,=\, \inf\{\, |x_1 - x_2|_* : x_1 \in E_1,\ x_2 \in E_2 \,\}\,.$$

The d_* r–neighbourhood of a subset E of $\mathbb{R}^3$ is the set

$$\mathcal{V}_*(E,r) \;=\; \{\, x \in \mathbb{R}^3 : d_*(x,E) < r \,\}\,.$$

The d_* diameter of a subset E of $\mathbb{R}^3$ is

$$\operatorname{diam}_* E = \sup\{\, |x-y|_* : x, y \in E \,\}\,.$$

We will generally work with the Euclidean distance d_2 on the continuous space $\mathbb{R}^3$ and with the distance d_∞ on the discrete lattice $\mathbb{Z}^3$. By default, when we speak of the diameter of a set E, without further precision, it means the d_∞ diameter $\operatorname{diam}_\infty E$.

Geometry. Let x be a point of $\mathbb{R}^3$ and let r be positive. The closed ball of center x and Euclidean radius r is denoted by $B(x,r)$. The sphere of center x and radius r is $\partial B(x,r)$. The unit sphere of $\mathbb{R}^3$ is denoted by S^2. The projective sphere PS^2 is obtained by identifying opposite points on S^2. Let w belong to S^2. By $\operatorname{hyp}(x,w)$ we denote the hyperplane containing x and orthogonal to w, i.e.,

$$\operatorname{hyp}(x,w) \;=\; \{\, y \in \mathbb{R}^3 : (y-x)\cdot w = 0 \,\}\,.$$

For r_1, r_2 in $\mathbb{R} \cup \{-\infty, +\infty\}$, we define

$$\operatorname{slab}(x,w,r_1,r_2) = \{\, y \in \mathbb{R}^3 : r_1 \le (y-x)\cdot w \le r_2 \,\}\,.$$

Let ν belong to S^2. We set

$$B_-(x,r,\nu) = B(x,r) \cap \operatorname{slab}(x,\nu,-r,0)\,,$$
$$B_+(x,r,\nu) = B(x,r) \cap \operatorname{slab}(x,\nu,0,+r)\,.$$

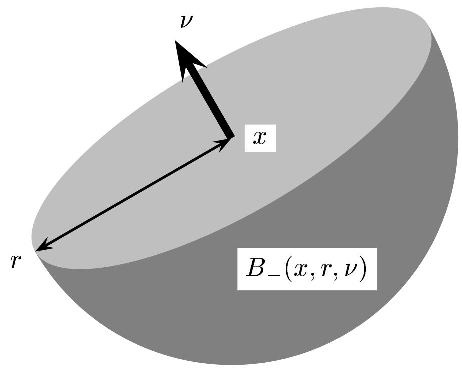

Figure 2: half–ball

Let u, v, w be an orthonormal basis of $\mathbb{R}^3$. By square (x, u, v, r) we denote the closed square centered at x of sides parallel to u and v and of side length r. By disc (x, r, w) we denote the closed disc centered at x of radius r and normal vector w.

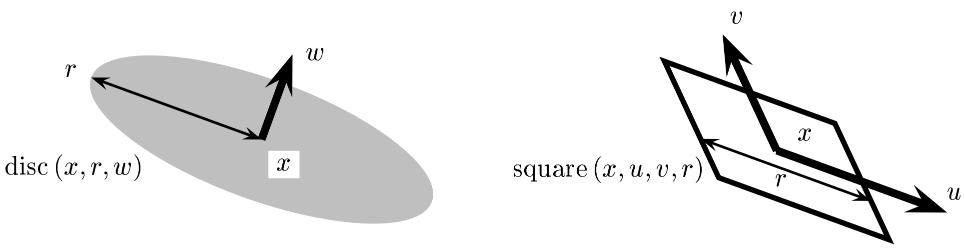

Figure 3: disc and square

By $\Lambda(x,r)$ we denote the cubic box of center $x = (x_1, x_2, x_3)$ and side length r defined by

$$\Lambda(x,r) = \{ y = (y_1, y_2, y_3) \in \mathbb{R}^3 : -r/2 < y_i - x_i \leq r/2 \} .$$

Notice that $\Lambda(x,r)$ has diameter r and is neither open nor closed. However, if $d_\infty(x,y) \geq r$ then $\Lambda(x,r)$ and $\Lambda(y,r)$ are disjoint. Let A be a subset of $\mathbb{R}^3$ of linear dimension 2, that is A spans an hyperplane of $\mathbb{R}^3$, which we denote by hyp A. By nor A we denote one of the two unit vectors orthogonal to hyp A, or equivalently the element of PS^2 orthogonal to hyp A. The cylinder of basis A is the set

$$\mathrm{cyl}\, A = \{ x + t \,\mathrm{nor}\, A : t \in \mathbb{R},\ x \in A \} .$$

We set also

$$\mathrm{cyl}\,(A, r) = \{ x + t \,\mathrm{nor}\, A : |t| \leq r,\ x \in A \} = \mathrm{cyl}\, A \cap \mathrm{slab}\,(x, \mathrm{nor}\, A, -r, r) .$$

Measure. We denote by $\mathcal{L}^3$ the three dimensional Lebesgue measure. A Borel set is said to be negligible if its Lebesgue measure is zero. When dealing with topological questions on the space $\mathcal{B}(\mathbb{R}^3)$, we consider the equivalence classes of the Borel sets modulo negligible sets. We denote by $\mathcal{H}^d$ the standard d–dimensional Hausdorff measure, for $d = 1, 2, 3$. We recall that for any subset A of $\mathbb{R}^3$, denoting by $\alpha(d)$ the volume of the unit ball of $\mathbb{R}^d$,

$$\mathcal{H}^d(A) = \sup_{\delta > 0} \inf \left\{ \alpha(d)\, 2^{-d} \sum_{i \in I} (\mathrm{diam}_2 E_i)^d : \sup_{i \in I} \mathrm{diam}_2 E_i \leq \delta,\ A \subset \bigcup_{i \in I} E_i \right\} .$$

Let E be a Borel subset of $\mathbb{R}^3$. A collection of sets $\mathcal{U}$ is called a Vitali class for E if for each x in E and δ positive there exists a set U in $\mathcal{U}$ containing x such that $0 < \mathrm{diam}_2 U < \delta$. We will use extensively the following result [34, Theorem 1.10].

The Vitali covering Theorem for $\mathcal{H}^2$. *Let E be an $\mathcal{H}^2$–measurable subset of $\mathbb{R}^3$ and let $\mathcal{U}$ be a Vitali class of closed sets for E. Then we may select a (finite or countable) disjoint sequence $(U_i)_{i\in I}$ from $\mathcal{U}$ such that either $\sum_{i\in I}(\mathrm{diam}_2 U_i)^2 = \infty$ or $\mathcal{H}^2(E\setminus\bigcup_{i\in I}U_i)=0$. If $\mathcal{H}^2(E)<\infty$ then, given $\varepsilon>0$, we may also require that $\mathcal{H}^2(E)\leq \pi 2^{-2}\sum_{i\in I}(\mathrm{diam}_2 U_i)^2+\varepsilon$.*

We will only use closed balls to perform our coverings. For Vitali classes which contain only balls, a similar Vitali covering result holds but for an arbitrary Radon measure (see for instance [77]).

We now describe the percolation model.

The lattice. We consider the cubic site lattice $\mathbb{Z}^3$. We turn $\mathbb{Z}^3$ into a graph by adding edges between all pairs of nearest neighbour sites; if x,y are two points of $\mathbb{Z}^3$ such that $|x-y|_1=1$, we denote by $\langle x,y\rangle$ the edge joining x and y. The edge $\langle x,y\rangle$ can be represented as the unit segment $[x,y]$ joining the sites x,y in $\mathbb{R}^3$. We denote by $\mathbb{E}^3$ the set of all edges of $\mathbb{Z}^3$. Two subgraphs of $(\mathbb{Z}^3,\mathbb{E}^3)$ are called disjoint if they have no vertices in common. A path in $(\mathbb{Z}^3,\mathbb{E}^3)$ is an alternating sequence $x_0,e_0,x_1,e_1,\dots,e_{n-1},x_n,\dots$ of distinct vertices x_i and edges e_i, where e_i is the edge between x_i and x_{i+1} (we adopt here the definition of [42], which is slightly different from the one used in [3]). If the path terminates at some vertex x_n it is said to connect x_0 to x_n and to have length n. Two paths are disjoint if they have no edges in common. Let D be a subset of $\mathbb{R}^3$. An edge $\langle x,y\rangle$ of $\mathbb{E}^3$ is said to be included in D if both points x,y belong to D. We denote by $\mathbb{Z}^3(D)$ the set $D\cap\mathbb{Z}^3$ and by $\mathbb{E}^3(D)$ the set of the edges of $\mathbb{E}^3$ included in D. For E a subset of $\mathbb{E}^3$, a formula like $E\subset\mathbb{E}^3(D)$ will be abbreviated into $E\subset D$.
We will need another graph structure $\mathbb{L}^3$ on $\mathbb{Z}^3$. We say that x,y are $\mathbb{L}^3$ adjacent if $|x-y|_\infty=1$ and we denote by $\mathbb{L}^3$ the corresponding set of edges.

Discrete topology. Let A be a subset of $\mathbb{Z}^3$. We define
- its edge boundary: $\partial_{\mathrm{e}}A = \{\,\langle x,y\rangle\in\mathbb{E}^3 : x\notin A,\, y\in A\,\}$,
- its inner vertex boundary: $\partial_{\mathrm{iv}}A = \{\,x\in A : \exists\, y\in\mathbb{Z}^3\setminus A,\ \langle x,y\rangle\in\mathbb{E}^3\,\}$,
- its outer vertex boundary: $\partial_{\mathrm{ov}}A = \{\,x\in\mathbb{Z}^3\setminus A : \exists\, y\in A,\ \langle x,y\rangle\in\mathbb{E}^3\,\}$.

These definitions are extended to the subsets of $\mathbb{R}^3$ by setting, for any E included in $\mathbb{R}^3$, $\partial_* E=\partial_*(\mathbb{Z}^3\cap E)$, where $*$ stands for e, iv or ov. The set A is said to be connected if the graph $(A,\mathbb{E}^3(A))$ is connected. The set A is said to be $\mathbb{L}^3$ connected if the graph having for vertex set A and for edge set the edges of $\mathbb{L}^3$ whose both endpoints belong to A is connected. A residual component of A is a connected component of the graph $(\mathbb{Z}^3\setminus A,\mathbb{E}^3(\mathbb{Z}^3\setminus A))$. If R is a residual component of a connected set A, then its inner and outer vertex boundaries $\partial_{\mathrm{ov}}R,\partial_{\mathrm{iv}}R$ are $\mathbb{L}^3$ connected (see [45, 64]).

The configuration space. The nearest neighbour Bernoulli bond percolation model on the cubic lattice at density p is defined by independently choosing each edge of $\mathbb{E}^3$ to be open with probability p or closed with probability $1-p$. We denote by P the corresponding product probability measure on the configuration space Ω consisting of all functions from $\mathbb{E}^3$ to $\{0,1\}$ (where 1 stands for open and 0 for closed). The restriction $\omega|_D$ of a configuration ω of Ω to a subset D of $\mathbb{R}^3$ is the function from $\mathbb{E}^3(D)$ to $\{0,1\}$ defined by $\omega|_D(e) = \omega(e)$ for all e in $\mathbb{E}^3(D)$.

Connection and separation. Two subsets S_1, S_2 of $\mathbb{R}^3$ are connected in the configuration ω if there is a path of open edges in ω connecting a site of $\mathbb{Z}^3(S_1)$ to a site of $\mathbb{Z}^3(S_2)$. The open clusters in ω are the connected components of the graph having vertex set $\mathbb{Z}^3$ and the open edges of ω only. We write $C(x)$ for the open cluster containing the vertex x. Let D, A_1, A_2 be three subsets of $\mathbb{R}^3$. A set of edges E of $\mathbb{E}^3$ is said to separate A_1 and A_2 in D if there is no path in the graph $(\mathbb{Z}^3(D), \mathbb{E}^3(D) \setminus E)$ from $\mathbb{Z}^3(A_1)$ to $\mathbb{Z}^3(A_2)$. A set of edges E of $\mathbb{E}^3$ is said to separate ∞ in D if the graph $(\mathbb{Z}^3(D), \mathbb{E}^3(D) \setminus E)$ has at least two unbounded connected components.

The supercritical regime. It is known that the Bernoulli bond percolation model has a phase transition at a value p_c strictly between 0 and 1: for $p < p_c$ the open clusters are finite and for $p > p_c$ there exists a unique infinite open cluster C_∞ [42]. We work with a fixed value $p > p_c$. The density of the infinite cluster is $\theta = P(\operatorname{card} C(0) = \infty)$. We denote by $\widehat{P}$ the probability measure P conditioned on the event that $C(0)$ is finite, that is, $\widehat{P}(\cdot) = P(\cdot / \operatorname{card} C(0) < \infty)$.

The Harris–FKG inequality. There is a natural order on Ω defined by the relation: $\omega_1 \leq \omega_2$ if and only if all open edges in ω_1 are open in ω_2. An event is said to be increasing (respectively decreasing) if its characteristic function is non decreasing (respectively non increasing) with respect to this partial order. Suppose the events A, B are both increasing or both decreasing. The Harris–FKG inequality [42] says that $P(A \cap B) \geq P(A)P(B)$.

CHAPTER 5

SURFACE TENSION

In this chapter, we build the surface tension associated to the Bernoulli bond percolation model on $\mathbb{Z}^3$. We study its properties and we finally prove some probabilistic estimates for the presence of a flat separating set of closed edges.

5.1. Existence

We will have to work with enlargements of continuous subsets of $\mathbb{R}^3$ so that they have a significative trace on the discrete lattice $\mathbb{Z}^3$. We fix a real number ζ larger than 5 and we enlarge a subset A of $\mathbb{R}^3$ by considering its ζ–neighbourhood $\mathcal{V}_2(A,\zeta)$. A minimal requirement to choose ζ is that, whenever A is an arcwise connected subset of $\mathbb{R}^3$, the graph $(\mathbb{Z}^3(\mathcal{V}_2(A,\zeta)),\mathbb{E}^3(\mathcal{V}_2(A,\zeta)))$ is also connected. Some of the constants appearing in the statements and the proofs depend on ζ. However the large deviation principles and the direction dependent surface tension are independent of the particular choice of ζ larger than 5.

We build now the surface tension in an arbitrary direction.

Definition 5.1. Let A be a closed planar set and let s be positive or infinite. We denote by $W(\partial A,s,\zeta)$ the event that there exists a finite set of closed edges separating ∞ in $\mathrm{cyl}\,A$, included in $\mathcal{V}_2(\mathrm{hyp}\,A,s)$ and whose intersection with $\mathcal{V}_2(\mathrm{cyl}\,\partial A,\zeta)$ is included in $\mathcal{V}_2(\mathrm{hyp}\,A,\zeta)$, i.e.,

$$\begin{aligned}W(\partial A,s,\zeta) = \big\{\,&\omega\in\Omega : \exists\, E\subset\mathbb{E}^3,\ \mathrm{card}\,E<\infty,\ \forall\, e\in E\quad \omega(e)=0,\\ &E\subset\mathcal{V}_2(\mathrm{hyp}\,A,s),\ E\cap\mathcal{V}_2(\mathrm{cyl}\,\partial A,\zeta)\subset\mathcal{V}_2(\mathrm{hyp}\,A,\zeta),\ E \text{ separates } \infty \text{ in } \mathrm{cyl}\,A\,\big\}.\end{aligned}$$

Remark. The event $W(\partial A,s,\zeta)$ is decreasing. Whenever this event occurs, it occurs inside $\mathrm{cyl}\,A\cap\mathcal{V}_2(\mathrm{hyp}\,A,s)$. Indeed, if E is a collection of edges realizing the event $W(\partial A,s,\zeta)$, then so does $E\cap\mathbb{E}^3(\mathrm{cyl}\,A\cap\mathcal{V}_2(\mathrm{hyp}\,A,s))$.

Remark. We will only study the event $W(\partial A,s,\zeta)$ for A a rectangle.

Remark. Recall the convention that $E\subset\mathcal{V}_2(\mathrm{hyp}\,A,s)$ means $E\subset\mathbb{E}^3(\mathcal{V}_2(\mathrm{hyp}\,A,s))$ and $E\cap\mathcal{V}_2(\mathrm{cyl}\,\partial A,\zeta)\subset\mathcal{V}_2(\mathrm{hyp}\,A,\zeta)$ means $E\cap\mathbb{E}^3(\mathcal{V}_2(\mathrm{cyl}\,\partial A,\zeta))\subset\mathbb{E}^3(\mathcal{V}_2(\mathrm{hyp}\,A,\zeta))$.

Proposition 5.2. *Let A be a planar rectangle such that $\mathcal{H}^2(A)$ is positive. Let $\phi(n)$ be a function from $\mathbb{N}$ to $\mathbb{R}^+ \cup \{\infty\}$ such that $\lim_{n\to\infty} \phi(n) = \infty$. The limit*

$$\lim_{n\to\infty} -\mathcal{H}^2(nA)^{-1} \ln P(W(\partial nA, \phi(n), \zeta))$$

exists and depends only on $\operatorname{nor} A$. *We denote it by $\tau(\operatorname{nor} A)$ and call it the surface tension in the direction* $\operatorname{nor} A$.

Remark. This statement tells nothing about the value $\tau(\operatorname{nor} A)$, which might be 0 or ∞.

Proof. This result is proved with the help of the same subadditivity argument used in [1, Proposition 2.4]. The only additional problem is that we work with curves whose position with respect to the discrete lattice $\mathbb{Z}^3$ is arbitrary. Let w be a unit vector of $\mathbb{R}^3$ and let A, A' be two planar rectangles such that $\operatorname{nor} A = \operatorname{nor} A' = w$, $\mathcal{H}^2(A)$ and $\mathcal{H}^2(A')$ are both positive. Let $\phi(n), \phi'(n)$ be two functions from $\mathbb{N}$ to $\mathbb{R}^+ \cup \{\infty\}$ such that $\lim_{n\to\infty} \phi(n) = \infty$, $\lim_{n\to\infty} \phi'(n) = \infty$. Let ζ, ζ' be two real numbers larger than 5. Let n, m be two integers such that $n\mathrm{diam}_2 A > m\,\mathrm{diam}_2 A'$ and

$$\min(n\,\mathrm{diam}_2 A, m\,\mathrm{diam}_2 A', \phi(n) - 2) \;>\; \max(\zeta, \zeta')\,.$$

Because we deal with rectangles, certainly there exists a collection of sets $(T(i), i \in I)$ such that: each set $T(i)$ is a translate of mA' and $d_2(T(i), \operatorname{hyp} nA \setminus nA) > \zeta + \zeta' + 4$; the sets $(T(i), i \in I)$ have pairwise disjoint interiors and are included in nA; their union cup $\{\, T(i) : i \in I \,\}$ contains the set

$$D(m,n) \;=\; \{\, x \in nA : d_2(x, n\partial A) > m\,\mathrm{diam}_2 A' + \zeta + \zeta' + 4 \,\}\,.$$

Since A is a rectangle, so is $D(m,n)$ and

$$\begin{aligned}\mathcal{H}^2(nA) - \mathcal{H}^1(n\partial A)(m\mathrm{diam}_2 A' + \zeta + \zeta' + 4) \;&\le\; \mathcal{H}^2(D(m,n))\\ &\le\; (\operatorname{card} I)\mathcal{H}^2(mA') \;\le\; \mathcal{H}^2(nA)\,.\end{aligned}$$

For each i in I, let $t(i)$ be a vector in $\mathbb{R}^3$ such that $|t(i)|_\infty \le 1$ (hence $|t(i)|_2 \le \sqrt{3}$) and $t(i) + T(i)$ is the image of mA' by an integer translation (a translation that leaves $\mathbb{Z}^3$ globally invariant). We denote by $T'(i)$ the set $t(i) + T(i)$. Let E_0 be the set of edges included in

$$\Big(\operatorname{cyl}(nA\setminus D(m,n)) \cap \mathcal{V}_2(\operatorname{hyp} nA, \zeta)\Big) \cup \bigcup_{i\in I} \Big(\mathcal{V}_2(\operatorname{cyl}\partial T'(i), \zeta'+2) \cap \mathcal{V}_2(\operatorname{hyp} nA, \zeta'+3)\Big)\,.$$

The Lebesgue measure of the d_2 1–neighbourhood of this set is less than

$$\begin{aligned}\mathcal{H}^1(n\partial A)(m\mathrm{diam}_2 A' + \zeta + \zeta' + 6)(2\zeta + 2) + 4\pi(2\zeta + 2)&\\ + \operatorname{card} I\left(\mathcal{H}^1(m\partial A')(2\zeta' + 8)^2 + 8\pi(\zeta' + 4)^3\right)&\,.\end{aligned}$$

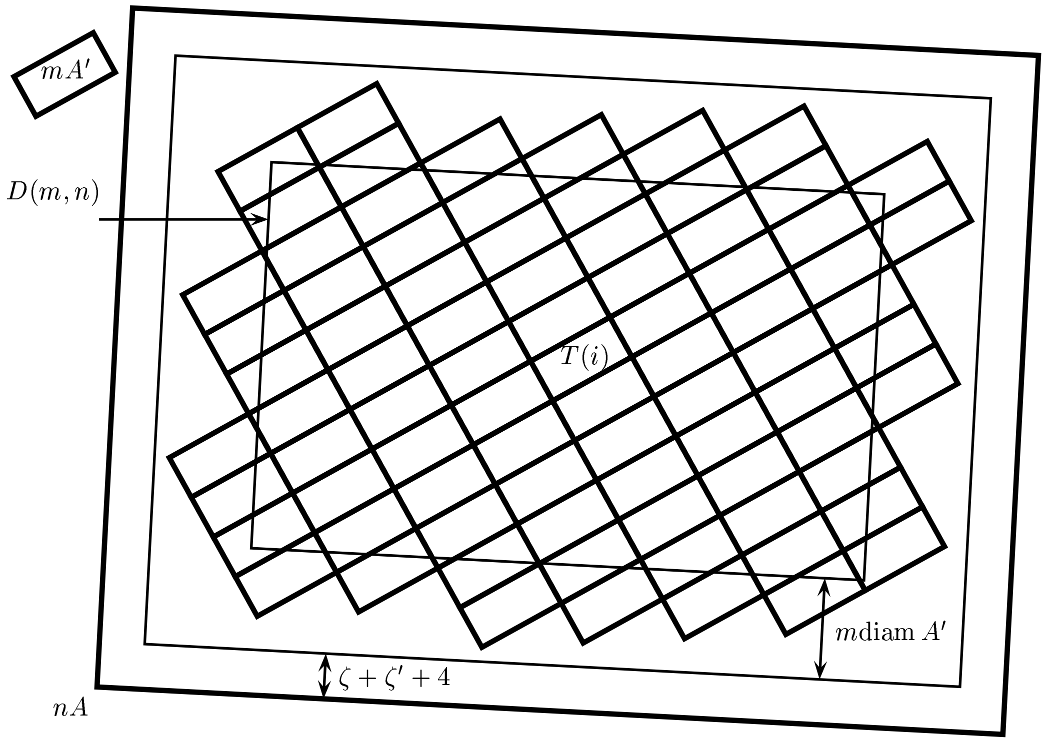

Figure 4: the plane hyp (nx, w)

Hence there exists a constant $c(\zeta,\zeta',A,A')$ depending only on ζ,ζ',A,A' such that

$$\operatorname{card} E_0 \;\leq\; c(\zeta,\zeta',A,A')(nm+n^2/m+1)\,.$$

Suppose that all the events

$$W(\partial T'(i),\phi(n)-2,\zeta'),\quad i\in I\,,$$

occur, and let $E(i)$, $i\in I$, be finite sets of closed edges realizing these events. Thus for i in I,

$$E(i)\subset\mathcal{V}_2(\operatorname{hyp} T'(i),\phi(n)-2)\cap\operatorname{cyl}\,T'(i),\quad E(i)\cap\mathcal{V}_2(\operatorname{cyl}\partial T'(i),\zeta)\subset\mathcal{V}_2(\operatorname{hyp} T'(i),\zeta)$$

and $E(i)$ separates ∞ inside $\operatorname{cyl} T'(i)$. Suppose in addition that all the edges in E_0 are closed. Let $E=E_0\cup\bigcup_{i\in I}E(i)$. We claim that the set of edges E realizes the event $W(\partial nA,\phi(n),\zeta)$. Firstly E is included in $\mathcal{V}_2(\operatorname{hyp} nA,\phi(n))$. Secondly, for each i in I, the d_2 distance between $\operatorname{cyl}\,T'(i)$ and $\operatorname{cyl}\partial nA$ is larger than $\zeta+\zeta'+4-\sqrt{3}$, so that neither $E(i)$ nor $\mathcal{V}_2(\operatorname{cyl}\partial T'(i),\zeta'+2)$ intersect $\mathcal{V}_2(\operatorname{cyl}\partial nA,\zeta)$, and $E_0\cap\mathcal{V}_2(\operatorname{cyl}\partial nA,\zeta)$

is included in $\mathcal{V}_2(\operatorname{hyp} nA, \zeta)$. Thirdly, let a belong to nA and let $x_0, \ldots, x_h$ be a path in $(\mathbb{Z}^3(\text{cyl } nA), \mathbb{E}^3(\text{cyl } nA))$ joining two sites x_0, x_h such that $(x_0 - a) \cdot w \leq -\phi(n)$ and $(x_h - a) \cdot w \geq \phi(n)$. Suppose that this path does not meet E. Since this path crosses $\operatorname{hyp} nA$ and

$$\mathcal{V}_2(nA \setminus \operatorname{cup}\{\operatorname{int} T(i) : i \in I\}, 5) \cap \operatorname{cyl} nA \subset \\ \big(\mathcal{V}_2(\operatorname{cyl}(nA \setminus D(m,n)), 5) \cap \operatorname{cyl} nA\big) \cup \bigcup_{i \in I} \mathcal{V}_2(\operatorname{cyl} \partial T(i), 5) \subset E$$

then the crossing of $\operatorname{hyp} nA$ cannot occur in $\mathcal{V}_2(nA \setminus \operatorname{cup}\{\operatorname{int} T(i) : i \in I\}, 2) \cap \operatorname{cyl} nA$; necessarily there exists i in I such that the path enters in $\operatorname{cyl} \operatorname{int} T(i)$ at a site y such that $(y - a) \cdot w < 0$ and exits it at a site z such that $(z - a) \cdot w > 0$. Hence there exist indices k_1, k_2 in $\{0, \ldots, h\}$ and i in I such that

- $\forall\, k \in \{k_1, \ldots, k_2\} \quad x_k \in \text{cyl int}\, T(i)$,
- either $k_1 = 0$ or $x_{k_1 - 1} \notin \text{cyl int}\, T(i)$,
- either $k_2 = h$ or $x_{k_2 + 1} \notin \text{cyl int}\, T(i)$,
- $(x_{k_1} - a) \cdot w < 0$ and $(x_{k_2} - a) \cdot w > 0$.

Since $\mathbb{E}^3(\mathcal{V}_2(\text{cyl } \partial T'(i), \zeta' + 2) \cap \mathcal{V}_2(\operatorname{hyp} nA, \zeta' + 3))$ is included in E_0, then no vertex of the subpath $x_{k_1}, \ldots, x_{k_2}$ belongs to $\mathcal{V}_2(\text{cyl } \partial T'(i), \zeta' + 1) \cap \mathcal{V}_2(\operatorname{hyp} nA, \zeta' + 2)$. Suppose that $k_1 \neq 0$. Then

$$x_{k_1 - 1} \notin \text{cyl int}\, T(i),\ x_{k_1} \in \text{cyl int}\, T(i),\ d_2(x_{k_1 - 1}, x_{k_1}) = 1,$$

whence $x_{k_1} \in \mathcal{V}_2(\operatorname{cyl} \partial T(i), 2) \subset \mathcal{V}_2(\text{cyl } \partial T'(i), \zeta' + 1)$. Thus

$$x_{k_1} \in \mathcal{V}_2(\operatorname{cyl} \partial T(i), 2) \cap \operatorname{slab}(a, w, -\infty, -\zeta' - 2)\,.$$

If $k_1 = 0$, then $x_{k_1} = x_0$ belongs to $\operatorname{slab}(a, w, -\infty, -\phi(n))$. In both cases, x_{k_1} belongs to the set $D_-(i)$ defined by

$$D_-(i) = \operatorname{slab}(a, w, -\infty, -\phi(n)) \cup \big(\mathcal{V}_2(\operatorname{cyl} \partial T(i), 3) \cap \operatorname{slab}(a, w, -\infty, -\zeta' - 2)\big)\,.$$

Similarly, x_{k_2} belongs to the set $D_+(i)$ defined by

$$D_+(i) = \operatorname{slab}(a, w, +\phi(n), +\infty) \cup \big(\mathcal{V}_2(\operatorname{cyl} \partial T(i), 3) \cap \operatorname{slab}(a, w, +\zeta' + 2, +\infty)\big)\,.$$

Let us define

$$l_2 = \min\{\, l : k_1 < l \leq k_2,\ x_l \in D_+(i)\,\} - 1\,,$$
$$l_1 = \max\{\, l : k_1 \leq l \leq l_2,\ x_l \in D_-(i)\,\} + 1\,.$$

Let l belong $\{\, l_1, \ldots, l_2\,\}$. By construction, x_l belongs to $\operatorname{cyl} \operatorname{int} T(i) \setminus (D_-(i) \cup D_+(i))$. Two cases can occur.

• if $d_2(x_l, \mathrm{hyp}\, nA) < \zeta' + 2$ then $d_2(x_l, \mathrm{cyl}\, \partial T'(i)) \geq \zeta' + 1$. Yet x_l is in the set $\mathcal{V}_2(\mathrm{cyl}\, T'(i), \sqrt{3})$ whence x_l belongs also to $\mathrm{cyl}\, T'(i)$ and $d_2(x_l, \mathbb{R}^3 \setminus \mathrm{cyl}\, T'(i)) > 1$.
• if $d_2(x_l, \mathrm{hyp}\, nA) \geq \zeta' + 2$ then $d_2(x_l, \mathbb{R}^3 \setminus \mathrm{cyl}\, T(i)) \geq 3$ and it follows that $d_2(x_l, \mathbb{R}^3 \setminus \mathrm{cyl}\, T'(i)) \geq 3 - \sqrt{3} > 1$.
Taking into account that $d_2(x_{l_1-1}, x_{l_1}) = 1$, $d_2(x_{l_2}, x_{l_2+1}) = 1$, we see that the subpath $x_{l_1-1}, \ldots, x_{l_2+1}$ is included in $\mathrm{int\, cyl}\, T'(i)$. Moreover x_{l_1-1} belongs to $D_-(i) \cap \mathrm{cyl}\, T'(i)$, which is by definition included in

$$\mathrm{slab}\,(a + t(i), w, -\infty, -\phi(n) + 2) \cup \big(\mathcal{V}_2(\mathrm{cyl}\, \partial T'(i), \zeta') \cap \mathrm{slab}\,(a + t(i), w, -\infty, -\zeta')\big)\,.$$

If x_{l_1-1} is in $\mathcal{V}_2(\mathrm{cyl}\, \partial T'(i), \zeta') \cap \mathrm{slab}\,(a+t(i), w, -\infty, -\zeta')$, since no edge of the latter set belongs to $E(i)$, then there exists a path γ_1 included in $\mathcal{V}_2(\mathrm{cyl}\, \partial T'(i), \zeta') \cap \mathrm{cyl}\ T'(i)$ connecting x_{l_1-1} to a site y_1 such that $\gamma_1 \cap E(i) = \varnothing$ and $(y_1 - a - t(i)) \cdot w \leq -\phi(n) + 2$. In case x_{l_1-1} is in $\mathrm{slab}\,(a+t(i), w, -\infty, -\phi(n)+2)$, we set $\gamma_1 = \varnothing$. We make a similar reasoning with x_{l_2+1} and the set $D_+(i)$ to get a path γ_2 in $\mathcal{V}_2(\mathrm{cyl}\, \partial T'(i), \zeta') \cap \mathrm{cyl}\ T'(i)$ connecting x_{l_2+1} to a site y_2 such that $\gamma_2 \cap E(i) = \varnothing$ and $(y_2 - a - t(i)) \cdot w \geq \phi(n) - 2$. In the end the path obtained by concatenating γ_1, $x_{l_1-1}, \ldots, x_{l_2+1}$, γ_2 is a path in $\mathrm{cyl}\ T'(i)$ which does not meet $E(i)$ and which connects $\mathrm{slab}\,(a+t(i), w, -\infty, -\phi(n)+2)$ and $\mathrm{slab}\,(a+t(i), w, +\phi(n)-2, +\infty)$. Since $E(i)$ is included in $\mathcal{V}_2(\mathrm{hyp}\, T'(i), \phi(n)-2)$, this stands in contradiction with the fact that $E(i)$ separates ∞ in $\mathrm{cyl}\ T'(i)$. Thus there exists no path in $(\mathbb{Z}^3(\mathrm{cyl}\ nA), \mathbb{E}^3(\mathrm{cyl}\ nA) \setminus E)$ joining $\mathrm{slab}\,(a, w, -\infty, -\phi(n))$ and $\mathrm{slab}\,(a, w, \phi(n), +\infty)$. We conclude that the set E separates ∞ in $\mathrm{cyl}\ nA$ and

$$\big\{\text{ all the edges of } E_0 \text{ are closed }\big\} \cap \bigcap_{i \in I} W(\partial T'(i), \phi(n) - 2, \zeta') \subset W(\partial nA, \phi(n), \zeta)\,.$$

Since all these events are decreasing, by the FKG inequality,

$$P(W(\partial nA, \phi(n), \zeta)) \geq (1-p)^{\mathrm{card}\, E_0} \prod_{i \in I} P(W(\partial T'(i), \phi(n) - 2, \zeta'))\ .$$

Since the model is invariant under the integer translations, for any i in I,

$$P(W(\partial T'(i), \phi(n) - 2, \zeta')) = P(W(\partial mA', \phi(n) - 2, \zeta'))\,.$$

Because $\phi(n)$ goes to ∞ as n goes to ∞,

$$\lim_{n \to \infty} P(W(\partial mA', \phi(n) - 2, \zeta')) = P(W(\partial mA', \infty, \zeta'))$$

whence, for n sufficiently large,

$$P(W(\partial mA', \phi(n) - 2, \zeta')) \geq (1/2) P(W(\partial mA', \infty, \zeta'))\,.$$

For such integers n, combining the preceding equations and passing to the logarithm,

$$\ln P(W(\partial nA,\phi(n),\zeta)) \geq \\ (\operatorname{card} I)\ln P(W(\partial mA',\phi'(m),\zeta')) \,+\, (\operatorname{card} E_0)\ln(1-p) \,-\, (\operatorname{card} I)\ln 2\,.$$

Using the previous inequalities on $\operatorname{card} I$ and $\operatorname{card} E_0$, we obtain

$$\mathcal{H}^2(nA)^{-1}\ln P(W(\partial nA,\phi(n),\zeta)) \,\geq\, \mathcal{H}^2(mA')^{-1}\ln P(W(\partial mA',\phi'(m),\zeta'))+ \\ c(\zeta,\zeta',A,A')\mathcal{H}^2(A)^{-1}(m/n+1/m+1/n^2)\ln(1-p)-\mathcal{H}^2(mA')^{-1}\ln 2\,.$$

Sending successively n to ∞ and then m to ∞ yields

$$\liminf_{n\to\infty}\mathcal{H}^2(nA)^{-1}\ln P(W(\partial nA,\phi(n),\zeta)) \\ \geq \limsup_{m\to\infty}\mathcal{H}^2(mA')^{-1}\ln P(W(\partial mA',\phi'(m),\zeta'))$$

which implies the result of the Proposition, since the inequality is valid for all rectangles A, A' such that nor $A =$ nor $A' = w$, $\mathcal{H}^2(A)>0$, $\mathcal{H}^2(A')>0$, for all functions $\phi(n)$, $\phi'(n)$ going to ∞ as n goes to ∞ and for all ζ,ζ' larger than 5. □

5.2. Properties

The surface tension τ inherits automatically some symmetry properties from the model. For instance, if f is a linear isometry of $\mathbb{R}^3$ such that $f(0)=0$ and $f(\mathbb{Z}^3)=\mathbb{Z}^3$ then $\tau\circ f=\tau$. Besides, the surface tension τ satisfies another important inequality called the weak triangle inequality. For details and results concerning this kind of inequalities, see [32,55,56].

Proposition 5.3. *(Weak triangle inequality)*
Let (ABC) be a non–degenerate triangle in $\mathbb{R}^3$ and let ν_A, ν_B, ν_C be the exterior normal unit vectors to the sides $[BC]$, $[AC]$, $[AB]$. Then

$$\mathcal{H}^1(BC)\tau(\nu_A) \,\leq\, \mathcal{H}^1(AC)\tau(\nu_B)+\mathcal{H}^1(AB)\tau(\nu_C)\,.$$

Proof. We consider first the case where $BA\cdot BC\geq 0$ and $CA\cdot CB\geq 0$. Let ν be a unit vector orthogonal to the triangle (ABC) and let ε, h be positive with $\varepsilon\leq 1\leq h$. We set $A'=A+h\nu$, $B'=B+h\nu$, $C'=C+h\nu$. Let E_0 be the set of the edges included in

$$\Big(\operatorname{cyl}\big(\mathcal{V}_2(\partial nBCC'B',4\varepsilon n)\cap\operatorname{hyp} nBCC'B'\big)\cap\mathcal{V}_2(\operatorname{hyp} nBCC'B',\zeta)\Big) \\ \cup\,\mathcal{V}_2(nABC,2\zeta)\,\cup\,\mathcal{V}_2(nA'B'C',2\zeta) \\ \cup\mathcal{V}_2(n[AA'],2\zeta)\,\cup\,\mathcal{V}_2(n[BB'],2\zeta)\,\cup\,\mathcal{V}_2(n[CC'],2\zeta)\,.$$

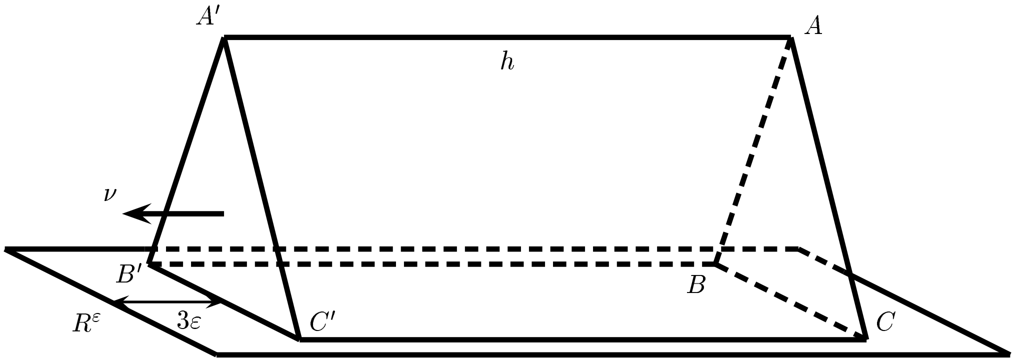

Figure 5

There exists a constant $c(\zeta)$ depending on ζ only such that

$$\operatorname{card} E_0 \leq c(\zeta)\big(n^2\mathcal{H}^2(ABC) + \varepsilon n^2(\mathcal{H}^1(BC) + h) + nh\big) .$$

Let R^ε be a rectangle in $\operatorname{hyp} BCC'B'$ such that $BCC'B' \subset R^\varepsilon \subset \mathcal{V}_2(BCC'B', 4\varepsilon)$ and $d_2(\partial R^\varepsilon, BCC'B') > 2\varepsilon$. For n large enough, so that $\varepsilon n > \zeta$, if the events

$$W(n\partial ABB'A', \varepsilon n, \zeta)\,, \quad W(n\partial ACC'A', \varepsilon n, \zeta)\,, \quad \{\text{all the edges of } E_0 \text{ are closed}\}$$

occur simultaneously, then the event $W(\partial nR^\varepsilon, \infty, \zeta)$ occurs as well; the argument here is similar to the one used in Proposition 5.2: thanks to the hypothesis $BA \cdot BC \geq 0$, $CA \cdot CB \geq 0$, the set $\mathcal{V}_2(nABB'A' \cup nACC'A', \varepsilon n)$ is included in the set $\mathcal{V}_2(\text{cyl } nBCC'B', \varepsilon n)$ and does not intersect $\mathcal{V}_2(\operatorname{cyl} \partial nR^\varepsilon, \zeta)$, so that the separating sets will be correctly localized to match the definition of the event $W(n\partial R^\varepsilon, \infty, \zeta)$. By the FKG inequality, this inclusion implies

$$(1-p)^{\operatorname{card} E_0} P(W(n\partial ABB'A', \varepsilon n, \zeta)) P(W(n\partial ACC'A', \varepsilon n, \zeta)) \leq P(W(n\partial R^\varepsilon, \infty, \zeta)).$$

Using the estimation of $\operatorname{card} E_0$ and Proposition 5.2, we get

$$\begin{aligned}\mathcal{H}^2(R^\varepsilon)\tau(\nu_A) \leq \mathcal{H}^2(ABB'A')\tau(\nu_C) + \mathcal{H}^2(ACC'A')\tau(\nu_B)&\\ -c(\zeta)\,(\mathcal{H}^2(ABC) + (\mathcal{H}^1(BC) + h)\varepsilon) \ln(1-p)&\,.\end{aligned}$$

Letting first h go to ∞ and then ε go to 0 yields the weak triangle inequality for the triangle (ABC).

Let now A, B, C be three points such that $BA \cdot BC < 0$, $CA \cdot CB \geq 0$. Let D be the orthogonal projection of B on $[AC]$. Then $BC \cdot BD > 0$, $DB \cdot DA = 0$, $BA \cdot BD > 0$. We apply the weak triangle inequality to the triangles (BCD) and (BDA):

$$\begin{aligned}\mathcal{H}^1(BC)\tau(\nu_A) &\leq \mathcal{H}^1(BD)\tau(\nu_{BD}) + \mathcal{H}^1(DC)\tau(\nu_B)\,,\\ \mathcal{H}^1(BD)\tau(\nu_{BD}) &\leq \mathcal{H}^1(AB)\tau(\nu_C) + \mathcal{H}^1(AD)\tau(\nu_B)\,,\end{aligned}$$

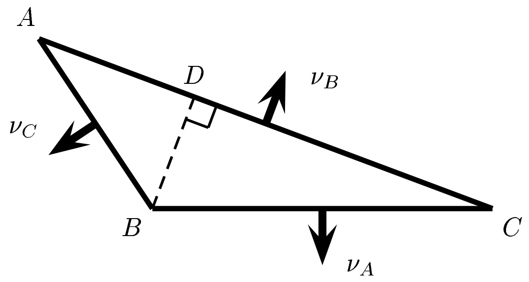

Figure 6: a triangle with $BA \cdot BC < 0$

where ν_{BD} is a unit vector orthogonal to $[BD]$. Combining the two inequalities, we get the weak triangle inequality for the triangle (ABC). The case $BA \cdot BC \geq 0$, $CA \cdot CB < 0$ is similar. $\square$

The weak triangle inequality implies a lot of nice properties for the surface tension.

Corollary 5.4. *The homogeneous extension τ_0 of τ to $\mathbb{R}^3$ defined by*

$$\forall\, w \in \mathbb{R}^3 \qquad \tau_0(w) \;=\; \begin{cases} |w|_2 \tau(w/|w|_2) & \text{if } w \neq 0 \\ 0 & \text{if } w = 0 \end{cases}$$

is a convex function.

The convexity of τ_0 is in fact equivalent to the weak triangle inequality.

Corollary 5.5. *The surface tension τ is bounded and continuous.*

Proof. Let w be a unit vector and let A be a unit square orthogonal to w. Let $E(n)$ be the set of the edges included in $\mathrm{cyl}\, nA \cap \mathcal{V}_2(\mathrm{hyp}\, nA, 5)$. Then $\mathrm{card}\, E(n) \leq 72(n+2)^2$ and

$$P(W(\partial nA, \infty, 5)) \;\geq\; P(\text{the edges of } E(n) \text{ are closed}) \;\geq\; (1-p)^{\mathrm{card}\, E(n)}\,.$$

Passing to the limit, we get $\tau(w) \leq -72 \ln(1-p)$. Since τ_0 is homogeneous, convex (Corollary 5.4) and bounded on S^2, it is finite everywhere. By a standard result of convex analysis [63, Corollary 10.1.1], it follows that τ_0 is continuous, as well as τ. $\square$

The next Corollary is a consequence of [32, Theorem 3.1]: the weak triangle inequality automatically implies the weak simplex inequality.

Corollary 5.6. *(Weak simplex inequality)*
Let $(ABCD)$ be a non–degenerate pyramid in $\mathbb{R}^3$. Let ν_A, ν_B, ν_C, ν_D be the external unit normal vectors to the faces (BCD), (ACD), (ABD), (ABC). Then

$$\mathcal{H}^2(BCD)\tau(\nu_A) \;\leq\; \mathcal{H}^2(ABC)\tau(\nu_D) + \mathcal{H}^2(ACD)\tau(\nu_B) + \mathcal{H}^2(ABD)\tau(\nu_C)\,.$$

Corollary 5.7. *For any fixed value of p in the supercritical regime $p > p_c$, the surface tension $\tau : S^2 \mapsto \mathbb{R}^+$ is bounded away from* 0.

Proof. Aizenman, Chayes, Chayes, Fröhlich and Russo have proved that $\tau(1,0,0)$ is positive in the supercritical regime $p > p_c$ (see [1, Theorem 1.1]). Suppose that $\tau(\nu) = 0$ for some ν in S^2. Let f_1, f_2 be two linear isometries of $\mathbb{R}^3$ such that $f_1(0) = f_2(0) = 0$, $f_1(\mathbb{Z}^3) = f_2(\mathbb{Z}^3) = \mathbb{Z}^3$ and $(f_1(\nu), f_2(\nu), \nu)$ is an orthonormal basis of $\mathbb{R}^3$. Then $\tau(f_1(\nu)) = \tau(f_2(\nu)) = 0$. Applying the weak simplex inequality to a pyramid having for basis a triangle orthogonal to $(1,0,0)$ and whose three other faces are orthogonal to $f_1(\nu), f_2(\nu), \nu$, we obtain that $\tau(1,0,0) = 0$, a contradiction. Thus the surface tension τ does not vanish on S^2. By Corollary 5.5, τ is continuous on S^2. Therefore τ is bounded away from 0 on S^2. □

The previous properties of τ can equivalently be described through its Wulff crystal

$$\mathcal{W}_\tau = \{\, x \in \mathbb{R}^3 : x \cdot w \leq \tau(w) \text{ for all } w \text{ in } S^2 \,\}\,.$$

Corollary 5.8. *The Wulff crystal $\mathcal{W}_\tau$ associated to τ is bounded, closed, convex and contains 0 in its interior. If f is a linear isometry of $\mathbb{R}^3$ such that $f(0) = 0$ and $f(\mathbb{Z}^3) = \mathbb{Z}^3$ then $f(\mathcal{W}_\tau) = \mathcal{W}_\tau$. The surface tension τ is the support function of its Wulff crystal, i.e.,*

$$\forall\, \nu \in S^2 \qquad \tau(\nu) = \sup\, \{ x \cdot \nu : x \in \mathcal{W}_\tau \,\}\,.$$

These properties are equivalent to the symmetry properties of τ and Corollaries 5.4, 5.5, 5.7. The function τ is the support function of $\mathcal{W}_\tau$ because τ_0 is convex and coincides with its bipolar, see for instance [63, Corollary 13.2.1], [37, Proposition 3.5] or [32, Theorem 2.1, Corollary 3.6].

5.3. Separating sets

With the help of the surface tension, we estimate next the probability of the occurrence of a separating set of closed edges near an hyperplane.

Definition 5.9. Let A be a planar set in $\mathbb{R}^3$ and let r be positive. We denote by $S(A,r)$ the event that there exists a set of closed edges in $\operatorname{cyl} A \cap \mathcal{V}_2(\operatorname{hyp} A, r)$ which separates ∞ in $\operatorname{cyl} A$, that is,

$$\begin{aligned} S(A,r) = \{\, \omega \in \Omega : \exists\, E \subset \mathbb{E}^3(\operatorname{cyl} A \cap \mathcal{V}_2(\operatorname{hyp} A, r)),& \\ \forall\, e \in E \quad \omega(e) = 0,\ E \text{ separates } \infty \text{ in } \operatorname{cyl} A \,\}\,.& \end{aligned}$$

Lemma 5.10. *There exists a positive constant $c(p,\zeta)$ depending on p and ζ only such that, for any x in $\mathbb{R}^3$, for any orthonormal basis u,v,w of $\mathbb{R}^3$ and for any positive r,δ,*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P(S(n\text{square}\,(x,u,v,\delta),nr))\ \le\ -\tau(w)\delta^2+c(p,\zeta)r\delta\,.$$

Proof. Let ε be positive smaller than both δ and r and let n be an integer such that $n\varepsilon>2\zeta$. Let E_0 be the set of the edges included in

$$\Big(\mathcal{V}_2(\text{cyl}\,n\partial\text{square}\,(x,u,v,\delta),\zeta)\cap\mathcal{V}_2(\text{hyp}\,(nx,w),nr)\Big)\cup \\ \Big(\text{cyl}\,(n\text{square}\,(x,u,v,\delta+\varepsilon)\setminus n\text{square}\,(x,u,v,\delta))\cap\mathcal{V}_2(\text{hyp}\,(nx,w),\zeta)\Big)\,.$$

Suppose that the event $S(n\text{square}\,(x,u,v,\delta),nr)$ occurs, and let E_S be a set of closed edges realizing it. Suppose also that all the edges of E_0 are closed. Define $E=E_0\cup E_S$. We claim that the set of closed edges E realizes the event $W(\partial n\text{square}\,(x,u,v,\delta+\varepsilon),nr)$. Firstly, $E\subset\mathcal{V}_2(\text{hyp}\,(nx,w),nr)$. Secondly,

$$E\cap\mathcal{V}_2(\text{cyl}\,\partial n\text{square}\,(x,u,v,\delta+\varepsilon),\zeta)\ \subset \\ E_0\cap\mathcal{V}_2(\text{cyl}\,\partial n\text{square}\,(x,u,v,\delta+\varepsilon),\zeta)\ \subset\ \mathcal{V}_2(\text{hyp}\,(nx,w),\zeta)\,.$$

Thirdly, let a in hyp (nx,w) and let $x_0,\dots,x_l$ be a path in cyl $n$square $(x,u,v,\delta+\varepsilon)$ joining two sites x_0,x_l such that $(x_0-a)\cdot w\le -nr$ and $(x_l-a)\cdot w\ge nr$. Suppose that this path does not meet E. Let

$$i_2=\min\{\,i:(x_i-a)\cdot w\ge nr\,\}-1\,,\quad i_1=\max\{\,i\le i_2:(x_i-a)\cdot w\le -nr\,\}+1\,.$$

Then the vertices of the subpath

$$\gamma(i_1,i_2)=x_{i_1},\dots,x_{i_2}$$

belong to $\mathcal{V}_2(\text{hyp}\,(nx,w),nr)$. Yet the edges of E_0 separate the sets

$$\text{cyl}\,(n\text{square}\,(x,u,v,\delta+\varepsilon)\setminus n\text{square}\,(x,u,v,\delta))\,,\quad \text{cyl}\;n\text{square}\,(x,u,v,\delta)$$

inside $\mathcal{V}_2(\text{hyp}\,(nx,w),nr)$. Two cases are possible.

• The subpath $\gamma(i_1,i_2)$ is included in cyl $(n\text{square}\,(x,u,v,\delta+\varepsilon)\setminus n\text{square}\,(x,u,v,\delta))$. However E_0 separates $\{\,y:(y-a)\cdot w<-\zeta\,\}$ and $\{\,y:(y-a)\cdot w>\zeta\,\}$ in this set. Thus this case cannot occur.

• The subpath $\gamma(i_1,i_2)$ is included in cyl $n$square (x,u,v,δ). Because no edge of $\gamma(i_1,i_2)$ belongs to E_0, the edges $\langle x_{i_1},x_{i_1+1}\rangle$ and $\langle x_{i_2-1},x_{i_2}\rangle$ are not in E_0. However they are in $\mathcal{V}_2(\text{hyp}\,(nx,w),nr)$, whence

$$d_2\big(\{x_{i_1},x_{i_2}\},\mathbb{R}^3\setminus\text{cyl}\;n\text{square}\,(x,u,v,\delta)\big)\ \ge\ \zeta-1\,,$$

and both x_{i_1-1} and x_{i_2+1} also belong to cyl $n$square(x,u,v,δ). Thus the path

$$x_{i_1-1},\ \langle x_{i_1-1},x_{i_1}\rangle, \gamma(i_1,i_2),\ \langle x_{i_2},x_{i_2+1}\rangle,\ x_{i_2+1}$$

joins slab$(nx,w,-\infty,-nr)$ and slab$(nx,w,+nr,+\infty)$ inside cyl $n$square(x,u,v,δ) without meeting E_S. This contradicts the fact that the set of edges E_S separates ∞ in cyl $n$square(x,u,v,δ) (recall that $E_S\subset\mathcal{V}_2(\mathrm{hyp}\,(nx,w),nr)$). Thus there exists no path in the graph

$$(\mathbb{Z}^3(\mathrm{cyl}\ n\mathrm{square}\,(x,u,v,\delta+\varepsilon)),\mathbb{E}^3(\mathrm{cyl}\ n\mathrm{square}\,(x,u,v,\delta+\varepsilon))\setminus E)$$

joining slab$(a,w,-\infty,-nr)$ and slab$(a,w,nr,+\infty)$. We conclude that the set E separates ∞ in cyl $n$square$(x,u,v,\delta+\varepsilon)$ and

$$\big\{\,\omega:\forall\, e\in E_0\quad \omega(e)=0\,\big\}\cap S(n\mathrm{square}\,(x,u,v,\delta),nr)\ \subset\ W(\partial n\mathrm{square}\,(x,u,v,\delta+\varepsilon),nr)\,.$$

Since all these events are decreasing, by the FKG inequality,

$$P(S(n\mathrm{square}\,(x,u,v,\delta),nr))\,(1-p)^{\mathrm{card}\,E_0}\ \leq\ P(W(\partial n\mathrm{square}\,(x,u,v,\delta+\varepsilon),nr))\,.$$

There exists a constant $c(\zeta)$ such that card$\,E_0\leq c(\zeta)n^2\delta(r+\varepsilon)$, whence, passing to the logarithm,

$$\ln P(S(n\mathrm{square}\,(x,u,v,\delta),nr))\leq \ln P(W(\partial n\mathrm{square}\,(x,u,v,\delta+\varepsilon),nr)) -c(\zeta)n^2\delta(r+\varepsilon)\ln(1-p)\,.$$

Letting n go to ∞, applying Proposition 5.2, and sending ε to 0, we obtain the desired inequality. $\square$

Remark. If $A_1,\dots,A_l$ are l disjoint subsets of a planar set A of $\mathbb{R}^3$, then, for any positive r, the sets cyl$(A_1,r),\dots,$cyl(A_l,r) are pairwise disjoint, hence the events $S(A_1,r),\dots,S(A_l,r)$ are independent so that

$$P(S(A,r))\leq P(S(A_1,r))\times\cdots\times P(S(A_l,r))\,.$$

Lemma 5.11. *Let O be a planar open set in $\mathbb{R}^3$ and let $\phi(n)$ be a function from $\mathbb{N}$ to $\mathbb{R}^+\cup\{\infty\}$ such that $\lim_{n\to\infty}\phi(n)=\infty$. We have*

$$\liminf_{n\to\infty}\frac{1}{n^2}\ln P(S(nO,\phi(n)))\ \geq\ -\mathcal{H}^2(O)\,\tau(\mathrm{nor}\,O)\,.$$

Proof. Let x belong to O and let (u,v) be an orthonormal basis of hyp$\,O$. Let δ be positive and set

$$M(\delta)\ =\ \{\,(i,j)\in\mathbb{Z}^2:\mathrm{square}\,(x+\delta(iu+jv),u,v,\delta)\cap O\neq\varnothing\,\}\,.$$

We see that the set

$$\bigcup_{(i,j)\in M(\delta)} \operatorname{square}(x+\delta(iu+jv),u,v,\delta)$$

contains the set O. Let E be the set of the edges included in the union

$$\bigcup_{(i,j)\in M(\delta)} \mathcal{V}_2(\operatorname{cyl}\partial n\operatorname{square}(x+\delta(iu+jv),u,v,\delta),\zeta+2)\cap\mathcal{V}_2(\operatorname{hyp} nO,\zeta+2)\,.$$

There exists a constant $c(\zeta)$ such that $\operatorname{card} E \leq c(\zeta)n\delta \operatorname{card} M(\delta)$. If all the events

$$W(\partial n\operatorname{square}(x+\delta(iu+jv),u,v,\delta),\phi(n)),\quad (i,j)\in M(\delta),$$

occur and all the edges of E are closed, then $S(nO,\phi(n))$ occurs as well, provided n is large enough so that $\phi(n)>\zeta+2$ (the proof of this assertion is similar to the one done in the course of the proof of Proposition 5.2). By the FKG inequality,

$$P(S(nO,\phi(n)))\ \geq\ (1-p)^{\operatorname{card} E}\prod_{(i,j)\in M(\delta)} P\big(W(\partial n\operatorname{square}(x+\delta(iu+jv),u,v,\delta),\phi(n))\big)$$

whence, by Proposition 5.2,

$$\liminf_{n\to\infty}\frac{1}{n^2}\ln P(S(nO,\phi(n)))\ \geq\ -\tau(\operatorname{nor} O)\,\delta^2\operatorname{card} M(\delta)\,.$$

However, we have $\mathcal{H}^2(O)\,=\,\inf_{\delta>0}\delta^2\operatorname{card} M(\delta)\,=\,\lim_{\delta\to 0}\delta^2\operatorname{card} M(\delta)$. Sending δ to zero in the previous inequality, we obtain the claim of the Lemma. ☐

Lemma 5.12. *There exists a positive constant $c(\zeta,p)$ depending on ζ and p only such that, for any planar open set O in $\mathbb{R}^3$, for any positive r,*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P(S(nO,nr))\ \leq$$
$$-\mathcal{H}^2(O)\tau(\operatorname{nor} O)+c(\zeta,p)\inf_{\delta>0}\Big(r\delta^{-1}\mathcal{H}^2(O)+\mathcal{H}^2(\{\,x\in O: d_2(x,\partial O)\leq 2\delta\,\})\Big)\,.$$

Proof. Let x belong to O and let (u,v) be an orthonormal basis of $\operatorname{hyp} O$. Let δ,ε be positive with $\varepsilon<\delta$ and set

$$N(\delta)\ =\ \{\,(i,j)\in\mathbb{Z}^2:\operatorname{square}(x+\delta(iu+jv),u,v,\delta)\subset O\,\}\,.$$

Since

$$S(nO,nr)\subset\bigcap_{(i,j)\in N(\delta)} S(n\operatorname{square}(x+\delta(iu+jv),u,v,\delta-\varepsilon),nr)$$

and since the sets $\text{square}\,(x+\delta(iu+jv),u,v,\delta-\varepsilon)$, $(i,j)\in N(\delta)$, are pairwise disjoint, then, using the independence of the associated events,

$$P(S(nO,nr)) \le \prod_{(i,j)\in N(\delta)} P(S(n\text{square}\,(x+\delta(iu+jv),u,v,\delta-\varepsilon),nr))\,.$$

Lemma 5.10 then implies

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P(S(nO,nr)) \le -\tau(\text{nor}\,O)\,(\delta-\varepsilon)^2\text{card}\,N(\delta)+c(p,\zeta)\,r(\delta-\varepsilon)\,\text{card}\,N(\delta).$$

We let ε go to 0 in this inequality. The fact that τ is bounded on S^2 (the bound depending only on p), together with the inequalities

$$\mathcal{H}^2(\{\,x\in O: d_2(x,\partial O)>2\delta\,\}) \le \delta^2\text{card}\,N(\delta) \le \mathcal{H}^2(O)$$

yield the desired result. □

Corollary 5.13. *For O a planar open set in $\mathbb{R}^3$ such that $\mathcal{H}^2(\partial O)=0$,*

$$\begin{aligned}\liminf_{r\to0}\ \liminf_{n\to\infty}\ \frac{1}{n^2}\ln P(S(nO,nr)) &= \limsup_{r\to0}\ \limsup_{n\to\infty}\ \frac{1}{n^2}\ln P(S(nO,nr))\\ &= -\mathcal{H}^2(O)\tau(\text{nor}\,O)\,.\end{aligned}$$

Corollary 5.14. *There exists a positive constant $c(\zeta,p)$ such that, for any x in $\mathbb{R}^3$, any positive ρ,η with $\eta\le\rho$, any w in S^2,*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P(S(n\text{disc}\,(x,\rho,w),n\eta)) \le -\tau(w)\pi\rho^2(1-c(\zeta,p)\sqrt{\eta/\rho})\,.$$

Proof. We apply Lemma 5.12 with $O=\text{disc}\,(x,\rho,w)$, $r=\eta$ and we choose $\delta=\sqrt{\rho\eta}$. We use also the fact that τ is bounded away from 0 (Corollary 5.7). □

CHAPTER 6

THE SURFACE ENERGY OF A CACCIOPPOLI SET

In this chapter we first recall some facts concerning the class of the Caccioppoli sets, introduced by Caccioppoli and subsequently studied by De Giorgi. Next, we define the surface energy associated to the surface tension of the model. We prove then two important approximation results.

6.1. Caccioppoli sets

For a complete account of the theory of Caccioppoli sets, see the references [24,25,26,27,33,35,41,52,77]. The perimeter of a Borel set E of $\mathbb{R}^3$ in an open set O is defined as

$$\mathcal{P}(E,O) \,=\, \sup\,\Big\{\,\int_E \operatorname{div} g(x)\, d\mathcal{L}^3(x) : g\in C_0^\infty(O,B(0,1))\,\Big\}$$

where $C_0^\infty(O,B(0,1))$ is the set of the C^∞ vector functions from $\mathbb{R}^3$ to $B(0,1)$ having a compact support included in O and div is the usual divergence operator. We define also $\mathcal{P}(E)=\mathcal{P}(E,\mathbb{R}^3)$. The set E is said to have finite perimeter in an open set O if $\mathcal{P}(E,O)$ is finite. The set E is said to be of locally finite perimeter or to be a Caccioppoli set if $\mathcal{P}(E,O)$ is finite for every bounded open set O of $\mathbb{R}^3$. The set E is of finite perimeter if $\mathcal{P}(E)=\mathcal{P}(E,\mathbb{R}^3)$ is finite. A set E has finite perimeter in an open set O if and only if its characteristic function χ_E is a function of bounded variation in O. The distributional derivative $\nabla\chi_E$ of χ_E is then a vector Radon measure and $\mathcal{P}(E,O)=||\nabla\chi_E||(O)$, where $||\nabla\chi_E||$ is the total variation measure of $\nabla\chi_E$. The perimeter $\mathcal{P}$ is lower semicontinuous on the space $(\mathcal{B}(\mathbb{R}^3),L^1_{\mathrm{loc}})$: if $(E_n)_{n\in\mathbb{N}}$ is a sequence of sets in $\mathcal{B}(\mathbb{R}^3)$ converging towards a set E in L^1_{loc}, then $\mathcal{P}(E,O)\le\liminf_{n\to\infty}\mathcal{P}(E_n,O)$ for any open set O.

Compactness property. For every bounded domain U and every $\lambda>0$, the collection of sets $\{\,E\in\mathcal{B}(U):\mathcal{P}(E)\le\lambda\,\}$ is compact for the topology L^1.

This result is crucial for our purposes. Indeed, this compactness property replaces in dimension three the combinatorial bound associated to skeletons in dimension two. Let us quote the original formulation of [27], Teorema 2.4, or [25], Teorema I.

Theorem 6.1. *Dati un insieme limitato $L \subset \mathbb{R}^3$, un numero positivo p, ed una successione $\{E_h\}$ di insiemi verificante le condizioni*

$$E_h \subset L\,, \qquad \mathcal{P}(E_h) \leq p\,, \qquad h = 1, 2, \ldots$$

esistono un insieme $E \subset \mathbb{R}^3$ ed una successione $\{E_{h_i}\}$ subordinata alla successione $\{E_h\}$, tali che risulti:

$$\lim_{i\to\infty} mis(E - E_{h_i}) + mis(E_{h_i} - E) \;=\; 0\,.$$

The compactness property is also an immediate consequence of the compactness Theorem stated in [52], chapter 2, p.70. Modern presentations are formulated through functions of bounded variations: if O is an open bounded domain with sufficiently regular boundary (say C^1), then a set of functions in $L^1(O)$ uniformly bounded in BV–norm is relatively compact in $L^1(O)$ (see any of the following references: [33], Section 5.2.3, [41], Theorem 1.19, [77], Corollary 5.3.4). To deduce the compactness result on sets of finite perimeter, we choose an open bounded domain O with regular boundary containing U in its interior. We embed $\mathcal{B}(U)$ in $L^1(O)$ by associating to a Borel set E of $\mathcal{B}(U)$ its characteristic function χ_E and we simply remark that the set $\{\,\chi_E : E \in \mathcal{B}(U)\,\}$ is a closed subset of $L^1(O)$.

Let E be a Caccioppoli set. Its reduced boundary $\partial^* E$ consists of the points x such that
• $||\nabla\chi_E||(B(x,r)) > 0$ for any $r > 0$,
• if $\nu_r(x) = -\nabla\chi_E(B(x,r))/||\nabla\chi_E||(B(x,r))$ then, as r goes to 0, $\nu_r(x)$ converges towards a limit $\nu_E(x)$ such that $|\nu_E(x)|_2 = 1$.
For a point x belonging to $\partial^* E$, the vector $\nu_E(x)$ is called the generalized exterior normal to E at x. A unit vector ν is called the measure theoretic exterior normal to E at x if

$$\lim_{r\to 0} r^{-3}\mathcal{L}^3(B_-(x,r,\nu)\setminus E) \;=\; 0\,, \quad \lim_{r\to 0} r^{-3}\mathcal{L}^3(B_+(x,r,\nu)\cap E) \;=\; 0\,.$$

At each point x of the reduced boundary $\partial^* E$ of E, the generalized exterior normal $\nu_E(x)$ is also the measure theoretic exterior normal to E at x and moreover

$$\forall\,\varepsilon > 0 \qquad \lim_{r\to 0} \quad (\pi r^2)^{-1}\mathcal{H}^2\big(\partial^* E\cap\{\,y \in B(x,r) : |(y-x)\cdot\nu_E(x)| \leq \varepsilon|y-x|_2\,\}\big) \;=\; 1\,,$$

$$\text{for } \mathcal{H}^2 \text{ almost all } x \text{ in } \partial^* E, \qquad \lim_{r\to 0} \quad (\pi r^2)^{-1}\mathcal{H}^2(B(x,r)\cap\partial^* E) \;=\; 1\,.$$

The map $x \in \partial^* E \mapsto \nu_E(x) \in S^2$ is $||\nabla\chi_E||$ measurable. For any Borel set A of $\mathbb{R}^3$,

$$||\nabla\chi_E||(A) = \mathcal{H}^2(A\cap\partial^* E)\,, \qquad \nabla\chi_E(A) = \int_{A\cap\partial^* E} -\nu_E(x)\,d\mathcal{H}^2(x)\,.$$

Let $f : \partial^* E \mapsto \mathbb{R}$ be a $||\nabla\chi_E||$ measurable bounded function. By the Besicovitch differentiation Theorem [6,11] applied to the measure $||\nabla\chi_E||$, for $\mathcal{H}^2$ almost all x in $\partial^* E$,

$$\lim_{r\to 0} \quad \frac{1}{\pi r^2}\int_{B(x,r)\cap\partial^* E} f(y)\, d\mathcal{H}^2(y) \;=\; f(x)\,.$$

Remark. The version of the differentiation Theorem for a general measure is due to Younovitch [76] and Besicovitch [11, 12]. The classical textbooks on measure theory usually propose very general versions of this Theorem (see for instance [44, Section 10.3, Theorem 1] or [35, Theorem 2.9.8]). Here we need only the version dealing with radon measures and Euclidean balls in $\mathbb{R}^3$ [77, Theorem 1.3.8]. Assouad and Quentin de Gromard [6] propose a proof of this result which is considerably simpler than the original proof of Besicovitch or its modern versions.

The reduced boundary $\partial^* E$ is countably 2–rectifiable, that is $\partial^* E \subset N \cup \bigcup_{i\in\mathbb{N}} M_i$ where $\mathcal{H}^2(N) = 0$ and each M_i is a 2-dimensional embedded C^1 submanifold of $\mathbb{R}^3$.

Remark. Since the reduced boundary is 2–rectifiable, some of the previous density and differentiation results follow from more general results of the theory of rectifiable sets. For instance, [35, Theorem 3.2.19] implies that $\partial^* E$ has 2 dimensional density 1 at $\mathcal{H}^2$ almost all its points. Whenever E is a set of finite perimeter, the balls $B(x,r)$, $x \in \partial^* E$, $r > 0$, are a $\mathcal{H}^2$ Vitali relation in the sense of Federer [35, 2.8.16] and we can apply the results of [35, chapter 2.9] to the space $\partial^* E$ endowed with the induced Euclidean metric; for instance the above statement on the differentiation of integrals is implied by [35, Theorem 2.9.8], or even by the fact that a $||\nabla\chi_E||$ measurable map is approximatively continuous $||\nabla\chi_E||$ almost everywhere [35, Corollary 2.9.13].

Next, we state a useful result (see [41, remark 2.14]). If E is a Caccioppoli set, then for any x in $\mathbb{R}^3$ and for $\mathcal{H}^1$ almost all positive radius r,

$$\mathcal{P}(E\cap B(x,r)) \;=\; \mathcal{P}(E, \operatorname{int} B(x,r)) \;+\; \mathcal{H}^2(E\cap\partial B(x,r))\,.$$

We recall finally the isoperimetric inequality and the Gauss–Green Theorem. There exist two constants $b_{\text{iso}}, c_{\text{iso}}$ depending only upon the dimension such that, for any Caccioppoli set E, any ball $B(x,r)$,

$$\min\big(\mathcal{L}^3(E\cap B(x,r)), \mathcal{L}^3((\mathbb{R}^3\setminus E)\cap B(x,r))\big) \;\leq\; b_{\text{iso}}\, \mathcal{P}(E, \operatorname{int} B(x,r))^{3/2}\,,$$
$$\min\big(\mathcal{L}^3(E), \mathcal{L}^3(\mathbb{R}^3\setminus E)\big) \;\leq\; c_{\text{iso}}\, \mathcal{P}(E)^{3/2}\,.$$

For any function f in $C^1_0(\mathbb{R}^3, \mathbb{R}^3)$, any Caccioppoli set E,

$$\int_E \operatorname{div}\, \mathrm{f}(x)\, d\mathcal{L}^3(x) \;=\; \int_{\partial^* E} f(x)\cdot\nu_E(x)\, d\mathcal{H}^2(x)\,.$$

6.2. The surface energy

The results of this section and the following one are valid for any function τ from S^2 to $\mathbb{R}^+$ satisfying the following hypothesis.

Hypothesis on τ. The homogeneous extension τ_0 of τ to $\mathbb{R}^3$ defined by

$$\forall\, x \in \mathbb{R}^3 \qquad \tau_0(x) \;=\; \begin{cases} |x|_2 \tau(x/|x|_2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is convex. The minimum $\tau_{\min} = \inf\{\tau(\nu) : \nu \in S^2\}$ of τ on S^2 is positive.

Remark. The above hypothesis implies in particular that τ is continuous. Indeed, the function τ_0 is convex and finite everywhere.

Remark. By Corollaries 5.4, 5.5, 5.7, the surface tension τ extracted from the three dimensional Bernoulli percolation model satisfies the above hypothesis.

The Wulff crystal of τ is the set

$$\mathcal{W}_\tau \;=\; \{\, x \in \mathbb{R}^3 : x \cdot w \leq \tau(w) \text{ for all } w \text{ in } S^2 \,\}\,.$$

Since τ is continuous and bounded away from 0 then its crystal $\mathcal{W}_\tau$ is convex, closed, bounded and contains the origin 0 in its interior [37, Proposition 3.5]. Moreover τ is the support function of $\mathcal{W}_\tau$, that is,

$$\forall\, \nu \in S^2 \qquad \tau(\nu) \;=\; \sup\{x \cdot \nu : x \in \mathcal{W}_\tau\,\}\,.$$

This last property stems from the fact that τ_0 coincides with its bipolar (this is not true in general, if τ_0 is not convex). For more details and the proofs of the above facts, see [37].

Definition 6.2. The surface energy $\mathcal{I}(A,O)$ of a Borel set A of $\mathbb{R}^3$ in an open set O is defined as

$$\mathcal{I}(A,O) \;=\; \sup\;\Big\{ \int_A \operatorname{div} f(x)\, d\mathcal{L}^3(x) : f \in C^1_0(O,\mathcal{W}_\tau) \Big\}$$

where $C^1_0(O,\mathcal{W}_\tau)$ is the set of the C^1 vector functions defined on $\mathbb{R}^3$ with values in $\mathcal{W}_\tau$ having compact support included in O and div is the usual divergence operator. We define also $\mathcal{I}(A) = \mathcal{I}(A,\mathbb{R}^3)$.

For a fixed function f in $C^1_0(O,\mathcal{W}_\tau)$, the map

$$A \in (\mathcal{B}(\mathbb{R}^3), L^1_{\text{loc}}) \mapsto \int_A \operatorname{div} f(x)\, d\mathcal{L}^3(x)$$

is continuous. Thus $\mathcal{I}(\cdot,O)$, being the supremum of all these maps, is lower semi-continuous.

Lemma 6.3. *Setting* $\tau_{\min} = \inf\{\tau(\nu) : \nu \in S^2\}$, *we have for any open set* O

$$\forall A \in \mathcal{B}(\mathbb{R}^3) \qquad \tau_{\min}\mathcal{P}(A,O) \le \mathcal{I}(A,O) \le ||\tau||_\infty \mathcal{P}(A,O)\,.$$

Proof. The hypothesis on τ implies that the crystal $\mathcal{W}_\tau$ is bounded and contains the origin 0 in its interior. More precisely, $B(0,\tau_{\min}) \subset \mathcal{W}_\tau \subset B(0,||\tau||_\infty)$. Indeed, let x belong to $B(0,\tau_{\min})$. Then for any ν in S^2, $x\cdot\nu \le \tau_{\min} \le \tau(\nu)$, hence x belongs to $\mathcal{W}_\tau$. Let x belong to $\mathcal{W}_\tau$. Then $|x|_2 = \sup\{\, x\cdot\nu : \nu \in S^2\} \le ||\tau||_\infty$ and x belongs to $B(0,||\tau||_\infty)$. Considering the definitions of $\mathcal{P}(A,O)$ and $\mathcal{I}(A,O)$, the desired inequalities follow easily. □

Corollary 6.4. *The functional* $\mathcal{I}$ *is a good rate function on the space* $(\mathcal{B}(\mathbb{R}^3), L^1_{\text{loc}})$, *i.e., for any* λ *in* $\mathbb{R}^+$, *the level set* $\{\, E \in \mathcal{B}(\mathbb{R}^3) : \mathcal{I}(E) \le \lambda\,\}$ *is compact for the topology* L^1_{loc}.

Proof. Let λ be positive and let $(E_n)_{n\in\mathbb{N}}$ be a sequence of Borel subsets of $\mathcal{B}(\mathbb{R}^3)$ such that $\mathcal{I}(E_n) \le \lambda$ for any n in $\mathbb{N}$. By Lemma 6.3, together with the identity on the perimeter recalled previously, for any n in $\mathbb{N}$, and $\mathcal{H}^1$ almost all positive r,

$$\mathcal{P}(E_n \cap B(0,r)) = \mathcal{P}(E_n, \operatorname{int} B(0,r)) + \mathcal{H}^2(E_n \cap \partial B(0,r)) \le ||\tau||_\infty(\lambda + 4\pi r^2)\,.$$

Let $(r_m)_{m\in\mathbb{N}}$ be an increasing sequence of positive real numbers going to ∞ and such that

$$\forall\, n,m \in \mathbb{N} \qquad \mathcal{P}(E_n \cap B(0,r_m)) \le ||\tau||_\infty(\lambda + 4\pi r_m^2)\,.$$

By the compactness property (Theorem 6.1), for each m, there exists a Borel subset A_m of $B(0,r_m)$ and a subsequence of $(E_n)_{n\in\mathbb{N}}$ converging to A_m. By a standard diagonal argument, we may extract a subsequence $(E_{\phi(n)})_{n\in\mathbb{N}}$ such that:

$$\forall\, m \in \mathbb{N} \qquad \lim_{n\to\infty} \mathcal{L}^3_{B(0,r_m)}\big(E_{\phi(n)} \Delta A_m\big) = 0\,.$$

In particular,

$$\forall\, m \in \mathbb{N} \qquad \mathcal{L}^3_{B(0,r_m)}\big(A_{m+1} \Delta A_m\big) = 0\,.$$

Let $A = \bigcup_{m\in\mathbb{N}} A_m$. The previous identities imply that the subsequence $(E_{\phi(n)})_{n\in\mathbb{N}}$ converges towards A for the topology L^1_{loc}. Since $\mathcal{I}$ is lower semicontinuous, the set A satisfies in addition $\mathcal{I}(A) \le \lambda$. □

We show now that the surface energy is the surface integral of τ on the reduced boundary.

Proposition 6.5. *The surface energy* $\mathcal{I}(A,O)$ *of a Borel set* A *of* $\mathbb{R}^3$ *of finite perimeter in an open set* O *is equal to*

$$\mathcal{I}(A,O) = \int_{\partial^* A \cap O} \tau(\nu_A(x))\, d\mathcal{H}^2(x)\,.$$

Proof. Let A be a Borel subset of $\mathbb{R}^3$ of finite perimeter in O, i.e., $\mathcal{P}(A,O)$ is finite and is equal to $\mathcal{H}^2(\partial^*A\cap O)$. By the Gauss–Green Theorem, for any function f in $C^1_0(O,\mathbb{R}^3)$,

$$\int_A \text{div f}(x)\,d\mathcal{L}^3(x) \;=\; \int_{\partial^*A} f(x)\cdot\nu_A(x)\,d\mathcal{H}^2(x)\,.$$

Taking the supremum over all functions f in $C^1_0(O,\mathcal{W}_\tau)$ yields

$$\mathcal{I}(A,O) \;=\; \sup\Big\{\int_{\partial^*A} f(x)\cdot\nu_A(x)\,d\mathcal{H}^2(x) : f\in C^1_0(O,\mathcal{W}_\tau)\Big\}\,.$$

Thus,

$$\mathcal{I}(A,O) \;\leq\; \int_{\partial^*A\cap O}\sup_{y\in\mathcal{W}_\tau} y\cdot\nu_A(x)\,d\mathcal{H}^2(x) \;=\; \int_{\partial^*A\cap O}\tau(\nu_A(x))\,d\mathcal{H}^2(x)\,.$$

Conversely, let ε belong to $]0,1/2[$. For $\mathcal{H}^2$ almost all x in $\partial^*A\cap O$,

$$\lim_{r\to 0}\;\;(\pi r^2)^{-1}\mathcal{H}^2(B(x,r)\cap\partial^*A) \;=\; 1\,.$$

Let $\partial^{**}A$ be the points of $\partial^*A\cap O$ where the above property holds. For any x in $\partial^{**}A$, there exists a positive $r_1(x,\varepsilon)$ such that $B(x,r_1(x,\varepsilon))\subset O$ and

$$\forall\, r<r_1(x,\varepsilon)\qquad \big|\mathcal{H}^2(B(x,r)\cap\partial^*A)-\pi r^2\big| \;\leq\; \varepsilon\,\pi r^2\,.$$

By Egoroff Theorem [65, Chapter 3, Exercise 16], there exists a compact set C included in $\partial^{**}A$ such that $\mathcal{H}^2(\partial^*A\cap O\setminus C)<\varepsilon$ and $\nabla\chi_A(B(x,r))/||\nabla\chi_A||(B(x,r))$ converges uniformly on C towards $\nu_A(x)$ as r goes to 0; then the restriction of ν_A to C is continuous. Since τ is also continuous, then for any x in C, there exists $r_2(x,\varepsilon)$ positive such that

$$\forall\, y\in C\cap B(x,r_2(x,\varepsilon))\qquad |\nu_A(y)-\nu_A(x)|_2\leq\varepsilon\,,\quad |\tau(\nu_A(y))-\tau(\nu_A(x))|_2\leq\varepsilon\,.$$

The family of balls $B(x,r)$, $x\in C$, $r<\min(r_1(x,\varepsilon),r_2(x,\varepsilon))$, is a Vitali relation for C. By the Vitali covering Theorem for $\mathcal{H}^2$ [34, Theorem 1.10, or chapter 4], we may select a finite or countable collection of disjoint balls $B(x_i,r_i)$, $i\in I$, such that: for any i in I, x_i belongs to C, $r_i<\min(r_1(x,\varepsilon),r_2(x,\varepsilon))$ and

$$\text{either}\qquad \mathcal{H}^2\Big(C\setminus\bigcup_{i\in I}B(x_i,r_i)\Big) \;=\; 0 \qquad\text{or}\qquad \sum_{i\in I}r_i^2 \;=\; \infty\,.$$

Because for each i in I, r_i is smaller than $r_1(x,\varepsilon)$,

$$(1-\varepsilon)\sum_{i\in I}\pi r_i^2 \;\leq\; \mathcal{H}^2(\partial^*A\cap O) \;=\; \mathcal{P}(A,O) \;<\; \infty$$

and therefore the first case occurs, so that we may select a finite subset J of I such that

$$\mathcal{H}^2\Big(C\setminus\bigcup_{i\in J}B(x_i,r_i)\Big)\,<\,\varepsilon\,.$$

Since τ is the support function of $\mathcal{W}_\tau$, then for each i in J there exists a vector y_i in $\mathcal{W}_\tau$ such that $y_i\cdot\nu_A(x_i)=\tau(\nu_A(x_i))$. The balls $B(x_i,r_i)$, $i\in J$, being closed and disjoint, certainly there exists a function f in $C^1_0(O,\mathcal{W}_\tau)$ such that

$$\forall\, i\in J\quad\forall\, x\in B(x_i,r_i)\qquad f(x_i)=y_i\,.$$

For such a function f,

$$\begin{aligned}\Big|\int_{\partial^*A}&f(x)\cdot\nu_A(x)\,d\mathcal{H}^2(x)-\int_{\partial^*A\cap O}\tau(\nu_A(x))\,d\mathcal{H}^2(x)\Big|\\ &\le\,2||\tau||_\infty\mathcal{H}^2(\partial^*A\cap O\setminus C)+\Big|\int_C\big(f(x)\cdot\nu_A(x)-\tau(\nu_A(x))\big)\,d\mathcal{H}^2(x)\Big|\\ &\qquad\le\,2||\tau||_\infty\Big(\mathcal{H}^2(\partial^*A\cap O\setminus C)+\mathcal{H}^2\Big(C\setminus\bigcup_{i\in J}B(x_i,r_i)\Big)\Big)\\ &\qquad\qquad+\sum_{i\in J}\Big|\int_{B(x_i,r_i)\cap C}\big(y_i\cdot\nu_A(x)-\tau(\nu_A(x))\big)\,d\mathcal{H}^2(x)\Big|\,.\end{aligned}$$

For i in J and x in $B(x_i,r_i)\cap C$, because $r_i<\min(r_1(x_i,\varepsilon),r_2(x_i,\varepsilon))$,

$$|y_i{\cdot}\nu_A(x){-}\tau(\nu_A(x))|\,\le\,|y_i{\cdot}(\nu_A(x){-}\nu_A(x_i))|{+}|\tau(\nu_A(x_i)){-}\tau(\nu_A(x))|\,\le\,(||\tau||_\infty{+}1)\,\varepsilon\,.$$

Integrating these inequalities,

$$\begin{aligned}\Big|\int_{\partial^*A}f(x)\cdot\nu_A(x)\,d\mathcal{H}^2(x)-\int_{\partial^*A\cap O}\tau(\nu_A(x))\,d\mathcal{H}^2(x)\Big|&\\ \le\,4\varepsilon||\tau||_\infty+\varepsilon\,(||\tau||_\infty+1)\mathcal{H}^2(\partial^*A\cap O)\,,&\end{aligned}$$

whence

$$\mathcal{I}(A,O)\,\ge\,\int_{\partial^*A\cap O}\tau(\nu_A(x))\,d\mathcal{H}^2(x)-4\varepsilon||\tau||_\infty-\varepsilon\,(||\tau||_\infty+1)\mathcal{P}(A,O)\,.$$

Letting ε go to zero, we obtain the converse inequality and the claim of the Proposition. $\square$

We next state an approximation result used for proving the large deviation upper bound.

Lemma 6.6. *Let A be a Caccioppoli set having finite perimeter in an open subset O of $\mathbb{R}^3$. For any positive ε, δ, there exists a finite collection of disjoint balls $B(x_i, r_i)$, $i \in I$, such that: for any i in I, x_i belongs to $\partial^* A$, r_i belongs to $]0,1[$, $B(x_i, r_i)$ is included in O,*

$$\mathcal{L}^3\big((A \cap B(x_i,r_i))\Delta B_-(x_i,r_i,\nu_A(x_i))\big) \leq \delta\, r_i^3\,,$$
$$\Big|\mathcal{I}(A,O) - \sum_{i\in I} \pi r_i^2 \tau(\nu_A(x_i))\Big| \leq \varepsilon\,.$$

Remark. It is logically equivalent to state the result with $\delta = \varepsilon$. However, in the course of the proof of the large deviation upper bound, we will first fix ε positive, and then apply the above result with a positive δ smaller than some δ_0 depending on ε.

Proof. Let ε, δ be positive, with $\varepsilon < 1/2$. Because a generalized normal vector is also a measure theoretic normal, for any x in $\partial^* A$, there exists a positive $r_1(x,\delta)$ such that, for any $r < r_1(x,\delta)$,

$$\mathcal{L}^3\big((A \cap B(x,r))\Delta B_-(x,r,\nu_A(x))\big) \leq \delta\, r^3\,.$$

The map $x \in \partial^* A \mapsto \nu_A(x) \in S^2$ is measurable with respect to the measure $\mathcal{H}^2|_{\partial^* A}$. Using the results stated in the section on Caccioppoli sets (at the beginning of chapter 6), for $\mathcal{H}^2$ almost all x in $\partial^* A$,

$$\lim_{r\to 0} \quad (\pi r^2)^{-1}\mathcal{H}^2(B(x,r)\cap \partial^* A) = 1\,,$$
$$\lim_{r\to 0} \quad \frac{1}{\pi r^2}\int_{B(x,r)\cap\partial^* A} \tau(\nu_A(y))\, d\mathcal{H}^2(y) = \tau(\nu_A(x))\,.$$

Let $\partial^{**}A$ be the set of the points of $\partial^* A$ where the two preceding identities hold simultaneously. Clearly $\mathcal{H}^2(\partial^* A \setminus \partial^{**} A) = 0$. For any x in $\partial^{**}A$, there exists a positive $r_2(x,\varepsilon)$ such that, for any $r < r_2(x,\varepsilon)$,

$$|\mathcal{H}^2(B(x,r)\cap\partial^* A) - \pi r^2| \leq \varepsilon\, \pi r^2\,,$$
$$\Big|\frac{1}{\pi r^2}\int_{B(x,r)\cap\partial^* A} \tau(\nu_A(y))\, d\mathcal{H}^2(y) - \tau(\nu_A(x))\Big| \leq \varepsilon\,.$$

The family of balls $B(x,r)$, $x \in \partial^{**}A \cap O$, $r < \min\big(r_1(x,\delta), r_2(x,\varepsilon), 1, d_2(x,\partial O)\big)$, is a Vitali relation for $\partial^{**}A\cap O$. By the Vitali covering Theorem for $\mathcal{H}^2$ [34, Theorem 1.10, or chapter 4], we may select a finite or countable collection of disjoint balls $B(x_i,r_i)$, $i \in I$, such that: for any i in I, x_i belongs to $\partial^{**}A\cap O$, $r_i < \min\big(r_1(x_i,\delta), r_2(x_i,\varepsilon), 1, d_2(x_i,\partial O)\big)$ and

$$\text{either} \qquad \mathcal{H}^2\Big((\partial^{**}A\cap O)\setminus \bigcup_{i\in I} B(x_i,r_i)\Big) = 0 \qquad \text{or} \qquad \sum_{i\in I} r_i^2 = \infty\,.$$

By hypothesis, $\mathcal{P}(A,O)=\mathcal{H}^2(\partial^* A\cap O)$ is finite. For each i in I, r_i is smaller than $r_2(x_i,\varepsilon)$, whence

$$(1-\varepsilon)\sum_{i\in I}\pi r_i^2\ \le\ \mathcal{H}^2(\partial^{**}A\cap O)\ <\ \infty$$

and therefore the first case occurs, so that we may select a finite subset J of I such that

$$\mathcal{H}^2\Big((\partial^{**}A\cap O)\setminus\bigcup_{i\in J}B(x_i,r_i)\Big)\ <\ \varepsilon\,\mathcal{H}^2(\partial^{**}A\cap O)\,.$$

We claim that the collection of balls $B(x_i,r_i), i\in J$, enjoys the desired properties. Indeed, there is only the last condition to be checked:

$$\begin{aligned}\Big|\mathcal{I}(A,O)-\sum_{i\in J}\pi r_i^2\tau(\nu_A(x_i))\Big|\ \le\ &\int_{(\partial^{**}A\cap O)\setminus\bigcup_{i\in J}B(x_i,r_i)}\tau(\nu_A(x))\,d\mathcal{H}^2(x)\\ &+\ \sum_{i\in J}\Big|\int_{\partial^{**}A\cap B(x_i,r_i)}\tau(\nu_A(x))\,d\mathcal{H}^2(x)-\pi r_i^2\tau(\nu_A(x_i))\Big|\,.\end{aligned}$$

The first integral of the right–hand member is less than $\varepsilon\,\mathcal{H}^2(\partial^{**}A\cap O)||\tau||_\infty$. For i in J,

$$\Big|\int_{\partial^{**}A\cap B(x_i,r_i)}\tau(\nu_A(x))\,d\mathcal{H}^2(x)-\pi r_i^2\tau(\nu_A(x_i))\Big|\ \le\ \varepsilon\,\pi r_i^2\ \le\ 2\varepsilon\mathcal{H}^2(B(x_i,r_i)\cap\partial^{**}A)$$

whence by summing over i in J,

$$\sum_{i\in J}\Big|\int_{\partial^{**}A\cap B(x_i,r_i)}\tau(\nu_A(x))\,d\mathcal{H}^2(x)-\pi r_i^2\tau(\nu_A(x_i))\Big|\ \le\ 2\varepsilon\,\mathcal{H}^2(\partial^{**}A\cap O).$$

Putting these inequalities together,

$$\Big|\mathcal{I}(A,O)-\sum_{i\in J}\pi r_i^2\tau(\nu_A(x_i))\Big|\ \le\ \varepsilon\,\mathcal{P}(A,O)\,(||\tau||_\infty+2)\,.$$

Since $(||\tau||_\infty+2)\mathcal{P}(A,O)$ does not depend on ε, we have the required estimate. □

The next Lemma bounds the surface energy of a Caccioppoli set intersected with a ball.

Lemma 6.7. *Let A be a Caccioppoli set. For x in $\mathbb{R}^3$, for $\mathcal{H}^1$ almost all positive radius r,*

$$\mathcal{I}(A\cap B(x,r))\ \le\ \mathcal{I}(A,\operatorname{int}B(x,r))\ +\ ||\tau||_\infty\mathcal{H}^2(A\cap\partial B(x,r))\,.$$

Proof. Since $\partial^*(A \cap B(x,r)) \cap \operatorname{int} B(x,r) = \partial^* A \cap \operatorname{int} B(x,r)$ and $\nu_{A\cap B(x,r)}(y) = \nu_A(y)$ for y in $\operatorname{int} B(x,r)$, using Proposition 6.5,

$$\begin{aligned}\mathcal{I}(A\cap B(x,r)) &= \int_{\partial^*(A\cap B(x,r))} \tau(\nu_{A\cap B(x,r)}(y))\, d\mathcal{H}^2(y)\\ &\leq \mathcal{I}(A, \operatorname{int} B(x,r)) + ||\tau||_\infty \mathcal{H}^2\big(\partial^*(A\cap B(x,r)) \cap \partial B(x,r)\big)\,.\end{aligned}$$

Next, for $\mathcal{H}^1$ almost all positive r (see the beginning of chapter 6 or [41, remark 2.14]),

$$\begin{aligned}\mathcal{H}^2\big(\partial^*(A\cap B(x,r))\cap \partial B(x,r)\big) &= \mathcal{P}(A\cap B(x,r)) - \mathcal{P}(A, \operatorname{int} B(x,r))\\ &= \mathcal{H}^2(A\cap \partial B(x,r))\,.\end{aligned}$$

Combining these inequalities, the result follows. □

In the remaining of the section we state an important approximation result which is the key for the proof of the large deviation lower bound.

Proposition 6.8. *Every bounded Caccioppoli set A can be approximated by a sequence $(A_n)_{n\in\mathbb{N}}$ of bounded C^∞ sets such that*

$$\lim_{n\to\infty} \mathcal{L}^3(A_n \Delta A) = 0\,, \quad \lim_{n\to\infty} \mathcal{I}(A_n) = \mathcal{I}(A)\,.$$

Proof. This result is a slight generalization of the corresponding result for the perimeters [41, Theorem 1.24]. Because the proof is quite long, we only sketch the essential points: it consists in adapting some minor details of [41, Theorem 1.24]. One needs a slight extension of the coarea formula [73]: for any function f of bounded variation,

$$\mathcal{I}(f) = \int_{-\infty}^{+\infty} \mathcal{I}\big(\{\, x\in\mathbb{R}^3 : f(x) < t\,\}\big)\, dt\,.$$

(For the definition of $\mathcal{I}(f)$, replace χ_A by f in the definition of $\mathcal{I}$). The first step consists in approximating the characteristic function χ_A by a regular function through a smoothing procedure (convolution with a mollifier) to obtain a sequence of C^∞ functions $(f_n)_{n\in\mathbb{N}}$ with values in $[0,1]$ such that

$$\lim_{n\to\infty} \int |f_n(x) - \chi_A(x)|\, d\mathcal{L}^3(x) = 0\,, \quad \lim_{n\to\infty} \mathcal{I}(f_n) = \mathcal{I}(A)\,.$$

The lower semicontinuity of $\mathcal{I}$ and the coarea formula imply that for $\mathcal{H}^1$ almost all t in $]0,1[$

$$\liminf_{n\to\infty} \mathcal{I}\big(\{\, x\in\mathbb{R}^3 : f_n(x) < t\,\}\big) = \mathcal{I}(A)\,.$$

By the Sard Lemma, for $\mathcal{H}^1$ almost all t in $]0,1[$, the boundary of the set $\{\, x\in\mathbb{R}^3 : f_n(x) < t\,\}$ is regular. Thus we may choose a value of t in $]0,1[$ such that the sets

$\{ x \in \mathbb{R}^3 : f_n(x) < t \}$, $n \in \mathbb{N}$, are all C^∞ and the infimum limit of the sequence $(\mathcal{I}(\{ x \in \mathbb{R}^3 : f_n(x) < t \}))_{n\in\mathbb{N}}$ is $\mathcal{I}(A)$. The sequence $(\{ x \in \mathbb{R}^3 : f_n(x) < t \})_{n\in\mathbb{N}}$ then contains a subsequence having the desired properties. □

A nice consequence of Proposition 6.8 is that $\mathcal{I}$ can alternatively be defined by

$$\mathcal{I}(A) \,=\, \inf \,\Big\{ \liminf_{n\to\infty} \int_{\partial A_n} \tau(\nu_{A_n}(x))\, d\mathcal{H}^2(x) \Big\}$$

where the infimum is taken over all the sequences of C^∞ sets $(A_n)_{n\in\mathbb{N}}$ converging towards A in L^1_{loc}. Thus $\mathcal{I}$ is the largest lower semicontinuous functional extending the surface integral $A \mapsto \int_{\partial A} \tau(\nu_A(x))\, d\mathcal{H}^2(x)$ from the C^∞ sets to the Borel sets.

To prove the large deviation lower bound, we will need another kind of approximation result, namely we will need to approximate the sets of finite perimeter with polyhedral sets. A Borel subset of $\mathbb{R}^3$ is polyhedral if its boundary is included in the union of a finite number of hyperplanes of $\mathbb{R}^3$.

Proposition 6.9. *Let A be a set of finite perimeter in $\mathbb{R}^3$. There exists a sequence $(A_n)_{n\in\mathbb{N}}$ of polyhedral sets of $\mathbb{R}^3$ converging to A for the topology L^1 such that $\mathcal{I}(A_n)$ converges to $\mathcal{I}(A)$ as n goes to ∞.*

Remark. We might assume in addition that the approximating sets $(A_n)_{n\in\mathbb{N}}$ are open, connected and that they contain the origin: by adding threads having arbitrary small surface and volume, it is possible to connect together a finite number of polyhedral connected sets and to include a neighbourhood of the origin. In case $\mathcal{L}^3(A)$ is finite, we might also assume that the sets $(A_n)_{n\in\mathbb{N}}$ are bounded.

Remark. This result yields a definition of the surface energy analogous to the original definition of the perimeter proposed by Caccioppoli [15,16], as the infimum of the limits of the perimeters of polyhedral approximations.

Remark. For the particular case $\tau = 1$, this result has been originally proved by De Giorgi [24]. Below we propose a different proof, with a special emphasis on the way to approximate a set limited by an hypersurface with a polyhedral set.

Proof. Let A be a set of finite perimeter. By the isoperimetric inequality, either $\mathcal{L}^3(A)$ or $\mathcal{L}^3(\mathbb{R}^3 \setminus A)$ is finite. If a sequence $(A_n)_{n\in\mathbb{N}}$ converges towards A in L^1 then the sequence $(\mathbb{R}^3 \setminus A_n)_{n\in\mathbb{N}}$ converges towards $\mathbb{R}^3 \setminus A$ in L^1. Moreover we have $\mathcal{I}(A) = \mathcal{I}(\mathbb{R}^3 \setminus A)$. Hence we need only to consider the case where $\mathcal{L}^3(A)$ is finite. For r positive, we set $A_r = A \cap B(0,r)$. Because $\mathcal{L}^3(A)$ is finite, the set A_r converges towards A for the topology L^1 as r goes to ∞. By Lemma 6.7, for $\mathcal{H}^1$ almost all r,

$$\mathcal{I}(A_r) \,\leq\, \int_{\partial^* A \cap \operatorname{int} B(0,r)} \tau(\nu_{A_r}(x))\, d\mathcal{H}^2(x) \;+\; ||\tau||_\infty \mathcal{H}^2(A \cap \partial B(0,r))\,.$$

Let ε be positive. Since $\mathcal{L}^3(A)$ is finite, then $\{ r > 0 : \mathcal{H}^2(A \cap \partial B(0,r)) \geq \varepsilon \}$ has finite $\mathcal{H}^1$ measure (less than $\mathcal{L}^3(A)/\varepsilon$). Certainly there exist arbitrarily large values of r such that the above inequality holds and $\mathcal{H}^2(A \cap \partial B(0,r)) < \varepsilon$, whence $\mathcal{I}(A_r) \leq \mathcal{I}(A) + \varepsilon ||\tau||_\infty$. For r large enough, we have also $\mathcal{L}^3(A \Delta A_r) < \varepsilon$. Therefore we need only to prove the approximation result for bounded sets of finite perimeter. By Proposition 6.8, it is then enough to consider the case of bounded C^∞ sets of finite perimeter.

The end of the proof uses a technical Lemma, that we state and prove now.

Lemma 6.10. *Let Γ be an hypersurface (that is a C^1 submanifold of $\mathbb{R}^3$ of codimension 1) and let K be a compact subset of Γ. There exists a positive $M = M(\Gamma, K)$ such that:*

$$\forall \varepsilon > 0 \quad \exists\, r > 0 \quad \forall\, x, y \in K \qquad |x-y|_2 \leq r \quad \Rightarrow \quad d_2(y, \tan(\Gamma, x)) \leq M\,\varepsilon\, |x-y|_2 \,.$$

($\tan(\Gamma, x)$ is the tangent plane of Γ at x).

Proof. By a standard compactness argument, it is enough to prove the following local property:

$$\forall\, x \in \Gamma \quad \exists\, M(x) > 0 \quad \forall \varepsilon > 0 \quad \exists\, r(x,\varepsilon) > 0 \quad \forall\, y, z \in \Gamma \cap B(x, r(x,\varepsilon)) \qquad d_2(y, \tan(\Gamma, z)) \leq M(x)\, \varepsilon\, |y-z|_2 \,.$$

Indeed, if this property holds, we cover K by the open balls $\operatorname{int} B(x, r(x,\varepsilon)/2)$, $x \in K$, we extract a finite subcovering $\operatorname{int} B(x_i, r(x_i,\varepsilon)/2)$, $1 \leq i \leq k$, and we set

$$M = \max\{\, M(x_i) : 1 \leq i \leq k \,\}\,, \quad r = \min\{\, r(x_i,\varepsilon)/2 : 1 \leq i \leq k \,\}\,.$$

Let now y, z belong to K with $|y-z|_2 \leq r$. Let i be such that y belongs to $B(x_i, r(x_i,\varepsilon)/2)$. Since $r < r(x_i,\varepsilon)/2$, then both y, z belong to the ball $B(x_i, r(x_i,\varepsilon))$ and it follows that $d_2(y, \tan(\Gamma, z)) \leq M(x_i)\, \varepsilon\, |y-z|_2 \leq M\, \varepsilon\, |y-z|_2$.

The proof of the local property relies on a classical lemma of differential calculus [51, I, 4, Corollary 2], which we state next.

Lemma. *Let E, F be two Banach spaces, let U be an open subset of E, x, z in U such that $[x,z] \subset U$ and let $f : U \mapsto F$ be a C^1 map. Then for any y in $[x,z]$,*

$$|f(z) - f(x) - df(y)(z-x)| \;\leq\; |z-x| \sup\{\, |df(\zeta) - df(y)| : \zeta \in [x,z] \,\}\,.$$

Proof of Lemma 6.10 **continued.** We turn now to the proof of the above local property. Since Γ is an hypersurface, for any x in Γ there exists a neighbourhood V of x in $\mathbb{R}^3$, a diffeomorphism $f : V \mapsto \mathbb{R}^3$ of class C^1 and a two dimensional vector space Z of $\mathbb{R}^3$ such that $Z \cap f(V) = f(\Gamma \cap V)$ (see for instance [35, 3.1.19]). Let A be a compact neighbourhood of x included in V. Since f is a diffeomorphism, the maps $y \in A \mapsto df(y) \in \mathrm{End}(\mathbb{R}^3)$, $u \in f(A) \mapsto df^{-1}(u) \in \mathrm{End}(\mathbb{R}^3)$ are continuous. Therefore they are bounded:

$$\exists\, M > 0 \quad \forall\, y \in A \quad ||df(y)|| \leq M\,, \quad \forall\, u \in f(A) \quad ||df^{-1}(u)|| \leq M$$

(here $||df(x)|| = \sup\{\,|df(x)(y)|_2 : |y|_2 \leq 1\,\}$ is the standard operator norm in $\mathrm{End}(\mathbb{R}^3)$). Since $f(A)$ is compact, the differential map df^{-1} is uniformly continuous on $f(A)$:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, u, v \in f(A) \qquad |u-v|_2 \leq \delta \quad \Rightarrow \quad ||df^{-1}(u) - df^{-1}(v)|| \leq \varepsilon\,.$$

Let ε be positive and let δ be associated to ε as above. Let ρ be positive and small enough so that $\rho < \delta/2$ and $B(f(x),\rho) \subset f(A)$ (since f is a C^1 diffeomorphism, $f(A)$ is a neighbourhood of $f(x)$). Let r be such that $0 < r < \rho/M$ and $B(x,r) \subset A$. We claim that M associated to x and r associated to ε, x answer the problem. Let y, z belong to $\Gamma \cap B(x,r)$. Since $[y,z] \subset B(x,r) \subset A$, and $||df(\zeta)|| \leq M$ on A, then

$$|f(y) - f(x)|_2 \leq M|y-x|_2 \leq Mr < \rho\,, \quad |f(z) - f(x)|_2 < \rho\,,$$
$$|f(y) - f(z)|_2 < \delta\,, \quad |f(y) - f(z)|_2 < M|y-z|_2\,.$$

We apply next the quoted Lemma to the map f^{-1} and the interval $[f(z), f(y)]$ (which is included in $B(f(x),\rho) \subset f(A)$) and the point $f(z)$:

$$|y - z - df^{-1}(f(z))(f(y) - f(z))|_2 \leq$$
$$|f(y) - f(z)|_2 \sup\{\,||df^{-1}(\zeta) - df^{-1}(f(z))|| : \zeta \in [f(z), f(y)]\,\}\,.$$

The right–hand member is less than $M|y-z|_2\,\varepsilon$. Since $z + df^{-1}(f(z))(f(y) - f(z))$ belongs to $\tan(\Gamma, z)$, we are done. $\square$

Proof of Proposition 6.9 continued. Let now A be a bounded C^∞ set of finite perimeter. In particular its boundary ∂A is an hypersurface. Since τ is continuous and ∂A is an hypersurface, for any x in ∂A,

$$\lim_{r \to 0} \quad (\pi r^2)^{-1} \mathcal{H}^2(B(x,r) \cap \partial A) \;=\; 1\,,$$
$$\lim_{r \to 0} \quad (\pi r^2)^{-1} \int_{B(x,r) \cap \partial A} \tau(\nu_A(y))\, d\mathcal{H}^2(y) \;=\; \tau(\nu_A(x))\,.$$

Since $\mathcal{H}^2(\partial A)$ is finite, for any x in ∂A, for $\mathcal{H}^1$ almost all positive r,

$$\mathcal{H}^2(\partial A \cap \partial B(x,r)) = 0\,.$$

Let M be associated to ∂A as in Lemma 6.10 (notice that ∂A is compact). We might assume that $M \geq 1$. Let ε belong to $]0, 1/2[$. Let $r(\partial A, \varepsilon)$ be associated to ∂A, ε as in Lemma 6.10. For x in ∂A, there exists a positive $r(x,\varepsilon)$ such that, for $r < r(x,\varepsilon)$,

$$|\mathcal{H}^2(B(x,r) \cap \partial A) - \pi r^2| \;\leq\; \varepsilon\, \pi r^2\,,$$
$$\Big|(\pi r^2)^{-1} \int_{B(x,r) \cap \partial A} \tau(\nu_A(y))\, d\mathcal{H}^2(y) - \tau(\nu_A(x))\Big| \;\leq\; \varepsilon\,.$$

The family of balls

$$B(x,r),\ x\in\partial A,\ r<\min(r(x,\varepsilon),r(\partial A,\varepsilon),\varepsilon),\ \mathcal{H}^2(\partial A\cap\partial B(x,r))=0\,,$$

is a Vitali relation for the set ∂A. By the standard Vitali covering Theorem [34, Theorem 1.10, or chapter 4], we may select a finite or countable collection of disjoint balls $B(x_i,r_i)$, $i\in I$, such that: for any i in I, $x_i\in\partial A$, $0<r_i<\min(r(x_i,\varepsilon),r(\partial A,\varepsilon),\varepsilon)$, $\mathcal{H}^2(\partial A\cap\partial B(x_i,r_i))=0$, and

$$\text{either}\qquad \mathcal{H}^2\Big(\partial A\setminus\bigcup_{i\in I}B(x_i,r_i)\Big)\,=\,0\qquad\text{or}\qquad\sum_{i\in I}r_i^2\,=\,\infty\,.$$

Because for each i in I, r_i is smaller than $r(x_i,\varepsilon)$,

$$(1-\varepsilon)\sum_{i\in I}\pi r_i^2\,\le\,\mathcal{H}^2(\partial A)\,<\,\infty$$

and therefore the first case occurs, so that we may select a finite subset I_0 of I such that

$$\mathcal{H}^2\Big(\partial A\setminus\bigcup_{i\in I_0}B(x_i,r_i)\Big)\,<\,\varepsilon\,\mathcal{H}^2(\partial A)\,.$$

By the very definition of the Hausdorff measure $\mathcal{H}^2$, there exists a collection of balls $B(y_j,s_j)$, $j\in J$, such that: for any j in J, $y_j\in\partial A$, $0<s_j<\varepsilon$,

$$\sum_{j\in J}\pi s_j^2\,\le\,\varepsilon(\mathcal{H}^2(\partial A)+1)\,,$$

$$\partial A\setminus\bigcup_{i\in I_0}\operatorname{int}B(x_i,r_i)\,\subset\,\bigcup_{j\in J}\operatorname{int}B(y_j,s_j)\,.$$

By compactness of ∂A, we might assume in addition that J is finite. For each i in I_0, let P_i be a convex open polygon inside the plane $\tan(\partial A,x_i)=\operatorname{hyp}(x_i,\nu_A(x_i))$ such that

$$\operatorname{disc}(x_i,r_i,\nu_A(x_i))\subset P_i\subset B(x_i,r_i(1+\varepsilon))\,,$$

$$|\mathcal{H}^1(\partial P_i)-2\pi r_i|\,\le\,2\varepsilon\pi r_i\,,\quad |\mathcal{H}^2(P_i)-\pi r_i^2|\le\varepsilon\pi r_i^2\,.$$

We set $\alpha=M\,\varepsilon\,(1+\varepsilon)$ and $F_i=\operatorname{cyl}(\operatorname{clo}P_i,\alpha r_i)$ for i in I_0. For each j in J, let Q_j be a polyhedral set such that $B(y_j,s_j)\subset Q_j\subset B(y_j,2s_j)$ and $\mathcal{H}^2(\partial Q_j)\le 8\pi s_j^2$. Let T be the set

$$T\,=\,A\cup\bigcup_{i\in I_0}F_i\cup\bigcup_{j\in J}Q_j\,.$$

We claim that T answers the problem. First $A\subset T\subset\mathcal{V}_2(A,2(M+1)\,\varepsilon)$. Secondly

$$T\setminus A\,\subset\,\bigcup_{i\in I_0}F_i\cup\bigcup_{j\in J}Q_j$$

whence

$$\mathcal{L}^3(T\setminus A)\ \le\ \sum_{i\in I_0} 2\pi r_i^2(1+\varepsilon)\alpha r_i+\sum_{j\in J}(4/3)\pi(2s_j)^3\,.$$

Yet $\sum_{i\in I_0}\pi r_i^2\le 2\mathcal{H}^2(\partial A)$ and $\sum_{j\in J}\pi s_j^2\le\varepsilon\,(\mathcal{H}^2(\partial A)+1)$ whence

$$\mathcal{L}^3(T\setminus A)\ \le\ 9M\mathcal{H}^2(\partial A)\,\varepsilon+11\,\varepsilon\,(\mathcal{H}^2(\partial A)+1)\,.$$

We show next that T is polyhedral. Indeed, for i in I_0, r_i is smaller than $r(\partial A,\varepsilon)$, so that

$$\forall\, x\in\partial A\cap B(x_i,r_i)\quad d_2(x,\mathrm{hyp}\,(x_i,\nu_A(x_i)))\le M\,\varepsilon\,|x-x_i|_2$$

whence $\partial A\cap B(x_i,r_i)\subset\mathrm{cyl}\,(P_i,M\,\varepsilon\,r_i)\subset\mathrm{int}\,F_i$. Moreover, letting $\beta=\sqrt{1-\alpha^2}$, the disc

$$G_i\ =\ \mathrm{disc}\,(x_i-\alpha r_i\nu_A(x_i),\beta r_i,\nu_A(x_i))$$

is included in the interior of A. Indeed, G_i is included in $B(x_i,r_i)\cap\partial F_i$ and therefore G_i does not intersect ∂A. It is included in $\mathrm{int}\,A$ because $\nu_A(x_i)$ is the exterior normal to A at x_i. Since in addition the sets F_i, $i\in I_0$, Q_j, $j\in J$ cover ∂A,

$$\partial T\ \subset\ \bigcup_{i\in I_0}\partial F_i\setminus G_i\cup\bigcup_{j\in J}\partial Q_j\,.$$

Thus ∂T is included in a finite union of hyperplanes and T is polyhedral. Finally

$$\begin{aligned}
\mathcal{I}(\partial T)\ &\le\ \sum_{i\in I_0}\mathcal{I}(\partial F_i\setminus G_i)+\sum_{j\in J}\mathcal{I}(\partial Q_j)\\
&\le\ \sum_{i\in I_0}\pi r_i^2(1+\varepsilon)\tau(\nu_A(x_i))+||\tau||_\infty\Big(\sum_{i\in I_0}\pi r_i^2\big(4(1+\varepsilon)\alpha+1+\varepsilon-\beta^2\big)+\sum_{j\in J}8\pi s_j^2\Big)\\
&\le\ (1+\varepsilon)\mathcal{I}(A)+4\varepsilon\mathcal{H}^2(\partial A)+||\tau||_\infty\Big(2\mathcal{H}^2(\partial A)\big(9M\varepsilon+\varepsilon+4M^2\varepsilon^2\big)+8\varepsilon(\mathcal{H}^2(\partial A)+1)\Big)\,.
\end{aligned}$$

Since M depends on A only, the set T approximates A as required. $\square$

6.3. The Wulff Theorem

We consider the following problem:

(P) minimize $\mathcal{I}(E)$ among all Borel sets E with $\mathcal{L}^3(E)=\mathcal{L}^3(\mathcal{W}_\tau)$.

In the case where τ is constant, the problem (P) reduces to the classical isoperimetric problem. In the case of anisotropic functions τ, the first attempts to solve (P) are due to Wulff, at the turn of the century [75]. Later Dinghas [30] proved that, among convex polyhedra, the Wulff crystal $\mathcal{W}_\tau$ is the solution to (P). Taylor obtained general existence and uniqueness results for (P) in the framework of the Geometric Measure Theory [69,70,71]. Recently, Fonseca and Müller reworked and slightly enhanced these results using the theory of the Caccioppoli sets [36,37,39]. All the above proofs rely on the Brunn–Minkowski Theorem. We restate next some results from [36,37,39].

Theorem 6.11. *For any set E of finite perimeter in $\mathbb{R}^3$ such that $\mathcal{L}^3(E) = \mathcal{L}^3(\mathcal{W}_\tau)$,*

$$\mathcal{I}(E) = \int_{\partial^* E} \tau(\nu_E(x))\, d\mathcal{H}^2(x) \geq \mathcal{I}(\mathcal{W}_\tau) = \int_{\partial^* \mathcal{W}_\tau} \tau(\nu_{\mathcal{W}_\tau}(x))\, d\mathcal{H}^2(x)\,.$$

Theorem 6.12. *Let E be a solution of the problem (P). Then $E = \mathcal{W}_\tau + c$ up to a negligible set, where $c = \mathcal{L}^3(\mathcal{W}_\tau)^{-1} \int_E x\, d\mathcal{L}^3(x)$.*

Proposition 6.13. *If $(E_n)_{n\in\mathbb{N}}$ is a minimizing sequence of the variational problem (P), then there exist translations $(E_n - a_n)_{n\in\mathbb{N}}$ such that $\mathcal{L}^3((E_n - a_n)\Delta\mathcal{W}_\tau)$ converges to 0 as n goes to ∞.*

Theorems 6.11, 6.12, Proposition 6.13 are specializations to $\mathbb{R}^3$ of [37, Theorem 4.2], [39, Theorem 3.3], [37, Proposition 5.1].

Corollary 6.14. *(Stability of the Wulff crystal)*
For any positive δ, there exists a positive η such that

$$\inf\Big\{\, \mathcal{I}(E) : E \in \mathcal{B}(\mathbb{R}^3),\, \mathcal{L}^3(\mathcal{W}_\tau) \leq \mathcal{L}^3(E) < \infty,\, \inf_{x\in\mathbb{R}^3} \mathcal{L}^3(E\Delta(x+\mathcal{W}_\tau)) \geq \delta \,\Big\} \geq \mathcal{I}(\mathcal{W}_\tau) + \eta\,.$$

Proof. If the result was false, there would exist a positive δ and a sequence $(E_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \mathcal{I}(E_n) = \mathcal{I}(\mathcal{W}_\tau)$ and for all n in $\mathbb{N}$

$$\mathcal{L}^3(\mathcal{W}_\tau) \leq \mathcal{L}^3(E_n) < \infty\,, \quad \forall\, x \in \mathbb{R}^3 \quad \mathcal{L}^3(E_n\Delta(x+\mathcal{W}_\tau)) \geq \delta\,.$$

For n in $\mathbb{N}$, let $\lambda_n = (\mathcal{L}^3(\mathcal{W}_\tau)/\mathcal{L}^3(E_n))^{1/3}$. On one hand, λ_n is smaller than 1; on the other hand, by Theorem 6.11, $\lambda_n^2 \mathcal{I}(E_n) = \mathcal{I}(\lambda_n E_n) \geq \mathcal{I}(\mathcal{W}_\tau)$, so that $\liminf_{n\to\infty} \lambda_n \geq 1$. Therefore $\lim_{n\to\infty} \lambda_n = 1$. It follows that $(\lambda_n E_n)_{n\in\mathbb{N}}$ is a minimizing sequence of the problem (P). Yet, for x in $\mathbb{R}^3$, using the hypothesis on E_n and the convexity of $\mathcal{W}_\tau$,

$$\begin{aligned}\mathcal{L}^3((\lambda_n E_n)\Delta(x+\mathcal{W}_\tau)) &\geq \lambda_n^3 \mathcal{L}^3(E_n\Delta(x+\mathcal{W}_\tau)) - \mathcal{L}^3((\lambda_n\mathcal{W}_\tau)\Delta\mathcal{W}_\tau)\\ &\geq \lambda_n^3\delta - \mathcal{L}^3(\mathcal{W}_\tau \setminus (\lambda_n\mathcal{W}_\tau))\,.\end{aligned}$$

Taking the infimum over x in $\mathbb{R}^3$ and passing to the limit, we get

$$\liminf_{n\to\infty}\ \inf_{x\in\mathbb{R}^3} \mathcal{L}^3\big((\lambda_n E_n)\Delta(x+\mathcal{W}_\tau)\big) \geq \delta\,.$$

This stands in contradiction with Proposition 6.13. □

CHAPTER 7

COARSE GRAINING

In this chapter, we restate some results of Pisztora [60] which are the fundamental ingredients of our proofs. Next, we give general results for building coarse graining estimates on the rescaled lattice.

7.1. The results of Pisztora

Let Λ be a cubic box of diameter n in $\mathbb{R}^3$, with sides parallel to the axis. We consider the configuration $\omega|_\Lambda$ restricted to the box Λ. A cluster of $\omega|_\Lambda$ is called crossing for Λ if it intersects the 6 faces of $\partial_{\rm iv}\Lambda$. Let g be an increasing function from $\mathbb{N}$ to $\mathbb{R}^+$ such that $g(n) \leq n$ for all n. Let also δ be a positive real number. We consider the configuration restricted to Λ and the following events:

• $U(\Lambda) = \{$ there exists a unique open crossing cluster in Λ, denoted by $C^*(\Lambda)\}$,

• $R(\Lambda, g(n)) = U(\Lambda) \cap \{$ every open path included in Λ of diameter larger than $g(n)$ is included in $C^*(\Lambda)\}$,

• $O(\Lambda, g(n)) = R(\Lambda, g(n)) \cap \{$ the crossing cluster $C^*(\Lambda)$ intersects every box of diameter $g(n)$ contained in $\Lambda\}$,

• $V(\Lambda, \delta) = U(\Lambda) \cap \{ (\theta - \delta)\,\mathrm{card}\,\Lambda < \mathrm{card}\,C^*(\Lambda) < (\theta + \delta)\,\mathrm{card}\,\Lambda \}$.

Theorem 7.1. *Let $p > p_c$ be fixed. For any x in $\mathbb{Z}^3$,*

$$\limsup_{n\to\infty} \frac{1}{n} \ln P\big(U(\Lambda(x,n))^c\big) \;<\; 0\,.$$

There exists a constant $\kappa = \kappa(p)$ such that if $g(n) \geq \kappa \ln n$ for all n then for any x in $\mathbb{Z}^3$,

$$\limsup_{n\to\infty} \frac{1}{g(n)} \ln P\big(O(\Lambda(x,n), g(n))^c\big) \;<\; 0\,.$$

Remark. Since we have the inclusion $O(\Lambda(x,n), g(n)) \subset R(\Lambda(x,n), g(n))$, Theorem 7.1 yields a similar bound for the probability of the event $R(\Lambda(x,n), g(n))$.

Lemma 7.2. *Let p belong to $[0,1]$. For any positive δ, any x in $\mathbb{Z}^3$,*

$$\limsup_{n\to\infty} \frac{1}{n^3} \ln P\Big(\sum_C \mathrm{card}\,C > (\theta+\delta)\,\mathrm{card}\,\Lambda(x,n)\Big) \;<\; 0$$

where the summation is over all the open clusters C intersecting $\partial_{\rm iv}\Lambda(x,n)$.

Lemma 7.3. *Let $p > p_c$ be fixed. For any positive δ, any x in $\mathbb{Z}^3$,*

$$\limsup_{n\to\infty} \frac{1}{n} \ln P\big(V(\Lambda(x,n),\delta)^c\big) \;<\; 0\,.$$

Remark. Theorem 7.1 is a specialization of [60, Theorem 3.1] to the case of three dimensional Bernoulli bond percolation, whereas Lemmata 7.2 and 7.3 correspond to [60, Lemmata 5.1 and 5.2]. In this case we have $\theta^f = \theta^w$ (percolation probabilities with free and wired boundary conditions, see the notation of [60]); moreover Grimmett and Marstrand have proved that the critical value $\widehat{p}_1$ (limit of percolation slab thresholds) coincides with p_c (see [43] or the second edition of [42]).

We have slightly altered the notation of [60], by adding the box Λ in $U(\Lambda)$, $R(\Lambda, g(n))$, $O(\Lambda, g(n))$, $V(\Lambda,\delta)$. The reason is that we will consider these events in different boxes of the lattice. Since our model is invariant under integer translations, the probabilities of the events $U(x+\Lambda)$, $R(x+\Lambda, g(n))$, $O(x+\Lambda, g(n))$ and $V(x+\Lambda,\delta)$ are the same for all x in $\mathbb{Z}^3$. Notice that all the clusters in the definitions of these events are clusters of the configuration restricted to the box $x+\Lambda$. Hence the occurrence of these events depends only on the edges of $x+\Lambda$ and two such events are independent when their associated boxes have no edges in common. We finally restate Theorem 7.1 and Lemma 7.3 in a form more convenient for our purposes.

Corollary 7.4. *Let $p > p_c$ be fixed. There exist two positive constants $b = b(p)$, $c = c(p)$ depending on p such that*

$$\forall\, n\in\mathbb{N}\quad \forall\, x\in\mathbb{Z}^3 \qquad P\big(U(\Lambda(x,n))^c\big) \;\leq\; b\exp -cn\,.$$

Let g be an increasing function from $\mathbb{N}$ to $\mathbb{R}^+$ such that $\kappa\ln n \leq g(n) \leq n$ for all n in $\mathbb{N}$, where $\kappa = \kappa(p)$ is the constant given by Theorem 7.1. There exist two positive constants $b = b(p)$, $c = c(p)$ depending on p only such that

$$\forall\, n\in\mathbb{N}\quad \forall\, x\in\mathbb{Z}^3$$
$$P\big(R(\Lambda(x,n), g(n))^c\big) \;\leq\; P\big(O(\Lambda(x,n), g(n))^c\big) \;\leq\; b\exp -cg(n)\,.$$

Remark. Under the hypothesis on $g(n)$, $O(\Lambda(x,n), g(n))^c \subset O(\Lambda(x,n), \kappa\ln n)^c$, hence the constants b, c are independent of the function g.

Corollary 7.5. *Let $p > p_c$ be fixed. Let δ be positive. There exist two positive constants $b = b(p,\delta)$, $c = c(p,\delta)$ depending on p and δ such that*

$$\forall\, n\in\mathbb{N}\quad \forall\, x\in\mathbb{Z}^3$$
$$P\Big(V(\Lambda(x,n),\delta)^c \cup \big(\sum_C \operatorname{card} C > (\theta+\delta)\operatorname{card}\Lambda(x,n)\big)^c\Big) \;\leq\; b\exp -cn$$

where the summation is over all the open clusters C intersecting $\partial_{\mathrm{iv}}\Lambda(x,n)$.

7.2. The rescaled lattice

Let h be a positive integer. We define the rescaled lattice of $\mathbb{Z}^3$ by a factor h to be the sublattice $h\mathbb{Z}^3$. The objects associated to the rescaled lattice $h\mathbb{Z}^3$, like points and random variables, will bear a h in superscript or subscript. The rescaled lattice $h\mathbb{Z}^3$ is identified with $\mathbb{Z}^3$ via the map: $x \in \mathbb{Z}^3 \mapsto hx \in h\mathbb{Z}^3$. This way it inherits the graph structures $\mathbb{L}^3$ and $\mathbb{E}^3$ defined on $\mathbb{Z}^3$. We denote by $h\mathbb{L}^3$ and $h\mathbb{E}^3$ the image graph structures on $h\mathbb{Z}^3$.

Definition 7.6. (Fat operator)
Let A be a subset of $\mathbb{Z}^3$. To A we associate a fattened set $\operatorname{fat}(A,h)$ on $h\mathbb{Z}^3$ defined by

$$\operatorname{fat}(A,h) \;=\; \big\{\, x^h \in h\mathbb{Z}^3 : \Lambda(x^h,h)\cap A \neq \varnothing \,\big\}\,.$$

Notice that the boxes $\Lambda(x^h,h)$, $x^h \in h\mathbb{Z}^3$, form a partition of $\mathbb{Z}^3$. Whenever A is a connected subset of the graph $(\mathbb{Z}^3,\mathbb{E}^3)$, then $\operatorname{fat}(A,h)$ is a connected subset of the graph $(h\mathbb{Z}^3,h\mathbb{E}^3)$. It is also clear that $\operatorname{card} A \leq h^3 \operatorname{card}\operatorname{fat}(A,h)$.

We suppose next that a random variable $X_h(x^h)$ with values in $\{0,1\}$ is associated to each vertex x^h of $h\mathbb{Z}^3$ and that the variable $X_h(x^h)$ is the indicator function of an unlikely event around x^h in the percolation configuration. More precisely, these random variables will satisfy an hypothesis of the following form.

Hypothesis $H(\varepsilon,\alpha)$. Let ε be a small positive real number and let α be a non–negative integer. The family of variables $(X_h(x^h),\, x^h \in h\mathbb{Z}^3)$ satisfies the hypothesis $H(\varepsilon,\alpha)$ if

- $\forall\, x^h \in h\mathbb{Z}^3 \qquad P(X_h(x^h)=1) \leq \varepsilon$
- $X_h(x^h)$ depends only on the configuration restricted to $\Lambda(x^h,\alpha h)$

The second property implies in particular that $X_h(y^h)$ and $X_h(z^h)$ are independent whenever $|y^h - z^h|_\infty \geq \alpha h$.

Lemma 7.7. *Under the hypothesis $H(\varepsilon,\alpha)$, for any subset A^h of $h\mathbb{Z}^3$,*

$$P\big(\forall\, x^h \in A^h \quad X_h(x^h)=1\big) \;\leq\; \exp(\alpha^{-3}\operatorname{card} A^h \ln\varepsilon)\,.$$

Proof. We define an equivalence relation $\equiv$ on $h\mathbb{Z}^3$:

$$(x_1^h,x_2^h,x_3^h) \equiv (y_1^h,y_2^h,y_3^h) \iff \alpha h \text{ divides } y_i - x_i,\; 1\leq i\leq 3\,.$$

Since the trace of the equivalence classes on A^h, i.e., the sets

$$\{\, y^h \in A^h : y^h \equiv x^h \,\}\,, \quad x^h \in \Lambda(0,\alpha h)\cap h\mathbb{Z}^3\,,$$

form a partition of A^h and $\operatorname{card}\Lambda(0,\alpha h)\cap h\mathbb{Z}^3 = \alpha^3$, certainly there exists x_0^h in $h\mathbb{Z}^3$ such that $\operatorname{card}\{\, x^h \in A^h : x^h \equiv x_0^h \,\} \geq \alpha^{-3}\operatorname{card} A^h$. Therefore

$$\begin{aligned} P\big(\forall\, x^h \in A^h \quad X_h(x^h)=1\big) &\leq P\big(\forall\, x^h \in \{\, y^h \in A^h : y^h \equiv x_0^h \,\} \quad X_h(x^h)=1\big) \\ &\leq \exp(\alpha^{-3}\operatorname{card} A^h \ln\varepsilon) \end{aligned}$$

since the variables $X_h(x^h)$, $x^h \in A^h$, $x^h \equiv x_0^h$, are independent. □

Corollary 7.8. *Under the hypothesis $H(\varepsilon,\alpha)$, for any subset A of $\mathbb{Z}^3$ and any positive s,*

$$P\big(\mathrm{card}\,\{\,x^h\in A\cap h\mathbb{Z}^3 : X_h(x^h)=1\,\}\geq s\big)\ \leq\ \sum_{i\geq s}\exp\Big(i\ln\mathrm{card}\,A\cap h\mathbb{Z}^3+i\alpha^{-3}\ln\varepsilon\Big)\,.$$

Proof. We decompose the event under consideration according to the set of points x^h such that $X_h(x^h)=1$:

$$P\big(\mathrm{card}\,\{\,x^h\in A\cap h\mathbb{Z}^3 : X_h(x^h)=1\,\}\geq s\big)\ \leq\ \sum_{i\geq s}\sum_{A^h}P\big(\forall\,x^h\in A^h\quad X_h(x^h)=1\big)$$

where the second summation extends over the sets A^h included in $A\cap h\mathbb{Z}^3$ of cardinality i; the number of such sets is less than $(\mathrm{card}\,A\cap h\mathbb{Z}^3)^i$. The conclusion follows from the bound given in Lemma 7.7. □

For a subset A of $\mathbb{Z}^3$ and a vertex x^h in $A\cap h\mathbb{Z}^3$, we denote by $C^h_{\mathbb{L}}(x^h,A,X_h)$ the $h\mathbb{L}^3$ component of x^h in the graph having for vertex set

$$\{\,y^h\in h\mathbb{Z}^3 : \Lambda(y^h,\alpha h)\subset A,\ X_h(y^h)=1\,\}$$

(in case $X_h(x^h)=0$, $C^h_{\mathbb{L}}(x^h,A,X_h)$ is the empty set).

Lemma 7.9. *There exists a positive constant b such that, under the hypothesis $H(\varepsilon,\alpha)$, for any subset A of $\mathbb{Z}^3$ such that* $\mathrm{card}\,A\cap h\mathbb{Z}^3\geq 2$, *for any positive real numbers β and s,*

$$P\big(\mathrm{card}\,\{\,x^h\in h\mathbb{Z}^3 : \mathrm{card}\ C^h_{\mathbb{L}}(x^h,A,X_h)\geq\beta\,\}\geq s\big)\ \leq$$
$$\sum_{i\geq s}2\exp\Big(2i\beta^{-1}\ln\mathrm{card}\,A\cap h\mathbb{Z}^3+i\ln b+i\alpha^{-3}\ln\varepsilon\Big)\,.$$

Proof. We decompose the event under consideration:

$$P\big(\mathrm{card}\,\{\,x^h\in h\mathbb{Z}^3 : \mathrm{card}\,C^h_{\mathbb{L}}(x^h,A,X_h)\geq\beta\,\}\geq s\big)\ \leq$$
$$\sum_{s\leq i\leq\mathrm{card}\,A\cap h\mathbb{Z}^3}\ \ \sum_{1\leq j\leq i/\beta}\ \ \sum_{i_1,\dots,i_j}\ \ \sum_{A^h_1,\dots,A^h_j}\ \ P\big(\forall\,x^h\in A^h_1\cup\cdots\cup A^h_j\quad X_h(x^h)=1\big)\,.$$

The penultimate summation extends over the integers $i_1,\dots,i_j$ larger or equal than β and such that $i_1+\cdots+i_j=i$ (this set is void unless $j\leq i\beta^{-1}$); the last summation extends over the pairwise disjoint sets $A^h_1,\dots,A^h_j$ such that, for any k in $\{1,\dots,j\}$, A^h_k is an $h\mathbb{L}^3$ connected subset of $A\cap h\mathbb{Z}^3$ of cardinality i_k. By Lemma 7.7, the probability appearing in the summation is less than $\exp(i\alpha^{-3}\ln\varepsilon)$. There exists a positive constant b such that the number of $\mathbb{L}^3$ connected sets of $\mathbb{Z}^3$ containing a fixed point and having cardinality i is less than b^i [50, Lemma 5.3]. For fixed values of $j,i_1,\dots,i_j$, the number of choices for the sets $A^h_1,\dots,A^h_j$ is less than $b^i(\mathrm{card}\,A\cap h\mathbb{Z}^3)^j$. For a fixed value of i, the number of terms involved in the last three summations is less than $2\exp(2i\beta^{-1}\ln\mathrm{card}\,A\cap h\mathbb{Z}^3+i\ln b)$. □

CHAPTER 8

THE CENTRAL LEMMA

This chapter is devoted to the central lemma, which is essential for the proof of the large deviation upper bound. This lemma gives a probabilistic estimate for the local presence of a collection of open clusters creating a small flat interface in the L^1 topology; moreover the estimate is uniform with respect to the localization, the size and the direction of the interface. This lemma is, in a sense, the heart of this work. Besides its technical formulation, this lemma provides a link between the discrete microscopic structure of the model and the continuous macroscopic effects. Starting with a situation where a flat piece of interface is observed on the macroscopic level, we analyze the most likely scenarios occurring on the microscopic level which induce the macroscopic interface and we relate them to the definition of the surface tension.

Notation. Let x belong to $\mathbb{R}^3$, w to S^2, n to $\mathbb{N}$ and let r, δ be positive real numbers. Let $\operatorname{sep}(n,x,r,w,\delta)$ be the event: there exists a collection $\mathcal{C}$ of open clusters in the configuration restricted to the ball $B(nx,nr)$ such that

$$\mathcal{L}^3(\mathcal{V}_\infty(\operatorname{cup}\mathcal{C},1/2)\Delta B_-(nx,nr,w)) \leq \delta r^3 n^3 .$$

Lemma 8.1. *(Central lemma)*
For any positive ε, there exists δ_0 positive such that: for any x in $\mathbb{R}^3$, w in S^2 and r in $]0,1[$,

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln P\big(\operatorname{sep}(n,x,r,w,\delta_0)\big) \leq -\pi r^2 \tau(w)\,(1-\varepsilon) .$$

Equivalently,

$$\lim_{\delta\to 0} \quad \sup_{x\in\mathbb{R}^3, w\in S^2, r\in]0,1[} \quad \limsup_{n\to\infty} \quad (n^2\pi r^2\tau(w))^{-1} \ln P\big(\operatorname{sep}(n,x,r,w,\delta)\big) \leq -1 .$$

Remark. The limit $\delta \to 0$ in the second statement exists because the quantity $P\big(\operatorname{sep}(n,x,r,w,\delta)\big)$ is a non–decreasing function of δ.

Remark. The uniformity with respect to x in $\mathbb{R}^3$, w in S^2 and r in $]0,1[$ is essential.

Remark. In the different context of the Ising model with Kac potentials, a procedure has been developed to get a probabilistic estimate for the local presence of an interface, whose spirit is similar to the technique of our proof (see the paragraph "minimal section" in [9]). In this context, a coarse–graining procedure specific to the Kac model is employed. The coarse–grained configuration is described by attributing a label $-1, 0, +1$ to each mesoscopic box. The labels $-1, +1$ correspond to boxes which have relaxed to the minus and plus phases, the label 0 to a box in an undeterminate status. Benois, Bodineau, Buttà, Presutti localize the interface between the minus and the plus phase and cover it with a collection of parallelepipeds (as we do with balls). Inside a parallelepiped, they map the configuration of labels on a configuration where there is no sequence of cubes of the same phase which crosses the parallelepiped, by modifying the labels in two strips, where the number of bad cubes is minimal (the bad cubes are the cubes having label 0 or cubes of the wrong phase). Hence the spirit of the proof of our central lemma is present there. An essential difference is, in our view, that the afore–mentioned work stays at the mesoscopic level of the boxes and does not go in depth to handle the microscopic structure of the model, so that the estimates are not precise when the range of interactions is finite. Probably the main reason is that no operational approach to define the surface tension on fixed finite interaction scales was available in [9]. The percolation approach for the definition of the surface tension turns out to be much more natural and it is crucial in order to recover a precise estimate with the help of Pisztora coarse–graining results, as we do here.

Proof. Let x belong to $\mathbb{R}^3$, w to S^2, r to $]0,1[$, δ to $]0,\theta/2[$. Since the event $\mathrm{sep}\,(n,x,r,w,\delta)$ depends only on the edges inside the ball $B(nx,nr)$, we need only to work with the configuration restricted to $B(nx,nr)$ in the whole proof. Suppose that the event $\mathrm{sep}\,(n,x,r,w,\delta)$ occurs and let $\mathcal{C}$ be a collection of open clusters of the configuration restricted to $B(nx,nr)$ realizing it. Let ρ,η be such that $0<\eta<\rho<r$, $0<2\eta<\sqrt{r^2-\rho^2}$ and $r^2\eta<1$ (ρ will be chosen later to be close to r). The set $\mathrm{cyl}\,(n\mathrm{disc}\,(x,\rho,w),n\eta)$ is included in the ball $B(nx,nr)$. Let n be large enough so that $4\eta n>\zeta$. Let C be an open cluster of the configuration restricted to $\mathrm{cyl}\,(n\mathrm{disc}\,(x,\rho,w),n\eta)$ joining the sets $\mathcal{V}_2(\mathrm{hyp}\,(nx-\eta nw,w),\zeta)$ and $\mathcal{V}_2(\mathrm{hyp}\,(nx+\eta nw,w),\zeta)$. Recall that ζ is a fixed constant larger than 5, used to define the surface tension (see chapter 5). We distinguish two cases:

• If $C\cap\mathrm{cup}\,\mathcal{C}=\varnothing$ then there exists an open cluster C^* in the configuration restricted to $\mathrm{cyl}\,(n\mathrm{disc}\,(x,\rho,w))\cap\mathrm{slab}\,(nx,w,-\eta n,0)$ connecting $\mathcal{V}_2(\mathrm{hyp}\,(nx,w),\zeta)$ and $\mathcal{V}_2(\mathrm{hyp}\,(nx-\eta n,w),\zeta)$ such that $C^*\subset C$ and therefore C^* is included in $\mathbb{Z}^3(B_-(nx,nr,w))\setminus\mathrm{cup}\,\mathcal{C}$.

• If $C\cap\mathrm{cup}\,\mathcal{C}\neq\varnothing$ then there exists an open cluster C^* in the configuration restricted to $\mathrm{cyl}\,(n\mathrm{disc}\,(x,\rho,w))\cap\mathrm{slab}\,(nx,w,1,+\eta n)$ connecting $\mathcal{V}_2(\mathrm{hyp}\,(nx,w),\zeta)$ and $\mathcal{V}_2(\mathrm{hyp}\,(nx+\eta nw,w),\zeta)$ such that $C^*\subset C$ and therefore C^* is included in $\mathrm{cup}\,\mathcal{C}\setminus\mathbb{Z}^3(B_-(nx,nr,w))$.

Let F be the union of all these clusters C^* (notice that several clusters C^* might correspond to the same cluster C). Since F is included in $\mathbb{Z}^3(B_-(nx,nr,w))\Delta\mathrm{cup}\,\mathcal{C}$

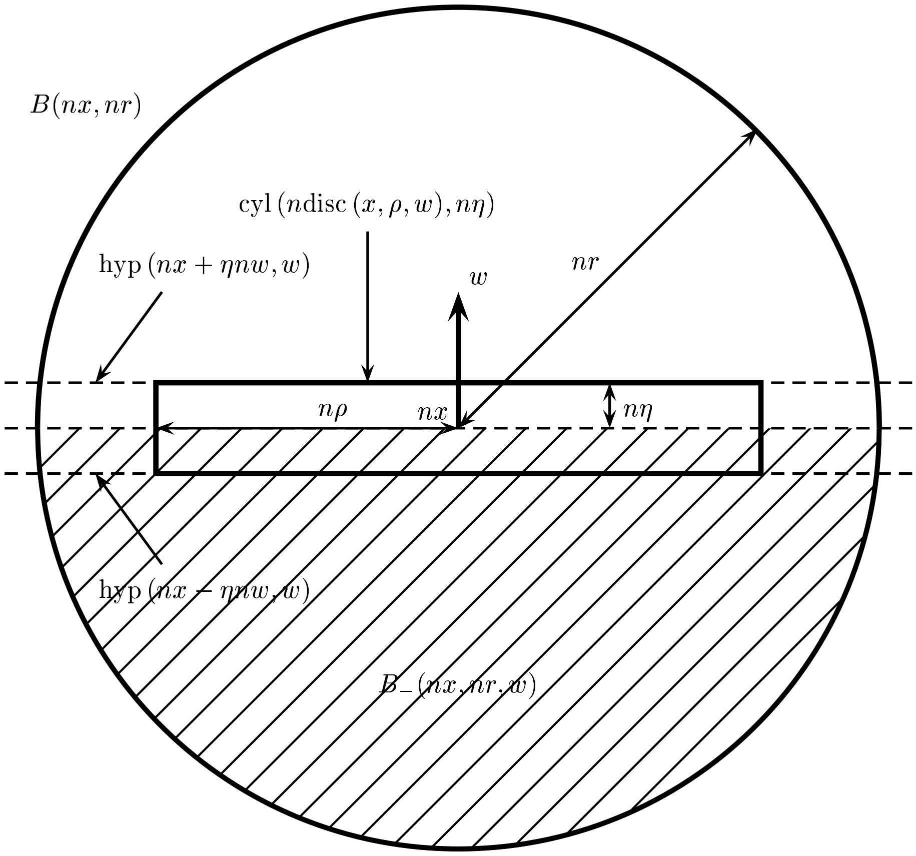

Figure 7

then

$$
\begin{aligned}
\operatorname{card} F &\leq \mathcal{L}^3\big(\mathcal{V}_\infty(\mathbb{Z}^3(B_-(nx,nr,w))\Delta \mathrm{cup}\,\mathcal{C}, 1/2)\big)\\
&= \mathcal{L}^3\big(\mathcal{V}_\infty(\mathbb{Z}^3\cap B_-(nx,nr,w),1/2)\Delta\mathcal{V}_\infty(\mathrm{cup}\,\mathcal{C},1/2)\big)\\
&\leq \mathcal{L}^3\big(B_-(nx,nr,w)\Delta\mathcal{V}_\infty(\mathrm{cup}\,\mathcal{C},1/2)\big)\\
&\qquad\qquad + \mathcal{L}^3\big(B_-(nx,nr,w)\Delta\mathcal{V}_\infty(\mathbb{Z}^3\cap B_-(nx,nr,w),1/2)\big)\\
&\leq \delta r^3n^3 + 3\sqrt{3}\pi r^2n^2 + 3\pi rn + 5\sqrt{3}\pi/4 \;\leq\; 2\delta r^3 n^3\,,
\end{aligned}
$$

the second inequality holds because we deal with subsets of $\mathbb{Z}^3$ and the last inequality is valid for n sufficiently large. We suppose in addition that n is large enough so that $\eta n/2 < (\eta n - 2\zeta)/\sqrt{3}$ and $\zeta < \kappa \ln n < \sqrt{n} < \eta n/14$, where κ is a large constant (several conditions will be imposed on κ during the computations). Let l be an integer, with $\kappa\ln n < l < \sqrt{n}$. The upper bound on l guarantees among other things that each open cluster contained in F has a d_∞ diameter larger than l. Let D_l be the set defined by

$$D_l \;=\; \big\{\, x^l \in l\mathbb{Z}^3 : \Lambda(x^l,3l) \subset \mathrm{slab}\,(nx,w,-\eta n,\eta n) \setminus \mathrm{slab}\,(nx,w,0,1)\,, \\ \Lambda(x^l,3l) \cap \mathrm{cyl}\,(n\mathrm{disc}\,(x,\rho,w)) \neq \varnothing \,\big\}\,.$$

Since $\rho^2+\eta^2<r^2$ and $r^2\eta<1$, for n large enough,

$$(n\rho+4\sqrt{n})^2+(n\eta+4\sqrt{n})^2<r^2n^2\,,\quad (n\rho+2\sqrt{n})^2(n\eta+2\sqrt{n})<n^3\,.$$

As a consequence, for any x^l in D_l, the box $\Lambda(x^l,3l)$ is still included in $B(nx,nr)$; moreover

$$\begin{aligned}\mathrm{card}\,D_l \;&\le\; l^{-3}\mathcal{L}^3(\mathcal{V}_\infty(D_l,l/2))\\ &\le\; l^{-3}\mathcal{L}^3(\mathrm{cyl}\,(\mathrm{disc}\,(nx,n\rho+2l,w),\eta n+2l)) \;\le\; 2\pi n^3 l^{-3}\,.\end{aligned}$$

We set $T(F)=D_l\cap \mathrm{fat}\,(F,l)$. For x^l in $l\mathbb{Z}^3$, let $X_l(x^l)$ be the indicator function of $R(\Lambda(x^l,3l),l-1)^c\cup V(\Lambda(x^l,l),\theta/2)^c$. With the help of the variables $X_l(x^l)$, $x^l\in l\mathbb{Z}^3$, we partition $T(F)$ in the three sets

$$\begin{aligned}T_{-1}(F) \;&=\; \{\, x^l\in T(F) : \Lambda(x^l,3l) \text{ is not included in } \mathrm{cyl}\,(n\mathrm{disc}\,(x,\rho,w))\,\}\,,\\ T_0(F) \;&=\; \{\, x^l\in T(F)\setminus T_{-1}(F) : X_l(x^l)=0\,\}\,,\\ T_1(F) \;&=\; \{\, x^l\in T(F)\setminus T_{-1}(F) : X_l(x^l)=1\,\}\,.\end{aligned}$$

We estimate next the cardinality of these sets. Firstly,

$$T_{-1}(F)\subset \mathrm{slab}\,(nx,w,-\eta n,\eta n)\cap \mathcal{V}_\infty(\partial\mathrm{cyl}\,(n\mathrm{disc}\,(x,\rho,w)),2l)$$

hence we obtain the deterministic bound

$$\mathrm{card}\,T_{-1}(F)\;\le\; 30\pi\rho\eta n^2/l^2\;\le\;100\eta n^2/l^2\,.$$

Secondly, let x^l belong to $T_0(F)$. Since F intersects $\Lambda(x^l,l)$ and the diameter of any open cluster contained in F is larger than l, then the box $\Lambda(x^l,3l)$ contains an open path of diameter larger than $l-1$ included in F. The occurrence of the event $R(\Lambda(x^l,3l),l-1)$ implies that this open path is contained in $C^*(\Lambda(x^l,3l))$, which in turn implies that the latter cluster is contained in F. Similarly, the crossing cluster $C^*(\Lambda(x^l,l))$ associated to the event $U(\Lambda(x^l,l))$ has diameter $l-1$ and it is thus contained in $C^*(\Lambda(x^l,3l))$. Since the event $V(\Lambda(x^l,l),\theta/2)$ occurs as well, then

$$\mathrm{card}\,F\cap\Lambda(x^l,l)\;\ge\;\mathrm{card}\,C^*(\Lambda(x^l,l))\;\ge\;\theta l^3/2\,.$$

Summing this inequality over $T_0(F)$ yields

$$\operatorname{card} F \;\geq\; \sum_{x^l \in T_0(F)} \operatorname{card} F \cap \Lambda(x^l, l) \;\geq\; (\theta/2) l^3 \operatorname{card} T_0(F)\,.$$

Since in addition $\operatorname{card} F < 2\delta r^3 n^3$, then

$$\operatorname{card} T_0(F) \;<\; 4\,\frac{\delta}{\theta}\Big(\frac{rn}{l}\Big)^3\,.$$

Thirdly, we have

$$\operatorname{card} T_1(F) \;\leq\; \operatorname{card}\big\{\, x^l \in D_l : X_l(x^l) = 1 \,\big\}\,.$$

Since $l \geq \kappa \ln n \geq \kappa \ln l$, for κ larger than the constant appearing in Theorem 7.1, by Corollaries 7.4, 7.5, there exist two positive constants b, c such that

$$\forall\, x^l \in D_l \qquad P(X_l(x^l) = 1) \;\leq\; b \exp(-cl)\,.$$

Applying Corollary 7.8 with $A = D_l$, $\alpha = 3$, $\varepsilon = b\exp(-cl)$, we get

$$P(\operatorname{card} T_1(F) \geq i) \;\leq\; \sum_{j \geq i} \exp\big(j \ln \operatorname{card} D_l + j(\ln b - cl)/27\big)\,.$$

Since $\operatorname{card} D_l \;\leq\; 2\pi n^3 l^{-3}$ and $l \;>\; \kappa \ln n$, for κ sufficiently large, there exist two positive constants b', c' such that, for any i in $\mathbb{N}$,

$$P(\operatorname{card} T_1(F) \geq i) \;\leq\; b' \exp(-c' l i)\,.$$

Let a be a positive real number. Using the previous estimates for the cardinality of $T_{-1}(F)$, $T_0(F)$ and $T_1(F)$, we write

$$\begin{aligned} P\big(\operatorname{sep}(n,x,r,w,\delta)\big) \;\leq\;& b' \exp(-c' a n^2) + \\ & P\big(\operatorname{sep}(n,x,r,w,\delta),\ \operatorname{card} T(F) \leq 100\eta n^2/l^2 + a n^2/l + 4\,\delta r^3 n^3/(\theta l^3)\big)\,. \end{aligned}$$

For i in $\mathbb{Z}$, we set

$$H(i) \;=\; \big\{\, y \in l\mathbb{Z}^3 : (y - nx)\cdot w \in [li, l(i+1)[\,\big\}\,.$$

For convenience, we omit the superscript l for the elements of the sets $H(i)$, although they belong to the rescaled lattice $l\mathbb{Z}^3$. The sets $H(i)$, $i \in \mathbb{Z}$, are pairwise disjoint. Hence for any subset I of $\mathbb{Z}$, the sum $\sum_{i\in I} \operatorname{card} T(F) \cap H(i)$ is less than $\operatorname{card} T(F)$ and there exists i in I such that $\operatorname{card} T(F) \cap H(i) \leq \operatorname{card} T(F)/\operatorname{card} I$. Applying the preceding remark to the sets $]-\eta n/l + 3, -3[\cap\mathbb{Z}$ and $]3, \eta n/l - 3[\cap\mathbb{Z}$, whose cardinalities are larger than $\eta n/l - 7 > \eta n/(2l)$, we find that there exist two random

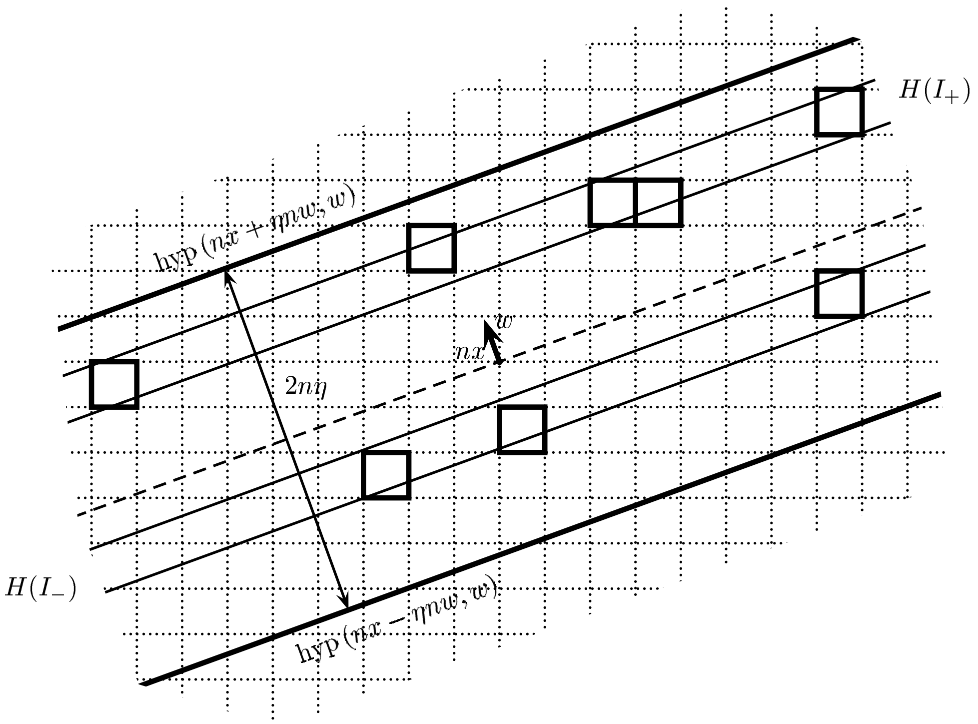

Figure 8

indices I_- and I_+ in $\mathbb{Z}$ such that $-\eta n/l + 3 < I_- < -3 < 3 < I_+ < \eta n/l - 3$ and both $T(F) \cap H(I_-)$, $T(F) \cap H(I_+)$ have a cardinality less than $2l(\eta n)^{-1} \operatorname{card} T(F)$. We decompose now the event $\operatorname{sep}(n, x, r, w, \delta)$ according to the possible values of the indices I_-, I_+ and the sets $T(F) \cap H(I_-)$, $T(F) \cap H(I_+)$:

$$P\big(\operatorname{sep}(n,x,r,w,\delta),\ \operatorname{card} T(F) \le 100\eta n^2/l^2 + an^2/l + 4\,\delta r^3 n^3/(\theta l^3)\big) \ \le$$

$$\sum P\Big(\operatorname{sep}(n,x,r,w,\delta),\ I_- = i_-,\ I_+ = i_+,$$
$$\operatorname{card} T(F) \cap H(I_-) = j_-,\ \operatorname{card} T(F) \cap H(I_+) = j_+,$$
$$T(F) \cap H(I_-) = \{\, y_-(1), \dots, y_-(j_-)\,\},\ T(F) \cap H(I_+) = \{\, y_+(1), \dots, y_+(j_+)\,\}\Big)$$

where the summation extends over the set:

$$-\eta n/l + 3 < i_- < -3 < 3 < i_+ < \eta n/l - 3\,, \quad 0 \le j_-, j_+ \le 200\frac{n}{l} + 2\frac{an}{\eta} + 8\frac{\delta r^3 n^2}{\theta \eta l^2}\,,$$
$$y_-(1), \dots, y_-(j_-) \in H(i_-) \cap D_l\,, \quad y_+(1), \dots, y_+(j_+) \in H(i_+) \cap D_l\,.$$

Since $\mathcal{V}_\infty(D_l, l/2) \subset B(nx, nr)$, for any i in $\mathbb{Z}$,

$$\begin{aligned}\operatorname{card} H(i) \cap D_l \ &\le l^{-3} \mathcal{L}^3(\mathcal{V}_\infty(H(i) \cap D_l, l/2)) \\ &\le l^{-3}\mathcal{L}^3(\operatorname{cyl}(\operatorname{disc}(nx, nr, w), l\sqrt{3}/2)) \ \le\ 2\pi (nr/l)^2\,.\end{aligned}$$

Therefore the number of terms in the summation is less than

$$\left(\frac{\eta n}{l}\right)^2\left(200\frac{n}{l}+2\frac{an}{\eta}+8\frac{\delta r^3n^2}{\theta\eta l^2}+1\right)^2$$
$$\times\exp\left(\left(400\frac{n}{l}+4\frac{an}{\eta}+16\frac{\delta r^3n^2}{\theta\eta l^2}\right)\ln(2\pi r^2n^2l^{-2})\right).$$

Let $\mathcal{E}$ be the event appearing in the summation, that is,

$$\Big\{\operatorname{sep}(n,x,r,w,\delta),\ I_-=i_-,\ I_+=i_+,$$
$$\operatorname{card}T(F)\cap H(I_-)=j_-,\ \operatorname{card}T(F)\cap H(I_+)=j_+,$$
$$T(F)\cap H(I_-)=\{\,y_-(1),\dots,y_-(j_-)\,\},\ T(F)\cap H(I_+)=\{\,y_+(1),\dots,y_+(j_+)\,\}\Big\}.$$

We next bound the probability of the event $\mathcal{E}$. By the definition of F, the event $\mathcal{E}$ is included in the event: any open cluster C of the configuration restricted to $\operatorname{cyl}(n\operatorname{disc}(x,\rho,w),n\eta)$ joining $\mathcal{V}_2(\operatorname{hyp}(nx-\eta nw,w),\zeta)$ and $\mathcal{V}_2(\operatorname{hyp}(nx+\eta nw,w),\zeta)$ satisfies

$$\begin{aligned}\text{either}\quad & C\cap\operatorname{cup}\{\,\Lambda(y,l):y\in H(i_-)\,\}\ \subset\ \operatorname{cup}\{\,\Lambda(y_-(j),l):1\le j\le j_-\,\}\\ \text{or}\quad & C\cap\operatorname{cup}\{\,\Lambda(y,l):y\in H(i_+)\,\}\ \subset\ \operatorname{cup}\{\,\Lambda(y_+(j),l):1\le j\le j_+\,\},\end{aligned}$$

depending on whether $C\cap\operatorname{cup}\mathcal{C}=\varnothing$ or $C\cap\operatorname{cup}\mathcal{C}\neq\varnothing$. Let A be the set of edges defined by

$$A\ =\ \bigcup\nolimits_{1\le j\le j_-}\partial_e\Lambda(y_-(j),l)\ \cup\ \bigcup\nolimits_{1\le j\le j_+}\partial_e\Lambda(y_+(j),l)\,.$$

To a configuration ω in $\mathcal{E}$, we associate a configuration $\Phi(A,\omega)$ defined by:

$$\forall\, e\in\mathbb{E}^3(B(nx,nr))\qquad \Phi(A,\omega)(e)\ =\ \begin{cases}0 & \text{if } e\in A\\ \omega(e) & \text{if } e\notin A\end{cases}$$

What we have done is to close by force all the edges in A. Doing so we have destroyed all the connections between $\mathcal{V}_2(\operatorname{hyp}(nx-\eta nw,w),\zeta)$ and $\mathcal{V}_2(\operatorname{hyp}(nx+\eta nw,w),\zeta)$ for the configuration $\Phi(A,\omega)$ restricted to $\operatorname{cyl}(n\operatorname{disc}(x,\rho,w),n\eta)$. Indeed any such connection which was present in ω had to go through an edge of A. More precisely any open cluster C of the configuration ω restricted to $\operatorname{cyl}(n\operatorname{disc}(x,\rho,w),n\eta)$ joining $\mathcal{V}_2(\operatorname{hyp}(nx-\eta nw,w),\zeta)$ and $\mathcal{V}_2(\operatorname{hyp}(nx+\eta nw,w),\zeta)$ is cut in the new configuration $\Phi(A,\omega)$ in several disjoint clusters (at least three), none of which intersects both sets $\operatorname{cup}\{\,\Lambda(y,l):y\in H(i_-)\,\}$ and $\operatorname{cup}\{\,\Lambda(y,l):y\in H(i_+)\,\}$. However the vertices of either of these two sets separate $\mathcal{V}_2(\operatorname{hyp}(nx-\eta nw,w),\zeta)$ and

$\mathcal{V}_2(\mathrm{hyp}\,(nx+\eta nw,w),\zeta)$ in $\mathrm{cyl}\,(n\mathrm{disc}\,(x,\rho,w),n\eta)$, that is, either for $i=i_-$ or for $i=i_+$, there exists no path in the graph

$$\mathbb{Z}^3\cap\mathrm{cyl}\,(n\mathrm{disc}\,(x,\rho,w),n\eta)\setminus\mathrm{cup}\,\{\,\Lambda(y,l):y\in H(i)\,\}\,,\quad \mathbb{E}^3\,(\mathrm{cyl}\,(n\mathrm{disc}\,(x,\rho,w),n\eta))$$

joining $\mathcal{V}_2(\mathrm{hyp}\,(nx-\eta nw,w),\zeta)$ and $\mathcal{V}_2(\mathrm{hyp}\,(nx+\eta nw,w),\zeta)$. Therefore the configuration $\Phi(A,\omega)$ realizes the event $S(n\mathrm{disc}\,(x,\rho,w),n\eta)$. The number of edges in A is less than $6(j_-+j_+)l^2$. Thus

$$\forall\,\omega\in\mathcal{E}\qquad P(\omega)\,\leq\,P(\Phi(A,\omega))\,\exp\Big(-6(j_-+j_+)l^2\ln(1-p)\Big)\,.$$

Summing over ω in $\mathcal{E}$,

$$P(\mathcal{E})\,\leq\,\exp\Big(-6(j_-+j_+)l^2\ln(1-p)\Big)\sum_{\omega\in\mathcal{E}}P(\Phi(A,\omega))\,.$$

Moreover

$$\sum_{\omega\in\mathcal{E}}P(\Phi(A,\omega))\,=\sum_{\omega\in\Phi(A,\mathcal{E})}P(\omega)\,\mathrm{card}\,(\Phi^{-1}(A,\omega)\cap\mathcal{E})$$

and for any ω in $\Phi(A,\mathcal{E})$, the cardinality of $\Phi^{-1}(A,\omega)\cap\mathcal{E}$ is less than $\exp(6(j_-+j_+)l^2\ln 2)$; thus

$$P(\mathcal{E})\,\leq\,\exp\Big(-6(j_-+j_+)l^2\ln((1-p)/2)\Big)P(\Phi(A,\mathcal{E}))\,.$$

Since $\Phi(A,\mathcal{E})$ is included in $S(n\mathrm{disc}\,(x,\rho,w),n\eta)$, using the upper bound on the values on j_-,j_+, we arrive at

$$P(\mathcal{E})\,\leq\,\exp\left(-6\Big(400\frac{n}{l}+4\frac{an}{\eta}+16\frac{\delta r^3n^2}{\theta\eta l^2}\Big)l^2\ln\frac{1-p}{2}\right)P(S(n\mathrm{disc}\,(x,\rho,w),n\eta))\,.$$

Coming back to the big summation,

$$\begin{aligned}P\big(\mathrm{sep}\,(n,x,r,w,\delta)\big)\,\leq\,b'\exp(-c'an^2)\,+\,\left(200\frac{\eta n^2}{l^2}+2\frac{an^2}{l}+8\frac{\delta r^3n^3}{\theta l^3}+\frac{\eta n}{l}\right)^2\times\\ \exp\left(\Big(400\frac{n}{l}+4\frac{an}{\eta}+16\frac{\delta r^3n^2}{\theta\eta l^2}\Big)\Big(\ln(2\pi r^2n^2l^{-2})-6l^2\ln\frac{1-p}{2}\Big)\right)\times\\ P(S(n\mathrm{disc}\,(x,\rho,w),n\eta))\,.\end{aligned}$$

Letting n and l go to ∞ in the regime where $\kappa\ln n<l$ and n/l^2 goes to ∞ and using Corollary 5.14, we get

$$\begin{aligned}\limsup_{n\to\infty}\quad(n^2\pi r^2\tau(w))^{-1}\ln P\big(\mathrm{sep}\,(n,x,r,w,\delta)\big)\,\leq\\ -\min\left\{\frac{c'a}{\pi r^2\tau(w)},\,\frac{\rho^2}{r^2}(1-c(\zeta,p)\sqrt{\eta/\rho})+\frac{96\delta r}{\pi\tau(w)\theta\eta}\ln\frac{1-p}{2}\right\}.\end{aligned}$$

We choose now $\eta = \sqrt{\delta}r/3$, $\rho = r\sqrt{1-\delta}$ and we let a go to ∞ in the preceding inequality. Because τ is bounded away from 0 (Corollary 5.7), there exists a constant $c'(\zeta,p)$ depending on ζ and p only such that

$$\limsup_{n\to\infty} \quad (n^2\pi r^2\tau(w))^{-1}\ln P\big(\mathrm{sep}\,(n,x,r,w,\delta)\big) \;\leq\; -1+\delta+c'(\zeta,p)\delta^{1/4}\,.$$

This inequality holds for all values of x in $\mathbb{R}^3$, r in $]0,1[$, δ in $]0,\theta/2[$ and w in S^2, and the right–hand side converges to -1 as δ goes to 0, uniformly with respect to x,w,r. $\square$

CHAPTER 9

PROOF OF THE LDP FOR A SINGLE CLUSTER

In this chapter, we prove the large deviation principle for a single cluster. After some elementary geometrical lemmata, we introduce two smoothing operators, fifa and aglu. We then prove the large deviation upper bound for the topology L^1_{loc}, the large deviation lower bound for the topology L^1, the enhanced large deviation upper bound and Proposition 2.4.

9.1. Geometrical Lemmata

We start with several geometrical lemmata about connected subsets of the lattice.

Lemma 9.1. *Let A be a finite $\mathbb{L}^3$ connected set of $\mathbb{Z}^3$. For any integer l larger than* 130*,*

$$\mathcal{L}^3(\mathcal{V}_\infty(A,l)) \;\leq\; 130\, l^2 \max(\operatorname{card} A,\, l)\,.$$

Proof. If $\operatorname{diam}_\infty A \leq l$, then $\mathcal{L}^3(\mathcal{V}_\infty(A,l)) \;\leq\; 27l^3$. Suppose that $l < \operatorname{diam}_\infty A < \infty$. Let $\{\, x_1,\dots,x_r \,\}$ be a collection of vertices of A of maximal cardinality such that

$$\forall\, i,j \in \{1,\dots,r\}, \quad i \neq j, \quad \Lambda(x_i,l)\cap\Lambda(x_j,l)=\varnothing\,.$$

The maximality of the collection implies that $A\subset\Lambda(x_1,2l)\cup\cdots\cup\Lambda(x_r,2l)$. Because r is necessarily larger or equal than 2 and A is $\mathbb{L}^3$ connected, for each i in $\{1,\dots,r\}$,

$$\operatorname{card} A\cap\Lambda(x_i,l) \;\geq\; d_\infty(x_i,\partial_{\mathrm{iv}}\Lambda(x_i,l)) \;\geq\; l/2-1$$

so that $\operatorname{card} A \geq r(l/2-1)$. Since $\mathcal{V}_\infty(A,l)$ is included in r boxes of diameter $4l$, we obtain $\mathcal{L}^3(\mathcal{V}_\infty(A,l)) \leq r(4l)^3 \;\leq\; (4l)^3(l/2-1)^{-1}\operatorname{card} A$, which gives the desired inequality for l larger than 130. $\square$

Lemma 9.2. *For any subset R of $\mathbb{Z}^3$, $\operatorname{card}\partial_{*v}R \leq \operatorname{card}\partial_{\mathrm{e}}R \leq 6\operatorname{card}\partial_{*v}R$, where $*$ stands either for o or for i.*

Lemma 9.3. *If R is a $\mathbb{L}^3$ or $\mathbb{E}^3$ connected subset of $\mathbb{Z}^3$ then $\operatorname{card} R \geq \operatorname{diam}_\infty R+1$, provided R is not empty.*

Lemma 9.4. *Let R be a $\mathbb{L}^3$ or $\mathbb{E}^3$ connected subset of $\mathbb{Z}^3$ having finite cardinality. Then* $4\mathrm{diam}_\infty R + 2 \leq \mathrm{card}\,\partial_{\mathrm{ov}} R$.

Proof. For i in $\mathbb{Z}$, set $H(i) = \{\,(i, x_2, x_3) : x_2, x_3 \in \mathbb{R}\,\}$. Then

$$\mathrm{card}\,\partial_{\mathrm{ov}} R \;=\; \sum_{i\in\mathbb{Z}} \mathrm{card}\,\partial_{\mathrm{ov}} R \cap H(i)\,.$$

Let $i_1 = \min\{\, i : H(i) \cap R \neq \varnothing\,\}$ and $i_2 = \max\{\, i : H(i) \cap R \neq \varnothing\,\}$. We may suppose that $i_2 - i_1 = \mathrm{diam}_\infty R$. For each i in $\{\, i_1, \dots, i_2\,\}$, $\mathrm{card}\,\partial_{\mathrm{ov}} R \cap H(i) \geq 4$; moreover $\partial_{\mathrm{ov}} R \cap H(i_1 - 1)$ and $\partial_{\mathrm{ov}} R \cap H(i_2 + 1)$ are not empty. Therefore $4\mathrm{diam}_\infty R + 2 \leq \mathrm{card}\,\partial_{\mathrm{ov}} R$. $\square$

Lemma 9.5. *Let A_1, A_2 be two disjoint subsets of $\mathbb{Z}^3$. We suppose that at least one set among A_1, A_2 has finite cardinality. For any positive r, the sets $\mathcal{V}_\infty(A_1, r)$ and $\mathcal{V}_\infty(A_2, r)$ are distinct.*

Proof. We need only to consider the case where both A_1 and A_2 have finite cardinality. Let x_1 (respectively x_2) be the smallest point of A_1 (respectively A_2) with respect to the lexicographic order on $\mathbb{Z}^3$. Necessarily x_1 and x_2 are distinct. Then $x_1 - (r, r, r)$ (respectively $x_2 - (r, r, r)$) is the smallest point of the closure of $\mathcal{V}_\infty(A_1, r)$ (respectively $\mathcal{V}_\infty(A_2, r)$). Therefore the sets $\mathcal{V}_\infty(A_1, r)$ and $\mathcal{V}_\infty(A_2, r)$ have distinct closures, hence they are distinct. $\square$

The next result is used several times in the proofs, sometimes without explicit reference.

Lemma 9.6. *Let x belong to $\mathbb{Z}^3$, let l_1, l_2 be two positive integers, with $l_1 < l_2$. Let A be a connected subset of $(\mathbb{Z}^3, \mathbb{E}^3)$ such that $A \cap \Lambda(x, l_1) \neq \varnothing$, $A \setminus \Lambda(x, l_2) \neq \varnothing$. Then $A \cap \Lambda(x, l_2)$ contains a connected subset of d_∞ diameter larger than $\lfloor (l_2 - l_1)/2 \rfloor$.*

Proof. Clearly the set A contains a path joining $\partial_{\mathrm{iv}} \Lambda(x, l_1)$ to $\partial_{\mathrm{iv}} \Lambda(x, l_2)$ inside $\Lambda(x, l_2)$. $\square$

We build next an operator which regularizes a connected set of the lattice, by performing successively a fattening and a filling step. We define first the fill operator.

Definition 9.7. (Fill operator)
For an $\mathbb{E}^3$ connected set A of $\mathbb{Z}^3$ and a positive real number s, we define

$$\mathrm{fill}\,(A, s) \;=\; A \cup \big\{\, R \text{ residual component of } A : \mathrm{card}\,\partial_{\mathrm{e}} R \leq s,\ \mathrm{card}\, R < \infty \,\big\}\,.$$

Notice that the definition is valid also for infinite sets A. The set $\mathrm{fill}\,(A, s)$ is obtained by filling all the finite residual components of A whose edge boundary has cardinality less than s. By construction, if R is a residual component of $\mathrm{fill}\,(A, s)$, then $\mathrm{card}\,\partial_{\mathrm{e}} R > s$, whence by Lemma 9.2, $\mathrm{card}\,\partial_{\mathrm{iv}} R > s/6$. Since for any residual component R of A, $\partial_{\mathrm{iv}}R$ is $\mathbb{L}^3$ connected and $\partial_{\mathrm{iv}}R = R \cap \partial_{\mathrm{ov}}A$, we conclude that any $\mathbb{L}^3$ component of the outer vertex boundary of $\mathrm{fill}\,(A,s)$ either meets the inner vertex boundary of an infinite residual component of A or has cardinality strictly larger than $s/6$. In the situation where A has diameter larger than $s/6$, then any $\mathbb{L}^3$ component of the outer vertex boundary of $\mathrm{fill}\,(A,s)$ has cardinality strictly larger than $s/6$.

Lemma 9.8. *Let A be a connected subset of $\mathbb{Z}^3$ (finite or infinite) and let h be a positive real number larger than* 2. *Then* $\mathrm{fill}\,(A,h)$ *is included in* $\mathcal{V}_\infty(A,h/2)$.

Proof. Let R be a finite residual component of A such that $\mathrm{card}\,\partial_{\mathrm{e}}R \leq h$. By Lemmata 9.2, 9.4, $\mathrm{diam}_\infty R \leq (h-2)/4 \leq h/2-1$. Since $d_\infty(A,R)=1$, we conclude that $R \subset \mathcal{V}_\infty(A,h/2)$. □

9.2. The fifa operation†

Throughout this section, we consider positive integers k,l satisfying $k \geq 6$, $l \geq 130$, $3k \leq l$. Let A be a connected subset of $\mathbb{Z}^3$ of diameter strictly larger than l. To A we associate a regularized set $\mathrm{fifa}\,(A,k,l)$ by performing successively filling and fattening operations on the rescaled lattice $k\mathbb{Z}^3$ and then coming back to $\mathbb{Z}^3$ (see definition 7.6 for the fat operator). More precisely, if $\mathrm{diam}_\infty A \leq l$, we set $\mathrm{fifa}\,(A,k,l) = \varnothing$; if $\mathrm{diam}_\infty A > l$, we set

$$\mathrm{fifa}\,(A,k,l) \;=\; \mathrm{cup}\,\{\,\Lambda(x^k,k) : x^k \in k\,\mathrm{fill}\,(k^{-1}\mathrm{fat}\,(A,k),l/k)\,\}\,.$$

Lemma 9.9. *For any connected subset A of $\mathbb{Z}^3$,* $\mathrm{fifa}\,(A,k,l)$ *is included in* $\mathcal{V}_\infty(A,l)$.

Proof. We need only to consider the case where $\mathrm{diam}_\infty A > l$. By Lemma 9.8, since $k^{-1}\mathrm{fat}\,(A,k)$ is connected and $l/k \geq 2$, then

$$\mathrm{fill}\,(k^{-1}\mathrm{fat}\,(A,k),l/k) \;\subset\; \mathcal{V}_\infty(k^{-1}\mathrm{fat}\,(A,k),l/2k)\,.$$

Coming back to the lattice $\mathbb{Z}^3$, we deduce that

$$\mathrm{fifa}\,(A,k,l) \;\subset\; \mathrm{cup}\,\{\,\Lambda(x^k,k) : x^k \in k\mathbb{Z}^3 \cap \mathcal{V}_\infty(\mathrm{fat}\,(A,k),l/2)\,\} \;\subset\; \mathcal{V}_\infty(A,l)\,. \quad \square$$

Lemma 9.10. *Let A be a connected subset of $\mathbb{Z}^3$ such that* $\mathrm{diam}_\infty A > l$. *For any compact set K of $\mathbb{R}^3$, any integer h larger than l,*

$$\mathcal{L}^3_K\big(\mathcal{V}_\infty(A,h) \setminus \mathrm{fifa}\,(A,k,l) \setminus \mathcal{V}_\infty(\partial K, l+h+1)\big) \;\leq\; 140h^3l^{-1}\mathcal{P}(\mathrm{fifa}\,(A,k,l), \mathrm{int}\,K)\,.$$

Proof. Let x in K be such that

$$0 < d_\infty(x, \mathrm{fifa}\,(A,k,l)) < h\,, \quad d_\infty(x, \mathbb{R}^3 \setminus K) \geq l+h+1\,.$$

†FIFA=Fédération Internationale de Football Association. France won the 1998 World Cup.

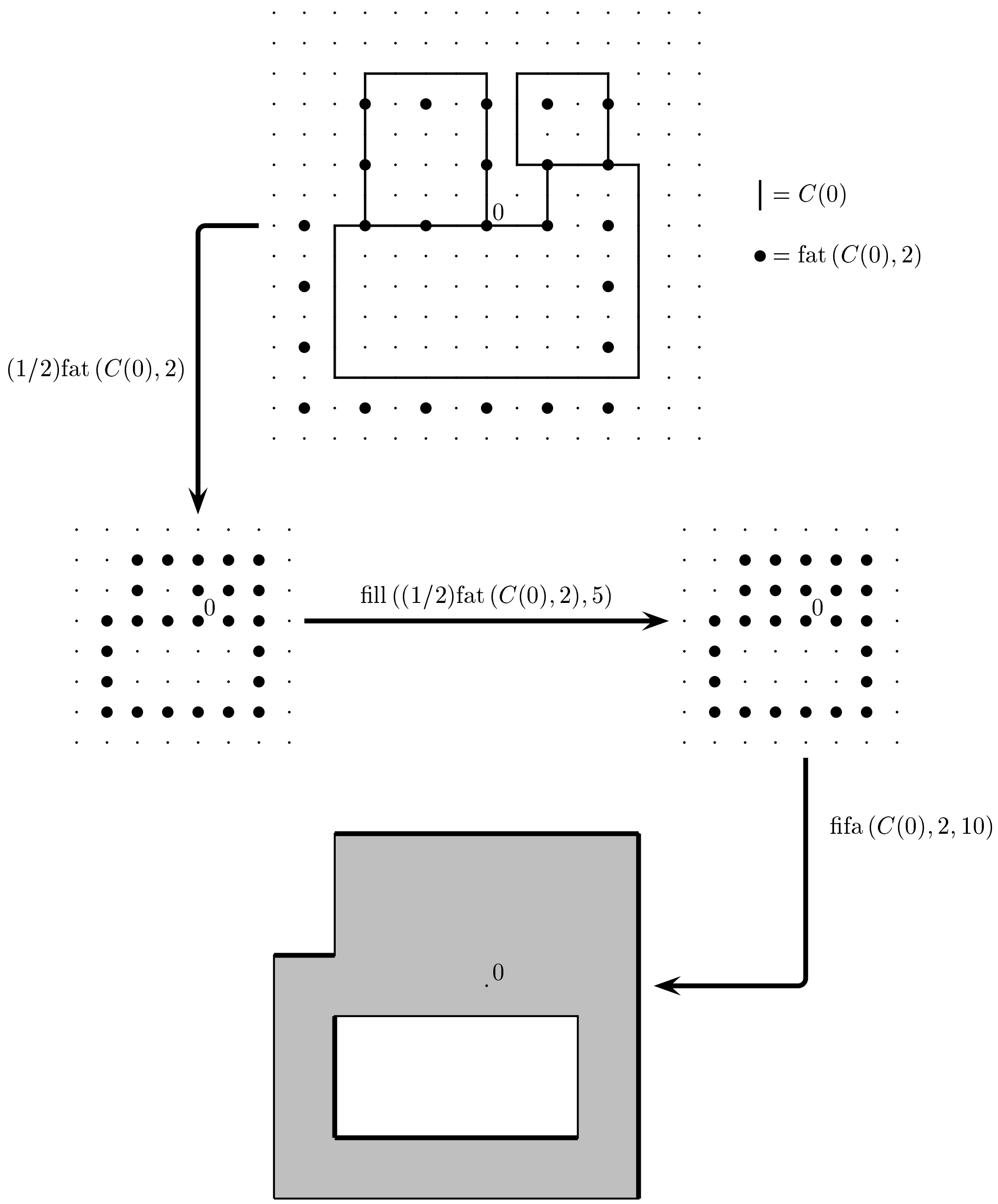

Figure 9: $\mathrm{fifa}\,(C(0), k, l)$

Because $\partial\mathrm{fifa}\,(A,k,l) \subset \mathrm{clo}\,\mathcal{V}_\infty(\partial_{\mathrm{ov}}\mathrm{fifa}\,(A,k,l),1)$ (recall that $\mathrm{fifa}\,(A,k,l)$ is an union of boxes $\Lambda(x^k,k)$ centered on $k\mathbb{Z}^3$), there exists a point y in $\partial_{\mathrm{ov}}\mathrm{fifa}\,(A,k,l)$ such that $d_\infty(x,y) < h+1$. Necessarily $d_\infty(y,\mathbb{R}^3\setminus K) \geq d_\infty(x,\mathbb{R}^3\setminus K) - d_\infty(x,y) > l$. Thus y belongs to the set

$$L \,=\, \{\, z \in \mathbb{Z}^3 : d_\infty(z,\mathbb{R}^3\setminus K) > l \,\}\,.$$

Since in addition $\mathcal{V}_\infty(A,h) \subset \mathcal{V}_\infty(\mathrm{fifa}\,(A,k,l),h)$, the preceding discussion shows that

$$\mathcal{V}_\infty(A,h)\setminus \mathrm{fifa}\,(A,k,l)\setminus \mathcal{V}_\infty(\mathbb{R}^3\setminus K, l+h+1) \,\subset\, \mathcal{V}_\infty(\partial_{\mathrm{ov}}\mathrm{fifa}\,(A,k,l)\cap L, h+1)\,.$$

By construction each $\mathbb{L}^3$ component of $\partial_{\mathrm{ov}}\mathrm{fill}\,(k^{-1}\mathrm{fat}\,(A,k),l/k)$ has cardinality strictly larger than $l/6k$ (see the discussion after Definition 9.7), hence each $\mathbb{L}^3$ component of $\partial_{\mathrm{ov}}\mathrm{fifa}\,(A,k,l)$ has cardinality larger than $lk/6$ (each vertex of the former set corresponds to at least k^2 vertices of the latter set). Let

$$M \,=\, \{\, x \in \mathbb{Z}^3 : d_\infty(x,\mathbb{R}^3\setminus K) > 1 \,\}\,.$$

Let $F_1,\dots,F_r$ be the $\mathbb{L}^3$ components of the set $\partial_{\mathrm{ov}}\mathrm{fifa}\,(A,k,l)\cap M$ which intersect L. Let i belong to $\{1,\dots,r\}$. If F_i does not intersect $\partial_{\mathrm{iv}}M$, then F_i is an $\mathbb{L}^3$ component of $\partial_{\mathrm{ov}}\mathrm{fifa}\,(A,k,l)$ and $\mathrm{card}\,F_i \geq lk/6 \geq l$ (by hypothesis $k\geq 6$). If F_i intersects $\partial_{\mathrm{iv}}M$, since $d_\infty(\partial K,\partial_{\mathrm{iv}}M) \leq 2$, then $\mathrm{diam}_\infty F_i > l-2$ whence $\mathrm{diam}_\infty F_i \geq l-1$ and $\mathrm{card}\,F_i \geq l$ by Lemma 9.3. Thus all the components $F_1,\dots,F_r$ have cardinality larger than l. By Lemma 9.1, using the bounds $130 \leq l \leq h$,

$$\begin{aligned}
\mathcal{L}^3(\mathcal{V}_\infty(\partial_{\mathrm{ov}}\mathrm{fifa}\,(A,k,l)\cap L,h+1)) \,&\leq\, \sum_{1\leq i\leq r} \mathcal{L}^3(\mathcal{V}_\infty(F_i,h+1))\\
&\leq\, \sum_{1\leq i\leq r} 130\,(h+1)^3 l^{-1}\mathrm{card}\,F_i\\
&\leq\, 130\,(h+1)^3 l^{-1}\mathrm{card}\,\big(\partial_{\mathrm{ov}}\mathrm{fifa}\,(A,k,l)\cap M\big)\\
&\leq\, 140\,h^3 l^{-1}\mathcal{P}(\mathrm{fifa}\,(A,k,l),\mathrm{int}\,K)\,.
\end{aligned}$$

The last inequality holds because $130(h+1)^3 \leq 140h^3$ (recall that $h \geq l \geq 130$) and each vertex of $\partial_{\mathrm{ov}}\mathrm{fifa}\,(A,k,l)\cap M$ contributes to at least one unit square on $\partial\mathrm{fifa}\,(A,k,l)\cap\mathrm{int}\,K$. $\square$

Corollary 9.11. *Let A be a finite connected subset of $\mathbb{Z}^3$ such that* $\mathrm{diam}_\infty A > l$ *and let h be an integer larger than l. Then*

$$\mathcal{L}^3(\mathcal{V}_\infty(A,h)\Delta\mathrm{fifa}\,(A,k,l)) \,\leq\, 140\,h^3 l^{-1}\mathcal{P}(\mathrm{fifa}\,(A,k,l))\,.$$

Proof. We choose a compact set K large enough to have $d_\infty(A,\mathbb{R}^3\setminus K) > l+2h+1$ and we apply Lemmata 9.9, 9.10. $\square$

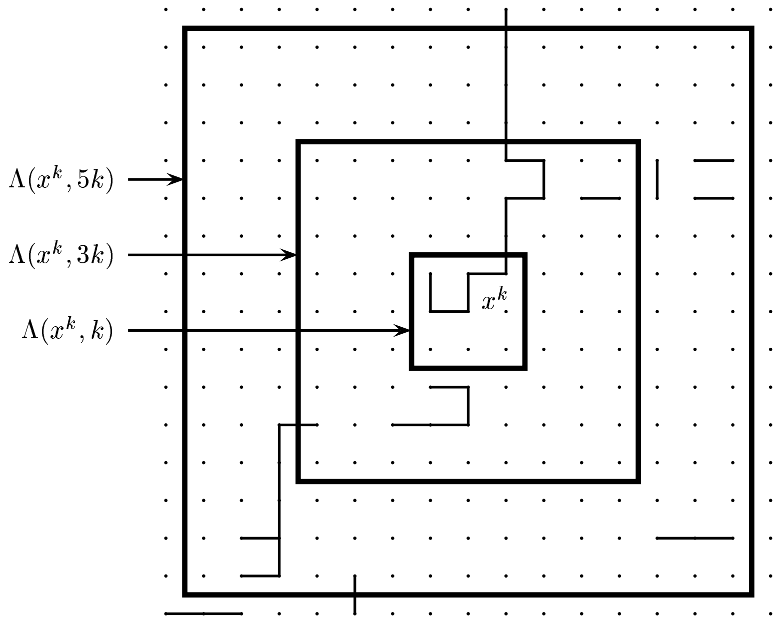

Figure 10: $X_k(x^k) = 1$

Proposition 9.12. *For k large enough, there exist two positive constants $b(k)$, $c(k)$ such that: for any l in $\mathbb{N}$, any s in $\mathbb{R}^+$, satisfying $5k \le l$, $k \ln s \le l$,*

$$\widehat{P}\big(\mathcal{P}(\text{fifa}\,(C(0), k, l)) \ge s\big) \;\le\; b(k) \exp(-c(k)s)\,.$$

Proof. To each vertex x^k of $k\mathbb{Z}^3$ we associate a random variable $X_k(x^k)$ with values in $\{0,1\}$ as follows: if in the configuration restricted to $\Lambda(x^k, 5k)$ there exists an open path from $\Lambda(x^k, 3k)$ to $\partial_{\text{iv}}\Lambda(x^k, 5k)$ which is not connected by an open path in $\Lambda(x^k, 5k)$ to $\Lambda(x^k, k)$, then $X_k(x^k) = 1$; otherwise $X_k(x^k) = 0$. Clearly the value of $X_k(x^k)$ depends only on the configuration restricted to $\Lambda(x^k, 5k)$. Because our model is translation invariant, the probability $P(X_k(x^k) = 1)$ is the same for all x^k in $k\mathbb{Z}^3$. By [50, Proposition 1], throughout the supercritical regime $p > p_c$, this probability goes to 0 as k goes to ∞. Corollary 7.4 yields the stronger estimate $P(X_k(x^k) = 1) \le b \exp -ck$ where b, c are two positive constants depending on p only. Let us pause to state an intermediate lemma.

Lemma 9.13. *Let C be an open cluster of diameter larger than $5k$. Let x^k belong to the outer vertex boundary of* $\text{fat}\,(C, k)$ *on $(k\mathbb{Z}^3, k\mathbb{E}^3)$. Then $X_k(x^k) = 1$.*

Remark. This fact was observed and exploited by Kesten and Zhang [50].

Proof. In this situation, the cluster C does not intersect the box $\Lambda(x^k, k)$ but it intersects $\Lambda(x^k, 3k)$, Since $\text{diam}_\infty C \ge 5k$, the cluster C contains an open path from

$\Lambda(x^k,3k)$ to $\partial_{\mathrm{iv}}\Lambda(x^k,5k)$; because $C\cap\Lambda(x^k,k)=\varnothing$, this path is not connected in $\Lambda(x^k,5k)$ to $\Lambda(x^k,k)$. It follows that $X_k(x^k)=1$. $\square$

Proof of Proposition 9.12 **continued.** We write

$$P\big(\mathcal{P}(\mathrm{fifa}\,(C(0),k,l))\geq s,\ \mathrm{card}\,C(0)<\infty\big)\ \leq \\ P\big(s<\mathrm{diam}_\infty C(0)<\infty\big)\ +\ P\big(\mathcal{P}(\mathrm{fifa}\,(C(0),k,l))\geq s\geq \mathrm{diam}_\infty C(0)\big)$$

and we handle separately each term of the right–hand side. We consider first the event $\{\,s<\mathrm{diam}_\infty C(0)<\infty\,\}$. Let i be an integer larger than $5k$ and suppose that $C(0)$ has a diameter equal to i. Let F be the inner vertex boundary of the unbounded residual component of $\mathrm{fat}\,(C(0),k)$ on $(k\mathbb{Z}^3,k\mathbb{E}^3)$. Then F is $k\mathbb{L}^3$ connected, it surrounds the origin and $\mathrm{card}\,F\geq i/k$. Since F is included in $\partial_{\mathrm{ov}}\mathrm{fat}\,(C(0),k)$, then $X_k(x^k)=1$ for any x^k in F by Lemma 9.13. Thus

$$P(\mathrm{diam}_\infty C(0)=i)\ \leq\ \sum_{j\geq i/k}\sum_F P(\forall\,x^k\in F\quad X_k(x^k)=1)\,,$$

where the second summation extends over all the subsets F of $k\mathbb{Z}^3$ of cardinality j which are $k\mathbb{L}^3$ connected and surround the origin. The number of such sets is less than jb^j for some constant b, whence, using the bound of Lemma 7.7 with $\alpha=5$, $\varepsilon=\varepsilon(k)=P(X_k(0)=1)$,

$$P(\mathrm{diam}_\infty C(0)=i)\ \leq\ \sum_{j\geq i/k}\exp\big(\ln j+j\ln b+j5^{-3}\ln\varepsilon(k)\big)\,.$$

Because $\varepsilon(k)$ goes to 0 as k goes to ∞, for k large enough, there exist two positive constants $b(k),c(k)$ such that for any positive s,

$$P(s\leq\mathrm{diam}_\infty C(0)<\infty)\ \leq\ b(k)\exp(-c(k)s)\,.$$

Let us pause again to state a little Lemma.

Lemma 9.14. *For any connected set A of $\mathbb{Z}^3$, we have the inequalities*

$$\mathcal{P}(\mathrm{fifa}\,(A,k,l))\ =\ k^2\mathrm{card}\,\partial_{\mathrm{e}}\mathrm{fill}\,\big(k^{-1}\mathrm{fat}\,(A,k),l/k\big) \\ \leq\ 6k^2\mathrm{card}\,\partial_{\mathrm{ov}}\mathrm{fill}\,\big(k^{-1}\mathrm{fat}\,(A,k),l/k\big)\,, \\ \mathrm{diam}_\infty\partial_{\mathrm{ov}}\mathrm{fill}\,(k^{-1}\mathrm{fat}\,(A,k),l/k)\ \leq\ k^{-1}\mathrm{diam}_\infty A+4\,.$$

Proof. These inequalities are direct consequences of the definition of the fifa operation together with the following facts. The boundary of $\mathrm{fifa}\,(A,k,l)$ is the union of squares of side length k, each of which is uniquely associated to an edge of

$\partial_{\mathrm{e}}\mathrm{fill}\,(k^{-1}\mathrm{fat}\,(A,k),l/k)$. Next, $\mathrm{diam}_\infty \mathrm{fat}\,(A,k) \leq \mathrm{diam}_\infty A + 2k$ which implies that $\mathrm{diam}_\infty \mathrm{fill}\,(k^{-1}\mathrm{fat}\,(A,k),l/k) \leq k^{-1}\mathrm{diam}_\infty A + 2$. $\square$

Proof of Proposition 9.12 **continued.** We study now the event

$$\{\,\mathcal{P}(\mathrm{fifa}\,(C(0),k,l)) \geq s \geq \mathrm{diam}_\infty C(0)\,\}\,.$$

Let $F = k\,\partial_{\mathrm{ov}}\mathrm{fill}\,(k^{-1}\mathrm{fat}\,(C(0),k),l/k)$. By construction the $k\mathbb{L}^3$ components of F on $k\mathbb{Z}^3$ (which are the inner vertex boundaries of the residual components of the set $k\,\mathrm{fill}\,(k^{-1}\mathrm{fat}\,(C(0),k),l/k)$ on $(k\mathbb{Z}^3,k\mathbb{E}^3)$) have a cardinality larger than $l/6k$ (see the discussion after Definition 9.7). Since the set F is included in $\partial_{\mathrm{ov}}\mathrm{fat}\,(C(0),k)$, then $X_k(x^k)=1$ for any x^k in F by Lemma 9.13, whenever $\mathrm{diam}_\infty C(0) > l \geq 5k$. By Lemma 9.14,

$$\mathcal{P}(\mathrm{fifa}\,(C(0),k,l)) \,\leq\, 6k^2\mathrm{card}\,F\,, \quad \mathrm{diam}_\infty F \,\leq\, \mathrm{diam}_\infty C(0) + 4k$$

and the boxes $\Lambda(x^k,5k)$, $x^k\in F$, are included in $\Lambda(0,2s+14k)$. Therefore the event $\{\,\mathcal{P}(\mathrm{fifa}\,(C(0),k,l)) \geq s \geq \mathrm{diam}_\infty C(0)\,\}$ is included in

$$\big\{\,\mathrm{card}\,\{\,x^k \in k\mathbb{Z}^3 : \mathrm{card}\,C^k_{\mathbb{L}}(x^k,\Lambda(0,2s+14k),X_k) \geq l/6k\,\} \geq sk^{-2}/6\,\big\}\,.$$

We use the bound given by Lemma 7.9 with

$$A=\Lambda(0,2s+14k)\,,\quad \alpha=5\,,\quad \beta=l/6k\,,\quad \varepsilon(k)=P(X_k(0)=1)\,.$$

Since $\mathrm{card}\,\Lambda(0,2s+14k)\cap k\mathbb{Z}^3 \leq (2s/k+16)^3$, we get

$$P\big(\mathcal{P}(\mathrm{fifa}\,(C(0),k,l)) \geq s \geq \mathrm{diam}_\infty C(0)\big) \;\leq\; \sum_{i\geq sk^{-2}/6} 2\exp\Big(36ikl^{-1}\ln(2s/k+16)+i\ln b+i5^{-3}\ln\varepsilon(k)\Big)\,.$$

We are interested in the regime where $k\ln s \leq l$. In this regime, $kl^{-1}\ln(2s/k+16)$ is bounded by a constant independent of k,l,s. Because $\varepsilon(k)$ goes to 0 as k goes to ∞, the above inequality implies that for k large enough there exist two positive constants $b(k),c(k)$ such that for any positive l,s with $5k\leq l$, $k\ln s\leq l$,

$$P\big(\mathcal{P}(\mathrm{fifa}\,(C(0),k,l)) \geq s \geq \mathrm{diam}_\infty C(0)\big) \;\leq\; b(k)\exp(-c(k)s)\,.$$

Hence we have obtained two estimates of the desired form. $\square$

We build now our second smoothing operation.

9.3. The aglu operation

Let l be a positive integer larger than 100. For each open cluster C of the configuration, we agglutinate the open clusters of diameter less than l at distance less than $3l$ from C into a new set $\mathrm{aglu}\,(C,l)$. More precisely, the set $\mathrm{aglu}\,(C,l)$ associated to C is

$$\mathcal{V}_\infty(C,1/2)\,\cup\,\mathrm{cup}\,\{\,\mathcal{V}_\infty(C',1/2):C'\text{ open cluster},\ \mathrm{diam}_\infty C'\le l,\ d_\infty(C,C')\le 3l\,\}\,.$$

Thus $\mathrm{aglu}\,(C,l)$ is the $1/2$–neighbourhood of the union of C and all the open clusters of diameter less than l whose d_∞ distance to C is less than $3l$. Notice that contrary to $\mathrm{fifa}\,(C,k,l)$ which is built through a deterministic procedure starting from the cluster C, the construction of the set $\mathrm{aglu}\,(C,l)$ involves the random configuration in the neighbourhood $\mathcal{V}_\infty(C,4l+2)$ of C.

Lemma 9.15. *For any open cluster C, any compact set K,*

$$\begin{aligned}\mathcal{L}^3_K\big(\mathcal{V}_\infty(C,l)\Delta\mathrm{aglu}\,(C,l)\big)\ \le&\\ &800l^3+3^3l^3\mathrm{card}\,\{\,x^l\in\mathrm{fat}\,(C,l)\cap\mathcal{V}_\infty(K,2l):R(\Lambda(x^l,6l),l)\ \textit{does not occur}\,\}\\ &\qquad+8960l^2\mathcal{P}\big(\mathrm{fifa}\,(C,k,l),\mathrm{int}\,K\big)+\mathcal{L}^3_K\big(\mathcal{V}_\infty(\partial K,5l+1)\cap\mathcal{V}_\infty(C,4l)\big)\,.\end{aligned}$$

Proof. If $\mathrm{diam}_\infty C\le l$ then $\mathcal{L}^3(\mathcal{V}_\infty(C,l))\le(3l)^3$ and $\mathcal{L}^3(\mathrm{aglu}\,(C,l))\le(9l+1)^3$ so that

$$\mathcal{L}^3(\mathcal{V}_\infty(C,l)\Delta\mathrm{aglu}\,(C,l))\ \le\ 800l^3\,.$$

Let us consider now an open cluster C such that $\mathrm{diam}_\infty C>l$. For each vertex x^l of $l\mathbb{Z}^3$ we denote by $X_l(x^l)$ the indicator function of the event $R(\Lambda(x^l,6l),l)$. Suppose that for some x^l in $l\mathbb{Z}^3$ we have $\Lambda(x^l,l)\cap C\ne\varnothing$ and $\mathcal{L}^3(\Lambda(x^l,4l-1)\setminus\mathrm{aglu}\,(C,l))>0$. Since

$$\Lambda(x^l,4l-1)\ =\ \mathrm{cup}\,\{\,\Lambda(x,1):x\in\mathbb{Z}^3\cap\Lambda(x^l,4l-1)\,\}\,,$$

then there exists x in $\mathbb{Z}^3\cap\Lambda(x^l,4l-1)$ such that $\mathcal{L}^3(\Lambda(x,1)\setminus\mathrm{aglu}\,(C,l))>0$ and necessarily $C(x)\cap C=\varnothing$ and $\mathrm{diam}_\infty C(x)>l$ (recall that $C(x)$ is the open cluster containing x for the configuration on the whole lattice $\mathbb{Z}^3$). Therefore the box $\Lambda(x^l,6l)$ contains two distinct open clusters of diameter larger than l and the event $R(\Lambda(x^l,6l),l)$ does not occur, so that $X_l(x^l)=1$. This discussion yields

$$\mathcal{L}^3\big(\mathrm{cup}\,\{\,\Lambda(x^l,4l-1):x^l\in\mathrm{fat}\,(C,l),\ X_l(x^l)=0\,\}\setminus\mathrm{aglu}\,(C,l)\big)\ =\ 0\,.$$

Since we have simultaneously $\mathcal{V}_\infty(C,l)\ \subset\ \mathrm{cup}\,\{\,\Lambda(x^l,3l):x^l\in\mathrm{fat}\,(C,l)\,\}$, then

$$\begin{aligned}\mathcal{L}^3_K\big(\mathcal{V}_\infty(C,l)\setminus\mathrm{aglu}\,(C,l)\big)\ &\le\ \mathcal{L}^3_K\big(\mathrm{cup}\,\{\,\Lambda(x^l,3l):x^l\in\mathrm{fat}\,(C,l),\ X_l(x^l)=1\,\}\big)\\ &\le\ 3^3l^3\mathrm{card}\,\{\,x^l\in\mathrm{fat}\,(C,l)\cap\mathcal{V}_\infty(K,2l):X_l(x^l)=1\,\}\,.\end{aligned}$$

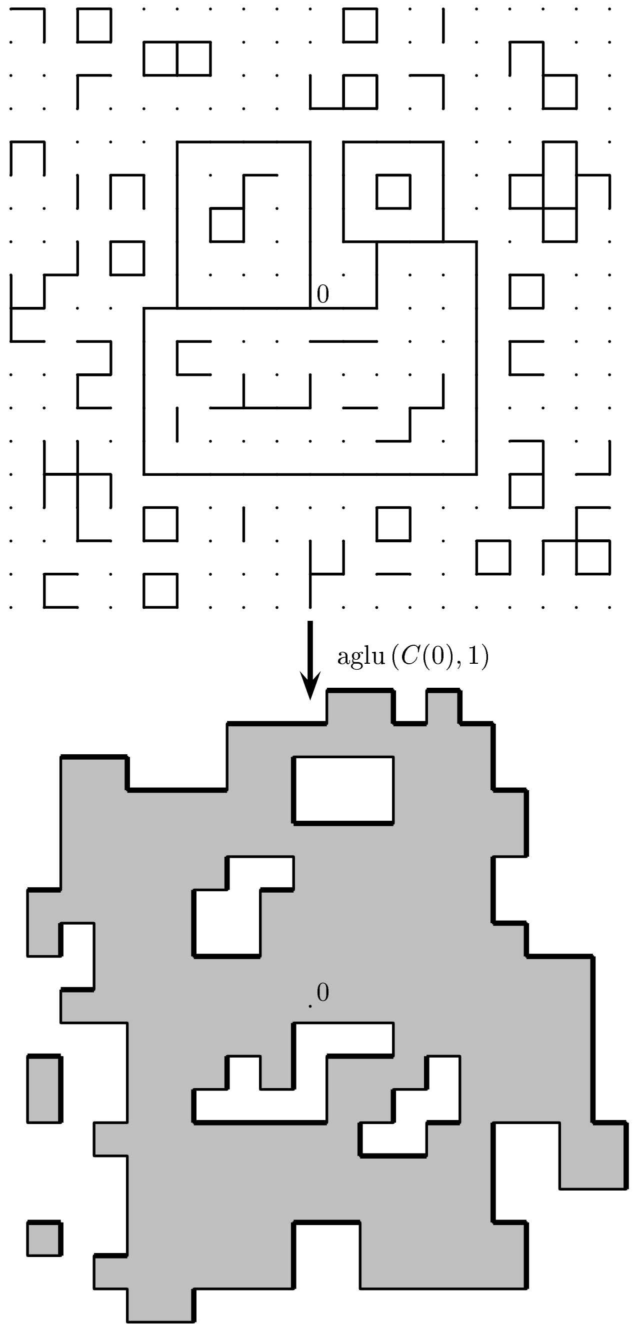

Figure 11: $\mathrm{aglu}\,(C(0), l)$

Conversely, the set aglu (C,l) is included in clo $\mathcal{V}_\infty(C,4l)$, whence by Lemmata 9.9 and 9.10,

$$\begin{aligned}\mathcal{L}^3_K(\mathrm{aglu}\,(C,l)\setminus\mathcal{V}_\infty(C,l)) \;&\leq\; \mathcal{L}^3_K(\mathcal{V}_\infty(C,4l)\setminus\mathrm{fifa}\,(C,k,l))\\ &\leq\; 8960l^2\mathcal{P}(\mathrm{fifa}\,(C,k,l),\mathrm{int}\,K)+\mathcal{L}^3_K(\mathcal{V}_\infty(\partial K,5l+1)\cap\mathcal{V}_\infty(C,4l))\,.\end{aligned}$$

Putting together the last two inequalities, we get the required estimation. □

Proposition 9.16. *There exist three positive constants κ, b, c such that: for any positive integer l and any positive real number s, satisfying $\kappa\ln s\leq l$, $2500l^3\leq s$,*

$$\widehat{P}\big(\mathcal{L}^3(\mathcal{V}_\infty(C(0),l)\Delta\mathrm{aglu}\,(C(0),l))\geq s\big)\;\leq\; b\exp(-csl^{-2})\,.$$

Proof. Let k be an integer large enough for the result of Proposition 9.12 to hold. By hypothesis $2500l^3\leq s$, so that $800l^3<s/3$. Applying the bound given in Lemma 9.15 with a compact set K containing $\mathcal{V}_\infty(C(0),10l)$ in its interior and remarking that $\mathrm{diam}_\infty C(0)\leq\mathcal{P}(\mathrm{fifa}\,(C(0),k,l))$ whenever fifa $(C(0),k,l)$ is a bounded non–empty set, we write

$$\begin{aligned}\widehat{P}\big(\mathcal{L}^3(\mathcal{V}_\infty(C(0),l)\Delta\mathrm{aglu}\,(C(0),l))\geq s\big)\leq \widehat{P}\big(8960l^2\mathcal{P}(\mathrm{fifa}\,(C(0),k,l))\geq s/3\big)+\\ P\Big(\mathrm{card}\,\big\{x^l\in\mathrm{fat}\,(C(0),l):R(\Lambda(x^l,6l),l)\text{ does not occur}\big\}\geq sl^{-3}3^{-4},\\ l^2\mathrm{diam}_\infty C(0)<s\Big)\,.\end{aligned}$$

For κ large enough, the inequalities $2500l^3\leq s$, $\kappa\ln s\leq l$ imply that $5k\leq l$, $k\ln s\leq l$ and Proposition 9.12 yields

$$\widehat{P}\big(l^2\mathcal{P}(\mathrm{fifa}\,(C(0),k,l))\geq s/26880\big)\;\leq\; b(k)\exp(-c(k)sl^{-2}/26880)\,.$$

We apply Corollaries 7.4, 7.8 to the indicator functions of the events $R(\Lambda(x^l,6l),l)$ and the box $\Lambda(0,2sl^{-2}+2l)$ (which contains fat $(C(0),l)$ whenever $\mathrm{diam}_\infty C(0)<sl^{-2}$) and we bound the second term by

$$\begin{aligned}P\Big(\mathrm{card}\,\big\{\,x^l\in\Lambda(0,2sl^{-2}+2l)\cap l\mathbb{Z}^3:R(\Lambda(x^l,6l),l)\text{ does not occur}\,\big\}\geq sl^{-3}3^{-4}\Big)\\ \leq\sum_{i\geq sl^{-3}3^{-4}}\exp\Big(3i\ln(2sl^{-3}+4)+i6^{-3}(\ln b-cl)\Big)\,.\end{aligned}$$

In the regime where $\kappa\ln s\leq l$, $2500l^3\leq s$ and κ is sufficiently large, we get again an upper bound of the form $b'\exp(-c'sl^{-2})$, with b',c' two positive constants. □

9.4. Proof of the upper bound

We prove here the upper bound of the LDP of Theorem 2.3, that is the LDP for the finite cluster shape in the topology L^1_{loc}. This upper bound implies also the upper bound of Theorem 2.1, that is the weak LDP in the topology L^1.

From now onwards, the integer k is fixed and large enough, so that the result of Proposition 9.12 holds, and we replace the integer l used in the previous constructions by $f(n)$, where $f(n)$ is a function from $\mathbb{N}$ to $\mathbb{N}$ such that

$$\lim_{n\to\infty} n/f(n)^2 \,=\, \infty\,, \qquad \forall\, n\in\mathbb{N} \quad f(n)\geq \kappa \ln n\,,$$

where $\kappa = \kappa(p)$ is a constant larger than those appearing in Theorem 7.1 and Proposition 9.16. We omit the variables k,l in the sequel, writing for instance $\operatorname{fifa} C$ instead of $\operatorname{fifa}(C,k,l)$. As a consequence of the estimates given in Corollary 9.11 and Propositions 9.12, 9.16, under the probability $\widehat{P}$, the sequences of random sets $n^{-1}\mathcal{V}_\infty(C(0),f(n))$, $n^{-1}\operatorname{fifa} C(0)$, $n^{-1}\operatorname{aglu} C(0)$ are exponentially contiguous for the topology L^1 (whence also for the weaker topology L^1_{loc}), that is, for any positive δ,

$$\limsup_{n\to\infty} \frac{1}{n^2}\ln \widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\Delta n^{-1}\operatorname{fifa} C(0)\big)\geq\delta\Big) \,=\, -\infty\,,$$
$$\limsup_{n\to\infty} \frac{1}{n^2}\ln \widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\Delta n^{-1}\operatorname{aglu} C(0)\big)\geq\delta\Big) \,=\, -\infty\,.$$

At the level of the large deviations of order n^2, we might work with either of these three objects. The proof of the large deviations upper bound follows a standard scheme. We first show that the law of the random set $n^{-1}\mathcal{V}_\infty(C(0),f(n))$ concentrates exponentially fast near the level sets of $\widehat{\mathcal{I}}$. Indeed, for any positive λ,δ, using Lemma 6.3, the fact that $\widehat{P}\big(\mathcal{L}^3(\operatorname{fifa} C(0))<\infty\big)=1$, and the estimate of Proposition 9.12, we have

$$\begin{aligned}&\widehat{P}\Big(\inf\Big\{\,\mathcal{L}^3\big(n^{-1}\operatorname{fifa} C(0)\Delta A\big) : A\in\mathcal{B}(\mathbb{R}^3),\,\widehat{\mathcal{I}}(A)\leq\lambda\Big\}\geq\delta\Big)\\ &\leq \widehat{P}\big(\widehat{\mathcal{I}}(n^{-1}\operatorname{fifa} C(0))\geq\lambda\big)\leq\widehat{P}\big(\mathcal{P}(\operatorname{fifa} C(0))\geq n^2\lambda/\|\tau\|_\infty\big)\leq b\exp\big(-cn^2\lambda/\|\tau\|_\infty\big)\end{aligned}$$

where b,c are two positive constants depending on k. Using the exponential contiguity between $n^{-1}\mathcal{V}_\infty(C(0),f(n))$ and $n^{-1}\operatorname{fifa} C(0)$, we obtain that there exists a positive constant $c'=c/\|\tau\|_\infty$ such that, for any positive λ,δ,

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\Big(\inf\Big\{\,\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\Delta A\big) : A\in\mathcal{B}(\mathbb{R}^3),\,\widehat{\mathcal{I}}(A)\leq\lambda\Big\}\geq\delta\Big)$$
$$\leq\,-c'\lambda.$$

Therefore the sequence of random sets $\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\big)_{n\in\mathbb{N}}$ is $\widehat{\mathcal{I}}$–tight on the space $(\mathcal{B}(\mathbb{R}^3),L^1)$. Because $\widehat{\mathcal{I}}$ is not a good rate function on the space $(\mathcal{B}(\mathbb{R}^3),L^1)$, the above estimate is not enough to get the full large deviation principle with respect to the L^1 topology. Yet it implies the weaker estimate: for any compact set K, for any positive δ, λ,

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\Big(\inf \Big\{ \mathcal{L}^3_K\big(n^{-1}\mathcal{V}_\infty(C(0), f(n))\Delta A\big) : A \in \mathcal{B}(\mathbb{R}^3),\, \widehat{\mathcal{I}}(A) \le \lambda \Big\} \ge \delta \Big) \le -c'\lambda.$$

Since the level sets of $\widehat{\mathcal{I}}$ are compact with respect to the topology L^1_{loc}, we need only to prove a local estimate to complete the proof of the large deviation upper bound of Theorem 2.3, like

$$\forall\, A \in \mathcal{B}(\mathbb{R}^3),\, \widehat{\mathcal{I}}(A) < \infty,$$
$$\limsup_{K\to\mathbb{R}^3, \delta\to 0}\ \limsup_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\Big(\mathcal{L}^3_K\big(n^{-1}\mathcal{V}_\infty(C(0), f(n))\Delta A\big) < \delta \Big) \ \le\ -\widehat{\mathcal{I}}(A)\,.$$

Let A be an element of $\mathcal{B}(\mathbb{R}^3)$ such that $\widehat{\mathcal{I}}(A)$ is finite (equivalently, $\mathcal{P}(A) + \mathcal{L}^3(A)$ is finite). Let ε be positive. Let δ_0 be associated to ε as in Lemma 8.1. Let $B(x_i, r_i)$, $i \in I$, be a finite collection of disjoint balls as given in Lemma 6.6, associated to A, ε and $\delta_0/3$. Let K be a compact set containing the balls $B(x_i, r_i)$, $i \in I$, in its interior. Let δ be a positive real number strictly smaller than $\min\{\, \delta_0 r_i^3/3 : i \in I \,\}$. Suppose the event $\{\, \mathcal{L}^3_K(n^{-1} \mathrm{aglu}\, C(0)\Delta A) < \delta \,\}$ occurs. Suppose also that n is large enough to have $4\pi\sqrt{3} r_i^2 n^2 + \pi\sqrt{3} < \delta_0 r_i^3 n^3/3$ for any i in I. Let i belong to I and let $\mathcal{C}(i)$ be the collection of the open clusters of the configuration restricted to $\mathrm{aglu}\, C(0) \cap B(nx_i, nr_i)$. Because of the definition of $\mathrm{aglu}\, C(0)$, the clusters of $\mathcal{C}(i)$ are open clusters of the configuration restricted to $B(nx_i, nr_i)$. Moreover

$$\begin{aligned}
&\mathcal{L}^3\Big(\mathcal{V}_\infty(\mathrm{cup}\, \mathcal{C}(i), 1/2)\Delta B_-(nx_i, nr_i, \nu_A(x_i))\Big)\\
&\quad\le \mathcal{L}^3\Big(\mathcal{V}_\infty(\mathrm{cup}\, \mathcal{C}(i), 1/2)\Delta(\mathrm{aglu}\, C(0) \cap B(nx_i, nr_i))\Big)\\
&\qquad\qquad\qquad + \mathcal{L}^3\Big((\mathrm{aglu}\, C(0) \cap B(nx_i, nr_i))\Delta B_-(nx_i, nr_i, \nu_A(x_i))\Big)\\
&\quad\le \mathcal{L}^3\big(B(nx_i, nr_i + \sqrt{3}/2) \setminus B(nx_i, nr_i - \sqrt{3}/2)\big)\\
&\qquad + \mathcal{L}^3_{nK}\big(\mathrm{aglu}\, C(0)\Delta nA\big) \ +\ \mathcal{L}^3\Big((nA \cap B(nx_i, nr_i))\Delta B_-(nx_i, nr_i, \nu_A(x_i))\Big)\\
&\quad< 4\pi\sqrt{3} r_i^2 n^2 + \pi\sqrt{3} + \delta n^3 + \delta_0 r_i^3 n^3/3 < \delta_0 r_i^3 n^3\,.
\end{aligned}$$

Therefore the collection $\mathcal{C}(i)$ realizes the event $\mathrm{sep}\,(n, x_i, r_i, \nu_A(x_i), \delta_0)$ for i in I and the event $\{\, \mathcal{L}^3_K(n^{-1}\mathrm{aglu}\, C(0)\Delta A) < \delta \,\}$ is included in $\bigcap_{i\in I} \mathrm{sep}\,(n, x_i, r_i, \nu_A(x_i), \delta_0)$. Since the balls $B(x_i, r_i)$, $i \in I$, are disjoint, these events are independent under P, hence

$$\widehat{P}\Big(\mathcal{L}^3_K\big(n^{-1}\mathrm{aglu}\, C(0)\Delta A\big) < \delta\Big) \ \le\ (1-\theta)^{-1} \prod_{i\in I} P\big(\mathrm{sep}\,(n, x_i, r_i, \nu_A(x_i), \delta_0)\big)$$

and by the choice of δ_0 (see Lemma 8.1) and of the balls $B(x_i, r_i)$ (see Lemma 6.6),

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\Big(\mathcal{L}_K^3(n^{-1}\operatorname{aglu} C(0)\Delta A) < \delta\Big) &\leq -\sum_{i\in I} \tau(\nu_A(x_i))\pi r_i^2(1-\varepsilon)\\ &\leq -\mathcal{I}(A)(1-\varepsilon)+\varepsilon\end{aligned}$$

Letting K grow to $\mathbb{R}^3$, δ and then ε go to zero in this inequality, we get the upper bound we were seeking for. $\square$

9.5. Proof of the lower bound

We prove here the lower bound of the LDP of Theorem 2.1, that is the LDP for the finite cluster shape in the topology L^1. This lower bound implies also the lower bound of Theorem 2.3, that is for the LDP in the topology L^1_{loc}. We first state two preparatory Lemmata.

Definition 9.17. Let U and O be two open connected subsets of $\mathbb{R}^3$ satisfying $d_2(U, \mathbb{R}^3 \setminus O) > 0$ (necessarily U is included in O). Let l be an integer. We define $\operatorname{full}(O, U, n, l)$ to be the following event: for the configuration restricted to nO, there exists an open cluster C such that nU is included in $\mathcal{V}_\infty(C, l)$ and no other open cluster of diameter larger than l intersects $\mathcal{V}_\infty(nU, l)$.

Lemma 9.18. *Let U, O be open connected subsets of $\mathbb{R}^3$ such that $d_2(U, \mathbb{R}^3 \setminus O) > 0$ and U is bounded. If $f(n)$ is a function from $\mathbb{N}$ to $\mathbb{N}$ such that*

$$\lim_{n\to\infty} n/f(n) = \infty\,, \qquad \forall n \in \mathbb{N} \quad f(n) \geq \kappa \ln n$$

where $\kappa = \kappa(p)$ is the constant given by Theorem 7.1, then

$$\lim_{n\to\infty} P\big(\operatorname{full}(O, U, n, f(n))\big) \;=\; 1\,.$$

Proof. Let δ be such that $0 < 3\delta < d_\infty(U, \mathbb{R}^3 \setminus O)$. The family of sets $\operatorname{int}\Lambda(x, \delta)$, $x \in \mathcal{V}_\infty(U, \delta)$, is an open covering of $\operatorname{clo}\mathcal{V}_\infty(U, \delta)$. By compactness of $\operatorname{clo}\mathcal{V}_\infty(U, \delta)$, we may extract a finite family $\operatorname{int}\Lambda(x_i, \delta)$, $x_i \in \mathcal{V}_\infty(U, \delta)$, $i \in I$, covering $\operatorname{clo}\mathcal{V}_\infty(U, \delta)$. Since U is connected, so is the set $\operatorname{cup}\{\operatorname{int}\Lambda(x_i, \delta) : i \in I\}$, as well as the set $\operatorname{cup}\{\Lambda(x_i, \delta) : i \in I\}$. A δ–chain connecting two points y, z is a sequence $x_1, \dots, x_r$ of points such that $x_1 = y$, $x_r = z$ and $d_\infty\big(x_l, x_{l+1}\big) < \delta$ for all l in $\{1, \dots, r-1\}$. Because $\mathcal{V}_\infty(U, \delta)$ is connected, the set $\{x_i : i \in I\}$ is δ–connected, i.e., for any j, k in I, there exists a δ–chain in $\{x_i : i \in I\}$ connecting x_j and x_k. Let us consider the boxes $\Lambda(x_i, 2\delta)$, $i \in I$. We have the following strict inclusions:

$$\mathcal{V}_\infty(U, \delta) \subsetneq \operatorname{cup}\{\Lambda(x_i, 2\delta) : i \in I\} \subsetneq O\,.$$

Each box centered in $\mathcal{V}_\infty(U,\delta)$ of diameter δ is contained in one of these boxes: indeed, if x belongs to $\mathcal{V}_\infty(U,\delta)$, there exists i in I such that $d_\infty(x,x_i)<\delta/2$, whence $\operatorname{clo}\Lambda(x,\delta)\subset\operatorname{int}\Lambda(x_i,2\delta)$.
Let n be large so that $\delta n>4f(n)+4$. Suppose that all the events $O(n\Lambda(x_i,2\delta),f(n))$, $i\in I$, occur simultaneously. We denote by $C^*(n\Lambda(x_i,2\delta))$ the open cluster in $n\Lambda(x_i,2\delta)$ associated to the event $O(n\Lambda(x_i,2\delta),f(n))$. Let i,j be two elements of I such that $d_\infty(x_i,x_j)<\delta$. The intersection of the boxes $\Lambda(x_i,2\delta)$ and $\Lambda(x_j,2\delta)$ contains some box of diameter δ, say $\Lambda(y,\delta)$ (for instance $y=(x_i+x_j)/2$). By definition of the event $O(n\Lambda(x_i,2\delta),f(n))$ and because $\delta n/2>f(n)$, the cluster $C^*(n\Lambda(x_i,2\delta))$ intersects $n\Lambda(y,\delta/2)$, as well as $n\Lambda(x_i,2\delta)\setminus n\Lambda(y,\delta)$. It follows that inside $n\Lambda(y,\delta)$, there exists an open path γ of diameter larger than $n\delta/4-1$, which is included in $C^*(n\Lambda(x_i,2\delta))$ since $n\delta/4-1>f(n)$. But γ is included in $n\Lambda(x_j,2\delta)$, and the occurrence of $O(n\Lambda(x_j,2\delta),f(n))$ forces the inclusion $\gamma\subset C^*(n\Lambda(x_j,2\delta))$. Hence the clusters $C^*(n\Lambda(x_i,2\delta))$ and $C^*(n\Lambda(x_j,2\delta))$ are not disjoint. Because the set $\{\,x_i: i\in I\,\}$ is δ–connected, the preceding remark implies the existence of a cluster C^* in the configuration restricted to $\operatorname{cup}\{\,n\Lambda(x_i,2\delta): i\in I\,\}$ containing all the clusters $C^*(n\Lambda(x_i,2\delta))$, $i\in I$. For i in I, the occurrence of $O(n\Lambda(x_i,2\delta),f(n))$ implies that $n\Lambda(x_i,\delta)$ is included in $\mathcal{V}_\infty(C^*(n\Lambda(x_i,2\delta)),f(n))$: otherwise there would exist x in $n\Lambda(x_i,\delta)$ such that $d_\infty(x,C^*(n\Lambda(x_i,2\delta)))\geq f(n)$ and $C^*(n\Lambda(x_i,2\delta))$ would not intersect the box $\Lambda(x,f(n))$, which is a box of diameter $f(n)$ included in $n\Lambda(x_i,2\delta)$. Let C^{**} be the open cluster of the configuration restricted to nO containing C^*. Then

$$nU\;\subset\;\operatorname{cup}\{\,n\Lambda(x_i,\delta): i\in I\,\}\;\subset\;\mathcal{V}_\infty(C^*,f(n))\;\subset\;\mathcal{V}_\infty(C^{**},f(n))\,.$$

Let now C be an open cluster of the configuration restricted to nO having a diameter larger than $f(n)$ and distinct from C^{**}. Suppose that C intersects $\mathcal{V}_\infty(nU,f(n))$. Let x be a vertex of C such that $d_\infty(x,nU)<f(n)$. The box $\Lambda(x,\delta n)$ contains an open path of diameter larger than $f(n)$ included in C (since $n\delta/2-1>f(n)$). Since $d_\infty(x/n,U)<f(n)/n<\delta/2$, there exists an index i in I such that $\Lambda(x/n,\delta)\subset\Lambda(x_i,2\delta)$. Then the box $n\Lambda(x_i,2\delta)$ contains an open path of diameter larger than $f(n)$ not intersecting $C^*(n\Lambda(x_i,2\delta))$, but this contradicts the occurrence of $O(n\Lambda(x_i,2\delta),f(n))$. Therefore, apart C^{**}, no open cluster of diameter larger than $f(n)$ of the configuration restricted to nO intersects $\mathcal{V}_\infty(nU,f(n))$. In conclusion,

$$\bigcap_{i\in I}O(n\Lambda(x_i,2\delta),f(n))\;\subset\;\operatorname{full}(O,U,n,f(n))\,.$$

Since $f(n)\geq\kappa\ln n$ for all n and since I is finite, then by Theorem 7.1

$$\limsup_{n\to\infty}\frac{1}{f(n)}\ln\Big(\sup_{i\in I}P\big(O(n\Lambda(x_i,2\delta),f(n))^c\big)\Big)\;<\;0\,.$$

Remark that the center of the box $n\Lambda(x_i, 2\delta)$ is not necessarily in $\mathbb{Z}^3$; however it is obvious that the results of Theorem 7.1 are still valid. This bound together with the previous inclusion implies that

$$\limsup_{n\to\infty} \frac{1}{f(n)} \ln P\big(\text{full}\,(O,U,n,f(n))^c\big) \;<\; 0\,.$$

Therefore the probability of the event full $(O,U,n,f(n))$ tends to 1 as n goes to ∞. $\square$

Let F be a planar polygonal set and let l be a positive integer. We define wall (nF, l) to be the event

$$S(nF,l) \,\cap\, \Big\{ \text{all the edges in } \mathcal{V}_2(\text{cyl}\, n\partial F, \zeta) \cap \mathcal{V}_2(\text{hyp}\, nF, l) \text{ are closed} \Big\}\,.$$

Lemma 9.19. *If $\phi(n)$ is a function from $\mathbb{N}$ to $\mathbb{R}^+$ such that both $\phi(n)$ and $n/\phi(n)$ go to ∞ when n goes to ∞, then*

$$\liminf_{n\to\infty} \frac{1}{n^2} \ln P\Big(\text{wall}\,(nF, \phi(n))\Big) \;\geq\; -\mathcal{H}^2(F)\tau(\text{nor}\, F)\,.$$

Proof. The number of edges in the set $\mathcal{V}_2(\text{cyl}\, n\partial F, \zeta) \cap \mathcal{V}_2(\text{hyp}\, nF, \phi(n))$ is less than $c(\zeta)\mathcal{H}^1(\partial F)n\phi(n)$ for some positive constant $c(\zeta)$. By the FKG inequality,

$$P\big(\text{wall}\,(nF,\phi(n))\big) \;\geq\; P\big(S(nF,\phi(n)\big) \exp\big(c(\zeta)\mathcal{H}^1(\partial F) n\phi(n) \ln(1-p)\big)\,.$$

This inequality, Lemma 5.11 and the hypothesis on $\phi(n)$ imply the desired result. $\square$

We turn now to the proof of the lower bound of the LDP of Theorem 2.1. In view of the approximation result stated in Proposition 6.9 and the remark thereafter, to prove the large deviation lower bound, we need only to show that for any polyhedral open connected bounded set A containing the origin, for any positive δ,

$$\liminf_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0), f(n))\Delta A\big) < \delta\Big) \;\geq\; -\mathcal{I}(A)\,.$$

Let now A be such a set. Its boundary ∂A is the union of a finite number of polygonal planar sets $F_1, \dots, F_r$ (necessarily $r \geq 4$). Let δ be positive and let U and O be open connected subsets of $\mathbb{R}^3$ such that

$$0 \in U\,, \quad U \subset O \subset A\,, \quad d_2(U, \mathbb{R}^3 \setminus O) > 0\,, \quad d_2(O, \mathbb{R}^3 \setminus A) > 0\,, \quad \mathcal{L}^3(U\Delta A) < \delta/2\,.$$

Let $\phi(n)$ be a function from $\mathbb{N}$ to $\mathbb{R}^+$ such that both $\phi(n)$ and $n/\phi(n)$ go to ∞ when n goes to ∞. Let n be large enough so that $\phi(n) > \zeta$ and $f(n) < nd_2(0, O\setminus U)$. Let $\mathcal{E}$ be the event: the edges of the three canonical axis contained in $B(0, f(n))$ are open. Suppose all the events

$$\mathcal{E}, \quad \text{full}\,(O,U,n,f(n)), \quad \text{wall}\,(nF_1,\phi(n)),\, \dots,\, \text{wall}\,(nF_r,\phi(n))$$

occur simultaneously. We denote by C the open cluster of the whole configuration containing the cluster realizing the event $\mathrm{full}\,(O,U,n,f(n))$. We have then $nU \subset \mathcal{V}_\infty(C,f(n))$. Moreover the occurrence of the events $\mathrm{wall}\,(nF_i,\phi(n))$, $1 \leq i \leq r$, implies that the cluster C is included in $\mathcal{V}_2(nA,\phi(n))$, whence $\mathcal{V}_\infty(C,f(n)) \subset \mathcal{V}_\infty(nA,f(n)+\phi(n))$. Since $\mathcal{E}$ occurs, then $\mathrm{diam}_\infty C(0) > f(n)$, but $B(0,f(n)) \subset nU$ and since $\mathrm{full}\,(O,U,n,f(n))$ occurs, then C is included in $C(0)$. Let η be such that $\mathcal{L}^3(\mathcal{V}_\infty(A,\eta)\setminus A) < \delta/2$. For n large enough, so that $f(n)+\phi(n) < \eta n$,

$$\mathcal{L}^3\big(\mathcal{V}_\infty(C(0),f(n))\Delta nA\big) \;\leq\; \mathcal{L}^3(\mathcal{V}_\infty(nA,n\eta)\setminus nA) + \mathcal{L}^3(nA\setminus nU) \;<\; \delta n^3\,.$$

Therefore

$$\begin{aligned}
&\widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\Delta A\big) < \delta\Big)\\
&\qquad = (1-\theta)^{-1}P\Big(\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\Delta A\big) < \delta,\ \mathrm{card}\,C(0) < \infty\Big)\\
&\qquad\qquad \geq (1-\theta)^{-1}P\Big(\mathcal{E}\cap \mathrm{full}\,(O,U,n,f(n)) \cap \bigcap_{1\leq i\leq r} \mathrm{wall}\,(nF_i,\phi(n))\Big)\,.
\end{aligned}$$

The event $\mathrm{wall}\,(nF_i,\phi(n))$ depends on the edges inside

$$\mathcal{V}_2(\mathrm{cyl}\,nF_i,\zeta)\cap\mathcal{V}_2(\mathrm{hyp}\,nF_i,\phi(n))$$

whereas the event $\mathcal{E}\cap\mathrm{full}\,(O,U,n,f(n))$ depends on the edges inside nO. For n large enough so that $\phi(n)+\zeta < nd_2(O,\mathbb{R}^3\setminus A)$, the sets nO and $\mathcal{V}_2(nF_1\cap\cdots\cap nF_r,\phi(n)+\zeta)$ are disjoint and the events

$$\mathcal{E}\cap\mathrm{full}\,(O,U,n,f(n)),\qquad \mathrm{wall}\,(nF_1,\phi(n))\cap\cdots\cap\mathrm{wall}\,(nF_r,\phi(n))$$

are independent, whence

$$\begin{aligned}
&\widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\Delta A\big) < \delta\Big) \;\geq\\
&\qquad (1-\theta)^{-1}P\big(\mathcal{E}\cap\mathrm{full}\,(O,U,n,f(n))\big)P\Big(\bigcap_{1\leq i\leq r}\mathrm{wall}\,(nF_i,\phi(n))\Big)\,.
\end{aligned}$$

By Lemma 9.18, $\lim_{n\to\infty}P(\mathrm{full}\,(O,U,n,f(n))) = 1$. Moreover $P(\mathcal{E}) \geq \exp((6f(n)+3)\ln p)$. Applying the FKG inequality (notice that the events $\mathrm{wall}\,(nF_i,\phi(n))$ are decreasing) and Lemma 9.19,

$$\liminf_{n\to\infty}\frac{1}{n^2}\ln P\Big(\bigcap_{1\leq i\leq r}\mathrm{wall}\,(nF_i,\phi(n))\Big) \;\geq\; -\sum_{1\leq i\leq r}\mathcal{H}^2(F_i)\tau(\mathrm{nor}\,F_i)\,.$$

The three previous inequalities together imply that

$$\liminf_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\Big(\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\Delta A\big) < \delta\Big) \;\geq\; -\mathcal{I}(A)\,.$$

Thus we are done. $\square$

9.6. Proof of the enhanced upper bound

We first state two preparatory Lemmata: a result of discrete geometry and a uniform large deviation upper bound.

Lemma 9.20. *Let X be a finite subset of $\mathbb{Z}^3$. There exists a subset E of $X \times \{1,\dots,3^{\operatorname{card} X}\}$ such that* $\operatorname{card} E \leq \operatorname{card} X$ *and*

- $\forall\,(a,r)\in E \quad B(a,r)\cap X \neq \varnothing\,,$
- $\operatorname{cup}\{\,B(x,1) : x\in X\,\} \subset \operatorname{cup}\{\,B(a,r) : (a,r)\in E\,\}\,,$
- $\forall\,(a,r),\,(b,s)\in E \quad (a,r)\neq(b,s) \Rightarrow B(a,r+1)\cap B(b,s+1)=\varnothing\,.$

Proof. We use an algorithm to build the set E starting from X. The algorithm works with a sequence of subsets of $X\times\{\,1,\dots,3^{\operatorname{card} X}\,\}$ which might violate only the last condition and it stops when this condition is fulfilled. We initialize the algorithm with the set $E(1) = \{\,(a,1) : a\in X\,\}$. Clearly the only condition which might be violated by $E(1)$ is the last one. Suppose $E(h)$ has been built for some $h\geq 1$. If $E(h)$ answers the problem, the algorithm stops. Otherwise the last condition is violated: there exists (a,r) and (b,s) in $E(h)$ such that $B(a,r+1)\cap B(b,s+1)\neq\varnothing$. We define then

$$E(h+1) \;=\; \{\,(a,r+2s+2)\,\}\cup E(h)\setminus\{\,(a,r),(b,s)\,\}\,.$$

Obviously $\operatorname{card} E(h+1)\leq \operatorname{card} E(h)-1$ and $E(h+1)$ satisfies the first two conditions when $E(h)$ does. Moreover

$$\max\{\,r : (a,r)\in E(h+1)\,\} \;\leq\; 3\max\{\,r : (a,r)\in E(h)\,\}+2\,.$$

Necessarily the algorithm stops at some step h less than $\operatorname{card} X$. The final set $E(h)$ answers the problem. $\square$

We introduce some notation necessary to state the next Lemma. Let m be an integer and let $U_1,\dots,U_m$ be m open connected bounded subsets of $\mathbb{R}^3$. We consider the product space $\mathcal{B}(U_1)\times\cdots\times\mathcal{B}(U_m)$ endowed with the product L^1 topology and the metric $\mathcal{L}^3_m$ defined by

$$\begin{aligned}&\forall\,(E_1,\dots,E_m),(F_1,\dots,F_m)\in\mathcal{B}(U_1)\times\cdots\times\mathcal{B}(U_m)\\ &\qquad\mathcal{L}^3_m\big((E_1,\dots,E_m),(F_1,\dots,F_m)\big) \;=\; \mathcal{L}^3(E_1\Delta F_1)+\cdots+\mathcal{L}^3(E_m\Delta F_m)\,.\end{aligned}$$

Let $\mathcal{I}^m$ be the map from $\mathcal{B}(U_1)\times\cdots\times\mathcal{B}(U_m)$ to $\mathbb{R}^+\cup\{\infty\}$ defined by:

$$\begin{aligned}&\forall\,(E_1,\dots,E_m)\in\mathcal{B}(U_1)\times\cdots\times\mathcal{B}(U_m),\\ &\qquad\qquad\mathcal{I}^m(E_1,\dots,E_m) \;=\; \mathcal{I}(E_1,U_1)+\cdots+\mathcal{I}(E_m,U_m)\,.\end{aligned}$$

For any open set O, the map $E\in\mathcal{B}(O)\mapsto\mathcal{I}(E,O)$ is lower semicontinuous, therefore $\mathcal{I}^m$ is also lower semicontinuous. Since in addition the sets $U_1,\dots,U_m$ are bounded,

the compactness result Theorem 6.1 implies that the level sets of $\mathcal{I}^m$ are compact, hence $\mathcal{I}^m$ is a good rate function on the space $(\mathcal{B}(U_1)\times\cdots\times\mathcal{B}(U_m),L^1)$.
Let $T(U_1,\dots,U_m)$ be the set of the m–uples of integer translations sending the sets $U_1,\dots,U_m$ onto pairwise disjoint sets, that is,

$$T(U_1,\dots,U_m) = \\ \big\{\,(x_1,\dots,x_m)\in(\mathbb{Z}^3)^m : x_1+U_1,\dots,x_m+U_m \text{ are pairwise disjoint}\,\big\}\,.$$

To $U_1,\dots,U_m$ and $(x_1,\dots,x_m)$ in $T(U_1,\dots,U_m)$ we associate the map $\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}$ from $\mathcal{B}(\mathbb{R}^3)$ to $\mathcal{B}(U_1)\times\cdots\times\mathcal{B}(U_m)$ defined by

$$\forall\, E\in\mathcal{B}(\mathbb{R}^3)\qquad \phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}(E)\,=\,\big(E\cap(U_1+x_1)-x_1,\dots,E\cap(U_m+x_m)-x_m\big)\,.$$

Lemma 9.21. *For any closed subset $\mathcal{F}^m$ of $\mathcal{B}(U_1)\times\cdots\times\mathcal{B}(U_m)$,*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\,\sup\Big\{\widehat{P}\Big(\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathcal{V}_\infty(C(0),f(n))\big)\in\mathcal{F}^m\Big)\,:\\ (x_1,\dots,x_m)\in T(U_1,\dots,U_m)\Big\}\\ \leq\,-\inf\,\big\{\,\mathcal{I}^m(F_1,\dots,F_m):(F_1,\dots,F_m)\in\mathcal{F}^m\,\big\}\,.$$

Proof. We prove the large deviations upper bound in a standard fashion, by proving the $\mathcal{I}^m$–tightness together with a local estimate. Moreover the estimates are uniform with respect to $(x_1,\dots,x_m)$ in $T(U_1,\dots,U_m)$.
• *$\mathcal{I}^m$–tightness.* Let λ,δ be positive. Denoting by $(\mathcal{I}^m)^{-1}([0,\lambda])$ the level set of $\mathcal{I}^m$ associated to λ, for any $(x_1,\dots,x_m)$ in $T(U_1,\dots,U_m)$, we have, using Lemma 6.3 and the fact that $\widehat{P}\big(\mathcal{L}^3(\text{fifa}\,C(0))<\infty\big)=1$,

$$\widehat{P}\Big(\inf\Big\{\mathcal{L}^3_m\big(\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\text{fifa}\,C(0)\big),(E_1,\dots,E_m)\big)\,:\\ (E_1,\dots,E_m)\in(\mathcal{I}^m)^{-1}([0,\lambda])\Big\}\geq\delta\Big)\\ \leq\,\widehat{P}\Big(\mathcal{I}^m\Big(\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\text{fifa}\,C(0)\big)\Big)\geq\lambda\Big)\\ \leq\,\widehat{P}\Big(\mathcal{I}(n^{-1}\text{fifa}\,C(0),x_1+U_1)+\cdots+\mathcal{I}(n^{-1}\text{fifa}\,C(0),x_m+U_m)\geq\lambda\Big)\\ \leq\,\widehat{P}\big(\widehat{\mathcal{I}}(n^{-1}\text{fifa}\,C(0))\geq\lambda\big)\,\leq\,\widehat{P}\big(\mathcal{P}(\text{fifa}\,C(0))\geq n^2\lambda/||\tau||_\infty\big)$$

Moreover we have

$$\widehat{P}\Big(\mathcal{L}^3_m\big(\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathcal{V}_\infty(C(0),f(n))\big),\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\text{fifa}\,C(0)\big)\big)\geq\delta\Big)\\ \leq\,\widehat{P}\Big(\sum_{1\leq i\leq m}\mathcal{L}^3_{x_i+U_i}\big(\frac{1}{n}\text{fifa}\,C(0)\Delta\frac{1}{n}\mathcal{V}_\infty(C(0),f(n))\big)\geq\delta\Big)\\ \leq\,\widehat{P}\Big(\mathcal{L}^3\big(\frac{1}{n}\mathcal{V}_\infty(C(0),f(n))\Delta\frac{1}{n}\text{fifa}\,C(0)\big)\geq\delta\Big)$$

These bounds are independent of $(x_1,\dots,x_m)$ in $T(U_1,\dots,U_m)$. Using the exponential contiguity between $n^{-1}\mathcal{V}_\infty(C(0),f(n))$ and $n^{-1}\mathrm{fifa}\,C(0)$ and Proposition 9.12, we deduce from the preceding bounds that there exists a positive constant $c'=c(k)/||\tau||_\infty$ (where $c(k)$ is the constant appearing in Proposition 9.12) such that for any positive λ,δ,

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\sup_{x_1,\dots,x_m}\widehat{P}\Big(\inf\Big\{\mathcal{L}^3_m\big(\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathcal{V}_\infty(C(0),f(n))\big),(E_1,\dots,E_m)\big):\\ (E_1,\dots,E_m)\in(\mathcal{I}^m)^{-1}([0,\lambda])\Big\}\geq\delta\Big)\ \leq\ -c'\lambda$$

where the supremum is taken over $(x_1,\dots,x_m)$ in $T(U_1,\dots,U_m)$.

• *Local estimate.* Let $(E_1,\dots,E_m)$ be an element in $\mathcal{B}(U_1)\times\cdots\times\mathcal{B}(U_m)$ such that $\mathcal{I}^m(E_1,\dots,E_m)<\infty$. We have to show that

$$\lim_{\delta\to 0}\ \limsup_{n\to\infty}\frac{1}{n^2}\ln\sup_{x_1,\dots,x_m}\widehat{P}\Big(\mathcal{L}^3_m\big(\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathcal{V}_\infty(C(0),f(n))\big),(E_1,\dots,E_m)\big)<\delta\Big)\\ \leq\ -\mathcal{I}^m(E_1,\dots,E_m)$$

where the supremum is taken over $(x_1,\dots,x_m)$ in $T(U_1,\dots,U_m)$. Let ε be positive. Let δ_0 be associated to ε as in Lemma 8.1. For each l in $\{1,\dots,m\}$, by Lemma 6.6 applied to the set E_l and the open set U_l, there exists a finite collection of disjoint balls $B(x^l_i,r^l_i)$, $i\in I(l)$, such that: for any i in $I(l)$, x^l_i belongs to $\partial^*E_l\cap U_l$, r^l_i belongs to $]0,1[$, $B(x^l_i,r^l_i)$ is included in U_l and

$$\mathcal{L}^3\big((E_l\cap B(x^l_i,r^l_i))\Delta B_-(x^l_i,r^l_i,\nu_{E_l}(x^l_i))\big)\ \leq\ \delta_0\,(r^l_i)^3/3\,,$$
$$\Big|\mathcal{I}(E_l,U_l)-\sum_{i\in I(l)}\pi(r^l_i)^2\tau(\nu_{E_l}(x^l_i))\Big|\ \leq\ \varepsilon\,.$$

Let δ be a positive real number strictly smaller than

$$\min\big\{\,\delta_0(r^l_i)^3/6:1\leq l\leq m,\,i\in I(l)\,\big\}\,.$$

Let $\mathcal{E}(\delta)$ be the event

$$\mathcal{E}(\delta)\ =\ \Big\{\,\mathcal{L}^3_m\big(\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}\big(n^{-1}\mathrm{aglu}\,C(0)\big),(E_1,\dots,E_m)\big)<2\delta\Big\}\,.$$

For $(x_1,\dots,x_m)$ in $T(U_1,\dots,U_m)$, we write

$$\widehat{P}\Big(\mathcal{L}^3_m\big(\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathcal{V}_\infty(C(0),f(n))\big),(E_1,\dots,E_m)\big)<\delta\Big)\\ \leq\ \widehat{P}\Big(\mathcal{L}^3_m\big(\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathcal{V}_\infty(C(0),f(n))\big),\phi^{U_1,\dots,U_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathrm{aglu}\,C(0)\big)\big)\geq\delta\Big)\ +\ \widehat{P}(\mathcal{E}(\delta))\\ \leq\ \widehat{P}\Big(\mathcal{L}^3\big(\frac{1}{n}\mathcal{V}_\infty(C(0),f(n))\Delta\frac{1}{n}\mathrm{aglu}\,C(0)\big)\geq\delta\Big)\ +\ \widehat{P}(\mathcal{E}(\delta))\,.$$

Since $n^{-1}\mathcal{V}_\infty(C(0),f(n))$ and $n^{-1}\text{aglu}\,C(0)$ are exponentially contiguous, we need only to estimate $\widehat{P}(\mathcal{E}(\delta))$. Suppose that $\mathcal{E}(\delta)$ occurs. Suppose also that n is large enough to have

$$\forall\, l\in\{1,\dots,m\}\quad \forall\, i\in I(l)\qquad 4\pi\sqrt{3}(r_i^l)^2n^2+\pi\sqrt{3}<\delta_0(r_i^l)^3n^3/3\,.$$

Let l belong to $\{1,\dots,m\}$, let i belong to $I(l)$ and let $\mathcal{C}(l,i)$ be the collection of the open clusters of the configuration restricted to $\text{aglu}\,C(0)\cap B(nx_l+nx_i^l,nr_i^l)$. Because of the definition of $\text{aglu}\,C(0)$, the clusters of $\mathcal{C}(l,i)$ are open clusters of the configuration restricted to $B(nx_l+nx_i^l,nr_i^l)$. Moreover

$$\begin{aligned}
&\mathcal{L}^3\Big(\mathcal{V}_\infty(\text{cup}\,\mathcal{C}(l,i),1/2)\Delta B_-(nx_l+nx_i^l,nr_i^l,\nu_{E_l}(x_i^l))\Big)\\
&\quad\le \mathcal{L}^3\Big(\mathcal{V}_\infty(\text{cup}\,\mathcal{C}(l,i),1/2)\Delta(\text{aglu}\,C(0)\cap B(nx_l+nx_i^l,nr_i^l))\Big)\\
&\qquad\qquad +\mathcal{L}^3\Big((\text{aglu}\,C(0)\cap B(nx_l+nx_i^l,nr_i^l))\Delta B_-(nx_l+nx_i^l,nr_i^l,\nu_{E_l}(x_i^l))\Big)\\
&\quad\le \mathcal{L}^3\big(B(nx_i^l,nr_i^l+\sqrt{3}/2)\setminus B(nx_i^l,nr_i^l-\sqrt{3}/2)\big)\\
&\qquad\qquad\qquad +\mathcal{L}^3\big((nU_l\cap(\text{aglu}\,C(0)-nx_l))\Delta nE_l\big)\\
&\qquad\qquad\qquad\qquad +\mathcal{L}^3\Big((nE_l\cap B(nx_i^l,nr_i^l))\Delta B_-(nx_i^l,nr_i^l,\nu_{E_l}(x_i^l))\Big)\\
&\quad< 4\pi\sqrt{3}(r_i^l)^2n^2+\pi\sqrt{3}+2\delta n^3+\delta_0(r_i^l)^3n^3/3<\delta_0(r_i^l)^3n^3\,.
\end{aligned}$$

Therefore the collection $\mathcal{C}(l,i)$ realizes the event $\text{sep}\,(n,x_l+x_i^l,r_i^l,\nu_{E_l}(x_i^l),\delta_0)$ and

$$\mathcal{E}(\delta)\ \subset\ \bigcap_{1\le l\le m}\ \bigcap_{i\in I(l)}\ \text{sep}\,(n,x_l+x_i^l,r_i^l,\nu_{E_l}(x_i^l),\delta_0)\,.$$

Since the balls $B(x_i^l,r_i^l)$, $i\in I(l)$, are disjoint for any l in $\{1,\dots,m\}$ and since $(x_1,\dots,x_m)$ belongs to $T(U_1,\dots,U_m)$, then the balls $B(nx_l+nx_i^l,nr_i^l)$, $i\in I(l)$, $1\le l\le m$, are still disjoint and the events $\text{sep}\,(n,x_l+x_i^l,r_i^l,\nu_{E_l}(x_i^l),\delta_0)$, $i\in I(l)$, $1\le l\le m$, are independent, whence

$$\widehat{P}(\mathcal{E}(\delta))\ \le\ \prod_{1\le l\le m}\ \prod_{i\in I(l)}\widehat{P}\big(\text{sep}\,(n,x_l+x_i^l,r_i^l,\nu_{E_l}(x_i^l),\delta_0)\big)\,.$$

Since the model is invariant under integer translations, then for l in $\{1,\dots,m\}$ and i in $I(l)$

$$\widehat{P}\big(\text{sep}\,(n,x_l+x_i^l,r_i^l,\nu_{E_l}(x_i^l),\delta_0)\big)\ =\ \widehat{P}\big(\text{sep}\,(n,x_i^l,r_i^l,\nu_{E_l}(x_i^l),\delta_0)\big)$$

and

$$\widehat{P}(\mathcal{E}(\delta))\ \le\ \prod_{1\le l\le m}\ \prod_{i\in I(l)}\widehat{P}\big(\text{sep}\,(n,x_i^l,r_i^l,\nu_{E_l}(x_i^l),\delta_0)\big)\,.$$

This last bound is independent of $(x_1,\dots,x_m)$ in $T(U_1,\dots,U_m)$. By the choice of δ_0 (see Lemma 8.1) and of the balls $B(x_i^l,r_i^l)$ (see Lemma 6.6)

$$\begin{aligned}\limsup_{n\to\infty}\ \frac{1}{n^2}\ln\ \sup\Big\{\widehat{P}(\mathcal{E}(\delta)) : (x_1,\dots,x_m)\in T(U_1,\dots,U_m)\Big\}\\ \leq -\sum_{1\leq l\leq m}\ \sum_{i\in I(l)}\tau(\nu_{E_l}(x_i^l))\pi(r_i^l)^2(1-\varepsilon)\\ \leq -\sum_{1\leq l\leq m}\Big(\mathcal{I}(E_l,U_l)\,(1-\varepsilon)-\varepsilon\Big)\\ = -\mathcal{I}^m(E_1,\dots,E_m)\,(1-\varepsilon)+m\varepsilon\,.\end{aligned}$$

Letting δ and then ε go to zero, we get the upper bound we were seeking for. $\square$

We turn now to the proof of the enhanced upper bound of Theorem 2.2. Let $\mathcal{F}$ be a closed subset of $(\mathcal{B}(\mathbb{R}^3),L^1)$. Let a,δ be positive with $\delta\leq 1\leq a$. We write

$$\begin{aligned}&\widehat{P}\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\in\mathcal{F}\big)\ \leq\\ &\quad\widehat{P}\big(\mathcal{P}(\operatorname{fifa}C(0))\geq an^2\big)\ +\ \widehat{P}\big(\mathcal{L}^3(\mathcal{V}_\infty(C(0),f(n))\Delta\operatorname{fifa}C(0))\geq\delta n^3\big)\ +\ \widehat{P}(\mathcal{E})\,,\end{aligned}$$

where $\mathcal{E}$ is the event

$$\Big\{\,n^{-1}\mathcal{V}_\infty(C(0),f(n))\in\mathcal{F},\ \mathcal{P}(\operatorname{fifa}C(0))<an^2,\\ \mathcal{L}^3(\mathcal{V}_\infty(C(0),f(n))\Delta\operatorname{fifa}C(0))<\delta n^3\,\Big\}\,.$$

Firstly by Proposition 9.12

$$\limsup_{n\to\infty}\ \frac{1}{n^2}\ln\widehat{P}\big(\mathcal{P}(\operatorname{fifa}C(0))\geq an^2\big)\ \leq\ -c(k)a\,.$$

Secondly, since $n^{-1}\mathcal{V}_\infty(C(0),f(n))$ and $n^{-1}\operatorname{fifa}C(0)$ are exponentially contiguous,

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\Big(\mathcal{L}^3\big(\mathcal{V}_\infty(C(0),f(n))\Delta\operatorname{fifa}C(0)\big)\geq\delta n^3\Big)\ =\ -\infty\,.$$

Thirdly, we examine the last term $\widehat{P}(\mathcal{E})$ of the inequality. Suppose that $\operatorname{card}C(0)$ is finite and that $\mathcal{P}(\operatorname{fifa}C(0))<an^2$. By the isoperimetric inequality, we have $\mathcal{L}^3(\operatorname{fifa}C(0))\leq c_{\text{iso}}a^{3/2}n^3$. Let X be the random subset of $\mathbb{Z}^3$ defined by

$$X\ =\ \{\,x\in\mathbb{Z}^3:\mathcal{L}^3(B(nx,n)\cap\operatorname{fifa}C(0))\geq\delta n^3\,\}\,.$$

Since $\operatorname{diam}\operatorname{fifa}C(0)\leq\mathcal{P}(\operatorname{fifa}C(0))<an^2$, then X is included in $B(0,2an^2)$. Since a point of $\mathbb{R}^3$ belongs to at most eight balls among the balls $B(x,1)$, $x\in\mathbb{Z}^3$, then

$$\delta n^3\operatorname{card}X\ \leq\ \sum_{x\in X}\mathcal{L}^3(B(nx,n)\cap\operatorname{fifa}C(0))\ \leq\ 8\mathcal{L}^3(\operatorname{fifa}C(0))\ \leq\ 8c_{\text{iso}}a^{3/2}n^3$$

and therefore $\operatorname{card} X \leq 8c_{\text{iso}}a^{3/2}/\delta$. For x in $\mathbb{Z}^3 \setminus X$, by the isoperimetric inequality relative to the ball $B(nx,n)$, taking into account that $\mathcal{L}^3(\operatorname{fifa} C(0) \cap B(nx,n)) < \delta n^3$,

$$\mathcal{L}^3\big(\operatorname{fifa} C(0) \cap B(nx,n)\big) \;\leq\; \delta^{1/3} n\, b_{\text{iso}}^{2/3} \mathcal{P}\big(\operatorname{fifa} C(0), \operatorname{int} B(nx,n)\big)\,.$$

Summing this inequality over x in $\mathbb{Z}^3 \setminus X$,

$$\begin{aligned}\mathcal{L}^3\big(\operatorname{fifa} C(0) \setminus \operatorname{cup}\{\, B(nx,n) : x \in X\,\}\big) \;&\leq\; \sum_{x\in\mathbb{Z}^3\setminus X} \mathcal{L}^3(B(nx,n) \cap \operatorname{fifa} C(0))\\ &\leq\; \delta^{1/3} n b_{\text{iso}}^{2/3} \sum_{x\in\mathbb{Z}^3\setminus X} \mathcal{P}\big(\operatorname{fifa} C(0), \operatorname{int} B(nx,n)\big)\\ &\leq\; 8\delta^{1/3} n b_{\text{iso}}^{2/3} \mathcal{P}\big(\operatorname{fifa} C(0)\big) \;\leq\; 8a\delta^{1/3} b_{\text{iso}}^{2/3} n^3\,.\end{aligned}$$

We set $M = 8c_{\text{iso}}a^{3/2}/\delta$ and $\eta = \delta + 8a\delta^{1/3}b_{\text{iso}}^{2/3}$. Whenever $\mathcal{E}$ occurs, we have therefore

$$X \subset B(0, 2an^2)\,, \quad \operatorname{card} X \leq M\,,$$
$$\mathcal{L}^3\big(\mathcal{V}_\infty(C(0), f(n)) \setminus \operatorname{cup}\{\, B(nx,n) : x \in X\,\}\big) \leq \eta n^3.$$

Let $E(X)$ be a subset of $X \times \{1, \dots, 3^{\operatorname{card} X}\}$ associated to X as in Lemma 9.20. We decompose the event $\mathcal{E}$ according to the possible values of the set $E(X)$:

$$\widehat{P}(\mathcal{E}) \;=\; \sum_{1\leq m\leq M} \;\sum_{r_1,\dots,r_m} \;\sum_{y_1,\dots,y_m} \widehat{P}\big(\mathcal{E},\, E(X) = \{\, (y_1, r_1), \dots, (y_m, r_m)\,\}\big)$$

where the second summation extends over the integers $r_1, \dots, r_m$ in $\{\,1, \dots, 3^M\,\}$ and the third summation extends over the points $y_1, \dots, y_m$ in $\mathbb{Z}^3 \cap B(0, 2an^2)$. The number of possible choices for $y_1, \dots, y_m$ is less than $\mathcal{L}^3(B(0, 2an^2 + 2))^m$, which is a polynomial function in n. The number of possible choices for the first two sums is less than $3^{M(M+1)}$. We estimate now the term inside the sums. Let $\{\, (y_1, r_1), \dots, (y_m, r_m)\,\}$ be a value for the random set $E(X)$ compatible with $\mathcal{E}$ (that is a value which occurs with positive probability). By the construction of $E(X)$, the balls $B(y_l, r_l + 1)$, $1 \leq l \leq m$, are pairwise disjoint so that $(y_1, \dots, y_m)$ belongs to $T(B(0, r_1 + 1), \dots, B(0, r_m + 1))$; moreover the balls $B(y_i, r_i)$, $1 \leq i \leq m$, cover the balls $B(x, 1)$, $x \in X$, whence

$$\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0), f(n)) \setminus B(y_1, r_1) \setminus \cdots \setminus B(y_m, r_m)\big) \;\leq\; \eta\,.$$

Let $\mathcal{F}_\eta^m(r_1, \dots, r_m)$ be the subset of $\mathcal{B}(B(0, r_1 + 1)) \times \cdots \times \mathcal{B}(B(0, r_m + 1))$ defined by

$$\begin{aligned}\mathcal{F}_\eta^m(r_1, \dots, r_m) \;=\; \Big\{\, &\phi_{z_1,\dots,z_m}^{B(0,r_1+1),\dots,B(0,r_m+1)}(E) : E \in \mathcal{F},\\ &(z_1, \dots, z_m) \in T(B(0, r_1 + 1), \dots, B(0, r_m + 1)),\, \mathcal{L}^3\big(E \setminus \bigcup_{1\leq i\leq m} B(z_i, r_i)\big) \leq \eta \,\Big\}\,.\end{aligned}$$

We write then

$$\begin{aligned}
&\widehat{P}\big(\mathcal{E},\, E(X)=\{\,(y_1,r_1),\dots,(y_m,r_m)\,\}\big)\\
&\quad\le\ \widehat{P}\Big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\in\mathcal{F},\,\mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\setminus\bigcup_{1\le i\le m}B(y_i,r_i)\big)\le\eta\Big)\\
&\quad\le\ \widehat{P}\Big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\in\big\{\,E\in\mathcal{F},\,\mathcal{L}^3\big(E\setminus B(y_1,r_1)\setminus\cdots\setminus B(y_m,r_m)\big)\le\eta\,\big\}\Big)\\
&\quad\le\ \widehat{P}\Big(\phi^{B(0,r_1+1),\dots,B(0,r_m+1)}_{y_1,\dots,y_m}\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\big)\in\mathcal{F}^m_\eta(r_1,\dots,r_m)\Big)\,.
\end{aligned}$$

But $\mathcal{F}^m_\eta(r_1,\dots,r_m)$ depends on $(r_1,\dots,r_m)$ and η only and not on $(y_1,\dots,y_m)$ in $T(B(0,r_1+1),\dots,B(0,r_m+1))$. Coming back to the innermost summation,

$$\begin{aligned}
\sum_{y_1,\dots,y_m}\widehat{P}\big(\mathcal{E},\, E(X)=\{\,(y_1,r_1),\dots,(y_m,r_m)\,\}\big)\ &\le\ \mathcal{L}^3(B(0,2an^2+2))^m\\
\times\sup_{y_1,\dots,y_m}\widehat{P}\Big(\phi^{B(0,r_1+1),\dots,B(0,r_m+1)}_{y_1,\dots,y_m}&\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\big)\in\mathcal{F}^m_\eta(r_1,\dots,r_m)\Big)
\end{aligned}$$

where the supremum is taken over $(y_1,\dots,y_m)$ in $T(B(0,r_1+1),\dots,B(0,r_m+1))$. This inequality and the upper bound of Lemma 9.21 yield

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n^2}\ln\sum_{y_1,\dots,y_m}\widehat{P}\big(\mathcal{E},\, E(X)=\{\,(y_1,r_1),\dots,(y_m,r_m)\,\}\big)\ \le\\
-\inf\big\{\,\mathcal{I}^m(F_1,\dots,F_m):(F_1,\dots,F_m)\in\operatorname{clo}\mathcal{F}^m_\eta(r_1,\dots,r_m)\,\big\}\,.
\end{aligned}$$

Since the number of terms involved in the first two sums is bounded by $3^{M(M+1)}$ which is independent of n, we conclude that

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}(\mathcal{E})\ \le\ -\inf\Big\{\,&\mathcal{I}^m(F_1,\dots,F_m):1\le m\le M,\\
&r_1,\dots,r_m\in\{\,1,\dots,3^M\,\},\,(F_1,\dots,F_m)\in\operatorname{clo}\mathcal{F}^m_\eta(r_1,\dots,r_m)\Big\}\,.
\end{aligned}$$

It remains to evaluate this infimum. Let m belong to $\{1,\dots,M\}$, let $r_1,\dots,r_m$ belong to $\{\,1,\dots,3^M\,\}$ and let $(F_1,\dots,F_m)$ belong to $\operatorname{clo}\mathcal{F}^m_\eta(r_1,\dots,r_m)$. By the very definition of $\operatorname{clo}\mathcal{F}^m_\eta(r_1,\dots,r_m)$, there exist a set E in $\mathcal{F}$ and a m–uple $(z_1,\dots,z_m)$ in $T(B(0,r_1+1),\dots,B(0,r_m+1))$ such that

$$\mathcal{L}^3\big(E\setminus\bigcup_{1\le i\le m}B(z_i,r_i)\big)\le\eta\,,\qquad\sum_{1\le i\le m}\mathcal{L}^3\big((E\cap B(z_i,r_i+1))\Delta(z_i+F_i)\big)\ <\ \eta\,.$$

Therefore

$$\sum_{1\le i\le m}\mathcal{L}^3\big(F_i\cap B(0,r_i+1)\setminus B(0,r_i)\big)\ \le\ 2\eta\,.$$

By Lemma 6.7, for i in $\{1,\dots,m\}$, for $\mathcal{H}^1$ almost all t in $]0,1[$,

$$\mathcal{I}(F_i\cap B(0,r_i+t)) \,\leq\, \mathcal{I}(F_i,\operatorname{int}B(0,r_i+t)) \,+\, ||\tau||_\infty\mathcal{H}^2(F_i\cap\partial B(0,r_i+t))\,.$$

Let T be the subset of $]0,1[$ where all the above inequalities hold simultaneously. Certainly $\mathcal{H}^1(T)=1$ and by integrating in polar coordinates

$$\sum_{1\leq i\leq m}\mathcal{L}^3\big(F_i\cap B(0,r_i+1)\setminus B(0,r_i)\big) \,=\, \int_T\sum_{1\leq i\leq m}\mathcal{H}^2\big(F_i\cap\partial B(0,r_i+t)\big)\,dt\,,$$

so that there exists t in T such that

$$\sum_{1\leq i\leq m}\mathcal{H}^2\big(F_i\cap\partial B(0,r_i+t)\big) \,\leq\, 2\eta\,.$$

Let F be the set

$$F \,=\, \bigcup_{1\leq i\leq m}\big(z_i+(F_i\cap B(0,r_i+t))\big)\,.$$

Then

$$\begin{aligned}&\mathcal{L}^3(E\Delta F)\,\leq\\ &\quad\mathcal{L}^3\Big((E\cup F)\setminus\bigcup_{1\leq i\leq m}B(z_i,r_i)\Big) \,+\, \sum_{1\leq i\leq m}\mathcal{L}^3\big((E\cap B(z_i,r_i+1))\Delta(z_i+F_i)\big) \,<\, 4\eta\end{aligned}$$

and because of the choice of t

$$\begin{aligned}\widehat{\mathcal{I}}(F) \,&=\, \sum_{1\leq i\leq m}\mathcal{I}\big(F_i\cap B(0,r_i+t)\big)\\ &\leq\sum_{1\leq i\leq m}\mathcal{I}(F_i,\operatorname{int}B(0,r_i+t)) \,+\, ||\tau||_\infty\sum_{1\leq i\leq m}\mathcal{H}^2(F_i\cap\partial B(0,r_i+t))\\ &\leq\mathcal{I}^m(F_1,\dots,F_m)+2\eta||\tau||_\infty\,.\end{aligned}$$

It follows that

$$\inf\big\{\,\widehat{\mathcal{I}}(F):F\in\mathcal{V}_{L^1}(\mathcal{F},4\eta)\,\big\} \,\leq\, \mathcal{I}^m(F_1,\dots,F_m)+2\eta||\tau||_\infty\,.$$

Passing to the infimum over $(F_1,\dots,F_m)$, $(r_1,\dots,r_m)$ and m,

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\big(\mathcal{E}\big) \,\leq\, -\inf\big\{\,\widehat{\mathcal{I}}(F):F\in\mathcal{V}_{L^1}(\mathcal{F},4\eta)\,\big\}+2\eta||\tau||_\infty\,.$$

Coming back to the initial inequality,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\big(n^{-1}\mathcal{V}_\infty(C(0),f(n))\in\mathcal{F}\big) \,\leq\\ &\qquad\qquad -\min\big(c(k)a,\,\inf\big\{\,\widehat{\mathcal{I}}(F):F\in\mathcal{V}_{L^1}(\mathcal{F},4\eta)\,\big\}-2\eta||\tau||_\infty\big)\,.\end{aligned}$$

Recalling that $\eta=\delta+8a\delta^{1/3}b_{\mathrm{iso}}^{2/3}$, sending first δ to 0 and then a to ∞, we get the enhanced upper bound we were seeking for. $\square$

9.7. Proof of Proposition 2.4

We end this chapter with the proof of Proposition 2.4.

Lemma 9.22. *Let η belong to $]0,\theta[$ and let k,l belong to $\mathbb{N}$ with $k \geq 6$, $l \geq 130$, $3k \leq l$. For x^l in $l\mathbb{Z}^3$, let $X_l^\eta(x^l)$ be the indicator function of the complement of the event*

$$V(\Lambda(x^l,l),\eta) \,\cap\, \Big(\sum_C \operatorname{card} C > (\theta+\eta)\operatorname{card}\Lambda(x^l,l)\Big) \,\cap\, R(\Lambda(x^l,5l),l)$$

where the summation is over all the open clusters C of the configuration restricted to $\Lambda(x^l,l)$ intersecting $\partial_{\mathrm{iv}}\Lambda(x^l,l)$. For any open cluster C of diameter larger than l, any compact set K, the difference $|\operatorname{card} C\cap K - \theta\mathcal{L}^3_K(\mathcal{V}_\infty(C,l))|$ is bounded by

$$\begin{aligned}\eta\mathcal{L}^3\big(\mathcal{V}_\infty(C,2l)\cap K\big) + l^3\operatorname{card}\big\{\, x^l\in l\mathbb{Z}^3\cap K\cap\mathcal{V}_\infty(C,2l) : X_l^\eta(x^l)=1\,\big\}\\ +1120l^2\mathcal{P}(\operatorname{fifa}(C,k,l),\operatorname{int}K) + 2\mathcal{L}^3\big(\mathcal{V}_\infty(\partial K,3l+1)\cap\mathcal{V}_\infty(C,2l)\big)\,.\end{aligned}$$

Proof. We start by rewriting the difference

$$\begin{aligned}&|\operatorname{card} C\cap K - \theta\mathcal{L}^3_K(\mathcal{V}_\infty(C,l))| \leq\\ &\theta\mathcal{L}^3_K\big(\mathcal{V}_\infty(C,l)\Delta\mathrm{cup}\,\{\,\Lambda(x^l,l) : x^l\in\operatorname{fat}(\operatorname{fifa}(C,k,l),l)\,\}\big)\\ &\qquad + \big|\operatorname{card} C\cap K - \theta\mathcal{L}^3_K\big(\mathrm{cup}\,\{\,\Lambda(x^l,l) : x^l\in\operatorname{fat}(\operatorname{fifa}(C,k,l),l)\,\}\big)\big|\\ &\leq \theta\mathcal{L}^3_K\big(\mathcal{V}_\infty(C,2l)\setminus\operatorname{fifa}(C,k,l)\big) + \sum_{x^l}|\operatorname{card}\big(C\cap K\cap\Lambda(x^l,l)\big) - \theta\mathcal{L}^3_K(\Lambda(x^l,l))|\end{aligned}$$

where we have used Lemma 9.9 to get the first term and the summation extends over x^l in $\operatorname{fat}(\operatorname{fifa}(C,k,l),l)\cap\operatorname{fat}(K,l)$. We decompose further this summation according to x^l and to the value of X_l^η as follows:

$$\sum_{x^l}\cdots \;=\; \sum_{x^l:\Lambda(x^l,l)\cap\partial K\neq\varnothing}\cdots \;+\; \sum_{x^l:\Lambda(x^l,l)\subset\operatorname{int}K,\, X_l^\eta(x^l)=0}\cdots \;+\; \sum_{x^l:\Lambda(x^l,l)\subset\operatorname{int}K,\, X_l^\eta(x^l)=1}\cdots\,.$$

Remarking that

$$\operatorname{fat}(\operatorname{fifa}(C,k,l),l)\cap\operatorname{fat}(K,l) \,\subset\, \mathcal{V}_\infty(C,3l/2)\cap\operatorname{clo}\mathcal{V}_\infty(K,l/2)\,,$$

we obtain

$$\sum_{x^l:\Lambda(x^l,l)\cap\partial K\neq\varnothing}\cdots \;\leq\; \mathcal{L}^3\big(\mathcal{V}_\infty(\partial K,2l)\cap\mathcal{V}_\infty(C,2l)\big)\,.$$

Next, whenever $X_l^\eta(x^l)=0$ for some x^l in $\operatorname{fat}(\operatorname{fifa}(C,k,l),l)$, then $d_\infty(x^l,C)<3l/2$ and C intersects $\Lambda(x^l,3l)$; in particular there exists an open path in $C\cap\Lambda(x^l,5l)$ of diameter larger than l; denoting by $C^*(\Lambda(x^l,l))$ the open crossing cluster in $\Lambda(x^l,l)$ realizing the event $V(\Lambda(x^l,l))$, we have

$$C^*(\Lambda(x^l,l)) \subset C\cap\Lambda(x^l,l) \subset \bigcup_C C$$

where the union is over all the open clusters C of the configuration restricted to $\Lambda(x^l,l)$ intersecting $\partial_{\mathrm{iv}}\Lambda(x^l,l)$. By definition of the event associated to $X_l^\eta(x^l)$, this implies $|\mathrm{card}\,\big(C\cap\Lambda(x^l,l)\big)-\theta l^3|\leq \eta l^3$ and

$$\sum_{x^l:\Lambda(x^l,l)\subset\mathrm{int}\,K,\ X_l^\eta(x^l)=0}\cdots \quad\leq \eta\mathcal{L}^3(\mathcal{V}_\infty(C,2l)\cap K)\,.$$

For the third sum,

$$\sum_{x^l:\Lambda(x^l,l)\subset\mathrm{int}\,K,\ X_l^\eta(x^l)=1}\cdots \quad\leq l^3\mathrm{card}\,\big\{\,x^l\in l\mathbb{Z}^3\cap K\cap\mathcal{V}_\infty(C,2l):X_l^\eta(x^l)=1\,\big\}\,.$$

By Lemma 9.10,

$$\begin{aligned}\theta\mathcal{L}^3_K\big(\mathcal{V}_\infty(C,2l)\setminus\mathrm{fifa}\,(C,k,l)\big) \leq\\ 1120l^2\mathcal{P}(\mathrm{fifa}\,(C,k,l),\mathrm{int}\,K)+\mathcal{L}^3_K(\mathcal{V}_\infty(\partial K,3l+1)\cap\mathcal{V}_\infty(C,2l))\,.\end{aligned}$$

Putting these estimates together yields the desired inequality. $\square$

Proof of Proposition 2.4. Let δ, a be positive. We write

$$\begin{aligned}\widehat{P}\big(|\mathrm{card}\,C(0)-\theta\mathcal{L}^3(\mathcal{V}_\infty(C(0),f(n)))|\geq\delta n^3\big) \leq \widehat{P}\big(\mathcal{L}^3(\mathcal{V}_\infty(C(0),2f(n)))\geq an^3\big)\\ +\widehat{P}\Big(|\mathrm{card}\,C(0)-\theta\mathcal{L}^3(\mathcal{V}_\infty(C(0),f(n)))|\geq\delta n^3,\,\mathcal{L}^3(\mathcal{V}_\infty(C(0),2f(n)))<an^3\Big)\,.\end{aligned}$$

We consider each term of the right–hand side separately. First

$$\begin{aligned}\widehat{P}\big(\mathcal{L}^3(\mathcal{V}_\infty(C(0),2f(n)))\geq an^3\big) \leq\\ \widehat{P}\big(\mathcal{L}^3(\mathcal{V}_\infty(C(0),2f(n))\Delta\mathrm{fifa}\,C(0))\geq an^3/2\big) + \widehat{P}\big(\mathcal{L}^3(\mathrm{fifa}\,C(0))\geq an^3/2\big)\,.\end{aligned}$$

Under $\widehat{P}$, the cluster $C(0)$ is bounded, whence by the isoperimetric inequality

$$\widehat{P}\big(\mathcal{L}^3(\mathrm{fifa}\,C(0))\geq an^3/2\big) \leq \widehat{P}\big(\mathcal{P}(\mathrm{fifa}\,C(0))\geq(a/(2c_{\mathrm{iso}}))^{2/3}n^2\big)\,.$$

Applying Corollary 9.11 and Proposition 9.12,

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\widehat{P}\big(\mathcal{L}^3(\mathcal{V}_\infty(C(0),2f(n)))\geq an^3\big) \leq -(a/2c_{\mathrm{iso}})^{2/3}c(k)\,.$$

Secondly, let η be such that $0 < \eta a < \delta/3$. We apply the inequality of Lemma 9.22 with $K = \operatorname{clo}\Lambda(0, 4an^3)$ and we suppose that n is large enough to have $27f(n)^3 < \delta n^3$ (so that $|\operatorname{card} C(0) - \theta\mathcal{L}^3(\mathcal{V}_\infty(C(0), f(n)))| \geq \delta n^3$ implies that $\operatorname{diam} C(0) \geq f(n)$) and $6f(n) < an^3$ (so that $\mathcal{L}^3(\mathcal{V}_\infty(C(0), 2f(n))) < an^3$ implies that $d_\infty(C(0), \partial K) > 6f(n)$). We get

$$\widehat{P}\Big(|\operatorname{card} C(0) - \theta\mathcal{L}^3(\mathcal{V}_\infty(C(0), f(n)))| \geq \delta n^3,\, \mathcal{L}^3(\mathcal{V}_\infty(C(0), 2f(n))) < an^3\Big) \leq$$
$$P\Big(f(n)^3 \operatorname{card}\{\, x \in \operatorname{clo}\Lambda(0, 4an^3) \cap f(n)\mathbb{Z}^3 : X^\eta(x) = 1 \,\} \geq \delta n^3/3\Big)$$
$$+ P\big(1120 f(n)^2 \mathcal{P}(\operatorname{fifa} C(0)) > \delta n^3/3\big)\,.$$

By Proposition 9.12, the second term of the right–hand side is less than

$$b(k)\exp(-c(k)(\delta/3360)n^3 f(n)^{-2})\,.$$

By Corollaries 7.4, 7.5, 7.8, the first term of the right–hand side is less than

$$\sum_{i \geq (\delta/3)(n/f(n))^3} \exp\Big(3i\ln(4an^3/f(n)+2) + i5^{-3}(\ln b(\eta) - c(\eta)f(n))\Big)\,.$$

Because $f(n)/\ln n$ goes to ∞ as n goes to ∞, for n large enough, the sum is less than $b'(\eta)\exp(-c'(\eta)(\delta/3)n^3 f(n)^{-2})$ for some positive constants $b'(\eta)$, $c'(\eta)$. The last two estimates imply that, for any δ, a positive,

$$\lim_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\Big(|\operatorname{card} C(0) - \theta\mathcal{L}^3(\mathcal{V}_\infty(C(0), f(n)))| \geq \delta n^3, \mathcal{L}^3(\mathcal{V}_\infty(C(0), 2f(n))) < an^3\Big) = -\infty$$

whence, coming back to the initial inequality,

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln \widehat{P}\Big(|\operatorname{card} C(0) - \theta\mathcal{L}^3(\mathcal{V}_\infty(C(0), f(n)))| \geq \delta n^3\Big) \leq -(a/2c_{\mathrm{iso}})^{2/3} c(k)\,.$$

Letting a go to ∞, we obtain the statement of the exponential contiguity. $\square$

CHAPTER 10

COLLECTIONS OF SETS

We denote by $\mathrm{BC}(\mathbb{R}^3)$ the set of all finite or countable collections of non empty Borel subsets of $\mathbb{R}^3$. We recall the basic set operations on collections of sets: for $\mathcal{A}$ a collection of sets, we denote the union of the sets of $\mathcal{A}$ by $\operatorname{cup}\mathcal{A}$, i.e.,

$$\operatorname{cup}\mathcal{A} \;=\; \bigcup\nolimits_{A\in\mathcal{A}} A\,,$$

and by $\operatorname{overlap}\mathcal{A}$ the union of the intersections of all pairs of distinct sets of $\mathcal{A}$, i.e.,

$$\operatorname{overlap}\mathcal{A} \;=\; \operatorname{cup}\big\{\, A_1\cap A_2 : A_1, A_2 \in \mathcal{A},\ A_1\neq A_2 \,\big\}\,.$$

When dealing with Borel collections, we identify the Borel sets whose symmetric difference is negligible, hence the elements of a Borel collection are pairwise distinct up to negligible sets. A Borel collection might be empty but it does not contain negligible sets. A Borel collection $\mathcal{A}$ of subsets of $\mathbb{R}^3$ is a Borel partition of $\mathbb{R}^3$ if and only if $\mathbb{R}^3\setminus\operatorname{cup}\mathcal{A}$ and $\operatorname{overlap}\mathcal{A}$ are negligible. We define a perimeter functional $\mathcal{P}$ on $\mathrm{BC}(\mathbb{R}^3)$: for any $\mathcal{A}$ in $\mathrm{BC}(\mathbb{R}^3)$, any open set O,

$$\mathcal{P}(\mathcal{A},O) \;=\; \sum_{A\in\mathcal{A}} \mathcal{P}(A,O)\,.$$

We define also $\mathcal{P}(\mathcal{A})=\mathcal{P}(\mathcal{A},\mathbb{R}^3)$. A Caccioppoli partition of $\mathbb{R}^3$ is a Borel partition $\mathcal{A}$ of $\mathbb{R}^3$ such that $\mathcal{P}(\mathcal{A},O)$ is finite for any bounded open set O. We denote by $\mathrm{CP}(\mathbb{R}^3)$ the set of all Caccioppoli partitions of $\mathbb{R}^3$.

Remark. The denomination "Caccioppoli partition" was introduced in [22,23] in relation with problems in image segmentation and the Mumford–Shah functional.

Definition 10.1. An arrangement $(A(i), i\in\mathbb{N})$ of an element $\mathcal{A}$ of $\mathrm{BC}(\mathbb{R}^3)$ is a sequence of sets in $\mathcal{A}\cup\{\varnothing\}$ such that each set of $\mathcal{A}$ appears exactly once in the sequence $(A(i), i\in\mathbb{N})$ and the empty set $\varnothing$ appears countably many times in the sequence $(A(i), i\in\mathbb{N})$.

Remark. If $\mathcal{A}$ is finite then $A(i)=\varnothing$ for i sufficiently large.

Lemma 10.2. *Let $\mathcal{A}$ belong to $BC(\mathbb{R}^3)$ and let $(A(i), i \in \mathbb{N})$, $(A'(i), i \in \mathbb{N})$ be two arrangements of $\mathcal{A}$. There exists a permutation ϕ of $\mathbb{N}$ such that $A'(i) = A(\phi(i))$ for all i in $\mathbb{N}$.*

Proof. The sets $\{ i \in \mathbb{N} : A(i) \neq \varnothing \}$ and $\{ i \in \mathbb{N} : A'(i) \neq \varnothing \}$ are in one to one correspondence with $\mathcal{A}$, while the sets $\{ i \in \mathbb{N} : A(i) = \varnothing \}$ and $\{ i \in \mathbb{N} : A'(i) = \varnothing \}$ are both countably infinite. $\square$

We define first a metric D on $\mathrm{BC}(\mathbb{R}^3)$: for $\mathcal{A}_1, \mathcal{A}_2$ in $\mathrm{BC}(\mathbb{R}^3)$,

$$D(\mathcal{A}_1, \mathcal{A}_2) = \inf \Big\{ \sup_{i \in \mathbb{N}} \mathcal{L}^3(A_1(i) \Delta A_2(i)) : (A_j(i), i \in \mathbb{N}) \text{ arrangement of } \mathcal{A}_j,\ j = 1,2 \Big\}.$$

We define next a family of pseudo–metrics D_K on $\mathrm{BC}(\mathbb{R}^3)$. For K a compact set and for $\mathcal{A}_1$, $\mathcal{A}_2$ in $\mathrm{BC}(\mathbb{R}^3)$ we set

$$D_K(\mathcal{A}_1, \mathcal{A}_2) = \inf \Big\{ \sup_{i \in \mathbb{N}} \mathcal{L}^3_K(A_1(i) \Delta A_2(i)) : (A_j(i), i \in \mathbb{N}) \text{ arrangement of } \mathcal{A}_j,\ j = 1,2 \Big\}$$

Corollary 10.3. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ belong to $BC(\mathbb{R}^3)$ and let $(A_1(i), i \in \mathbb{N})$ be an arrangement of $\mathcal{A}_1$. For any compact set K,*

$$D_K(\mathcal{A}_1, \mathcal{A}_2) = \inf \Big\{ \sup_{i \in \mathbb{N}} \mathcal{L}^3_K(A_1(i) \Delta A_2(i)) : (A_2(i), i \in \mathbb{N}) \text{ arrangement of } \mathcal{A}_2 \Big\}.$$

Proof. By the definition of D_K, for any positive ε, there exist arrangements $(A'_1(i), i \in \mathbb{N})$, $(A'_2(i), i \in \mathbb{N})$ of $\mathcal{A}_1$ and $\mathcal{A}_2$ such that

$$\sup_{i \in \mathbb{N}} \mathcal{L}^3_K(A'_1(i) \Delta A'_2(i)) \leq D_K(\mathcal{A}_1, \mathcal{A}_2) + \varepsilon.$$

By Lemma 10.2, there exists a permutation ϕ of $\mathbb{N}$ such that $A'_1(i) = A_1(\phi(i))$ for all i in $\mathbb{N}$. Let $(A_2(i), i \in \mathbb{N})$ be the arrangement of $\mathcal{A}_2$ defined by $A_2(i) = A'_2(\phi^{-1}(i))$ for i in $\mathbb{N}$. Then

$$\sup_{i \in \mathbb{N}} \mathcal{L}^3_K(A_1(i) \Delta A_2(i)) = \sup_{i \in \mathbb{N}} \mathcal{L}^3_K(A'_1(i) \Delta A'_2(i)) \leq D_K(\mathcal{A}_1, \mathcal{A}_2) + \varepsilon.$$

Taking the infimum of the left–hand side with respect to all arrangements $(A_2(i), i \in \mathbb{N})$ of $\mathcal{A}_2$ and letting ε go to 0 yields the result. $\square$

We construct a metric D_{loc} on $\mathrm{BC}(\mathbb{R}^3)$ by using an increasing sequence of compact sets filling $\mathbb{R}^3$, for instance the closed balls $(B(0,j), j \in \mathbb{N})$, and setting

$$\forall\, \mathcal{A}_1, \mathcal{A}_2 \in \mathrm{BC}(\mathbb{R}^3) \quad D_{\mathrm{loc}}(\mathcal{A}_1, \mathcal{A}_2) = \sum_{j \geq 1} j^{-5} D_{B(0,j)}(\mathcal{A}_1, \mathcal{A}_2).$$

A sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of Borel collections of subsets of $\mathbb{R}^3$ converges with respect to D_{loc} towards a Borel collection $\mathcal{A}$ if and only if for any compact set K

$$\lim_{n\to\infty} D_K(\mathcal{A}_n,\mathcal{A}) \;=\; 0\,.$$

This notion of convergence corresponds to the topology on $\mathrm{BC}(\mathbb{R}^3)$ generated by the following system of neighbourhoods: a fundamental system of neighbourhoods of a Borel collection $\mathcal{A}$ of subsets of $\mathbb{R}^3$ is

$$V(\mathcal{A},K,\varepsilon) \;=\; \big\{\,\mathcal{A}'\in\mathrm{BC}(\mathbb{R}^3): D_K(\mathcal{A}',\mathcal{A})<\varepsilon\,\big\}\,,\quad K \text{ compact}\,,\ \varepsilon>0\,.$$

The sets $V(\mathcal{A},K,\varepsilon)$ are open; indeed, for any compact set K, by Corollary 10.3,

$$\forall\,\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\in\mathrm{BC}(\mathbb{R}^3)\qquad D_K(\mathcal{A}_1,\mathcal{A}_3)\le D_K(\mathcal{A}_1,\mathcal{A}_2)+D_K(\mathcal{A}_2,\mathcal{A}_3)\,.$$

Another way to build a metric on $\mathrm{BC}(\mathbb{R}^3)$ is to use the measure ψ defining the topology L^1_{loc} on $\mathcal{B}(\mathbb{R}^3)$. We recall that $\psi(A)=\sum_{j\ge1} j^{-5}\mathcal{L}^3(A\cap B(0,j))$ for a Borel set A. We set for any $\mathcal{A}_1,\mathcal{A}_2$ in $\mathrm{BC}(\mathbb{R}^3)$

$$D_\psi(\mathcal{A}_1,\mathcal{A}_2)=$$
$$\inf\Big\{\sup_{i\in\mathbb{N}}\psi(A_1(i)\Delta A_2(i)):(A_j(i),i\in\mathbb{N})\text{ arrangement of }\mathcal{A}_j,\ j=1,2\Big\}\,.$$

That D_ψ and D_{loc} define the same topology on $\mathrm{BC}(\mathbb{R}^3)$ is a consequence of the following inequalities, valid for any $\mathcal{A}_1,\mathcal{A}_2$ in $\mathrm{BC}(\mathbb{R}^3)$ and any j in $\mathbb{N}$,

$$D_{\mathrm{loc}}(\mathcal{A}_1,\mathcal{A}_2)\;\le\;D_\psi(\mathcal{A}_1,\mathcal{A}_2)\;\le\;2D_{B(0,j)}(\mathcal{A}_1,\mathcal{A}_2)+4\pi/(3j)\,.$$

Lemma 10.4. *Let $\mathcal{A}$ belong to $BC(\mathbb{R}^3)$ and let $(A(i),i\in\mathbb{N})$ be an arrangement of $\mathcal{A}$. A sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ in $BC(\mathbb{R}^3)$ converges to $\mathcal{A}$ for the metric D_{loc} if and only if there exist arrangements $(A_n(i),i\in\mathbb{N})_{n\in\mathbb{N}}$ of $(\mathcal{A}_n)_{n\in\mathbb{N}}$ such that for any compact set K*

$$\lim_{n\to\infty}\sup_{i\in\mathbb{N}}\mathcal{L}^3_K(A_n(i)\Delta A(i))\;=\;0\,.$$

Remark. The interesting point is that the arrangements chosen for the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ are independent of the compact sets K.

Proof. In case arrangements exist such that the above limit occurs, then clearly the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ converges to $\mathcal{A}$ for the metric D_{loc}. Conversely, suppose that the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ converges to $\mathcal{A}$ for the metric D_{loc}. Let for n in $\mathbb{N}$

$$\phi(n)\;=\;\max\big\{\,m\in\mathbb{N}:D_{B(0,m)}(\mathcal{A}_n,\mathcal{A})\le 1/m\,\big\}\,.$$

By the definition of the metric D_{loc}, for any m in $\mathbb{N}$, the quantity $D_{B(0,m)}(\mathcal{A}_n,\mathcal{A})$ goes to 0 as n goes to ∞, so that $\lim_{n\to\infty}\phi(n)=\infty$. By Corollary 10.3, for each n in $\mathbb{N}$, there exists an arrangement $(A_n(i), i\in\mathbb{N})$ of $\mathcal{A}_n$ such that

$$\sup_{i\in\mathbb{N}}\mathcal{L}^3_{B(0,\phi(n))}(A_n(i)\Delta A(i))\;\leq\;D_{B(0,\phi(n))}(\mathcal{A}_n,\mathcal{A})+1/\phi(n)\,.$$

We claim that these arrangements answer the problem. Let indeed K be any compact set. For n sufficiently large, the ball $B(0,\phi(n))$ contains K whence

$$\sup_{i\in\mathbb{N}}\mathcal{L}^3_{K}(A_n(i)\Delta A(i))\;\leq\;D_{B(0,\phi(n))}(\mathcal{A}_n,\mathcal{A})+1/\phi(n)\;\leq\;2/\phi(n)\,.$$

Therefore the left–hand quantity converges to 0 as n goes to ∞. $\square$

Lemma 10.5. *Let f be a lower semicontinuous map from $(\mathcal{B}(\mathbb{R}^3),D_{\mathrm{loc}})$ to $\mathbb{R}^+\cup\{\infty\}$, that is, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets in $\mathcal{B}(\mathbb{R}^3)$ converging towards some set A in $\mathcal{B}(\mathbb{R}^3)$ for the topology L^1_{loc}, then $\liminf_{n\to\infty}f(A_n)\geq f(A)$. The functional on $(BC(\mathbb{R}^3),D_{\mathrm{loc}})$ defined by*

$$\forall\,\mathcal{A}\in BC(\mathbb{R}^3)\qquad f(\mathcal{A})\;=\;\sum_{A\in\mathcal{A}}f(A)$$

is lower semicontinuous: if $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is a sequence in $BC(\mathbb{R}^3)$ converging to $\mathcal{A}$ in $BC(\mathbb{R}^3)$ for the metric D_{loc}, then $\liminf_{n\to\infty}f(\mathcal{A}_n)\geq f(\mathcal{A})$.

Remark. In particular, for any open set O, the perimeter functional $\mathcal{P}(\cdot,O)$ is lower semicontinuous on $(\mathrm{BC}(\mathbb{R}^3),D_{\mathrm{loc}})$.

Proof. Since f is lower semicontinuous, for each set A in $\mathcal{B}(\mathbb{R}^3)$ and ε positive, there exist a compact set $K(A,\varepsilon)$ and a positive $\eta(A,\varepsilon)$ such that:

$$\forall\,A'\in\mathcal{B}(\mathbb{R}^3)\qquad\mathcal{L}^3_{K(A,\varepsilon)}(A\Delta A')<\eta(A,\varepsilon)\quad\Rightarrow\quad f(A')\geq\min(f(A)-\varepsilon,1/\varepsilon)\,.$$

Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{BC}(\mathbb{R}^3)$ converging to $\mathcal{A}$ for the metric D_{loc}. Let ε be positive. Let $A(1),\dots,A(r)$ be distinct sets in $\mathcal{A}$ such that

$$\sum_{i=1}^{r}f(A(i))\;\geq\;\min(f(\mathcal{A})-\varepsilon,1/\varepsilon)\,.$$

Let

$$K=\mathrm{cup}\,\{\,K(A(i),\varepsilon/r):1\leq i\leq r\,\}\,,$$
$$\eta=\min\Big(\min\,\{\,\eta(A(i),\varepsilon/r):1\leq i\leq r\,\},\min\,\{\,\frac{1}{2}\mathcal{L}^3_K(A(i)):1\leq i\leq r\,\}\Big)\,.$$

There exists an integer N such that $D_K(\mathcal{A}_n,\mathcal{A})<\eta$ for $n\geq N$. Fix some n larger than N. There exist r distinct sets $A_n(i)$, $1\leq i\leq r$, in the collection $\mathcal{A}_n$ such that

$$\max_{1\leq i\leq r}\mathcal{L}^3_K(A_n(i)\Delta A(i))\;<\;\eta\,.$$

The definition of K,η yields also that $f(A_n(i)) \geq \min(f(A(i)) - \varepsilon/r, r/\varepsilon)$ for i in $\{1,\dots,r\}$. Therefore, for any n larger than N,

$$f(\mathcal{A}_n) \geq \sum_{i=1}^{r} f(A_n(i)) \geq \min(f(\mathcal{A}) - \varepsilon, 1/\varepsilon) - \varepsilon\,.$$

Letting successively n go to ∞ and ε go to 0 ends the proof. $\square$

Definition 10.6. To a Borel collection $\mathcal{A}$ of $\mathrm{BC}(\mathbb{R}^3)$ we associate the partition $\operatorname{part}\mathcal{A}$ of $\mathbb{R}^3$ defined by

$$\operatorname{part}\mathcal{A} = \{\, A \setminus \operatorname{overlap}\mathcal{A} : A \in \mathcal{A}\,\} \cup \{\operatorname{overlap}\mathcal{A}\,\} \cup \{\,\mathbb{R}^3 \setminus \operatorname{cup}\mathcal{A}\,\}\,.$$

Lemma 10.7. *For any Borel collection $\mathcal{A}$ of $BC(\mathbb{R}^3)$, any compact set K, we have*

$$D_K(\operatorname{part}\mathcal{A}, \mathcal{A}) \leq \max\Big(\mathcal{L}^3_K\big(\operatorname{overlap}\mathcal{A}\big)\,, \mathcal{L}^3\big(K \setminus \operatorname{cup}\mathcal{A}\big)\Big)\,.$$

Proof. Let $(A(i), i \in \mathbb{N})$ be an arrangement of $\mathcal{A}$. For i in $\mathbb{N}$, we set $A'(i) = A(i) \setminus \operatorname{overlap}\mathcal{A}$. Let i_1, i_2 be two indices such that $A(i_1) = A(i_2) = \varnothing$. We set $A'(i_1) = \mathbb{R}^3 \setminus \operatorname{cup}\mathcal{A}$. In case $\operatorname{overlap}\mathcal{A}$ is not already an element of $\mathcal{A}$, we set $A'(i_2) = \operatorname{overlap}\mathcal{A}$, otherwise we set $A'(i_2) = \varnothing$. Then $(A'(i))_{i\in\mathbb{N}}$ is an arrangement of $\operatorname{part}\mathcal{A}$. Moreover

$$\forall\, i \in \mathbb{N} \qquad \mathcal{L}^3_K\big(A(i)\Delta A'(i)\big) \leq \max\Big(\mathcal{L}^3_K\big(\operatorname{overlap}\mathcal{A}\big)\,, \mathcal{L}^3\big(K \setminus \operatorname{cup}\mathcal{A}\big)\Big)\,.$$

This implies the claim of the Lemma. $\square$

The next Theorem is a slightly simplified version of the compactness result of Congedo and Tamanini [23, Theorem 1.6], which deals with a stronger topology and includes the possibility of associating weights to the sets of the collections.

Theorem 10.8. *For any λ in $\mathbb{R}^+$, the λ level set $\{\,\mathcal{A} \in CP(\mathbb{R}^3) : \mathcal{P}(\mathcal{A}) \leq \lambda\,\}$ of the perimeter functional restricted to the Caccioppoli partitions is compact with respect to the metric D_{loc}.*

Remark. Notice that the set $\mathrm{CP}(\mathbb{R}^3)$ of the Caccioppoli partitions of $\mathbb{R}^3$ is not a closed subset of $(\mathrm{BC}(\mathbb{R}^3), D_{\mathrm{loc}})$. Indeed, a bounded set having infinite perimeter can be approximated by a sequence of Caccioppoli sets. Even the set of the Borel partitions of $\mathbb{R}^3$ is not a closed subset of $\mathrm{BC}(\mathbb{R}^3)$, because there might be a loss of mass when passing to the limit without any control on the perimeter. Using the lower semicontinuity of the volume and Lemma 10.5, we already know that if $(\mathcal{A}_n)_{n\in\mathbb{N}}$ converges towards $\mathcal{A}$, then $\liminf_{n\to\infty} \sum_{A\in\mathcal{A}_n} \mathcal{L}^3(A\cap O) \geq \sum_{A\in\mathcal{A}} \mathcal{L}^3(A\cap O)$ for any open set O. However the inequality might be strict in general. For instance, the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ defined by

$$\forall n \in \mathbb{N} \qquad \mathcal{A}_n = \Big\{ \Lambda\big((i/n, j/n, k/n), 1/n\big) : (i,j,k) \in \mathbb{Z}^3 \Big\}$$

converges towards the empty collection, yet $\mathcal{L}^3(\mathcal{A}_n) = \infty$ for each n.

Proof. For the reader convenience and for the sake of completeness, we reproduce here the proof of [23, Theorem 1.6] within our context and with our weaker topology. We recall that $\psi(A) = \sum_{j\geq 1} j^{-5}\mathcal{L}^3(A\cap B(0,j))$ for a Borel set A. Let λ belong to $\mathbb{R}^+$ and let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{CP}(\mathbb{R}^3)$ such that $\mathcal{P}(\mathcal{A}_n) \leq \lambda$ for all n in $\mathbb{N}$. For t positive, there is a finite number of sets A in $\mathcal{A}_n$ such that $\psi(A) + \mathcal{P}(A) > t$. Therefore there exists an arrangement $(A_n(i), i \in \mathbb{N})$ of $\mathcal{A}_n$ such that, for any non–negative integer i,

$$\psi(A_n(2i)) + \mathcal{P}(A_n(2i)) \geq \psi(A_n(2i+2)) + \mathcal{P}(A_n(2i+2))\,, \quad A_n(2i+1) = \varnothing\,.$$

By Lemma 6.3 and Corollary 6.4, the space $\{\, E \in \mathcal{B}(\mathbb{R}^3) : \mathcal{P}(E) \leq \lambda \,\}$ is compact with respect to the topology L^1_{loc}. By a standard diagonal argument, we can extract from the sequence of arrangements $(A_n(i), i \in \mathbb{N})_{n\in\mathbb{N}}$ a subsequence (not relabeled) such that: for each i in $\mathbb{N}$, there exists a Borel set $A(2i)$ in $\mathcal{B}(\mathbb{R}^3)$ such that $A_n(2i)$ converges to $A(2i)$ for the topology L^1_{loc} (equivalently $\psi(A_n(2i)\Delta A(2i))$ converges to 0 as n goes to ∞). For i odd we set $A(i) = \varnothing$. For any $i_1 \neq i_2$ and n in $\mathbb{N}$, any compact set K,

$$\mathcal{L}^3_K(A(i_1)\cap A(i_2)) \leq \mathcal{L}^3_K(A(i_1)\Delta A_n(i_1)) + \mathcal{L}^3_K(A_n(i_2)\Delta A(i_2))\,.$$

Letting first n go to ∞ and then K grow to $\mathbb{R}^3$, we obtain that $A(i_1)\cap A(i_2)$ is negligible for any $i_1 \neq i_2$. Let $\mathcal{A}$ be the collection of the non negligible sets of the sequence $(A(i),\ i \in \mathbb{N})$, that is, $\mathcal{A} = \{\, A(i) : i \in \mathbb{N} \,\} \setminus \{\varnothing\}$. Then $(A(i), i \in \mathbb{N})$ is an arrangement of $\mathcal{A}$. Moreover $\mathrm{overlap}\,\mathcal{A}$ is negligible. Next, we have

$$\forall n, i \in \mathbb{N} \qquad \psi(\mathbb{R}^3) + \lambda \geq i\mathcal{P}(A_n(2i)) + i\psi(A_n(2i))\,.$$

We set $a = \psi(\mathbb{R}^3) + \lambda$. Let B be a ball in $\mathbb{R}^3$. There exists a positive constant c depending on B such that

$$\forall A \in \mathcal{B}(B) \quad \psi(A) \geq c\,\mathcal{L}^3(A)\,.$$

Let i_0 be such that

$$a/(ci_0) < \mathcal{L}^3(B)/2\,.$$

For all n in $\mathbb{N}$ and $i \geq i_0$, we have

$$\mathcal{L}^3(A_n(2i)\cap B) \leq \frac{1}{c}\,\psi(A_n(2i)\cap B) \leq \frac{a}{ci} < \frac{1}{2}\mathcal{L}^3(B)\,,$$

so that, by the isoperimetric inequality relative to the ball B,

$$\mathcal{L}^3(A_n(2i)\cap B)\ \le\ b_{\mathrm{iso}}\mathcal{P}(A_n(2i),B)^{3/2}\ \le\ b_{\mathrm{iso}}(a/i)^{3/2}\,.$$

This inequality yields that for any n in $\mathbb{N}$ and i larger than i_0

$$D_B(\mathcal{A}_n,\mathcal{A})\ \le\ \max\Big(\max_{0\le k\le 2i}\mathcal{L}^3_B(A_n(k)\Delta A(k)),\,2b_{\mathrm{iso}}(a/i)^{3/2}\Big)\,.$$

Letting first n and then i go to ∞ in this inequality, we get $\lim_{n\to\infty}D_B(\mathcal{A}_n,\mathcal{A})\ =\ 0$. Since the ball B is arbitrary, then the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ converges towards $\mathcal{A}$ in $(\mathrm{BC}(\mathbb{R}^3),D_{\mathrm{loc}})$. By summing the above isoperimetric inequality, for any i larger than i_0,

$$\forall\, n\in\mathbb{N}\qquad \mathcal{L}^3(B)\ =\ \sum_{A\in\mathcal{A}_n}\mathcal{L}^3_B(A)\ \le\ \sum_{0\le k\le 2i}\mathcal{L}^3_B(A_n(k))+2b_{\mathrm{iso}}\,a^{3/2}i^{-1/2}\,.$$

By letting successively n and i go to infinity, we get $\mathcal{L}^3(B)\le\sum_{A\in\mathcal{A}}\mathcal{L}^3_B(A)$. Since the ball B is arbitrary and overlap $\mathcal{A}$ is negligible, then $\mathbb{R}^3\setminus$cup $\mathcal{A}$ is negligible. Thus $\mathcal{A}$ is a Borel partition of $\mathbb{R}^3$. By Lemma 10.5, the perimeter of $\mathcal{A}$ is less than λ. □

CHAPTER 11

THE SURFACE ENERGY OF A CACCIOPPOLI PARTITION

In this chapter, we define the surface energy of a collection of Borel sets. We prove two approximation results. The approximation through polyhedral partitions is quite delicate.

11.1. Definition of the surface energy

Let $\mathcal{A}$ be a Caccioppoli partition of $\mathbb{R}^3$. The reduced boundary $\partial^*\mathcal{A}$ of $\mathcal{A}$ is the set

$$\partial^*\mathcal{A} \;=\; \mathrm{cup}\,\{\,\partial^* A_1 \cap \partial^* A_2 : A_1, A_2 \in \mathcal{A},\; A_1 \neq A_2\,\}\,.$$

By [23, Lemma 1.4], for any Borel set E,

$$2\mathcal{H}^2(\partial^*\mathcal{A} \cap E) \;=\; \sum_{A\in\mathcal{A}} \mathcal{H}^2(\partial^* A \cap E)\,.$$

For a point x of $\partial^*\mathcal{A}$, the pair (A_1, A_2) of elements of $\mathcal{A}$ satisfying $A_1 \neq A_2$, $x \in \partial^* A_1 \cap \partial^* A_2$, is unique up to the order (both sets A_1 and A_2 have density $1/2$ at x); also, the generalized normal vectors of A_1 and A_2 at x satisfy $\nu_{A_1}(x) + \nu_{A_2}(x) = 0$. In particular, the sets $\partial^* A_1 \cap \partial^* A_2$, $A_1, A_2 \in \mathcal{A}$, $A_1 \neq A_2$, are pairwise disjoint. On the space $\mathcal{A} \times \mathcal{A} \times S^2$, we define an equivalence relation $\sim$ by

$$(A_1, A_2, \nu) \sim (A_1', A_2', \nu') \iff$$
$$(A_1, A_2, \nu) = (A_1', A_2', \nu') \text{ or } (A_1, A_2, \nu) = (A_2', A_1', -\nu')\,.$$

Hence in the quotient space $\mathcal{A}\times\mathcal{A}\times S^2/\sim$, (A_1, A_2, ν) and $(A_2, A_1, -\nu)$ are identified. We can then define uniquely a map

$$x \in \partial^*\mathcal{A} \mapsto (A_1(x), A_2(x), \nu(x)) \in \mathcal{A} \times \mathcal{A} \times S^2/\sim$$

such that, for any x in $\partial^*\mathcal{A}$,

$$A_1(x) \neq A_2(x)\,, \quad x \in \partial^* A_1(x) \cap \partial^* A_2(x)\,, \quad \nu(x) = \nu_{A_1(x)}(x) = -\nu_{A_2(x)}(x)\,.$$

We set $\nu(\mathcal{A}, x)$ to be the projection of $\nu(x)$ on the projective sphere PS^2. The surface energy of a Caccioppoli partition $\mathcal{A}$ in an open set O is then defined to be

$$\mathcal{I}(\mathcal{A}, O) \;=\; \int_{\partial^*\mathcal{A}\cap O} \tau(\nu(\mathcal{A}, x))\, d\mathcal{H}^2(x)\,.$$

Notice that the symmetry of the surface tension τ allows to define it on the projective sphere PS^2, so that $\tau(\nu(\mathcal{A},x))$ makes sense. We extend $\mathcal{I}$ to $\mathrm{BC}(\mathbb{R}^3)$ by setting $\mathcal{I}(\mathcal{A},O)=\infty$ for all $\mathcal{A}$ in $\mathrm{BC}(\mathbb{R}^3)\setminus\mathrm{CP}(\mathbb{R}^3)$. We define also $\mathcal{I}(\mathcal{A})=\mathcal{I}(\mathcal{A},\mathbb{R}^3)$.

Remark. If $\mathcal{A}$ is a Caccioppoli partition, then $\mathcal{I}(\mathcal{A},O)$ is equal to $\frac{1}{2}\sum_{A\in\mathcal{A}}\mathcal{I}(A,O)$.

Lemma 11.1. *For any collection $\mathcal{A}$ in $BC(\mathbb{R}^3)$, any open set O,*

$$\tau_{\min}\mathcal{P}(\mathcal{A},O)\;\leq\;2\mathcal{I}(\mathcal{A},O)\;\leq\;||\tau||_\infty\mathcal{P}(\mathcal{A},O)\,.$$

Proof. This result is a consequence of the previous remark and of Lemma 6.3. □

We say that a Caccioppoli partition $\mathcal{A}$ has finite surface energy, or finite perimeter, if $\mathcal{I}(\mathcal{A})$ is finite (or equivalently $\mathcal{P}(\mathcal{A})$ is finite).

Lemma 11.2. *Let $\mathcal{A}$ be a Caccioppoli partition. Let $f:\partial^*\mathcal{A}\mapsto\mathbb{R}$ be a $\mathcal{H}^2|_{\partial^*\mathcal{A}}$ measurable bounded function. For $\mathcal{H}^2$ almost all x in $\partial^*\mathcal{A}$,*

$$\lim_{r\to0}\quad\frac{1}{\pi r^2}\int_{B(x,r)\cap\partial^*\mathcal{A}}f(y)\,d\mathcal{H}^2(y)\;=\;f(x)\,.$$

Proof. Since $\mathcal{A}$ is a Caccioppoli partition, then for any bounded open set O, $\mathcal{H}^2(\partial^*\mathcal{A}\cap O)$ is finite, whence for $\mathcal{H}^2$ almost all x in $\partial^*\mathcal{A}$ [34, Corollary 2.5],

$$\limsup_{r\to0}\quad(\pi r^2)^{-1}\mathcal{H}^2(B(x,r)\cap\partial^*\mathcal{A})\;\leq\;1\,.$$

By [23, Lemma 1.4 and formula (1.5)], for any A in $\mathcal{A}$,

$$\mathcal{H}^2(\partial^*A)\;=\;\sum_{E\in\mathcal{A}\setminus\{A\}}\mathcal{H}^2(\partial^*A\cap\partial^*E)\,.$$

It follows that for $\mathcal{H}^2$ almost all x in ∂^*A,

$$\liminf_{r\to0}\;(\pi r^2)^{-1}\mathcal{H}^2(B(x,r)\cap\partial^*\mathcal{A})\geq\liminf_{r\to0}\;(\pi r^2)^{-1}\mathcal{H}^2(B(x,r)\cap\partial^*A)\;=\;1\,,$$
$$\lim_{r\to0}\;(\pi r^2)^{-1}\mathcal{H}^2\big(B(x,r)\cap(\partial^*A\Delta\partial^*\mathcal{A})\big)\;=\;0\,.$$

Since $\mathcal{A}$ is countable, then for $\mathcal{H}^2$ almost all x in $\partial^*\mathcal{A}$,

$$\lim_{r\to0}\quad(\pi r^2)^{-1}\mathcal{H}^2(B(x,r)\cap\partial^*\mathcal{A})\;=\;1\,,$$
$$\lim_{r\to0}\quad(\pi r^2)^{-1}\mathcal{H}^2\big(B(x,r)\cap(\partial^*\mathcal{A}\Delta\partial^*A_1(x))\big)\;=\;0\,.$$

Using the results on Caccioppoli sets (see chapter 6), for $\mathcal{H}^2$ almost all x in $\partial^*\mathcal{A}$,

$$\lim_{r\to0}\quad\frac{1}{\pi r^2}\int_{B(x,r)\cap\partial^*A_1(x)}f(y)\,d\mathcal{H}^2(y)\;=\;f(x)\,.$$

By decomposing $B(x,r)\cap\partial^*\mathcal{A}$ as

$$\Big(B(x,r)\cap\partial^*A_1(x)\Big)\cup\Big(B(x,r)\cap\big(\partial^*\mathcal{A}\setminus\partial^*A_1(x)\big)\Big)\setminus\Big(B(x,r)\cap\big(\partial^*A_1(x)\setminus\partial^*\mathcal{A}\big)\Big)$$

and integrating f separately over each of these sets, with the help of the previous density results, we obtain the claim of the Lemma. □

Lemma 11.3. *Let $\mathcal{A}$ be a Caccioppoli partition of $\mathbb{R}^3$ having finite perimeter. Then $\mathcal{A}$ contains exactly one set A_∞ such that $\mathcal{L}^3(A_\infty)=\infty$. Moreover $\mathcal{L}^3(\mathbb{R}^3\setminus A_\infty)<\infty$.*

Proof. For any A in $\mathcal{A}$, we have $\mathcal{P}(\mathcal{A})\geq\mathcal{P}(A)$ and by the isoperimetric inequality,

$$\forall\, A\in\mathcal{A}\qquad \min\big(\mathcal{L}^3(A),\mathcal{L}^3(\mathbb{R}^3\setminus A)\big)\;\leq\; c_{\mathrm{iso}}\,\mathcal{P}(\mathcal{A})^{3/2}\,.$$

Suppose that $\mathcal{L}^3(A)$ is finite for all A in $\mathcal{A}$. Then

$$\forall\, A\in\mathcal{A}\qquad \mathcal{L}^3(A)\;\leq\; c_{\mathrm{iso}}\mathcal{P}(\mathcal{A})^{1/2}\mathcal{P}(A)$$

and by summing $\mathcal{L}^3(\mathcal{A})\leq c_{\mathrm{iso}}\mathcal{P}(\mathcal{A})^{3/2}$, contradicting the fact that $\mathcal{A}$ is a partition of $\mathbb{R}^3$. Hence there exists a set A_∞ in $\mathcal{A}$ such that $\mathcal{L}^3(A_\infty)=\infty$. Yet $\mathcal{L}^3(\mathbb{R}^3\setminus A_\infty)\leq c_{\mathrm{iso}}\,\mathcal{P}(\mathcal{A})^{3/2}$ and the other sets of $\mathcal{A}$ are included in $\mathbb{R}^3\setminus A_\infty$. $\square$

Proposition 11.4. *The map $\mathcal{A}\in BC(\mathbb{R}^3)\mapsto\mathcal{I}(\mathcal{A})\in\mathbb{R}^+\cup\{\infty\}$ is lower semi-continuous and its level sets $\{\,\mathcal{A}\in BC(\mathbb{R}^3):\mathcal{I}(\mathcal{A})\leq\lambda\,\}$, $\lambda\in\mathbb{R}^+$, are compact with respect to the metric D_{loc}.*

Proof. Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{BC}(\mathbb{R}^3)$ converging towards an element $\mathcal{A}$ of $\mathrm{BC}(\mathbb{R}^3)$ for the metric D_{loc} and such that $\liminf_{n\to\infty}\mathcal{I}(\mathcal{A}_n)$ is finite. Up to the extraction of a subsequence, we may assume that $\lim_{n\to\infty}\mathcal{I}(\mathcal{A}_n)$ exists and that the sequence $(\mathcal{I}(\mathcal{A}_n))_{n\in\mathbb{N}}$ is bounded. In particular, the collections $\mathcal{A}_n$, $n\in\mathbb{N}$, belong to $\mathrm{CP}(\mathbb{R}^3)$ and by Lemma 11.1, their perimeters are uniformly bounded. Theorem 10.8 implies that the limit $\mathcal{A}$ also belongs to $\mathrm{CP}(\mathbb{R}^3)$. Thus $\mathcal{I}(\mathcal{A})=\frac{1}{2}\sum_{A\in\mathcal{A}}\mathcal{I}(A)$. Since $\mathcal{I}$ is lower semicontinuous on $(\mathcal{B}(\mathbb{R}^3),L^1_{\mathrm{loc}})$, Lemma 10.5 implies that $\mathcal{I}(\mathcal{A})\leq\lim_{n\to\infty}\mathcal{I}(\mathcal{A}_n)$. Therefore the surface energy $\mathcal{I}$ is lower semicontinuous on $(\mathrm{BC}(\mathbb{R}^3),D_{\mathrm{loc}})$. A further application of Theorem 10.8, together with Lemma 11.1, shows that the level sets of $\mathcal{I}$ are compact. $\square$

11.2. Approximation of the surface energy

By definition of the surface energy, for any open set O,

$$\forall\,\mathcal{A}\in\mathrm{CP}(\mathbb{R}^3)\qquad \mathcal{I}(\mathcal{A},O)\;=\;\int_{\partial^*\mathcal{A}\cap O}\tau(\nu(\mathcal{A},x))\,d\mathcal{H}^2(x)\,.$$

We first approximate the integral by an elementary sum. The next Lemma is the analog for Caccioppoli partitions of Lemma 6.6. It is needed to prove the large deviation upper bound of Theorem 2.7.

Lemma 11.5. *Let O be an open subset of $\mathbb{R}^3$. Let $\mathcal{A}$ be a Caccioppoli partition of $\mathbb{R}^3$ having finite perimeter in O. For any positive ε,δ, there exists a finite collection of disjoint balls $B(x_i,r_i)$, $i\in I$, such that: for any i in I, x_i belongs to $\partial^*\mathcal{A}\cap O$,*

r_i belongs to $]0,1[$, $B(x_i,r_i)$ is included in O,

$$\mathcal{L}^3\big((A_1(x_i)\cap B(x_i,r_i))\Delta B_-(x_i,r_i,\nu(x_i))\big) \,\leq\, \delta\, r_i^3\,,$$
$$\mathcal{L}^3\big((A_2(x_i)\cap B(x_i,r_i))\Delta B_+(x_i,r_i,\nu(x_i))\big) \,\leq\, \delta\, r_i^3\,,$$
$$\Big|\mathcal{I}(\mathcal{A},O)-\sum_{i\in I}\pi r_i^2\tau(\nu(\mathcal{A},x_i))\Big| \,\leq\, \varepsilon\,.$$

(For the definition of the maps $x\in\partial^*\mathcal{A}\cap O\mapsto(A_1(x),A_2(x),\nu(x))\in\mathcal{A}\times\mathcal{A}\times S^2/\sim$ and $x\in\partial^*\mathcal{A}\cap O\mapsto\nu(\mathcal{A},x)\in PS^2$, see the beginning of the chapter.)

Remark. We could take $\varepsilon=\delta$ in the statement of the Lemma. However, it is more convenient for later use to work with two distinct variables here.

Proof of Lemma 11.5. Let ε,δ be positive with $\varepsilon<1/2$, $\delta<1$. Because a generalized normal vector is also a measure theoretic normal, for any x in $\partial^*\mathcal{A}\cap O$, there exists a positive $r_1(x,\delta)$ such that, for any $r<r_1(x,\delta)$,

$$\mathcal{L}^3\big((A_1(x)\cap B(x,r))\Delta B_-(x,r,\nu(x))\big) \,\leq\, \delta\, r^3\,,$$
$$\mathcal{L}^3\big((A_2(x)\cap B(x,r))\Delta B_+(x,r,\nu(x))\big) \,\leq\, \delta\, r^3\,.$$

For A in $\mathcal{A}$, the map $x\in\partial^*A\mapsto\nu_A(x)\in S^2$ is measurable with respect to the measure $\mathcal{H}^2|_{\partial^*A}$, therefore the map $x\in\partial^*\mathcal{A}\cap O\mapsto\nu(\mathcal{A},x)\in PS^2$ is measurable with respect to $\mathcal{H}^2|_{\partial^*\mathcal{A}\cap O}$. By Lemma 11.2, for $\mathcal{H}^2$ almost all x in $\partial^*\mathcal{A}\cap O$,

$$\lim_{r\to 0}\ (\pi r^2)^{-1}\mathcal{H}^2(B(x,r)\cap\partial^*\mathcal{A}) \,=\, 1\,,$$
$$\lim_{r\to 0}\ \frac{1}{\pi r^2}\int_{B(x,r)\cap\partial^*\mathcal{A}}\tau(\nu(\mathcal{A},y))\,d\mathcal{H}^2(y) \,=\, \tau(\nu(\mathcal{A},x))\,.$$

Let $\partial^{**}\mathcal{A}$ be the set of the points of $\partial^*\mathcal{A}\cap O$ where the two preceding identities hold simultaneously. Clearly $\mathcal{H}^2(\partial^*\mathcal{A}\cap O\setminus\partial^{**}\mathcal{A})=0$. For any x in $\partial^{**}\mathcal{A}$, there exists a positive $r_2(x,\varepsilon)$ such that, for any $r<r_2(x,\varepsilon)$,

$$|\mathcal{H}^2(B(x,r)\cap\partial^*\mathcal{A})-\pi r^2| \,\leq\, \varepsilon\,\pi r^2\,,$$
$$\Big|\frac{1}{\pi r^2}\int_{B(x,r)\cap\partial^*\mathcal{A}}\tau(\nu(\mathcal{A},y))\,d\mathcal{H}^2(y)-\tau(\nu(\mathcal{A},x))\Big| \,\leq\, \varepsilon\,.$$

The family of balls $B(x,r)$, $x\in\partial^{**}\mathcal{A}$, $r<\min\big(r_1(x,\delta),r_2(x,\varepsilon),1,d_2(x,\partial O)\big)$, is a Vitali relation for $\partial^{**}\mathcal{A}$. By the standard Vitali covering Theorem [34, Theorem 1.10, or chapter 4], we may select a finite or countable collection of disjoint balls $B(x_i,r_i)$, $i\in I$, such that: for any i in I, x_i belongs to $\partial^{**}\mathcal{A}$, $r_i<\min\big(r_1(x_i,\delta),r_2(x_i,\varepsilon),1,d_2(x_i,\partial O)\big)$ and

$$\text{either}\qquad \mathcal{H}^2\Big(\partial^{**}\mathcal{A}\setminus\bigcup_{i\in I}B(x_i,r_i)\Big) \,=\, 0\qquad\text{or}\qquad \sum_{i\in I}r_i^2 \,=\, \infty\,.$$

Because for each i in I, r_i is smaller than $r_2(x_i,\varepsilon)$,

$$(1-\varepsilon)\sum_{i\in I}\pi r_i^2 \;\leq\; \mathcal{H}^2(\partial^{**}\mathcal{A}) \;=\; \mathcal{P}(\mathcal{A},O)/2 \;<\; \infty$$

and therefore the first case occurs, so that we may select a finite subset J of I such that

$$\mathcal{H}^2\Big(\partial^{**}\mathcal{A}\setminus\bigcup_{i\in J}B(x_i,r_i)\Big) \;<\; \varepsilon\,\mathcal{H}^2(\partial^{**}\mathcal{A})\,.$$

We claim that the collection of balls $B(x_i,r_i)$, $i\in J$, enjoys the desired properties. Indeed,

$$\begin{aligned}\Big|\int_{\partial^*\mathcal{A}\cap O}\tau(\nu(\mathcal{A},x))\,d\mathcal{H}^2(x)-\sum_{i\in J}\pi r_i^2\tau(\nu(\mathcal{A},x_i))\Big| &\leq \int_{\partial^{**}\mathcal{A}\setminus\bigcup_{i\in J}B(x_i,r_i)}\tau(\nu(\mathcal{A},x))\,d\mathcal{H}^2(x)\\ &\quad+\;\sum_{i\in J}\Big|\int_{\partial^{**}\mathcal{A}\cap B(x_i,r_i)}\tau(\nu(\mathcal{A},x))\,d\mathcal{H}^2(x)-\pi r_i^2\tau(\nu(\mathcal{A},x_i))\Big|\,.\end{aligned}$$

The first integral of the right–hand member is less than $\varepsilon\,\mathcal{H}^2(\partial^{**}\mathcal{A})||\tau||_\infty$. For any i in J,

$$\Big|\int_{\partial^{**}\mathcal{A}\cap B(x_i,r_i)}\tau(\nu(\mathcal{A},x))\,d\mathcal{H}^2(x)-\pi r_i^2\tau(\nu(\mathcal{A},x_i))\Big| \;\leq\; 2\varepsilon\,\mathcal{H}^2(B(x_i,r_i)\cap\partial^{**}\mathcal{A})\,,$$

whence by summing over i in J,

$$\sum_{i\in J}\Big|\int_{\partial^{**}\mathcal{A}\cap B(x_i,r_i)}\tau(\nu(\mathcal{A},x))\,d\mathcal{H}^2(x)-\pi r_i^2\tau(\nu(\mathcal{A},x_i))\Big| \;\leq\; 2\varepsilon\,\mathcal{H}^2(\partial^{**}\mathcal{A}).$$

By Lemma 11.1, $\mathcal{H}^2(\partial^{**}\mathcal{A})=\mathcal{P}(\mathcal{A},O)/2\leq\mathcal{I}(\mathcal{A},O)/\tau_{\min}$. Putting these inequalities together,

$$\Big|\mathcal{I}(\mathcal{A},O)-\sum_{i\in J}\pi r_i^2\tau(\nu(\mathcal{A},x_i))\Big| \;\leq\; \varepsilon\,\mathcal{I}(\mathcal{A},O)\,(2+||\tau||_\infty)/\tau_{\min}\,.$$

Since $(2+||\tau||_\infty)/\tau_{\min}$ is a fixed constant, we have the required estimate. □

We state next an approximation result which is a specialization of a result of Quentin de Gromard [61, Théorème A] to the three dimensional case. The detailed proof should appear soon in [62]. An hypersurface is a C^1 submanifold of $\mathbb{R}^3$ of codimension 1.

Theorem 11.6. *(Strong approximation of sets of finite perimeter)*
Let O be an open set in $\mathbb{R}^3$. Let E be a set of finite perimeter in O and let ε be

positive. There exists a set L of finite perimeter in O such that $O \cap \partial L$ is included in a finite union of hypersurfaces and

$$\mathcal{H}^2((\partial^* E \Delta \partial^* L) \cap O) < \varepsilon\,, \quad \mathcal{L}^3((E\Delta L)\cap O) < \varepsilon\,,$$
$$L \subset \mathcal{V}_2(E,\varepsilon)\,, \quad O \setminus L \subset \mathcal{V}_2(O\setminus E,\varepsilon)\,.$$

A careful inspection of the proof yields the more precise following statement (with $O=\mathbb{R}^3$). Let E be a set of finite perimeter in $\mathbb{R}^3$ and let ε be positive. There exists a set L of finite perimeter, a C^1 function $f:\mathbb{R}^3\mapsto\mathbb{R}$, a compact set C, an open set V and an open bounded set B such that, setting $F=\{\,x\in\mathbb{R}^3 : f(x)\geq 0\,\}$, the set $V\cap\partial F$ is the hypersurface $\{\,x\in V : f(x)=0\,\}$ and

$$C\subset B\subset V\subset \{\,x\in\mathbb{R}^3 : df(x)\neq 0\,\}\,, \quad C\subset \partial^* E\cap\partial F\,,$$
$$L\cap B = F\cap B\,, \quad V\cap\partial^* F = V\cap \partial F\,,$$
$$\forall\, x\in C \quad \nu_E(x)=\nu_F(x) = -|df(x)|^{-1} df(x)\,,$$
$$\mathcal{L}^3(V)<\varepsilon\,, \quad \mathcal{L}^3(E\Delta L)<\varepsilon\,,$$
$$\mathcal{H}^2(\partial F\cap (V\setminus C))<\varepsilon\,, \quad \mathcal{H}^2(\partial^* E\setminus C)<\varepsilon\,, \quad \mathcal{H}^2(\partial^* E\Delta\partial L)<\varepsilon\,,$$
$$L\subset \mathcal{V}_2(E,\varepsilon)\,, \quad \mathbb{R}^3\setminus L\subset\mathcal{V}_2(\mathbb{R}^3\setminus E,\varepsilon)\,.$$

The slight improvement here compared to [61] is the estimate $\mathcal{H}^2(\partial^* E\Delta\partial L)<\varepsilon$, instead of $\mathcal{H}^2(\partial^* E\Delta\partial^* L)<\varepsilon$. Let us show that the set L introduced in the proof of [61,62] satisfies this stronger inequality.

Proof. During this check, we use the objects and notation of the proof of [61,62] (except for the reduced boundary, which is still denoted by $\partial^* E$). First, since $L\cap B = F\cap B$, where B is open, and $\partial F\cap B=\partial^* F\cap B$, then $\partial^* L\cap B=\partial L\cap B$. Secondly, $L\cap A = G\cap A$ and $\partial L\cap A=\partial G\cap A$, implying $\partial^* L\cap A=\partial L\cap A$. It remains to study $\mathcal{H}^2((\partial L\setminus\partial^* L)\cap\Gamma)$. Since L is closed, then $\partial^* L\subset L$ and $\mathcal{H}^2(\partial^* L\cap\Gamma)=\mathcal{H}^2(\partial^* L\cap L\cap\Gamma)$. Rather

$$\mathcal{H}^2(\partial L\cap\Gamma)=\mathcal{H}^2(\partial L\cap\Gamma\cap(F\cup G))\leq \mathcal{H}^2(\Gamma\cap F\cap G\cap\partial L)+\mathcal{H}^2(\Gamma\cap(F\Delta G)\cap\partial L)\,.$$

Yet $F\cap G\subset\partial F\cup\partial G\cup(\operatorname{int} F\cap\operatorname{int} G)$. Moreover $\mathcal{H}^2(\Gamma\cap\partial G)=0$, $\mathcal{H}^2(\Gamma\cap\partial F)=0$, $\operatorname{int} F\cap\operatorname{int} G\cap\Gamma\cap\partial L=\varnothing$. Thus $\mathcal{H}^2(\Gamma\cap F\cap G\cap\partial L)=0$. Following the final steps of the proof, we get $\mathcal{H}^2(\Gamma\cap(F\Delta G))\leq(1+6K(3)\mathcal{P}(E)/\omega_2+4\sqrt{3}K(3))\varepsilon$. $\square$

Definition 11.7. A collection of Borel subsets of $\mathbb{R}^3$ is said to be polyhedral if all the sets of the collection are polyhedral. It is said to be finite if its cardinality is finite. A partition is said to be bounded if all its sets except one are bounded.

The proof of the large deviation lower bound relies on the following approximation result.

Theorem 11.8. *Let $\mathcal{A}$ be a Caccioppoli partition of $\mathbb{R}^3$ such that $\mathcal{I}(\mathcal{A})$ is finite. There exists a sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of bounded finite polyhedral partitions of $\mathbb{R}^3$ converging to $\mathcal{A}$ with respect to the metric D_{loc} such that $\mathcal{I}(\mathcal{A}_n)$ converges to $\mathcal{I}(\mathcal{A})$ as n goes to ∞.*

Remark. A finite polyhedral partition having finite perimeter is bounded.

Remark. It seems that this approximation problem cannot be solved easily with the help of the corresponding approximation result for one Caccioppoli set (Proposition 6.9). The main obstacle is that it is not possible in general to approximate a Caccioppoli set by polyhedral or C^∞ sets from the outside or from the inside: see [41, Remark 1.27] and [40]. If we start with a Caccioppoli partition and we approximate separately each set of the partition by a polyhedral set with Proposition 6.9, then we get a collection of sets whose union might have plenty of holes and we are not able to control the surface energy of these holes. This is the reason why we use the strong approximation result of Theorem 11.6.

Proof. Let $\mathcal{A}$ be a Caccioppoli partition of $\mathbb{R}^3$ such that $\mathcal{I}(\mathcal{A})$ is finite. By Lemma 11.3, the collection $\mathcal{A}$ contains exactly one set A_∞ such that $\mathcal{L}^3(A_\infty)=\infty$. This set satisfies in addition $\mathcal{L}^3(\mathbb{R}^3\setminus A_\infty)<\infty$. It follows that

$$\sum_{A\in\mathcal{A}\setminus\{A_\infty\}}\mathcal{L}^3(A)\,=\,\mathcal{L}^3(\mathbb{R}^3\setminus A_\infty)\,<\,\infty\,.$$

Therefore, for each positive t, there exists at most a finite number of sets A in $\mathcal{A}$ such that $\mathcal{L}^3(A)>t$. Thus there exists an arrangement $(A(i),i\in\mathbb{N})$ of $\mathcal{A}$ such that

$$\forall\, i\in\mathbb{N}\qquad \mathcal{L}^3(A(2i))\,\geq\,\mathcal{L}^3(A(2i+2))\,,\quad A(2i+1)=\varnothing\,.$$

For n in $\mathbb{N}$, let $\mathcal{A}_n$ be the collection of sets

$$\big\{\,A(i):0\leq i\leq n\,\big\}\,\cup\,\big\{\,\mathrm{cup}\,\{\,A(i):i>n\,\}\,\big\}\,\setminus\,\{\,\varnothing\,\}\,.$$

Each $\mathcal{A}_n$ is a Caccioppoli partition of $\mathbb{R}^3$ having finite cardinality. Moreover

$$D(\mathcal{A}_n,\mathcal{A})\,\leq\,\sum_{i>n}\mathcal{L}^3(A(i))\,,\qquad \mathcal{I}(\mathcal{A}_n)\,\leq\,\mathcal{I}(\mathcal{A})\,.$$

The sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is a sequence of finite Caccioppoli partitions of $\mathbb{R}^3$ converging towards $\mathcal{A}$ and such that $(\mathcal{I}(\mathcal{A}_n))_{n\in\mathbb{N}}$ converges towards $\mathcal{I}(\mathcal{A})$. It is therefore enough to prove the claim of the Theorem for $\mathcal{A}$ a finite Caccioppoli partition.
For r positive, let $\mathcal{A}_r$ be the bounded Caccioppoli partition defined by

$$\mathcal{A}_r\,=\,\big\{\,A\cap B(0,r):A\in\mathcal{A},\,A\neq A_\infty\,\big\}\,\cup\,\big\{\,A_\infty\cup(\mathbb{R}^3\setminus B(0,r))\,\big\}\,.$$

Clearly $\mathcal{A}_r$ converges to $\mathcal{A}$ for the metric D_{loc} as r goes to ∞. By Lemma 6.7, for $\mathcal{H}^1$ almost all positive r,

$$\mathcal{I}\big((\mathbb{R}^3 \setminus A_\infty) \cap B(0,r)\big) \;\leq\; \mathcal{I}(\mathbb{R}^3 \setminus A_\infty) + ||\tau||_\infty \mathcal{H}^2\big((\mathbb{R}^3 \setminus A_\infty) \cap \partial B(0,r)\big)$$

and for any A in $\mathcal{A} \setminus \{A_\infty\}$, for $\mathcal{H}^1$ almost all r,

$$\mathcal{I}(A \cap B(0,r)) \;\leq\; \mathcal{I}(A) + ||\tau||_\infty \mathcal{H}^2(A \cap \partial B(0,r))\,.$$

Moreover $\mathcal{I}\big((\mathbb{R}^3 \setminus A_\infty) \cap B(0,r)\big) = \mathcal{I}\big(A_\infty \cup (\mathbb{R}^3 \setminus B(0,r))\big)$. Summing all the previous inequalities, we obtain that, for $\mathcal{H}^1$ almost all r,

$$\mathcal{I}(\mathcal{A}_r) \;\leq\; \mathcal{I}(\mathcal{A}) \;+\; ||\tau||_\infty \mathcal{H}^2\big((\mathbb{R}^3 \setminus A_\infty) \cap \partial B(0,r)\big)\,.$$

Let ε be positive. Since $\mathcal{L}^3(\mathbb{R}^3 \setminus A_\infty)$ is finite, then the set $\big\{\, r > 0 : \mathcal{H}^2\big((\mathbb{R}^3 \setminus A_\infty) \cap \partial B(0,r)\big) \geq \varepsilon \,\big\}$ has finite $\mathcal{H}^1$ measure (less than $\mathcal{L}^3(\mathbb{R}^3 \setminus A_\infty)/\varepsilon$). Certainly there exist arbitrarily large values of r such that the above inequality holds and $\mathcal{H}^2\big((\mathbb{R}^3 \setminus A_\infty) \cap \partial B(0,r)\big) < \varepsilon$, whence $\mathcal{I}(\mathcal{A}_r) \leq \mathcal{I}(\mathcal{A}) + \varepsilon ||\tau||_\infty$. For r large enough, we have also $D_{\mathrm{loc}}(\mathcal{A}_r, \mathcal{A}) < \varepsilon$. Therefore we need only to prove the approximation result for bounded Caccioppoli partitions.

Let us consider now a finite bounded Caccioppoli partition $\mathcal{A}$ such that $\mathcal{I}(\mathcal{A})$ is finite. We write $\mathcal{A} = \{\, A_0, A_1, \dots, A_h \,\}$ where $A_1, \dots, A_h$ are bounded and the sets $A_0, \dots, A_h$ have finite perimeter. Let ε belong to $]0, 1/2[$. We apply the approximation result of Quentin de Gromard to each set $A_0, \dots, A_h$. For each i in $\{0, \dots, h\}$, there exists a set L_i of finite perimeter, a C^1 function $f_i : \mathbb{R}^3 \mapsto \mathbb{R}$, a compact set C_i, an open set V_i and an open bounded set B_i such that, setting $F_i = \{\, x \in \mathbb{R}^3 : f_i(x) \geq 0 \,\}$, the set $V_i \cap \partial F_i$ is the hypersurface $\{\, x \in V_i : f_i(x) = 0 \,\}$ and

$$\begin{gathered}
C_i \subset B_i \subset V_i \subset \{\, x \in \mathbb{R}^3 : df_i(x) \neq 0 \,\}\,, \quad C_i \subset \partial^* A_i \cap \partial F_i\,, \\
L_i \cap B_i = F_i \cap B_i\,, \quad V_i \cap \partial^* F_i = V_i \cap \partial F_i\,, \\
\forall\, x \in C_i \quad \nu_{A_i}(x) = \nu_{F_i}(x) = -|df_i(x)|^{-1} df_i(x)\,, \\
\mathcal{L}^3(V_i) < \varepsilon\,, \quad \mathcal{L}^3(A_i \Delta L_i) < \varepsilon\,, \\
\mathcal{H}^2(\partial F_i \cap (V_i \setminus C_i)) < \varepsilon\,, \quad \mathcal{H}^2(\partial^* A_i \setminus C_i) < \varepsilon\,, \quad \mathcal{H}^2(\partial^* A_i \Delta \partial L_i) < \varepsilon\,, \\
L_i \subset \mathcal{V}_2(A_i, \varepsilon)\,, \quad \mathbb{R}^3 \setminus L_i \subset \mathcal{V}_2(\mathbb{R}^3 \setminus A_i, \varepsilon)\,.
\end{gathered}$$

Since $\mathcal{A}$ is a Caccioppoli partition, then the sets $\partial^* A_i \cap \partial^* A_j$, $0 \leq i < j \leq h$, are pairwise disjoint, and so are the sets $C_i \cap C_j$, $0 \leq i < j \leq h$. It is possible to choose the open sets V_i, $0 \leq i \leq h$, in such a way that the sets $V_i \cap V_j$, $0 \leq i < j \leq h$, are also disjoint. At the beginning of the proof of Theorem 11.6, the compact sets C_i, $0 \leq i \leq h$, are chosen by applying Egoroff Theorem; using the exterior regularity of the measures $\mathcal{H}^2|_{\partial^* A_i}$, they are then approximated from outside by the sets V_i,

$0 \leq i \leq h$. We perform simultaneously this step for all the sets and we impose that each set V_i is close enough to C_i. If we set

$$\psi = (1/3) \min \left\{ d_2(C_i \cap C_j, C_k \cap C_l) : 0 \leq i, j, k, l \leq h,\, i < j,\, k < l,\, (i,j) \neq (k,l) \right\},$$

then ψ is positive (the sets $C_i \cap C_j$, $0 \leq i < j \leq h$, are disjoint and compact) and it is enough to require that V_i is included in $\mathcal{V}_2(C_i, \psi)$ for each i in $\{0, \ldots, h\}$.
For any i in $\{0, \ldots, h\}$,

$$\mathcal{H}^2(\partial L_i \setminus C_i) \;\leq\; \mathcal{H}^2(\partial L_i \setminus \partial^* A_i) + \mathcal{H}^2(\partial^* A_i \setminus C_i) \;\leq\; 2\varepsilon\,.$$

Therefore, if we set $H = \operatorname{cup}\{\, \partial L_i \setminus B_i : 0 \leq i \leq h \,\}$, then $\mathcal{H}^2(H) \leq 2(h+1)\varepsilon$.
Setting $C = \operatorname{cup}\{\, C_i \cap C_j : 0 \leq i < j \leq h \,\}$,

$$\begin{aligned}
\mathcal{H}^2(\partial L_i \setminus C) \;&\leq\; \mathcal{H}^2(\partial L_i \setminus C_i) \;+\; \mathcal{H}^2\big(C_i \setminus \operatorname{cup}\{\, C_j : 0 \leq j \leq h, j \neq i \,\}\big) \\
&\leq\; 2\varepsilon + \mathcal{H}^2\big(C_i \setminus \operatorname{cup}\{\, \partial^* A_j : 0 \leq j \leq h, j \neq i \,\}\big) \\
&\qquad\qquad + \mathcal{H}^2\big(\operatorname{cup}\{\, \partial^* A_j \setminus C_j : 0 \leq j \leq h, j \neq i \,\}\big) \\
&\leq\; (h+2)\varepsilon\,.
\end{aligned}$$

Setting $G = \operatorname{cup}\{\, \partial L_i : 0 \leq i \leq h \,\}$, we get $\mathcal{H}^2(G \setminus C) \leq (h+1)(h+2)\varepsilon$. Moreover $\mathbb{R}^3 \setminus A_0, A_1, \ldots, A_h$ are bounded, and so are $\partial L_0, \ldots, \partial L_h$. It follows that G is compact. For each i in $\{0, \ldots, h\}$, we apply Lemma 6.10 to the set $\partial F_i \cap \operatorname{clo} B_i$ and the hypersurface $\partial F_i \cap V_i$ (since B_i is bounded, then $\partial F_i \cap \operatorname{clo} B_i$ is compact):

$$\begin{aligned}
&\exists\, M_i > 0 \quad \forall\, \delta > 0 \quad \exists\, \eta_i > 0 \quad \forall\, x, y \in \partial F_i \cap \operatorname{clo} B_i \\
&\qquad\qquad\qquad |x - y|_2 \leq \eta_i \quad \Rightarrow \quad d_2\big(y, \tan(\partial F_i \cap V_i, x)\big) \leq M_i \delta |x - y|_2\,.
\end{aligned}$$

For a point x belonging to C_i, the tangent plane of $\partial F_i \cap V_i$ at x is precisely $\operatorname{hyp}(x, \nu(\mathcal{A}, x))$. Let M be the maximum $\max\{\, 1, M_0, \ldots, M_h \,\}$ and let δ in $]0, 1/2[$ be such that $2\delta M < \varepsilon$. For i in $\{0, \ldots, h\}$, let η_i be a positive real number associated to δ as in the above formula and let $\eta = \min\{\, \eta_0, \ldots, \eta_h \,\}$. Let also

$$\rho = (1/6) \min\{\, d_2(C_i, \mathbb{R}^3 \setminus B_i) : 0 \leq i \leq h \,\}\,.$$

Since each set C_i is a compact subset of the open set B_i, then ρ is positive. For each A in $\mathcal{A}$, the map $x \in \partial^* A \mapsto \nu_A(x) \in S^2$ is measurable with respect to the measure $\mathcal{H}^2|_{\partial^* A}$, therefore the map $x \in \partial^* \mathcal{A} \mapsto \nu(\mathcal{A}, x) \in PS^2$ is measurable with respect to $\mathcal{H}^2|_{\partial^* \mathcal{A}}$. By Lemma 11.2, for $\mathcal{H}^2$ almost all x in $\partial^* \mathcal{A}$,

$$\lim_{r \to 0} \quad (\pi r^2)^{-1} \mathcal{H}^2(B(x,r) \cap \partial^* \mathcal{A}) \;=\; 1\,,$$

$$\lim_{r \to 0} \quad \frac{1}{\pi r^2} \int_{B(x,r) \cap \partial^* \mathcal{A}} \tau(\nu(\mathcal{A}, y))\, d\mathcal{H}^2(y) \;=\; \tau(\nu(\mathcal{A}, x))\,.$$

Let $\partial^{**}\mathcal{A}$ be the set of the points of $\partial^*\mathcal{A}$ where the two preceding identities hold simultaneously. Clearly $\mathcal{H}^2(\partial^*\mathcal{A} \setminus \partial^{**}\mathcal{A}) = 0$. For any x in $\partial^{**}\mathcal{A}$, there exists a positive $r(x,\varepsilon)$ such that, for any $r < r(x,\varepsilon)$,

$$|\mathcal{H}^2(B(x,r) \cap \partial^*\mathcal{A}) - \pi r^2| \leq \varepsilon\, \pi r^2 \,,$$
$$\Big|\frac{1}{\pi r^2}\int_{B(x,r)\cap\partial^*\mathcal{A}} \tau(\nu(\mathcal{A},y))\, d\mathcal{H}^2(y) - \tau(\nu(\mathcal{A},x))\Big| \leq \varepsilon \,.$$

The family of balls $B(x,r)$, $x \in \partial^{**}\mathcal{A}$, $r < \min(r(x,\varepsilon),\varepsilon,\eta,\rho)$, is a Vitali relation for C. By the standard Vitali covering Theorem [34, Theorem 1.10, or chapter 4], we may select a finite or countable collection of disjoint balls $B(x_i,r_i)$, $i \in I$, such that: for any i in I, x_i belongs to C, $r_i < \min(r(x_i,\varepsilon),\varepsilon,\eta,\rho)$ and

$$\text{either} \qquad \mathcal{H}^2\Big(C \setminus \bigcup_{i\in I} B(x_i,r_i)\Big) = 0 \qquad \text{or} \qquad \sum_{i\in I} r_i^2 = \infty \,.$$

Because for each i in I, r_i is smaller than $r(x_i,\varepsilon)$,

$$\pi(1-\varepsilon)\sum_{i\in I} r_i^2 \leq \mathcal{H}^2(\partial^*\mathcal{A}) < \infty$$

and therefore the first case occurs, so that we may select a finite subset I_0 of I such that

$$\mathcal{H}^2\Big(C \setminus \bigcup_{i\in I_0} B(x_i,r_i)\Big) < \varepsilon \,.$$

We have a finite number of disjoint closed balls $B(x_i,r_i)$, $i \in I_0$. By increasing slightly all the radii r_i, we can keep the balls disjoint, each r_i strictly smaller than $\min(r(x_i,\varepsilon),\varepsilon,\eta,\rho)$ for i in I_0, and get the stronger inequality

$$\mathcal{H}^2\Big(C \setminus \bigcup_{i\in I_0} \operatorname{int} B(x_i,r_i)\Big) < \varepsilon \,.$$

We deduce from the preceding inequalities that

$$\begin{aligned}\mathcal{H}^2\Big(\big(G \setminus \bigcup_{i\in I_0} \operatorname{int} B(x_i,r_i)\big) \cup H\Big) &\leq \mathcal{H}^2(G \setminus C) + \varepsilon + \mathcal{H}^2(H)\\ &\leq (h+1)(h+2)\varepsilon + \varepsilon + 2(h+1)\varepsilon \,.\end{aligned}$$

By the very definition of the Hausdorff measure $\mathcal{H}^2$, there exists a collection of balls $B(y_j,s_j)$, $j \in J$, such that: for any j in J, $y_j \in \big(G \setminus \bigcup_{i\in I_0} \operatorname{int} B(x_i,r_i)\big) \cup H$, $0 < s_j < \varepsilon$,

$$\sum_{j\in J} \pi s_j^2 \leq (h+1)(h+2)\varepsilon + \varepsilon + 2(h+1)\varepsilon + \varepsilon = (h+2)(h+3)\varepsilon \,,$$

$$\Big(G \setminus \bigcup_{i\in I_0} \operatorname{int} B(x_i,r_i)\Big) \cup H \subset \bigcup_{j\in J} \operatorname{int} B(y_j,s_j) \,.$$

Since the set appearing on the left–hand side of the above inclusion is compact, we might assume in addition that J is finite. For each i in I_0, let P_i be a convex open polygon inside the plane $\operatorname{hyp}(x_i, \nu(\mathcal{A}, x_i))$ such that

$$\operatorname{disc}(x_i, r_i, \nu(\mathcal{A}, x_i)) \subset P_i \subset \operatorname{disc}(x_i, r_i(1+\delta), \nu(\mathcal{A}, x_i)),$$
$$|\mathcal{H}^1(\partial P_i) - 2\pi r_i| \leq 2\delta\pi r_i, \quad |\mathcal{H}^2(P_i) - \pi r_i^2| \leq \delta\pi r_i^2.$$

We set $\alpha = M\delta(1+\delta)$ (thus $\alpha < 1$). For i in I_0, let D_i be the cylinder $\operatorname{cyl}(P_i, \alpha r_i)$. The sets $\operatorname{cup}\{ D_i : i \in I_0,\, x_i \in C_k \cap C_l \}$, $0 \leq k < l \leq h$, are pairwise disjoint. Indeed, let i be an index in I_0 such that x_i is in $C_k \cap C_l$; because $r_i < \rho$,

$$D_i \subset B(x_i, 3\rho) \subset B_k \cap B_l \subset V_k \cap V_l$$

and the sets $V_k \cap V_l$, $0 \leq k < l \leq h$, are disjoint.
For each j in J, let Q_j be an open polyhedral set such that

$$B(y_j, s_j) \subset Q_j \subset B(y_j, 2s_j), \qquad \mathcal{H}^2(\partial Q_j) \leq 8\pi s_j^2.$$

We set for l in $\{0, \dots, h\}$

$$T_l = L_l \cup \bigcup_{0\leq k<l} \operatorname{cup}\{ D_i : i \in I_0,\, x_i \in C_k \cap C_l \} \setminus \bigcup_{l<k\leq h} \operatorname{cup}\{ D_i : i \in I_0,\, x_i \in C_k \cap C_l \} \setminus \bigcup_{j\in J} Q_j$$

and $T_{h+1} = \mathbb{R}^3 \setminus \operatorname{cup}\{ T_l : 0 \leq l \leq h \}$.
We claim that the collection $\mathcal{T} = \{ T_0, \dots, T_{h+1} \}$ almost answers the problem. First $\operatorname{cup}\mathcal{T} = \mathbb{R}^3$. Secondly for each l in $\{ 0, \dots, h \}$,

$$T_l \Delta A_l \subset (A_l \Delta L_l) \cup \bigcup_{i\in I_0} D_i \cup \bigcup_{j\in J} Q_j$$

whence

$$\mathcal{L}^3(T_l \Delta A_l) \leq \varepsilon + \sum_{i\in I_0} 2\pi r_i^2 (1+\delta)\alpha r_i + \sum_{j\in J} (4/3)\pi (2s_j)^3.$$

Yet each r_i is smaller than ε, $\sum_{i\in I_0} \pi r_i^2 \leq 2\mathcal{H}^2(\partial^*\mathcal{A})$, $\sum_{j\in J} \pi s_j^2 \leq (h+2)(h+3)\varepsilon$, whence

$$\mathcal{L}^3(T_l \Delta A_l) \leq \varepsilon + 6\varepsilon\mathcal{H}^2(\partial^*\mathcal{A}) + 11(h+2)(h+3)\varepsilon.$$

Moreover T_{h+1} is included in $\operatorname{cup}\{ Q_j : j \in J \} \cup \operatorname{cup}\{ A_l \setminus L_l : 0 \leq l \leq h \}$ whence

$$\mathcal{L}^3(T_{h+1}) \leq \sum_{0\leq l\leq h} \mathcal{L}^3(A_l \Delta L_l) + \sum_{j\in J} (4/3)\pi(2s_j)^3 \leq (h+1)\varepsilon + 11(h+2)(h+3)\varepsilon.$$

It follows that

$$D(\mathcal{A},\mathcal{T}) \;\le\; (h+2)\varepsilon + 6\varepsilon\mathcal{H}^2(\partial^*\mathcal{A}) + 11(h+2)(h+3)\varepsilon$$

and the collection $\mathcal{T}$ approximates the collection $\mathcal{A}$ with respect to the metric D. We show next that $\mathcal{T}$ is polyhedral. Indeed, for any $k<l$ in $\{0,\dots,h\}$, any i in I_0 such that x_i belongs to $C_k\cap C_l$, r_i is smaller than ρ and η, so that $B(x_i,r_i)\subset B_k\cap B_l$,

$$\begin{gathered}\partial L_k\cap B(x_i,r_i)=\partial F_k\cap B(x_i,r_i)\,,\quad \partial L_l\cap B(x_i,r_i)=\partial F_l\cap B(x_i,r_i)\,,\\ \forall\, x\in(\partial F_k\cup\partial F_l)\cap B(x_i,r_i)\quad d_2\big(x,\mathrm{hyp}\,(x_i,\nu(\mathcal{A},x_i))\big)\le M\delta|x-x_i|_2\,,\end{gathered}$$

whence

$$(\partial L_k\cup\partial L_l)\cap B(x_i,r_i)\;\subset\;\mathrm{cyl}\,\big(\mathrm{disc}\,(x_i,r_i,\nu(\mathcal{A},x_i)),M\delta r_i\big)\;\subset\;\mathrm{int}\,D_i\,.$$

Next, for m distinct from k,l, since $B(x_i,r_i)\subset B_k\cap B_l$ and $B_k\cap B_l\cap B_m=\varnothing$, then $\partial L_m\cap B(x_i,r_i)\subset\partial L_m\setminus B_m\subset H$. Thus

$$G\cap B(x_i,r_i)\;\subset\;\big((\partial L_k\cup\partial L_l)\cap B(x_i,r_i)\big)\cup H\;\subset\;\mathrm{int}\,D_i\cup H\,.$$

The sets Q_j, $j\in J$, cover $\big(G\setminus\bigcup_{i\in I_0}\mathrm{int}\,B(x_i,r_i)\big)\cup H$, hence the sets $\mathrm{int}\,D_i$, $i\in I_0$, Q_j, $j\in J$, cover G. The definition of the sets T_m, $0\le m\le h+1$, implies that

$$\bigcup_{0\le m\le h+1}\partial T_m\;\subset\;\Big(G\setminus\Big(\bigcup_{i\in I_0}\mathrm{int}\,D_i\cup\bigcup_{j\in J}Q_j\Big)\Big)\cup\bigcup_{i\in I_0}\partial D_i\cup\bigcup_{j\in J}\partial Q_j\;=\;\bigcup_{i\in I_0}\partial D_i\cup\bigcup_{j\in J}\partial Q_j$$

and the collection $\mathcal{T}$ is polyhedral. We next refine the above inclusion in order to estimate the surface energy of $\mathcal{T}$. Let us again consider $k<l$ in $\{0,\dots,h\}$ and i in I_0 such that x_i belongs to $C_k\cap C_l$. Let $\beta=\sqrt{1-\alpha^2}$. We set

$$G_i\;=\;\mathrm{disc}\,\big(x_i+\alpha r_i\nu_{L_k}(x_i),\beta r_i,\nu(\mathcal{A},x_i)\big)\;=\;\mathrm{disc}\,\big(x_i-\alpha r_i\nu_{L_l}(x_i),\beta r_i,\nu(\mathcal{A},x_i)\big)\,.$$

We claim that G_i is included in the interior of L_l and in the interior of $\mathbb{R}^3\setminus L_k$. Indeed, G_i is included in $B(x_i,r_i)\cap\partial D_i$ and therefore G_i does not intersect $\partial L_k\cup\partial L_l$. Since $\nu_{L_k}(x_i)=\nu_{A_k}(x_i)=-\nu_{L_l}(x_i)$ is the exterior normal vector to L_k at x_i and the interior normal vector to L_l at x_i then G_i is included in $\mathrm{int}\,L_l\cap\mathrm{int}\,(\mathbb{R}^3\setminus L_k)$. The sets

$$\mathrm{cup}\,\{\,D_i : i\in I_0,\,x_i\in C_{k'}\cap C_{l'}\,\}\,,\quad 0\le k'<l'\le h$$

being closed and disjoint, looking at the definition of T_k and T_l, we see that for a sufficiently small neighbourhood W_i of G_i

$$W_i\cap T_l=W_i\cap L_l\setminus\bigcup_{j\in J}Q_j\,,\qquad W_i\cap T_k=W_i\cap L_k=\varnothing\,.$$

It follows that

$$\partial T_l \cap G_i \subset \bigcup_{j\in J} \partial Q_j\,, \qquad \partial T_k \cap G_i = \varnothing\,.$$

Therefore

$$\partial^* \mathcal{T} \subset \Big(\bigcup_{0\le m\le h} \partial T_m \setminus \bigcup_{i\in I_0} G_i \Big) \cup \Big(\bigcup_{0\le m\le h} \partial T_m \cap \bigcup_{i\in I_0} G_i \Big) \subset \bigcup_{i\in I_0} (\partial D_i \setminus G_i) \cup \bigcup_{j\in J} \partial Q_j\,.$$

Finally

$$\begin{aligned}
\mathcal{I}(\partial^*\mathcal{T}) \;&\le\; \sum_{i\in I_0} \mathcal{I}(\partial D_i \setminus G_i) + \sum_{j\in J} \mathcal{I}(\partial Q_j) \\
&\le \sum_{i\in I_0} \pi r_i^2 (1+\delta)\tau(\nu(\mathcal{A},x_i)) + ||\tau||_\infty \Big(\sum_{i\in I_0} \pi r_i^2 \big(4(1+\delta)\alpha + 1 + \delta - \beta^2\big) + \sum_{j\in J} 8\pi s_j^2 \Big) \\
&\le\; (1+\delta)\mathcal{I}(\mathcal{A}) + 4\varepsilon \mathcal{H}^2(\partial^*\mathcal{A}) + ||\tau||_\infty \Big(2\mathcal{H}^2(\partial^*\mathcal{A})\big(11M\delta + \delta\big) + 8(h+2)(h+3)\varepsilon \Big) \\
&\le\; \mathcal{I}(\mathcal{A}) + \varepsilon \Big(\mathcal{I}(\mathcal{A}) + \mathcal{I}(\mathcal{A})\big(4 + 14||\tau||_\infty\big)/\tau_{\min} + 8||\tau||_\infty (h+2)(h+3) \Big)\,,
\end{aligned}$$

where we have used the condition $2\delta \le 2M\delta < \varepsilon$ in the last step. The only remaining problem is that the collection $\mathcal{T}$ does not necessarily satisfy $\mathcal{L}^3(\mathrm{overlap}\,\mathcal{T}) = 0$. This issue is solved through the next Lemma.

Lemma 11.9. *Let $\mathcal{T}$ be a finite collection of bounded polyhedral subsets of $\mathbb{R}^3$. There exists a finite collection $\mathcal{U}$ of bounded polyhedral subsets of $\mathbb{R}^3$ such that*

$$\begin{gathered}
\mathrm{cup}\,\mathcal{U} = \mathrm{cup}\,\mathcal{T}, \quad \mathcal{I}(\mathcal{U}) \le \mathcal{I}(\mathcal{T}), \quad \mathcal{L}^3(\mathrm{overlap}\,\mathcal{U}) = 0, \\
D(\mathcal{U},\mathcal{T}) \;\le\; (\mathrm{card}\,\mathcal{T} - 1)\,\mathrm{card}\,\mathcal{T}\,\mathcal{L}^3(\mathrm{overlap}\,\mathcal{T})\,.
\end{gathered}$$

Proof of Theorem 11.8 **continued.** We apply Lemma 11.9 to the collection $\mathcal{T}$. Because of the choice of T_{h+1}, we have $\mathrm{cup}\,\mathcal{T} = \mathbb{R}^3$. Moreover

$$\begin{aligned}
\mathcal{L}^3(\mathrm{overlap}\,\mathcal{T}) \;&\le\; \sum_{0\le k<l\le h} \mathcal{L}^3(T_k \cap T_l) \\
&\le \sum_{0\le k<l\le h} \mathcal{L}^3(T_k \Delta A_k) + \mathcal{L}^3(T_l \Delta A_l) \;\le\; h(h+1) D(\mathcal{T},\mathcal{A})\,.
\end{aligned}$$

Therefore the resulting collection $\mathcal{U}$ is a bounded finite polyhedral Caccioppoli partition satisfying

$$\begin{aligned}
D(\mathcal{U},\mathcal{A}) \;&\le\; \varepsilon\Big((h+2) + 6\mathcal{I}(\mathcal{A})/\tau_{\min} + 11(h+2)(h+3) \Big) \Big(1 + h(h+1)^2(h+2) \Big)\,, \\
\mathcal{I}(\mathcal{U}) \;&\le\; \mathcal{I}(\mathcal{A}) + \varepsilon\Big(\mathcal{I}(\mathcal{A}) + \mathcal{I}(\mathcal{A})\big(4 + 14||\tau||_\infty\big)/\tau_{\min} + 8||\tau||_\infty (h+2)(h+3) \Big)\,.
\end{aligned}$$

Since $\mathcal{I}(\mathcal{A})$ is finite and h is fixed, we have the required approximation. $\square$

Proof of Lemma 11.9. We use an algorithm to build the collection $\mathcal{U}$ starting from $\mathcal{T}$. Since we work up to Lebesgue negligible sets, we might assume that each polyhedral set has positive density at each of its points. In particular, for any element T of $\mathcal{T}$, we have then $\mathcal{H}^2(\partial T \setminus \partial^* T) = 0$. We initialize the algorithm with the collection $\mathcal{T}^0 = \mathcal{T}$. We describe next the h–step of the algorithm. Suppose that we have built the collection $\mathcal{T}^h$ for some h in $\mathbb{N}$. If $\mathcal{L}^3(\operatorname{overlap} \mathcal{T}^h) = 0$, the algorithm stops. Otherwise, let T_1, T_2 be two sets of the collection $\mathcal{T}^h$ such that $\mathcal{L}^3(T_1 \cap T_2) > 0$. Let $T_1' = T_1 \setminus T_2$ and $T_2' = T_2 \setminus T_1$. Since $\mathcal{L}^3(T_1 \cap T_2) \leq \mathcal{L}^3(\operatorname{overlap} \mathcal{T}^h)$, then $\mathcal{L}^3(T_1' \Delta T_1) \leq \mathcal{L}^3(\operatorname{overlap} \mathcal{T}^h)$ and $\mathcal{L}^3(T_2' \Delta T_2) \leq \mathcal{L}^3(\operatorname{overlap} \mathcal{T}^h)$. Moreover

$$\nu_{T_1 \setminus T_2}(x) = \begin{cases} \nu_{T_1}(x) & \text{if } x \in \partial^* T_1 \setminus \operatorname{clo} T_2 \\ -\nu_{T_2}(x) & \text{if } x \in \partial^* T_2 \cap \operatorname{int} T_1 \\ \nu_{T_1}(x) & \text{if } x \in \partial^* T_2 \cap \partial^* T_1 \text{ and } \nu_{T_1}(x) + \nu_{T_2}(x) = 0 \end{cases}$$

This result is quite direct here because we deal with polyhedral sets. The remaining set where $\nu_{T_1 \setminus T_2}(x)$ might be non–zero is included in $(\partial T_1 \setminus \partial^* T_1) \cup (\partial T_2 \setminus \partial^* T_2)$, which has zero $\mathcal{H}^2$ measure. See [74] for a more general result. Using the symmetry and the positivity of τ,

$$\mathcal{I}(T_1') \leq \int_{\partial^* T_1 \setminus \operatorname{clo} T_2} \tau(\nu_{T_1}(x))\, d\mathcal{H}^2(x) + \int_{\partial^* T_2 \cap \operatorname{int} T_1} \tau(\nu_{T_2}(x))\, d\mathcal{H}^2(x) + \int_{\partial^* T_1 \cap \partial^* T_2} \tau(\nu_{T_1}(x))\, d\mathcal{H}^2(x)\,,$$
$$\mathcal{I}(T_2') \leq \int_{\partial^* T_2 \setminus \operatorname{clo} T_1} \tau(\nu_{T_2}(x))\, d\mathcal{H}^2(x) + \int_{\partial^* T_1 \cap \operatorname{int} T_2} \tau(\nu_{T_1}(x))\, d\mathcal{H}^2(x) + \int_{\partial^* T_2 \cap \partial^* T_1} \tau(\nu_{T_2}(x))\, d\mathcal{H}^2(x)\,.$$

Summing the two inequalities yields $\mathcal{I}(T_1') + \mathcal{I}(T_2') = \mathcal{I}(T_1) + \mathcal{I}(T_2)$. Two cases can occur.

- If $\mathcal{I}(T_1') \leq \mathcal{I}(T_1)$, then we set $\mathcal{T}^{h+1} = \{T_1'\} \cup \mathcal{T}^h \setminus \{T_1\}$.
- If $\mathcal{I}(T_2') \leq \mathcal{I}(T_2)$, then we set $\mathcal{T}^{h+1} = \{T_2'\} \cup \mathcal{T}^h \setminus \{T_2\}$.

The collection $\mathcal{T}^{h+1}$ satisfies

$$\mathcal{I}(\mathcal{T}^{h+1}) \leq \mathcal{I}(\mathcal{T}^h), \quad \operatorname{cup} \mathcal{T}^{h+1} = \operatorname{cup} \mathcal{T}^h, \quad \operatorname{overlap} \mathcal{T}^{h+1} \subset \operatorname{overlap} \mathcal{T}^h,$$
$$D(\mathcal{T}^{h+1}, \mathcal{T}) \leq D(\mathcal{T}^{h+1}, \mathcal{T}^h) + D(\mathcal{T}^h, \mathcal{T}) \leq \mathcal{L}^3(\operatorname{overlap} \mathcal{T}^h) + D(\mathcal{T}^h, \mathcal{T})\,.$$

Necessarily the algorithm stops at some step h less than $(\operatorname{card} \mathcal{T} - 1) \operatorname{card} \mathcal{T}$. Let $\mathcal{U}$ be the final collection obtained at the end of the algorithm. Then $\mathcal{U}$ is a finite collection of bounded polyhedral subsets of $\mathbb{R}^3$ satisfying $\operatorname{cup} \mathcal{U} = \operatorname{cup} \mathcal{T}$, $\mathcal{I}(\mathcal{U}) \leq \mathcal{I}(\mathcal{T})$ and

$$D(\mathcal{U}, \mathcal{T}) \leq (\operatorname{card} \mathcal{T} - 1) \operatorname{card} \mathcal{T}\, \mathcal{L}^3(\operatorname{overlap} \mathcal{T})\,. \quad \square$$

11.3. The Wulff Theorem for Caccioppoli partitions

Let Λ be a cubic box in $\mathbb{R}^3$ such that $\mathcal{L}^3(\Lambda)=1$. Let λ in $]0,1[$ be such that some translate of the set $(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau$ is included in the interior of Λ. Let us define

$$\mathbb{E}(\lambda) \;=\; \Big\{\, \mathcal{A}\in \mathrm{BC}(\mathbb{R}^3) : \forall\, A\in\mathcal{A} \quad \text{either } \mathcal{L}^3(A)<\infty \text{ or } \mathcal{L}^3\big(\Lambda\cap A\big)\leq 1-\lambda \,\Big\}\,.$$

Notice that $\mathbb{E}(\lambda)$ is closed in $\mathrm{BC}(\mathbb{R}^3)$ for the metric D. We consider the following problem:

$$(\mathbb{P}) \qquad \text{minimize } \mathcal{I}(\mathcal{A}) \text{ among all Borel collections } \mathcal{A} \text{ in } \mathbb{E}(\lambda)\,.$$

Theorem 11.10. *The solutions to the problem $(\mathbb{P})$ are the collections*

$$\mathcal{A}_{\mathcal{W}_\tau}(x,\lambda) \;=\; \Big\{\, x+(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau,\, \mathbb{R}^3\setminus\big(x+(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau\big)\,\Big\}\,,\qquad x\in\Lambda(\lambda)\,,$$

where $\Lambda(\lambda) \;=\; \big\{\, x\in\Lambda : x+(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau\subset \operatorname{clo}\Lambda\,\big\}$.

Proof. We first compute the value of the infimum of the problem $(\mathbb{P})$. Let $\mathcal{A}$ be an element of $\mathbb{E}(\lambda)$ such that $\mathcal{I}(\mathcal{A})$ is finite. In particular, $\mathcal{A}$ is a Caccioppoli partition of $\mathbb{R}^3$ containing exactly one set A_∞ such that $\mathcal{L}^3(A_\infty)=\infty$ (by Lemma 11.3). Since $\mathcal{A}$ belongs to $\mathbb{E}(\lambda)$, then $\mathcal{L}^3(A_\infty\cap\Lambda)\leq 1-\lambda$ and $\mathcal{L}^3(\mathbb{R}^3\setminus A_\infty)\geq\lambda$. By the Wulff Isoperimetric Theorem 6.11,

$$\mathcal{I}(\mathcal{A}) \;\geq\; \mathcal{I}(\mathbb{R}^3\setminus A_\infty) \;\geq\; (\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}\mathcal{I}(\mathcal{W}_\tau)\,.$$

Therefore the infimum of the problem $(\mathbb{P})$ is larger than the above value. This value is realized by the elements $\mathcal{A}_{\mathcal{W}_\tau}(x,\lambda)$, $x\in\Lambda(\lambda)$. Therefore these elements are solutions to the problem $(\mathbb{P})$. It remains to prove that they are the only solutions. Let $\mathcal{A}$ be a solution to the problem $(\mathbb{P})$. Then $\mathcal{I}(\mathcal{A})$ is equal to $(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}\mathcal{I}(\mathcal{W}_\tau)$ and $\mathcal{A}$ is a Caccioppoli partition of $\mathbb{R}^3$ containing exactly one set A_∞ such that $\mathcal{L}^3(A_\infty)=\infty$. Since $\mathcal{A}$ belongs to $\mathbb{E}(\lambda)$, then $\mathcal{L}^3(A_\infty\cap\Lambda)\leq 1-\lambda$ and $\mathcal{L}^3(\mathbb{R}^3\setminus A_\infty)\geq\lambda$. However

$$(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}\mathcal{I}(\mathcal{W}_\tau) \;\geq\; \mathcal{I}(\mathcal{A}) \;\geq\; \mathcal{I}(A_\infty) \;=\; \mathcal{I}(\mathbb{R}^3\setminus A_\infty)\,.$$

By the uniqueness statement of the Wulff Isoperimetric Theorem 6.12, up to a negligible set, the set $\mathbb{R}^3\setminus A_\infty$ is a translate of the suitable dilation of the Wulff crystal, that is, there exists x in $\mathbb{R}^3$ such that

$$\mathcal{L}^3\big((\mathbb{R}^3\setminus A_\infty)\,\Delta\,(x+(\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau)\big) \;=\; 0\,.$$

Thus $\mathcal{L}^3(\mathbb{R}^3\setminus A_\infty)=\lambda$. Since $\mathcal{L}^3\big(\Lambda\cap(\mathbb{R}^3\setminus A_\infty)\big)\geq\lambda$, then $\mathbb{R}^3\setminus A_\infty\setminus\Lambda$ is negligible and x belongs to $\Lambda(\lambda)$. Let A be a set in $\mathcal{A}$ distinct from A_∞ (by definition of a collection, $\mathcal{L}^3(A)\neq 0$). Necessarily A is included in $\mathbb{R}^3\setminus A_\infty$ and

$$\mathcal{I}(\mathcal{A}) \;\geq\; \mathcal{I}(A_\infty)+\int_{\partial^*A\cap\operatorname{int}\mathbb{R}^3\setminus A_\infty}\tau(\nu_A(x))\,d\mathcal{H}^2(x) \;\geq\; \mathcal{I}(A_\infty)+\tau_{\min}\mathcal{P}(A,\operatorname{int}\mathbb{R}^3\setminus A_\infty)\,.$$

Thus $\mathcal{P}(A, \operatorname{int} \mathbb{R}^3 \setminus A_\infty) = 0$. By the isoperimetric inequality relative to the balls, for each ball B included in $\mathbb{R}^3 \setminus A_\infty$, either $A \setminus B$ or $B \setminus A$ is negligible. Since $\mathbb{R}^3 \setminus A_\infty = x + (\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau$ is convex and A is not negligible, necessarily $\mathcal{L}^3(A\Delta(\mathbb{R}^3 \setminus A_\infty)) = 0$. In conclusion, up to negligible sets, the collection $\mathcal{A}$ consists of two sets, the translated and dilated Wulff crystal $x + (\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau$ and its complement, that is, $\mathcal{A} = \mathcal{A}_{\mathcal{W}_\tau}(x, \lambda)$ where x belongs to $\Lambda(\lambda)$. $\square$

Proposition 11.11. *If $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is a minimizing sequence of the variational problem $(\mathbb{P})$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\Lambda(\lambda)$ such that $D\big(\mathcal{A}_n, \mathcal{A}_{\mathcal{W}_\tau}(x_n, \lambda)\big)$ converges to 0 as n goes to ∞.*

Proof. Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a minimizing sequence of the variational problem $(\mathbb{P})$. We might suppose that the sequence $(\mathcal{I}(\mathcal{A}_n))_{n\in\mathbb{N}}$ is bounded. Then each collection $\mathcal{A}_n$ is a Caccioppoli partition of $\mathbb{R}^3$ and contains exactly one set $A_{n,\infty}$ having infinite volume (by Lemma 11.3). The sequence $(A_{n,\infty})_{n\in\mathbb{N}}$ is such that

- $\forall n \in \mathbb{N} \qquad \mathcal{L}^3(\Lambda \cap A_{n,\infty}) \leq 1 - \lambda\,.$
- $\limsup_{n\to\infty} \mathcal{I}(A_{n,\infty}) \leq (\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{2/3}\mathcal{I}(\mathcal{W}_\tau)\,.$

Therefore the sequence

$$\big((\mathcal{L}^3(\mathcal{W}_\tau)/\mathcal{L}^3(\mathbb{R}^3 \setminus A_{n,\infty}))^{1/3}(\mathbb{R}^3 \setminus A_{n,\infty})\big)_{n\in\mathbb{N}}$$

is a minimizing sequence for the problem (P) defined at the end of chapter 6. By Proposition 6.13, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\begin{aligned}\mathcal{L}^3\big((\mathbb{R}^3 \setminus A_{n,\infty})\Delta(x_n + (\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau)\big) \;=&\\ \mathcal{L}^3\big(A_{n,\infty}\Delta(\mathbb{R}^3 \setminus (x_n + (\lambda/\mathcal{L}^3(\mathcal{W}_\tau))^{1/3}\mathcal{W}_\tau))\big)&\end{aligned}$$

converges to 0 as n goes to ∞. Necessarily the distance between x_n and $\Lambda(\lambda)$ goes to 0 and $\mathcal{L}^3(\mathbb{R}^3 \setminus \Lambda \setminus A_{n,\infty})$ goes to 0 as well. Let $A_{n,1}$ be a set of maximal volume in $\mathcal{A}_n \setminus \{A_{n,\infty}\}$, and let $A_{n,2}$ be the union of all the remaining sets of $\mathcal{A}_n$, that is,

$$A_{n,2} \;=\; \operatorname{cup}\big(\mathcal{A}_n \setminus \{A_{n,\infty},\, A_{n,1}\}\big)\,.$$

Since $\mathcal{A}_n$ is a partition, then $\mathcal{L}^3(A_{n,i} \setminus \Lambda)$, $i = 1, 2$, go to 0.
We claim first that $\liminf_{n\to\infty} \mathcal{L}^3(A_{n,1})$ is positive. By absurd, suppose that it is not the case. Then we can extract a subsequence (not relabeled) such that

$$\lim_{n\to\infty} \sup_{A\in\mathcal{A}_n\setminus\{A_{n,\infty}\}} \mathcal{L}^3(A \cap \Lambda) \;=\; 0\,.$$

By Theorem 10.8, we can reextract a subsequence converging with respect to the metric D_{loc} towards a Caccioppoli partition $\mathcal{A}$ having a finite surface energy. Let A_∞ be the unique element of $\mathcal{A}$ having infinite volume. Then we have simultaneously $\mathcal{L}^3(A_\infty \cap \Lambda) \leq 1 - \lambda$ (because A_∞ is the limit of a subsequence of $(A_{n,\infty})_{n\in\mathbb{N}}$) and

for any A in $\mathcal{A} \setminus \{A_\infty\}$, $\mathcal{L}^3(A \cap \Lambda) = 0$, which is impossible.
We claim next that $\mathcal{L}^3(A_{n,2})$ converges to 0. By absurd, suppose that it is not the case. Then we can extract a subsequence (not relabeled) such that

$$\liminf_{n \to \infty} \mathcal{L}^3(A_{n,2} \cap \Lambda) \; > \; 0 \,.$$

Let us consider the sequences $(A_{n,1} \cap \Lambda)_{n \in \mathbb{N}}$, $(A_{n,2} \cap \Lambda)_{n \in \mathbb{N}}$. These are subsequences of Borel subsets of the bounded set Λ, moreover their perimeters are uniformly bounded:

$$\mathcal{P}(A_{n,i} \cap \Lambda) \; \leq \; \mathcal{P}(A_{n,i}, \operatorname{int} \Lambda) + \mathcal{P}(\Lambda) \; \leq \; \mathcal{P}(\mathcal{A}_n) + \mathcal{P}(\Lambda) \,, \qquad i = 1, 2 \,.$$

By Theorem 6.1, we can reextract converging subsequences (not relabeled): there exist two Borel subsets E_1, E_2 of Λ such that

$$\lim_{n \to \infty} \mathcal{L}^3\big((A_{n,i} \cap \Lambda) \Delta E_i\big) = 0 \,, \quad i = 1, 2 \,.$$

Let $\mathcal{E}$ be the collection $\{\, E_1, E_2, \mathbb{R}^3 \setminus (E_1 \cup E_2) \,\}$. Then $\mathcal{E}$ is a Caccioppoli partition having finite perimeter and

$$\lim_{n \to \infty} \mathcal{L}^3\big(A_{n,i} \Delta E_i\big) = 0 \,, \quad i = 1, 2 \,,$$
$$\lim_{n \to \infty} \mathcal{L}^3\big(A_{n,\infty} \Delta (\mathbb{R}^3 \setminus (E_1 \cup E_2))\big) = 0$$

whence $\mathcal{L}^3\big(\Lambda \cap \mathbb{R}^3 \setminus (E_1 \cup E_2)\big) \leq 1 - \lambda$. By the lower semicontinuity of $\mathcal{I}$, we have

$$\mathcal{I}(\mathcal{E}) \; \leq \; \liminf_{n \to \infty} \frac{1}{2} \big(\mathcal{I}(A_{n,\infty}) + \mathcal{I}(A_{n,1}) + \mathcal{I}(A_{n,2})\big) \; \leq \; \liminf_{n \to \infty} \mathcal{I}(\mathcal{A}_n) \,.$$

Therefore $\mathcal{E}$ is a solution to the problem $(\mathbb{P})$. However neither E_1 nor E_2 is negligible, which contradicts Theorem 11.10. In conclusion, we have $\lim_{n \to \infty} \mathcal{L}^3(A_{n,2}) = 0$ and

$$\lim_{n \to \infty} \mathcal{L}^3\big(A_{n,1} \Delta (x_n + (\lambda / \mathcal{L}^3(\mathcal{W}_\tau))^{1/3} \mathcal{W}_\tau)\big) \; = \; 0$$

so that $\lim_{n \to \infty} D\big(\mathcal{A}_n, \mathcal{A}_{\mathcal{W}_\tau}(x_n, \lambda)\big) \; = \; 0$. $\square$

Corollary 11.12. *For any positive δ, there exist positive ε, η such that*

$$\inf \Big\{ \mathcal{I}(\mathcal{A}) : \mathcal{A} \in \mathbb{E}(\lambda - \varepsilon), \; \inf_{x \in \Lambda(\lambda)} D\big(\mathcal{A}, \mathcal{A}_{\mathcal{W}_\tau}(x, \lambda)\big) \geq \delta \Big\} \geq (\lambda / \mathcal{L}^3(\mathcal{W}_\tau))^{2/3} \mathcal{I}(\mathcal{W}_\tau) + \eta \,.$$

Proof. If the result was false, there would exist a positive δ and a sequence $(\mathcal{A}_n)_{n \in \mathbb{N}}$ such that for all n in $\mathbb{N}$

$$\mathcal{I}(\mathcal{A}_n) \; \leq \; (\lambda / \mathcal{L}^3(\mathcal{W}_\tau))^{2/3} \mathcal{I}(\mathcal{W}_\tau) + 1/n \,, \quad \mathcal{A}_n \in \mathbb{E}(\lambda - 1/n) \,,$$
$$\forall \, x \in \Lambda(\lambda) \quad D\big(\mathcal{A}_n, \mathcal{A}_{\mathcal{W}_\tau}(x, \lambda)\big) \geq \delta \,.$$

For n in $\mathbb{N}$, let $\lambda_n = (\lambda - 1/n)^{-3}\lambda^3$. For any positive ρ, the sequence $(\lambda_n \mathcal{A}_n)_{n\in\mathbb{N}}$ is a minimizing sequence of the problem $(\mathbb{P})$ associated to λ and the enlarged box $\Lambda' = (1+\rho)\Lambda$ (the collection $\lambda_n \mathcal{A}_n$ is the collection $\{\, \lambda_n A_n : A_n \in \mathcal{A}_n \,\}$). We choose ρ small enough so that

$$\forall\, x \in \Lambda'(\lambda) \quad \exists\, y \in \Lambda(\lambda) \quad D\big(\mathcal{A}_{\mathcal{W}_\tau}(x,\lambda), \mathcal{A}_{\mathcal{W}_\tau}(y,\lambda)\big) \,\leq\, \delta/3\,.$$

Yet, for x in $\Lambda(\lambda)$, using the hypothesis on $\mathcal{A}_n$,

$$\begin{aligned} D\big(\lambda_n \mathcal{A}_n, \mathcal{A}_{\mathcal{W}_\tau}(x,\lambda)\big) \,&\geq\, D\big(\lambda_n \mathcal{A}_n, \lambda_n \mathcal{A}_{\mathcal{W}_\tau}(x,\lambda)\big) - D\big(\lambda_n \mathcal{A}_{\mathcal{W}_\tau}(x,\lambda), \mathcal{A}_{\mathcal{W}_\tau}(x,\lambda)\big) \\ &\geq\, \lambda_n^3 \delta - D\big(\lambda_n \mathcal{A}_{\mathcal{W}_\tau}(x,\lambda), \mathcal{A}_{\mathcal{W}_\tau}(x,\lambda)\big)\,. \end{aligned}$$

The last term converges towards δ as n goes to ∞. Hence for n large enough,

$$\forall\, x \in \Lambda'(\lambda) \qquad D\big(\lambda_n \mathcal{A}_n, \mathcal{A}_{\mathcal{W}_\tau}(x,\lambda)\big) \,\geq\, \delta/3\,.$$

This stands in contradiction with Proposition 11.11. $\square$

CHAPTER 12

PROOF OF THE LDP FOR THE WHOLE CONFIGURATION

In this chapter, we prove the large deviation results for the whole configuration. After some preparatory results, we prove the large deviation upper bound and the large deviation lower bound, and we end with the proof of Proposition 2.10.

12.1. Preparatory results

Throughout the chapter, we will use extensively the notation and the constructions of chapter 9. We consider positive integers k, l with $k \geq 6$, $l \geq 130$ and $3k \leq l$. We define for $\mathcal{A}$ a collection of connected subsets of $\mathbb{Z}^3$

$$\operatorname{fifa}(\mathcal{A}, k, l) = \big\{ \operatorname{fifa}(A, k, l) : A \in \mathcal{A} \big\} .$$

Proposition 12.1. *For k large enough, there exist positive constants $a(k)$, $b(k)$, $c(k)$, such that for any compact set K of diameter larger than 2, any s in $\mathbb{R}^+$, any l in $\mathbb{N}$ satisfying $\ln \operatorname{diam}_\infty K \leq l/k$, $3k \leq l$, $a(k) \ln l \leq l$, we have*

$$P\big(\mathcal{P}\big(\operatorname{fifa}(\mathcal{C}, k, l), \operatorname{int} K\big) \geq s\big) \leq b(k) \exp(-c(k)s) .$$

Proof. Suppose that $\big\{ \mathcal{P}(\operatorname{fifa}(\mathcal{C}, k, l), \operatorname{int} K) \geq s \big\}$ occurs. Then there exist open disjoint clusters $C_1, \dots, C_r$ such that

$$\forall\, i \in \{1, \dots, r\} \quad \partial \operatorname{fifa}(C_i, k, l) \cap \operatorname{int} K \neq \varnothing ,$$
$$\sum_{1 \leq i \leq r} \mathcal{P}(\operatorname{fifa}(C_i, k, l), \operatorname{int} K) \geq s .$$

Let Λ be the smallest cubic box containing $\mathcal{V}_\infty(K, k)$. For any connected set A of $\mathbb{Z}^3$, we have the inequality (see Lemma 9.14)

$$\mathcal{P}(\operatorname{fifa}(A, k, l), \operatorname{int} K) \leq 6k^2 \operatorname{card}\big(\big(k\, \partial_{\mathrm{ov}} \operatorname{fill}\big(k^{-1} \operatorname{fat}(A, k), l/k\big)\big) \cap \Lambda\big) .$$

Therefore, if we set $F_i = k\, \partial_{\mathrm{ov}} \operatorname{fill}(k^{-1} \operatorname{fat}(C_i, k), l/k)$ for i in $\{1, \dots, r\}$, then

$$\sum_{1 \leq i \leq r} \operatorname{card} F_i \cap \Lambda \geq sk^{-2}/6 .$$

For each i, the set F_i is included in $\mathrm{clo}\,\mathcal{V}_\infty(C_i, 3k/2)$. Hence if a point x belongs to F_i, then C_i intersects the box $\Lambda(x, 4k)$. The number of distinct clusters intersecting a box $\Lambda(x, 4k)$ is certainly less than $(4k)^3$, whence

$$\mathrm{card}\,(F_1 \cup \cdots \cup F_r) \cap \Lambda \;\geq\; (4k)^{-3} \sum_{1 \leq i \leq r} \mathrm{card}\, F_i \cap \Lambda \;\geq\; sk^{-5}/384\,.$$

By construction, the $k\mathbb{L}^3$ components of $F_1 \cup \cdots \cup F_r$ on $k\mathbb{Z}^3$ have a cardinality larger than $l/6k$, because this is already true for each set $F_1, \ldots, F_r$ (see the discussion after Definition 9.7). Thus the $k\mathbb{L}^3$ components of $(F_1 \cup \cdots \cup F_r) \cap \mathcal{V}_\infty(\Lambda, k + l/6)$ which intersect Λ have also a cardinality larger than $l/6k$. Indeed, let R be such a component. If R intersects $\mathcal{V}_\infty(\Lambda, k + l/6) \setminus \mathcal{V}_\infty(\Lambda, l/6)$, then its diameter is larger than $l/6$, and by Lemma 9.3, $\mathrm{card}\, R \geq l/6k$; if R does not intersect $\mathcal{V}_\infty(\Lambda, k + l/6) \setminus \mathcal{V}_\infty(\Lambda, l/6)$, then it is a $k\mathbb{L}^3$ component of $F_1 \cup \cdots \cup F_r$ on $k\mathbb{Z}^3$, whence $\mathrm{card}\, R \geq l/6k$. We use now the variables X_k defined at the beginning of the proof of Proposition 9.12. By Lemma 9.13, $X_k(x^k) = 1$ for any x^k in $(F_1 \cup \cdots \cup F_r) \cap \mathcal{V}_\infty(\Lambda, k + l/6)$. By Corollary 7.4, the probability of the event $\{\, X_k(x^k) = 1 \,\}$ is less than $b \exp(-ck)$ for any x^k in $k\mathbb{Z}^3$, where b, c are two positive constants. Moreover for any x^k in $\mathcal{V}_\infty(\Lambda, k + l/6)$, the box $\Lambda(x^k, 5k)$ is included in $\mathcal{V}_\infty(\Lambda, 4k + l/6)$. Therefore the event $\{\, \mathcal{P}(\mathrm{fifa}\,(\mathcal{C}, k, l), \mathrm{int}\, K) \geq s \,\}$ is included in

$$\Big\{ \mathrm{card} \,\{\, x^k \in k\mathbb{Z}^3 : \mathrm{card}\, C^k_{\mathbb{L}}(x^k, \mathcal{V}_\infty(\Lambda, 4k + l/6), X_k) \geq l/6k \,\} \geq sk^{-5}/384 \Big\}\,.$$

We use now the bound given by Lemma 7.9 with

$$A = \mathcal{V}_\infty(\Lambda, 4k + l/6)\,, \quad \alpha = 5\,, \quad \beta = l/6k\,, \quad \varepsilon = b \exp(-ck)$$

to get

$$\begin{aligned}
&P\big(\mathcal{P}\big(\mathrm{fifa}\,(\mathcal{C}, k, l), \mathrm{int}\, K\big) \geq s\big) \\
&\quad \leq P\Big(\mathrm{card} \,\{\, x^k \in k\mathbb{Z}^3 : \mathrm{card}\, C^k_{\mathbb{L}}(x^k, \mathcal{V}_\infty(\Lambda, 4k + l/6), X_k) \geq l/6k \,\} \geq sk^{-5}/384\Big) \\
&\leq \sum_{i \geq sk^{-5}/384} 2 \exp\Big(i(12k/l) \ln \mathrm{card}\, \big(\mathcal{V}_\infty(\Lambda, 4k{+}l/6) {\cap} k\mathbb{Z}^3\big) + i \ln b + i5^{-3}(\ln b - ck)\Big)\,.
\end{aligned}$$

For any compact set K of diameter larger than 2 and any integer k larger than 6,

$$\begin{aligned}
\mathrm{card}\, \mathcal{V}_\infty(\Lambda, 4k + l/6) \cap k\mathbb{Z}^3 \;&\leq\; k^{-3} \mathcal{L}^3(\mathcal{V}_\infty(\Lambda, 5k + l/6)) \\
&\leq\; k^{-3}(\mathrm{diam}_\infty K + 12k + l/3)^3 \\
&\leq\; 64 l^3 (\mathrm{diam}_\infty K)^3 \;\leq\; l^5 (\mathrm{diam}_\infty K)^3\,.
\end{aligned}$$

We are interested in the regime where $k \ln \mathrm{diam}_\infty K \leq l$. Combining the preceding inequalities,

$$P\big(\mathcal{P}\big(\mathrm{fifa}\,(\mathcal{C}, k, l), \mathrm{int}\, K\big) \geq s\big) \;\leq \sum_{i \geq sk^{-5}/384} 2 \exp\Big(36i + 60ik \ln l/l + i \ln b + i5^{-3}(\ln b - ck)\Big).$$

The preceding inequality yields the desired bound for k and $l/\ln l$ sufficiently large. □

Lemma 12.2. *Let U be a bounded Borel set in $\mathbb{R}^3$. For k large enough, there exist two positive constants $b(k)$, $c(k)$ such that: for any l in $\mathbb{N}$, any s in $\mathbb{R}^+$, satisfying $5k \leq l$, $k \ln s \leq l$,*

$$P\Big(\exists\, C \in \mathcal{C},\, C \cap \mathcal{V}_\infty(U,l) \neq \varnothing,\, s \leq \mathrm{diam}_\infty C < \infty\Big) \\ \leq\ (\mathrm{diam}_\infty U + l)^3\, b(k) \exp(-c(k)s).$$

Proof. Using the translation invariance of the model, we have

$$P\Big(\exists\, C \in \mathcal{C},\, C \cap \mathcal{V}_\infty(U,l) \neq \varnothing,\, s \leq \mathrm{diam}_\infty C < \infty\Big) \\ \leq \sum_{x \in \mathcal{V}_\infty(U,l)\cap\mathbb{Z}^3} P\big(s \leq \mathrm{diam}_\infty C(x) < \infty\big)\ \leq\ (\mathrm{diam}_\infty U + l)^3\, \widehat{P}\big(\mathrm{diam}_\infty C(0) \geq s\big)\,.$$

By the hypothesis, we are in position to apply Proposition 9.12, which yields the desired upper bound. $\square$

Notation. In the following statements, for y^l in $l\mathbb{Z}^3$, we denote by $Y_l(y^l)$ the indicator function of the event $O(\Lambda(y^l,6l),l)^c$.

Lemma 12.3. *Let C_1, C_2 be two distinct open clusters of diameter strictly larger than l. If y^l belongs to* $\mathrm{fat}\,\big(\mathcal{V}_\infty(C_1,l)\cap\mathcal{V}_\infty(C_2,l),l\big)$ *then* $Y_l(y^l)=1$.

Proof. If y^l belongs to $\mathrm{fat}\,\big(\mathcal{V}_\infty(C_1,l)\cap\mathcal{V}_\infty(C_2,l),l\big)$, then both C_1 and C_2 intersect the box $\Lambda(y^l,3l)$. It follows that the box $\Lambda(y^l,6l)$ contains two distinct clusters of diameter larger than l and the event $R(\Lambda(y^l,6l),l)$ does not occur. $\square$

Lemma 12.4. *For any compact set K,*

$$D_K\big(\mathcal{V}_\infty(\mathcal{C},l),\mathrm{fifa}\,(\mathcal{C},k,l)\big)\ \leq\ l^3 \mathrm{card}\,\big\{\, y^l \in \mathrm{fat}\,(K,l) : Y_l(y^l) = 1 \,\big\} \\ + 140 l^2 \mathcal{P}\big(\mathrm{fifa}\,(\mathcal{C},k,l), \mathrm{int}\,K\big) + \mathcal{L}^3_K(\mathcal{V}_\infty(\partial K, 2l+1))\,.$$

Proof. Let $(C(i), i \in \mathbb{N})$ be an arrangement of the collection $\mathcal{C}$. Let i belong to $\mathbb{N}$. We define $C'(i) = \mathcal{V}_\infty(C(i),l)$ if $\mathrm{diam}_\infty C(i) > l$ and $\mathcal{V}_\infty(C(j),l) \neq \mathcal{V}_\infty(C(i),l)$ for all j in $\{\,0,\ldots,i-1\,\}$ and we set $C'(i) = \varnothing$ otherwise. We set also $C''(i) = \mathrm{fifa}\,(C(i),k,l)$ if $\mathrm{fifa}\,(C(j),k,l) \neq \mathrm{fifa}\,(C(i),k,l)$ for all j in $\{\,0,\ldots,i-1\,\}$ and $C''(i) = \varnothing$ otherwise. The sequences $(C'(i), i \in \mathbb{N})$ and $(C''(i), i \in \mathbb{N})$ are arrangements of the collections $\mathcal{V}_\infty(\mathcal{C},l)$ and $\mathrm{fifa}\,(\mathcal{C},k,l)$ respectively. We next compute $\mathcal{L}^3_K(C'(i)\Delta C''(i))$ for i in $\mathbb{N}$. We distinguish four cases:

- $d_\infty(C(i),K) > l$ or $C'(i) = C''(i) = \varnothing$. In this case $\mathcal{L}^3_K(C'(i)\Delta C''(i))$ vanishes.

In the remaining cases, we suppose that $d_\infty(C(i),K) \leq l$ and that $\mathrm{diam}_\infty C(i) > l$.

- $C'(i) \neq \varnothing$, $C''(i) \neq \varnothing$. By Lemmata 9.9, 9.10

$$\mathcal{L}^3_K(C'(i)\Delta C''(i))\ \leq\ 140 l^2 \mathcal{P}\big(\mathrm{fifa}\,(C(i),k,l), \mathrm{int}\,K\big) \\ + \mathcal{L}^3_K\big(\mathcal{V}_\infty(\partial K, 2l+1) \cap \mathcal{V}_\infty(C(i),l)\big)\,.$$

• $C'(i)=\varnothing$. Then there exists $j<i$ such that $\mathcal{V}_\infty(C(j),l)=\mathcal{V}_\infty(C(i),l)$. But the clusters $C(i)$, $C(j)$ are disjoint, by Lemma 9.5 this case cannot happen.
• $C''(i)=\varnothing$. Then $\mathrm{fifa}\,(C(j),k,l)=\mathrm{fifa}\,(C(i),k,l)$ for some $j<i$. By Lemma 9.9, $\mathrm{fifa}\,(C(i),k,l)$ is included in $\mathcal{V}_\infty(C(i),l)$. Applying Lemma 12.3,

$$\begin{aligned}\mathcal{L}^3_K(\mathrm{fifa}\,(C(i),k,l)) &\le l^3\mathrm{card}\,\big(\mathrm{fat}\,(K,l)\cap\mathrm{fat}\,\big(\mathrm{fifa}\,(C(i),k,l)\cap\mathrm{fifa}\,(C(j),k,l),l\big)\big)\\ &\le l^3\mathrm{card}\,\big(\mathrm{fat}\,(K,l)\cap\mathrm{fat}\,\big(\mathcal{V}_\infty(C(i),l)\cap\mathcal{V}_\infty(C(j),l),l\big)\big)\\ &\le l^3\mathrm{card}\,\{\,y^l\in\mathrm{fat}\,(K,l):Y_l(y^l)=1\,\}\,.\end{aligned}$$

We conclude finally that for any i in $\mathbb{N}$, the quantity $\mathcal{L}^3_K(C'(i)\Delta C''(i))$ is smaller than

$$l^3\mathrm{card}\{\,y^l\in\mathrm{fat}\,(K,l):Y_l(y^l)=1\,\}+140l^2\mathcal{P}\big(\mathrm{fifa}\,(\mathcal{C},k,l),\mathrm{int}\,K\big)+\mathcal{L}^3_K(\mathcal{V}_\infty(\partial K,2l+1))$$

This gives the inequality stated in the Lemma. □

Proposition 12.5. *For any compact set K,*

$$\begin{aligned}\mathcal{L}^3_K\big(\mathrm{overlap}\,\mathrm{fifa}\,(\mathcal{C},k,l)\big) &\le \mathcal{L}^3_K\big(\mathrm{overlap}\,\mathcal{V}_\infty(\mathcal{C},l)\big)\\ &\qquad\le l^3\mathrm{card}\,\{\,y^l\in\mathrm{fat}\,(K,l):Y_l(y^l)=1\,\}\,,\\ \mathcal{L}^3_K\big(\mathrm{fifa}\,(\mathcal{C},k,l)\big) &\ge \mathcal{L}^3(K)-l^3\mathrm{card}\,\{\,y^l\in\mathrm{fat}\,(K,l):Y_l(y^l)=1\,\}\\ &\qquad -140l^2\mathcal{P}\big(\mathrm{fifa}\,(\mathcal{C},k,l),\mathrm{int}\,K\big)-\mathcal{L}^3_K\big(\mathcal{V}_\infty(\partial K,2l+1)\big)\,.\end{aligned}$$

Proof. Lemmata 9.9, 12.3 imply that

$$\mathrm{overlap}\,\mathrm{fifa}\,(\mathcal{C},k,l)\,\subset\,\mathrm{overlap}\,\mathcal{V}_\infty(\mathcal{C},l)\,\subset\,\mathrm{cup}\,\{\,\Lambda(y^l,l):y^l\in l\mathbb{Z}^3,\,Y_l(y^l)=1\,\}\,,$$

which yields the first inequality stated in the Proposition.
Next, by Lemma 9.9, for any connected subset A of $\mathbb{Z}^3$ such that $\mathrm{diam}_\infty A>l$, $\mathrm{fifa}\,(A,k,l)$ is included in $\mathcal{V}_\infty(A,l)$, whence

$$\mathrm{cup}\,\mathrm{fifa}\,(\mathcal{C},k,l)\,\subset\,\mathrm{cup}\,\mathcal{V}_\infty(\mathcal{C},l)\,.$$

For y^l in $l\mathbb{Z}^3$, if $\Lambda(y^l,l)$ is not included in $\mathrm{cup}\,\mathcal{V}_\infty(\mathcal{C},l)$, then either there is no unique crossing cluster in $\Lambda(y^l,6l)$ or the crossing cluster $C^*(\Lambda(y^l,6l))$ does not intersect $\Lambda(y^l,l)$, so that $O(\Lambda(y^l,6l),l)$ does not happen and $Y_l(y^l)=1$. Hence

$$\mathcal{L}^3_K\big(\mathrm{cup}\,\mathcal{V}_\infty(\mathcal{C},l)\big)\ge\mathcal{L}^3(K)-l^3\mathrm{card}\,\{\,y^l\in\mathrm{fat}\,(K,l):Y_l(y^l)=1\,\}\,.$$

Moreover, by Lemma 9.10,

$$\begin{aligned}&\mathcal{L}^3_K\Big(\mathrm{cup}\,\mathcal{V}_\infty(\mathcal{C},l)\setminus\mathrm{cup}\,\mathrm{fifa}\,(\mathcal{C},k,l)\Big)\\ &\quad\le\mathcal{L}^3_K\big(\mathcal{V}_\infty(\partial K,2l+1)\big)+\mathcal{L}^3_K\Big(\mathrm{cup}\,\mathcal{V}_\infty(\mathcal{C},l)\setminus\mathrm{cup}\,\mathrm{fifa}\,(\mathcal{C},k,l)\setminus\mathcal{V}_\infty(\partial K,2l+1)\Big)\\ &\le\mathcal{L}^3_K\big(\mathcal{V}_\infty(\partial K,2l+1)\big)+\sum_{C\in\mathcal{C},\,\mathrm{diam}_\infty C>l}\mathcal{L}^3_K\big(\mathcal{V}_\infty(C,l)\setminus\mathrm{fifa}\,(C,k,l)\setminus\mathcal{V}_\infty(\partial K,2l+1)\big)\\ &\qquad\le 140l^2\mathcal{P}\big(\mathrm{fifa}\,(\mathcal{C},k,l),\mathrm{int}\,K\big)+\mathcal{L}^3_K\big(\mathcal{V}_\infty(\partial K,2l+1)\big)\,.\end{aligned}$$

These inequalities yield the desired lower bound for $\mathcal{L}^3_K\big(\mathrm{fifa}\,(\mathcal{C},k,l)\big)$. $\square$

Of course, for a collection $\mathcal{C}$ of open clusters, we define

$$\mathrm{aglu}\,(\mathcal{C},l) \,=\, \big\{\,\mathrm{aglu}\,(C,l) : C\in\mathcal{C}\,\big\}\,.$$

Lemma 12.6. *For any compact set K,*

$$\begin{aligned} D_K\big(\mathcal{V}_\infty(\mathcal{C},l),\mathrm{aglu}\,(\mathcal{C},l)\big) \,\leq\, & 3^3l^3\mathrm{card}\,\big\{\,y^l\in\mathcal{V}_\infty(K,2l) : Y_l(y^l)=1\,\big\} \\ & +8960l^2\mathcal{P}\big(\mathrm{fifa}\,(\mathcal{C},k,l),\mathrm{int}\,K\big)+\mathcal{L}^3_K\big(\mathcal{V}_\infty(\partial K,5l+1)\big)\,. \end{aligned}$$

Proof. Let $(C(i),i\in\mathbb{N})$ be an arrangement of the collection $\mathcal{C}$. Let i belong to $\mathbb{N}$. We define $C'(i)=\mathcal{V}_\infty(C(i),l)$ if $\mathrm{diam}_\infty C(i)>l$ and $\mathcal{V}_\infty(C(j),l)\neq\mathcal{V}_\infty(C(i),l)$ for all j in $\{\,0,\dots,i-1\,\}$ and we set $C'(i)=\varnothing$ otherwise. We set also $C''(i)=\mathrm{aglu}\,(C(i),l)$ if $\mathrm{aglu}\,(C(j),l)\neq\mathrm{aglu}\,(C(i),l)$ for all j in $\{\,0,\dots,i-1\,\}$ and $C''(i)=\varnothing$ otherwise. The sequences $(C'(i),i\in\mathbb{N})$ and $(C''(i),i\in\mathbb{N})$ are arrangements of the collections $\mathcal{V}_\infty(\mathcal{C},l)$ and $\mathrm{aglu}\,(\mathcal{C},l)$ respectively. We next compute $\mathcal{L}^3_K(C'(i)\Delta C''(i))$ for i in $\mathbb{N}$. We distinguish four cases:

- $d_\infty(C(i),K)>5l$ or $C'(i)=C''(i)=\varnothing$. In this case $\mathcal{L}^3_K(C'(i)\Delta C''(i))$ vanishes.

In the remaining cases, we suppose that $d_\infty(C(i),K)\leq 5l$ and that $\mathrm{diam}_\infty C(i)>l$.

- $C'(i)\neq\varnothing$, $C''(i)\neq\varnothing$. We simply use the bound given by Lemma 9.15 (remark that since $\mathrm{diam}_\infty C(i)>l$, we do not need to take into account the first term $800l^3$).
- $C'(i)=\varnothing$. Then there exists $j<i$ such that $\mathcal{V}_\infty(C(j),l)=\mathcal{V}_\infty(C(i),l)$. But the clusters $C(i)$, $C(j)$ are disjoint, by Lemma 9.5 this case cannot happen.
- $C''(i)=\varnothing$. Then there exists j in $\{\,0,\dots,i-1\,\}$ such that $\mathrm{aglu}\,(C(j),l)=\mathrm{aglu}\,(C(i),l)$. However the region $\mathrm{aglu}\,(C(i),l)$ contains exactly one open cluster of diameter strictly larger than l, which is precisely $C(i)$. And so does the region $\mathrm{aglu}\,(C(j),l)$. Hence $C(i)=C(j)$, which is impossible and this case does not occur.

We conclude finally that for any i in $\mathbb{N}$, the quantity $\mathcal{L}^3_K(C'(i)\Delta C''(i))$ is less than the right–hand member of the inequality stated in the Lemma. $\square$

12.2. Proof of the upper bound

From now onwards, the integer k is fixed and large (so that Proposition 12.1 holds) and we replace the integer l in the previous constructions by a function $f(n)$ from $\mathbb{N}$ to $\mathbb{N}$ such that

$$\lim_{n\to\infty} n/f(n)^2 \,=\, \infty\,,\qquad \forall\,n\in\mathbb{N}\quad f(n)\geq\kappa\ln n\,,$$

where $\kappa=\kappa(p)$ is a large constant (larger than the one given by Theorem 7.1, larger than $3k$ and large enough for the next Lemma to hold). We omit the variables k,l in the sequel, writing for instance $\mathrm{fifa}\,\mathcal{C}$, $X(x)$, $Y(x)$ instead of $\mathrm{fifa}\,(\mathcal{C},k,l)$, $X_k(x^k)$, $Y_l(x^l)$.

Lemma 12.7. *For κ large enough, for any positive r,δ,*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\Big(\, f(n)^3\big(1+\mathrm{card}\,\{x\in\Lambda(0,rn^2)\cap f(n)\mathbb{Z}^3 : Y(x)=1\}\big)\\ +f(n)^2\mathcal{P}\big(\mathrm{fifa}\,\mathcal{C},\mathrm{int}\,\Lambda(0,rn)\big)\geq \delta n^3\,\Big)\;=\;-\infty\,.$$

Proof. First $n/f(n)^2$ tends to ∞ as n goes to ∞, hence $f(n)^3$ is smaller than $\delta n^3/3$ for n large enough. Secondly, provided κ is large enough, Corollaries 7.4, 7.8 together yield the existence of two positive constants b,c such that for any n in $\mathbb{N}$,

$$P\Big(f(n)^3\mathrm{card}\,\{x\in\Lambda(0,rn^2)\cap f(n)\mathbb{Z}^3 : Y(x)=1\}\geq\delta n^3/3\Big)\\ \leq b\exp\big(-(c\delta/3)n^2(n/f(n)^2)\big)\,.$$

Thirdly, by Proposition 12.1, for some positive constants $b(k),c(k)$,

$$P\big(f(n)^2\mathcal{P}\big(\mathrm{fifa}\,\mathcal{C},\mathrm{int}\,\Lambda(0,rn)\big)\geq\delta n^3/3\big)\;\leq\;b(k)\exp\big(-c(k)(\delta/3)n^2(n/f(n)^2)\big)\,.$$

These three facts imply the claim of the Corollary. □

Using Lemmata 10.7, 12.4, 12.6, 12.7 and Proposition 12.5, we see that the random collections of sets

$$n^{-1}\mathcal{V}_\infty(\mathcal{C},f(n))\,,\quad n^{-1}\mathrm{fifa}\,\mathcal{C}\,,\quad n^{-1}\mathrm{aglu}\,\mathcal{C}\,,\quad n^{-1}\mathrm{part\,fifa}\,\mathcal{C}$$

are exponentially contiguous with respect to the metric D_{loc}, that is, for any compact set K and any positive δ,

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\Big(D_K\big(n^{-1}\mathcal{V}_\infty(\mathcal{C},f(n)),\,n^{-1****}\mathcal{C}\big)\geq\delta\Big)\;=\;-\infty\,,$$

where $****$ stands for either among fifa , aglu , part fifa . At the level of the large deviations of order n^2, we might work with either of these four objects. We prove the large deviations upper bound in a standard fashion, by proving $\mathcal{I}$–tightness together with a local estimate.

• *$\mathcal{I}$–tightness.* Let λ,δ be positive and let K be a compact set. We write, for any positive a,

$$P\Big(\inf\Big\{\,D_K\big(n^{-1}\mathcal{V}_\infty(\mathcal{C},f(n)),\mathcal{A}\big):\mathcal{A}\in\mathrm{BC}(\mathbb{R}^3),\,\mathcal{I}(\mathcal{A})\leq\lambda\Big\}\geq\delta\Big)\\ \leq P\Big(D_K\big(n^{-1}\mathcal{V}_\infty(\mathcal{C},f(n)),n^{-1}\mathrm{fifa}\,\mathcal{C}\big)\geq\delta/3\Big)\\ +P\Big(D_K\big(n^{-1}\mathrm{fifa}\,\mathcal{C},n^{-1}\mathrm{part\,fifa}\,\mathcal{C}\big)\geq\delta/3\Big)\\ +P\Big(\exists\, C\in\mathcal{C},\,C\cap\mathcal{V}_\infty(nK,f(n))\neq\varnothing,\,an^2\leq\mathrm{diam}_\infty C<\infty\Big)\\ +P\left(\begin{array}{l}\inf\Big\{\,D_K\big(n^{-1}\mathrm{part\,fifa}\,\mathcal{C},\mathcal{A}\big):\mathcal{A}\in\mathrm{BC}(\mathbb{R}^3),\,\mathcal{I}(\mathcal{A})\leq\lambda\Big\}\geq\delta/3,\text{ any open}\\ \text{finite cluster } C \text{ intersecting } \mathcal{V}_\infty(nK,f(n)) \text{ has diameter less than } an^2\end{array}\right).$$

We consider successively each term of the right–hand side. We have already controlled the first term with Lemmata 12.4, 12.7. For the second term, we have, using Lemma 10.7,

$$P\Big(D_K\big(n^{-1}\mathrm{fifa}\,\mathcal{C}, n^{-1}\mathrm{part\,fifa}\,\mathcal{C}\big)\geq\delta/3\Big)$$
$$\leq P\Big(\max\Big(\mathcal{L}^3_K\big(\mathrm{overlap}\,n^{-1}\mathrm{fifa}\,\mathcal{C}\big),\mathcal{L}^3\big(K\setminus\mathrm{cup}\,n^{-1}\mathrm{fifa}\,\mathcal{C}\big)\Big)\geq\delta/3\Big)\,.$$

Proposition 12.5 and Lemma 12.7 imply then that

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\Big(D_K\big(n^{-1}\mathrm{fifa}\,\mathcal{C}, n^{-1}\mathrm{part\,fifa}\,\mathcal{C}\big)\geq\delta/3\Big)\;=\;-\infty\,.$$

For the third term, we apply Lemma 12.2 to get

$$P\Big(\exists\, C\in\mathcal{C},\, C\cap\mathcal{V}_\infty(nK,f(n))\neq\varnothing,\, an^2\leq\mathrm{diam}_\infty C<\infty\Big)\;\leq$$
$$(n\mathrm{diam}_\infty K+f(n))^3\,b\exp(-can^2)$$

where b,c are two positive constants depending on k. We finally consider the last term. Let $\mathcal{E}$ be the collection

$$\mathcal{E}\;=\;\mathrm{part}\Big\{\,\mathrm{fifa}\,C: C\in\mathcal{C}, \mathrm{card}\,C<\infty,\, C\cap\mathcal{V}_\infty(nK,f(n))\neq\varnothing\,\Big\}\,.$$

Certainly, we have $D_K\big(n^{-1}\mathcal{E},n^{-1}\mathrm{part\,fifa}\,\mathcal{C}\big)=0$. Thus, if

$$\inf\Big\{\,D_K\big(n^{-1}\mathrm{part\,fifa}\,\mathcal{C},\mathcal{A}\big):\mathcal{A}\in\mathrm{BC}(\mathbb{R}^3),\,\mathcal{I}(\mathcal{A})\leq\lambda\,\Big\}\geq\delta/3$$

then $\mathcal{I}(\mathcal{E})\geq n^2\lambda$. Suppose next that any open finite cluster C intersecting the set $\mathcal{V}_\infty(nK,f(n))$ has diameter less than an^2. Then, using Lemma 11.1, we have

$$\mathcal{I}(\mathcal{E})\;\leq\;||\tau||_\infty\mathcal{P}\big(\mathrm{fifa}\,\mathcal{C},\mathrm{int}\,\mathcal{V}_\infty(nK,2f(n)+an^2)\big)\,.$$

For n sufficiently large, we have

$$\ln\big(n\mathrm{diam}_\infty K+2f(n)+an^2\big)\leq f(n)/k,\quad 3k\leq f(n),\quad a(k)\ln f(n)\leq f(n)$$

($a(k)$ is the constant appearing in the hypothesis of Proposition 12.1). For such large values of n, combining the previous observations and applying Proposition 12.1, we see that

$$P\begin{pmatrix}\inf\Big\{\,D_K\big(n^{-1}\mathrm{part\,fifa}\,\mathcal{C},\mathcal{A}\big):\mathcal{A}\in\mathrm{BC}(\mathbb{R}^3),\,\mathcal{I}(\mathcal{A})\leq\lambda\,\Big\}\geq\delta/3,\text{ any open}\\ \text{finite cluster } C \text{ intersecting } \mathcal{V}_\infty(nK,f(n)) \text{ has diameter less than } an^2\end{pmatrix}$$
$$\leq P\Big(\mathcal{P}\big(\mathrm{fifa}\,\mathcal{C},\mathrm{int}\,\mathcal{V}_\infty(nK,2f(n)+an^2)\big)\geq n^2\lambda/||\tau||_\infty\Big)\;\leq\;b\exp\big(-cn^2\lambda/||\tau||_\infty\big)$$

where b, c are two positive constants depending on k. Putting together the previous estimates, sending successively n and a to ∞, we obtain that there exists a positive constant $c' = c/||\tau||_\infty$ such that, for any positive λ, δ, any compact set K,

$$\limsup_{n\to\infty} \frac{1}{n^2} \ln P\Big(\inf\Big\{ D_K\big(n^{-1}\mathcal{V}_\infty(\mathcal{C}, f(n)), \mathcal{A}\big) : \mathcal{A} \in \mathrm{BC}(\mathbb{R}^3),\, \mathcal{I}(\mathcal{A}) \le \lambda \Big\} \ge \delta \Big) \le -c'\lambda\,.$$

• *Local estimate.* We complete the proof of the large deviations upper bound with the help of the following local estimate:

$\forall\, \mathcal{A} \in \mathrm{BC}(\mathbb{R}^3)$, $\mathcal{I}(\mathcal{A}) < \infty$,

$$\lim_{K\to\mathbb{R}^3, \delta\to 0} \limsup_{n\to\infty} \frac{1}{n^2} \ln P\Big(D_K\big(n^{-1}\mathcal{V}_\infty(\mathcal{C}, f(n)), \mathcal{A}\big) < \delta \Big) \le -\mathcal{I}(\mathcal{A})\,.$$

Let finally $\mathcal{A}$ be a Caccioppoli partition of finite perimeter. Let ε be positive. Let δ_0 be associated to ε as in Lemma 8.1. Let $B(x_i, r_i)$, $i \in I$, be a finite collection of disjoint balls as given in Lemma 11.5, associated to $\mathcal{A}$, ε and $\delta_0/3$. Let K be a compact set containing in its interior all the balls $B(x_i, r_i)$, $i \in I$. Let δ be a positive real number strictly smaller than $\min\big\{ \delta_0 r_i^3/3,\, \mathcal{L}^3_K(A_1(x_i)) : i \in I \big\}$. Suppose the event $\big\{ D_K\big(n^{-1}\mathrm{aglu}\,\mathcal{C}, \mathcal{A}\big) < \delta \big\}$ occurs. Suppose also that n is large enough to have $4\pi\sqrt{3} r_i^2 n^2 + \pi\sqrt{3} < \delta_0 r_i^3 n^3/3$ for any i in I. Let i belong to I. There exists a set $C_n(x_i)$ belonging to $\mathrm{aglu}\,\mathcal{C}$ such that

$$\mathcal{L}^3_{nK}\big(nA_1(x_i)\Delta C_n(x_i)\big) < \delta\, n^3\,.$$

Let $\mathcal{C}(i)$ be the collection of the open clusters of the configuration restricted to the set $C_n(x_i) \cap B(nx_i, nr_i)$. Because of the definition of $\mathrm{aglu}\,\mathcal{C}$, the clusters of $\mathcal{C}(i)$ are open clusters of the configuration restricted to $B(nx_i, nr_i)$. Moreover

$$\begin{aligned}
&\mathcal{L}^3\Big(\mathcal{V}_\infty(\mathrm{cup}\,\mathcal{C}(i), 1/2)\Delta B_-(nx_i, nr_i, \nu(x_i))\Big)\\
&\quad\le \mathcal{L}^3\Big(\mathcal{V}_\infty(\mathrm{cup}\,\mathcal{C}(i), 1/2)\Delta\big(C_n(x_i) \cap B(nx_i, nr_i)\big)\Big)\\
&\qquad\qquad + \mathcal{L}^3\Big(\big(C_n(x_i) \cap B(nx_i, nr_i)\big)\Delta B_-(nx_i, nr_i, \nu(x_i))\Big)\\
&\quad\le \mathcal{L}^3\big(B(nx_i, nr_i + \sqrt{3}/2) \setminus B(nx_i, nr_i - \sqrt{3}/2)\big)\\
&\quad + \mathcal{L}^3_{nK}\big(C_n(x_i)\Delta nA_1(x_i)\big) + \mathcal{L}^3_{nK}\Big(\big(nA_1(x_i) \cap B(nx_i, nr_i)\big)\Delta B_-(nx_i, nr_i, \nu(x_i))\Big)\\
&\quad< 4\pi\sqrt{3} r_i^2 n^2 + \pi\sqrt{3} + \delta n^3 + \delta_0 r_i^3 n^3/3 < \delta_0 r_i^3 n^3\,.
\end{aligned}$$

Therefore the collection $\mathcal{C}(i)$ realizes the event $\mathrm{sep}\,(n, x_i, r_i, \nu(x_i), \delta_0)$ for i in I and the event $\big\{ D_K\big(n^{-1}\mathrm{aglu}\,\mathcal{C}, \mathcal{A}\big) < \delta \big\}$ is included in $\bigcap_{i\in I} \mathrm{sep}\,(n, x_i, r_i, \nu(x_i), \delta_0)$. By independence (recall that the balls $B(x_i, r_i)$, $i \in I$, are disjoint),

$$P\Big(D_K\big(n^{-1}\mathrm{aglu}\,\mathcal{C}, \mathcal{A}\big) < \delta\Big) \le \prod_{i\in I} P\big(\mathrm{sep}\,(n, x_i, r_i, \nu(x_i), \delta_0)\big)$$

and by the choice of δ_0 (see Lemma 8.1) and of the balls $B(x_i,r_i)$ (see Lemma 11.5),

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n^2}\ln P\Big(D_K\big(n^{-1}\mathrm{aglu}\,\mathcal{C},\mathcal{A}\big)<\delta\Big) &\leq -\sum_{i\in I}\tau(\nu(x_i))\pi r_i^2(1-\varepsilon)\\ &\leq -\mathcal{I}(\mathcal{A})(1-\varepsilon)+\varepsilon\,.\end{aligned}$$

Letting first K grow to $\mathbb{R}^3$, δ go to 0 and then ε go to zero in this inequality, we get the upper bound we were seeking for. $\square$

12.3. Proof of the lower bound

In view of the approximation result stated in Theorem 11.8, to prove the large deviation lower bound we need only to show that for any finite bounded polyhedral Caccioppoli partition $\mathcal{A}$, any compact set K and any positive δ,

$$\liminf_{n\to\infty}\frac{1}{n^2}\ln P\Big(D_K\big(n^{-1}\mathcal{V}_\infty(\mathcal{C},f(n)),\mathcal{A}\big)<\delta\Big)\geq -\mathcal{I}(\mathcal{A})\,.$$

We might even assume that the sets of $\mathcal{A}$ are open and connected. Let $\mathcal{A}=\{A_1,\dots,A_r\}$ be a bounded finite polyhedral Caccioppoli partition whose elements are open and connected. We suppose that A_1 is the only unbounded set of the partition. The reduced boundary $\partial^*\mathcal{A}$ of $\mathcal{A}$ is the union of a finite number of polygonal planar sets $F_1,\dots,F_s$. Let K be a compact set containing $\partial^*\mathcal{A}$ in its interior. In particular $A_2,\dots,A_r$ are included in $\mathrm{int}\,K$ and $A_1\cap\mathrm{int}\,K$ is connected. Let δ be positive and let $U_1,\dots,U_r$ and $O_1,\dots,O_r$ be open connected subsets of $\mathrm{int}\,K$ such that for any i in $\{1,\dots,r\}$,

$$U_i\subset O_i\subset A_i\,,\quad d_2(U_i,K\setminus O_i)>0\,,\quad d_2(O_i,K\setminus A_i)>0\,,\quad \mathcal{L}^3_K(U_i\Delta A_i)<\delta/r\,.$$

Let $\phi(n)$ be a function from $\mathbb{N}$ to $\mathbb{R}^+$ such that both $\phi(n)$ and $n/\phi(n)^2$ go to ∞ when n goes to ∞. Let n be large enough so that $\phi(n)>\zeta$. For i in $\{1,\dots,s\}$, the event $\mathrm{wall}\,(nF_i,\phi(n))$ depends on the edges inside $\mathcal{V}_2(nF_i,\phi(n))$ whereas for i in $\{1,\dots,r\}$, the event $\mathrm{full}\,(O_i,U_i,n,f(n))$ depends on the edges inside nO_i. Since the distance between the sets $F_1\cup\dots\cup F_s$ and $O_1\cup\dots\cup O_r$ is strictly positive and $\phi(n)/n$ goes to 0 as n goes to ∞, for n large enough, we have

$$\phi(n)/n<\min\big\{\,d_\infty(O_i,K\setminus A_i):1\leq i\leq r\,\big\}$$

and the events

$$\begin{gathered}\mathrm{full}\,(O_1,U_1,n,f(n)),\dots,\mathrm{full}\,(O_r,U_r,n,f(n)),\\ \mathrm{wall}\,(nF_1,\phi(n))\cap\dots\cap\mathrm{wall}\,(nF_s,\phi(n))\end{gathered}$$

are independent. Suppose all these events occur simultaneously. For i in $\{1,\dots,r\}$, we denote by C_i the open cluster of the whole configuration containing the cluster realizing the event full$(O_i, U_i, n, f(n))$. We have then $nU_i \subset \mathcal{V}_\infty(C_i, f(n))$. Moreover the occurrence of the events wall$(nF_i, \phi(n))$, $1 \le i \le s$, precludes that an open path connects two distinct sets O_i, so that the clusters C_i, $1 \le i \le r$, are distinct and satisfy $C_i \subset \mathcal{V}_\infty(A_i, \phi(n))$, whence

$$\forall i \in \{1, \dots, r\} \qquad \mathcal{V}_\infty(C_i, f(n)) \subset \mathcal{V}_\infty(nA_i, f(n) + \phi(n))\,.$$

Let η be such that $\mathcal{L}^3_K(\mathcal{V}_\infty(A_i, \eta) \setminus A_i) < \delta/2$ for i in $\{1, \dots, r\}$. Suppose that n is large enough so that $f(n) + \phi(n) < \eta n$ and $f(n) < n \min\big\{ d_\infty(U_i, K \setminus O_i) : 1 \le i \le r \big\} - 1$. Then

$$\forall i \in \{1, \dots, r\} \qquad \mathcal{L}^3_K\big(n^{-1}\mathcal{V}_\infty(C_i, f(n))\Delta A_i\big) \;\le\; \mathcal{L}^3_K(\mathcal{V}_\infty(A_i, \eta) \setminus U_i) \;<\; \delta\,.$$

Let i belong to $\{1, \dots, r\}$. We claim that there is no open cluster (of the whole configuration) disjoint from C_i of diameter strictly larger than $f(n)$ and intersecting $\mathcal{V}_\infty(nU_i, f(n))$. Indeed, if C is such a cluster, then in the configuration restricted to nO_i there exists an open cluster included in C of diameter strictly larger than $f(n)$ and intersecting $\mathcal{V}_\infty(nU_i, f(n))$, which contradicts the fact that full$(O_i, U_i, n, f(n))$ occurs. Thus, if C is an open cluster of diameter strictly larger than $f(n)$ and distinct from $C_1, \dots, C_r$, then

$$n^{-1}\mathcal{V}_\infty(C, f(n)) \cap K \;\subset\; K \setminus \mathrm{cup}\,\{\, U_i : 1 \le i \le r\,\}$$

whence

$$\mathcal{L}^3_K\big(n^{-1}\mathcal{V}_\infty(C, f(n))\big) \;\le\; \sum_{1 \le i \le r} \mathcal{L}^3_K(A_i \setminus U_i) \;<\; \delta\,.$$

We get finally that $D_K\big(n^{-1}\mathcal{V}_\infty(\mathcal{C}, f(n)), \mathcal{A}\big) \;<\; \delta$. Therefore

$$\begin{aligned} P\Big(D_K\big(n^{-1}\mathcal{V}_\infty(\mathcal{C}, f(n)), \mathcal{A}\big) < \delta\Big) & \\ \ge P\Big(\bigcap_{1 \le i \le r} &\mathrm{full}\,(O_i, U_i, n, f(n)) \cap \bigcap_{1 \le i \le s} \mathrm{wall}\,(nF_i, \phi(n))\Big) \\ \ge \prod_{1 \le i \le r} &P\big(\mathrm{full}\,(O_i, U_i, n, f(n))\big) \times P\Big(\bigcap_{1 \le i \le s} \mathrm{wall}\,(nF_i, \phi(n))\Big)\,. \end{aligned}$$

By Lemma 9.18, the first product goes to 1 as n goes to ∞. Applying the FKG inequality (notice that the events wall$(nF_i, \phi(n))$ are decreasing) and Lemma 9.19, we obtain

$$\liminf_{n \to \infty} \frac{1}{n^2} \ln P\Big(\bigcap_{1 \le i \le s} \mathrm{wall}\,(nF_i, \phi(n))\Big) \;\ge\; -\sum_{1 \le i \le s} \mathcal{H}^2(F_i)\tau(\mathrm{nor}\,F_i)\,.$$

The two previous inequalities together imply that

$$\liminf_{n \to \infty} \frac{1}{n^2} \ln P\Big(D_K\big(n^{-1}\mathcal{V}_\infty(\mathcal{C}, f(n)), \mathcal{A}\big) < \delta\Big) \;\ge\; -\mathcal{I}(\mathcal{A})\,.$$

Thus we are done. $\square$

12.4. Proof of Proposition 2.8

We prove the first part of the Proposition. Let Λ be a cubic box containing K. The first inequality of Proposition 12.5 yields

$$\mathcal{L}^3_{nK}\big(\mathrm{overlap}\,\mathcal{V}_\infty(\mathcal{C},f(n))\big)\ \le\ f(n)^3\mathrm{card}\,\big\{\,x\in\mathrm{fat}\,(n\Lambda,f(n)):Y(x)=1\,\big\}\,.$$

Since $\mathrm{cup}\,\mathcal{V}_\infty(\mathcal{C},f(n))$ contains $\mathrm{cup}\,\mathrm{fifa}\,\mathcal{C}$, applying the second result of Proposition 12.5, we have

$$\begin{aligned}\mathcal{L}^3\big(nK\setminus\mathrm{cup}\,\mathcal{V}_\infty(\mathcal{C},f(n))\big)\ &\le\ \mathcal{L}^3\big(n\Lambda\setminus\mathrm{cup}\,\mathcal{V}_\infty(\mathcal{C},f(n))\big)\ \le\ \mathcal{L}^3\big(n\Lambda\setminus\mathrm{cup}\,\mathrm{fifa}\,\mathcal{C}\big)\\ \le\ f(n)^3&\mathrm{card}\,\big\{\,x\in\mathrm{fat}\,(n\Lambda,f(n)):Y(x)=1\,\big\}\\ &\qquad+140f(n)^2\mathcal{P}(\mathrm{fifa}\,\mathcal{C},\mathrm{int}\,n\Lambda)+\mathcal{L}^3\big(\mathcal{V}_\infty(n\partial\Lambda,2f(n)+1)\big)\,.\end{aligned}$$

The conclusion follows from Lemma 12.7.
We now prove the second part of the Proposition. We write, for any positive a,

$$\begin{aligned}&P\big(\mathcal{L}^3(\mathrm{overlap}\,\mathcal{V}_\infty\mathcal{C}(nU,f(n)))>\delta n^3\big)\ \le\\ &P\Big(\exists\, C\in\mathcal{C},\ C\cap\mathcal{V}_\infty(nU,f(n))\neq\varnothing,\ an^2\le\mathrm{diam}_\infty C<\infty\Big)\\ &\qquad\qquad+P\begin{pmatrix}\mathcal{L}^3(\mathrm{overlap}\,\mathcal{V}_\infty\mathcal{C}(nU,f(n)))>\delta n^3\text{ and any open finite clus–}\\ \text{ter intersecting }\mathcal{V}_\infty(nU,f(n))\text{ has diameter less than }an^2\end{pmatrix}.\end{aligned}$$

By Lemma 12.2, the first term is bounded by $(n\mathrm{diam}_\infty U+f(n))^3b(k)\exp(-c(k)an^2)$. Let us examine the second term. Suppose that every open cluster intersecting $\mathcal{V}_\infty(nU,f(n))$ has diameter less than an^2. Then

$$\mathrm{overlap}\,\mathcal{V}_\infty\mathcal{C}(nU,f(n))\ \subset\ \mathcal{V}_\infty(nU,an^2+2f(n))\,.$$

We apply the first result of Proposition 12.5 with $K=\mathrm{clo}\,\Lambda(0,2an^2)$. For n large enough, the compact set K certainly contains $\mathcal{V}_\infty(nU,an^2+2f(n))$ and

$$\begin{aligned}\mathcal{L}^3\big(\mathrm{overlap}\,\mathcal{V}_\infty\mathcal{C}(nU,f(n))\big)\ &\le\ \mathcal{L}^3_K\big(\mathrm{overlap}\,\mathcal{V}_\infty(\mathcal{C},f(n))\big)\\ &\le\ f(n)^3\mathrm{card}\,\big\{\,x\in\Lambda(0,2an^2+f(n))\cap f(n)\mathbb{Z}^3:Y(x)=1\,\big\}\,.\end{aligned}$$

By Lemma 12.7 we conclude that, for any positive a, the limsup

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\begin{pmatrix}\mathcal{L}^3(\mathrm{overlap}\,\mathcal{V}_\infty\mathcal{C}(nU,f(n)))>\delta n^3\text{ and any open finite clus–}\\ \text{ter intersecting }\mathcal{V}_\infty(nU,f(n))\text{ has diameter less than }an^2\end{pmatrix}$$

is equal to $-\infty$. The previous estimates yield that, for any positive a,

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\big(\mathcal{L}^3(\mathrm{overlap}\,\mathcal{V}_\infty\mathcal{C}(nU,f(n)))>\delta n^3\big)\ \le\ -ca$$

where c is a positive constant depending on k. The desired claim follows by letting a go to ∞. $\square$

12.5. Proof of Proposition 2.10

Let Λ be a cubic box and let δ be positive. Let η be such that $4\eta\mathcal{L}^3(\Lambda) < \delta$. We apply the inequality of Lemma 9.22 to $n\Lambda$ and an arbitrary open cluster C of diameter larger than $f(n)$:

$$\begin{aligned}\big|\mathrm{card}\, C\cap n\Lambda - \theta\mathcal{L}^3_{n\Lambda}(\mathcal{V}_\infty(C,f(n)))\big| \leq&\\ \eta\mathcal{L}^3(n\Lambda) + f(n)^3\mathrm{card}\,\{\, x\in f(n)\mathbb{Z}^3\cap n\Lambda : X^\eta(x)=1\,\}&\\ +1120 f(n)^2\mathcal{P}\big(\mathrm{fifa}\,(\mathcal{C},k,f(n)),\mathrm{int}\, n\Lambda\big) + 2\mathcal{L}^3\big(\mathcal{V}_\infty(\partial n\Lambda, 3f(n)+1)\big)\,.&\end{aligned}$$

Notice that the upper bound does not depend on the cluster C. We suppose that n is large enough so that $2\mathcal{L}^3\big(\mathcal{V}_\infty(\partial n\Lambda, 3f(n)+1)\big) < \delta n^3/4$. Then

$$\begin{aligned}P\Big(\sup_{C\in\mathcal{C}}\big|\mathrm{card}\, C\cap n\Lambda - \theta\mathcal{L}^3_{n\Lambda}(\mathcal{V}_\infty(C,f(n)))\big| \geq \delta n^3\Big) \leq&\\ P\big(f(n)^3\mathrm{card}\,\{\, x\in f(n)\mathbb{Z}^3\cap n\Lambda : X^\eta(x)=1\,\}\geq \delta n^3/4\big)&\\ +P\big(1120 f(n)^2\mathcal{P}\big(\mathrm{fifa}\,(\mathcal{C},k,f(n)),\mathrm{int}\, n\Lambda\big)\geq \delta n^3/4\big)\,.&\end{aligned}$$

By Proposition 12.1, for n large enough, the second term of the right–hand side is less than

$$b(k)\exp\big(-c(k)(\delta/4480)n^3 f(n)^{-2}\big)\,.$$

By Corollaries $7.4, 7.5, 7.8$, the first term of the right–hand side is less than

$$\sum_{i\geq(\delta/4)(n/f(n))^3}\exp\Big(i\ln\big(8n^3(\mathcal{L}^3(\Lambda)+1)\big)+i5^{-3}\big(\ln b(\eta)-c(\eta)f(n)\big)\Big)\,.$$

Because $f(n)/\ln n$ goes to ∞ as n goes to ∞, for n large enough, the sum is less than $b'(\eta)\exp\big(-c'(\eta)(\delta/4)n^3 f(n)^{-2}\big)$ for some positive constants $b'(\eta)$, $c'(\eta)$. $\square$

12.6. Proof of the enhanced upper bound

We introduce some notation necessary to state the next results. Let B be a closed ball in $\mathbb{R}^3$. We define $\mathrm{BC}(B)$ as the subset of $\mathrm{BC}(\mathbb{R}^3)$ consisting of the Borel collections $\mathcal{A}$ such that:

$$\forall\, A\in\mathcal{A}\qquad \text{either } \mathcal{L}^3(A\setminus B)=0 \quad\text{or}\quad \mathcal{L}^3(\mathbb{R}^3\setminus A\setminus B)=0\,.$$

That is, up to negligible sets, the elements of a Borel collection $\mathcal{A}$ in $\mathrm{BC}(B)$ are either included in B or contain $\mathbb{R}^3\setminus B$.

Lemma 12.8. *The set $BC(B)$ is a closed subset of $(BC(\mathbb{R}^3), D_{\mathrm{loc}})$.*

Remark. The set $\mathrm{BC}(B)$ is a fortiori closed for the stronger metric D.

Proof. Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{BC}(B)$ converging towards an element $\mathcal{A}$ of $\mathrm{BC}(\mathbb{R}^3)$ with respect to the metric D_{loc}. Let A belong to $\mathcal{A}$. By Lemma 10.4, there exists a sequence $(A_n)_{n\in\mathbb{N}}$ such that A_n belongs to $\mathcal{A}_n$ for each n in $\mathbb{N}$ and for each compact set K, $\lim_{n\to\infty}\mathcal{L}^3_K(A\Delta A_n)=0$. Up to the extraction of a subsequence, we have

• either $\mathcal{L}^3(A_n\setminus B)=0$ for all n in $\mathbb{N}$, whence $\mathcal{L}^3_K(A\setminus B)=0$ for each compact set K;

• or $\mathcal{L}^3(\mathbb{R}^3\setminus A_n\setminus B)=0$ for all n in $\mathbb{N}$, whence $\mathcal{L}^3_K(\mathbb{R}^3\setminus A\setminus B)=0$ for each compact set K.

Letting K grow to $\mathbb{Z}^3$, we see that either $\mathcal{L}^3(A\setminus B)=0$ or $\mathcal{L}^3(\mathbb{R}^3\setminus A\setminus B)=0$. Therefore $\mathcal{A}$ belongs to $\mathrm{BC}(B)$. $\square$

Let m be an integer and let $B_1,\dots,B_m$ be m closed balls in $\mathbb{R}^3$. We consider the product space $\mathrm{BC}(B_1)\times\cdots\times\mathrm{BC}(B_m)$ endowed with the metric D_m defined by

$$\forall\,(\mathcal{A}_1,\dots,\mathcal{A}_m),(\mathcal{A}'_1,\dots,\mathcal{A}'_m)\in\mathrm{BC}(B_1)\times\cdots\times\mathrm{BC}(B_m)$$
$$D_m\big((\mathcal{A}_1,\dots,\mathcal{A}_m),(\mathcal{A}'_1,\dots,\mathcal{A}'_m)\big)\;=\;\max_{1\le i\le m}D(\mathcal{A}_i,\mathcal{A}'_i)\,.$$

Let $\mathcal{I}^m$ be the map from $\mathrm{BC}(B_1)\times\cdots\times\mathrm{BC}(B_m)$ to $\mathbb{R}^+\cup\{\infty\}$ defined by:

$$\forall\,(\mathcal{A}_1,\dots,\mathcal{A}_m)\in\mathrm{BC}(B_1)\times\cdots\times\mathrm{BC}(B_m)$$
$$\mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m)\;=\;\mathcal{I}(\mathcal{A}_1,\mathrm{int}\,B_1)+\cdots+\mathcal{I}(\mathcal{A}_m,\mathrm{int}\,B_m)\,.$$

Notice that $\mathcal{I}(\mathcal{A}_1,\dots,\mathcal{A}_m)$ is infinite whenever one of the Borel collections among $\mathcal{A}_1,\dots,\mathcal{A}_m$ is not a Caccioppoli partition of $\mathbb{R}^3$.

Proposition 12.9. *The map $\mathcal{I}^m$ is a good rate function on the space $BC(B_1)\times\cdots\times BC(B_m)$ endowed with the metric D_m.*

Proof. It is enough to consider the case of one ball B (i.e., $m=1$) and to prove that for any positive λ, the level set

$$\big\{\,\mathcal{A}\in\mathrm{BC}(B):\mathcal{I}(\mathcal{A},\mathrm{int}\,B)\le\lambda\,\big\}$$

is compact with respect to the metric D. Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{BC}(B)$ such that $\mathcal{I}(\mathcal{A}_n,\mathrm{int}\,B)\le\lambda$ for all n in $\mathbb{N}$. Then for each n in $\mathbb{N}$,

$$\mathcal{I}(\mathcal{A}_n)\;\le\;\mathcal{I}(\mathcal{A}_n,\mathrm{int}\,B)+||\tau||_\infty\mathcal{H}^2(\partial B)$$

and, by Lemma 11.1, the sequence of perimeters $(\mathcal{P}(\mathcal{A}_n))_{n\in\mathbb{N}}$ is bounded. Theorem 10.8 implies that, up to the extraction of a subsequence, the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ converges towards an element $\mathcal{A}$ of $\mathrm{CP}(\mathbb{R}^3)$ with respect to the metric D_{loc}. By Lemma 12.8, the collection $\mathcal{A}$ belongs to $\mathrm{BC}(B)$. Moreover we have

$$\mathcal{I}(\mathcal{A},\mathrm{int}\,B)\;=\;\frac{1}{2}\sum_{A\in\mathcal{A}}\mathcal{I}(A,\mathrm{int}\,B)\,.$$

Since the map $A \in \mathcal{B}(\mathbb{R}^3) \mapsto \mathcal{I}(A, \operatorname{int} B)$ is lower semicontinuous on $(\mathcal{B}(\mathbb{R}^3), L^1_{\text{loc}})$ (see just after Definition 6.2), Lemma 10.5 implies that

$$\mathcal{I}(\mathcal{A}, \operatorname{int} B) \leq \liminf_{n\to\infty} \mathcal{I}(\mathcal{A}_n, \operatorname{int} B).$$

It remains to show that the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ converges towards $\mathcal{A}$ with respect to the metric D. Let ε be positive and let K be a compact set containing B and such that $\mathcal{L}^3(K \setminus B) > 2\varepsilon$. There exists n_0 such that for $n \geq n_0$, we have $D_K(\mathcal{A}_n, \mathcal{A}) < \varepsilon$. Let n be larger than n_0 and let $(A(i),\, i \in \mathbb{N})$, $(A_n(i),\, i \in \mathbb{N})$ be arrangements of $\mathcal{A}$ and $\mathcal{A}_n$ such that $\sup_{i\in\mathbb{N}} \mathcal{L}^3_K(A(i)\Delta A_n(i)) \leq \varepsilon$. Let i belong to $\mathbb{N}$. Since we deal with elements of $\mathrm{BC}(B)$, four cases can occur:

- $\mathcal{L}^3(A(i)\setminus B) = \mathcal{L}^3(\mathbb{R}^3 \setminus A_n(i)\setminus B) = 0$. Then $\mathcal{L}^3_K(A(i)\Delta A_n(i)) \geq \mathcal{L}^3(K\setminus B) > 2\varepsilon$, which is impossible.
- $\mathcal{L}^3(\mathbb{R}^3 \setminus A(i)\setminus B) = \mathcal{L}^3(A_n(i)\setminus B) = 0$. Then $\mathcal{L}^3_K(A(i)\Delta A_n(i)) \geq \mathcal{L}^3(K\setminus B) > 2\varepsilon$, which is impossible.
- $\mathcal{L}^3(A(i) \setminus B) = \mathcal{L}^3(A_n(i) \setminus B) = 0$. Then $\mathcal{L}^3(A(i)\Delta A_n(i)) \leq \varepsilon$.
- $\mathcal{L}^3(\mathbb{R}^3 \setminus A(i) \setminus B) = \mathcal{L}^3(\mathbb{R}^3 \setminus A_n(i) \setminus B) = 0$. Then $\mathcal{L}^3(A(i)\Delta A_n(i)) \leq \varepsilon$.

Thus we obtain that $\sup_{i\in\mathbb{N}} \mathcal{L}^3(A(i)\Delta A_n(i)) \leq \varepsilon$ whenever n is larger than n_0, which implies that $D(\mathcal{A}_n, \mathcal{A}) \leq \varepsilon$. Therefore the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ converges towards $\mathcal{A}$ with respect to the metric D. $\square$

Let $T(B_1, \ldots, B_m)$ be the set of the m–uples of integer translations sending $B_1, \ldots, B_m$ onto pairwise disjoint balls, that is,

$$T(B_1, \ldots, B_m) = \\ \big\{\, (x_1, \ldots, x_m) \in (\mathbb{Z}^3)^m : x_1 + B_1, \ldots, x_m + B_m \text{ are pairwise disjoint} \,\big\}.$$

If $\mathcal{A}$ is a collection of Borel sets of $\mathbb{R}^3$ and B a ball in $\mathbb{R}^3$, we define $\mathcal{A} \cap B$ to be the collection of $\mathrm{BC}(B)$ given by

$$\mathcal{A} \cap B = \\ \big\{\, (A \cap B) \cup (\mathbb{R}^3 \setminus B) : A \in \mathcal{A},\, \mathcal{L}^3(A) = \infty \,\big\} \cup \big\{\, A \cap B : A \in \mathcal{A},\, \mathcal{L}^3(A) < \infty \,\big\} \setminus \{\varnothing\}.$$

In this definition, as usual, we consider equivalence classes of sets modulo negligible sets. Notice that if $\mathcal{A}$ is a Caccioppoli partition of $\mathbb{R}^3$ having finite perimeter, so is $\mathcal{A} \cap B$. To $B_1, \ldots, B_m$ and $(x_1, \ldots, x_m)$ in $T(B_1, \ldots, B_m)$ we associate the map $\phi^{B_1,\ldots,B_m}_{x_1,\ldots,x_m}$ from $\mathrm{BC}(\mathbb{R}^3)$ to $\mathrm{BC}(B_1) \times \cdots \times \mathrm{BC}(B_m)$ defined by

$$\forall\, \mathcal{A} \in \mathrm{BC}(\mathbb{R}^3) \qquad \phi^{B_1,\ldots,B_m}_{x_1,\ldots,x_m}(\mathcal{A}) = \big(\mathcal{A} \cap (B_1 + x_1) - x_1, \ldots, \mathcal{A} \cap (B_m + x_m) - x_m\big).$$

Lemma 12.10. *For any $\mathcal{A}_1$, $\mathcal{A}_2$ in $BC(\mathbb{R}^3)$,*

$$D_m\Big(\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}(\mathcal{A}_1),\,\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}(\mathcal{A}_2)\Big)\ \le\ \mathcal{L}^3(\mathrm{overlap}\,\mathcal{A}_1)+D(\mathcal{A}_1,\mathcal{A}_2)+\mathcal{L}^3(\mathrm{overlap}\,\mathcal{A}_2)\,.$$

Proof. For $j=1,2$, let $(A_j(i), i\in\mathbb{N})$ be an arrangement of $\mathcal{A}_j$. For each h in $\{1,\dots,m\}$ and for $j=1,2$, we set for i in $\mathbb{N}$

$$A_{j,h}(i)\ =\ \begin{cases} A_j(i)\cap(B_h+x_h)-x_h & \text{if } \mathcal{L}^3(A_j(i))<\infty\\ \big(\mathbb{R}^3\setminus(B_h+x_h)\big)\cup\big(A_j(i)\cap(B_h+x_h)\big)-x_h & \text{if } \mathcal{L}^3(A_j(i))=\infty\end{cases}$$

and then

$$A'_{j,h}(i)\ =\ \begin{cases}\varnothing & \text{if there exists } k<i \text{ such that } A_{j,h}(k)=A_{j,h}(i)\\ A_{j,h}(i) & \text{otherwise}\end{cases}\,.$$

For $j=1,2$, the sequence $(A'_{j,h}(i))_{i\in\mathbb{N}}$ is an arrangement of $\mathcal{A}_j\cap(B_h+x_h)-x_h$. Let h belong to $\{1,\dots,m\}$ and let us evaluate $\mathcal{L}^3(A'_{1,h}(i)\Delta A'_{2,h}(i))$ for i in $\mathbb{N}$. First, we have, for any i in $\mathbb{N}$,

$$\mathcal{L}^3(A'_{1,h}(i)\Delta A'_{2,h}(i))\ \le\ \mathcal{L}^3(\mathrm{overlap}\,\mathcal{A}_1)+\mathcal{L}^3(A_{1,h}(i)\Delta A_{2,h}(i))+\mathcal{L}^3(\mathrm{overlap}\,\mathcal{A}_2)\,.$$

We next evaluate $\mathcal{L}^3(A_{1,h}(i)\Delta A_{2,h}(i))$. We distinguish four cases.

- $\mathcal{L}^3(A_1(i))=\infty$ and $\mathcal{L}^3(A_2(i))<\infty$. Then $\mathcal{L}^3(A_1(i)\Delta A_2(i))=\infty$.
- $\mathcal{L}^3(A_1(i))<\infty$ and $\mathcal{L}^3(A_2(i))=\infty$. Then $\mathcal{L}^3(A_1(i)\Delta A_2(i))=\infty$.
- $\mathcal{L}^3(A_1(i))=\mathcal{L}^3(A_2(i))=\infty$. Then

$$\mathcal{L}^3(A_{1,h}(i)\Delta A_{2,h}(i))\ =\ \mathcal{L}^3_{x_h+B_h}(A_1(i)\Delta A_2(i))\ \le\ \mathcal{L}^3(A_1(i)\Delta A_2(i))\,.$$

- $\mathcal{L}^3(A_1(i))<\infty$ and $\mathcal{L}^3(A_2(i))<\infty$. Then

$$\mathcal{L}^3(A_{1,h}(i)\Delta A_{2,h}(i))\ =\ \mathcal{L}^3_{x_h+B_h}(A_1(i)\Delta A_2(i))\ \le\ \mathcal{L}^3(A_1(i)\Delta A_2(i))\,.$$

Therefore, for any h in $\{1,\dots,m\}$, we have

$$\sup_{i\in\mathbb{N}}\mathcal{L}^3(A'_{1,h}(i)\Delta A'_{2,h}(i))\ \le\ \mathcal{L}^3(\mathrm{overlap}\,\mathcal{A}_1)+\sup_{i\in\mathbb{N}}\mathcal{L}^3(A_1(i)\Delta A_2(i))+\mathcal{L}^3(\mathrm{overlap}\,\mathcal{A}_2)\,.$$

Taking the infimum over all possible arrangements, we get

$$\max_{1\le h\le m} D\big(\mathcal{A}_1\cap(B_h+x_h)-x_h,\mathcal{A}_2\cap(B_h+x_h)-x_h\big)\ \le\ D(\mathcal{A}_1,\mathcal{A}_2)$$

which is the claim of the Lemma. $\square$

Let U be a bounded subset of $\mathbb{R}^3$. We recall that

$$\mathcal{C}(nU, f(n)) \;=\; \big\{\, C \in \mathcal{C},\, f(n) < \operatorname{diam}_\infty C < \infty,\, C \cap nU \neq \varnothing \,\big\}$$

and we define the following random sets and Borel collections:

$$\begin{aligned}
\mathcal{V}_\infty\mathcal{C}(nU)_0 \;&=\; \big\{\, \mathcal{V}_\infty(C, f(n)) : C \in \mathcal{C}(nU, f(n)) \,\big\}\,,\\
\mathcal{V}_\infty\mathcal{C}(nU)_\infty \;&=\; \mathbb{R}^3 \setminus \operatorname{cup} \mathcal{V}_\infty\mathcal{C}(nU)_0\,,\\
\mathcal{V}_\infty\mathcal{C}(nU) \;&=\; \mathcal{V}_\infty\mathcal{C}(nU)_0 \,\cup\, \{\, \mathcal{V}_\infty\mathcal{C}(nU)_\infty \,\}\,,
\end{aligned}$$

$$\begin{aligned}
\operatorname{fifa}\mathcal{C}(nU)_0 \;&=\; \big\{\, \operatorname{fifa} C : C \in \mathcal{C}(nU, f(n)) \,\big\}\,,\\
\operatorname{fifa}\mathcal{C}(nU)_\infty \;&=\; \mathbb{R}^3 \setminus \operatorname{cup} \operatorname{fifa}\mathcal{C}(nU)_0\,,\\
\operatorname{fifa}\mathcal{C}(nU) \;&=\; \operatorname{fifa}\mathcal{C}(nU)_0 \,\cup\, \{\, \operatorname{fifa}\mathcal{C}(nU)_\infty \,\}\,,
\end{aligned}$$

$$\begin{aligned}
\operatorname{aglu}\mathcal{C}(nU)_0 \;&=\; \big\{\, \operatorname{aglu} C : C \in \mathcal{C}(nU, f(n)) \,\big\}\,,\\
\operatorname{aglu}\mathcal{C}(nU)_\infty \;&=\; \mathbb{R}^3 \setminus \operatorname{cup} \operatorname{aglu}\mathcal{C}(nU)_0\,,\\
\operatorname{aglu}\mathcal{C}(nU) \;&=\; \operatorname{aglu}\mathcal{C}(nU)_0 \,\cup\, \{\, \operatorname{aglu}\mathcal{C}(nU)_\infty \,\}\,.
\end{aligned}$$

The notation $\mathcal{V}_\infty\mathcal{C}(nU)$ corresponds to $\mathcal{V}_\infty\mathcal{C}(nU, f(n))$ in the statements of Theorems 2.6 and 2.7. We drop $f(n)$ to simplify the notation.

Proposition 12.11. *Let U be a bounded Borel subset of $\mathbb{R}^3$. The sequences of random Borel collections*

$$n^{-1}\mathcal{V}_\infty\mathcal{C}(nU), \quad n^{-1}\operatorname{fifa}\mathcal{C}(nU), \quad n^{-1}\operatorname{aglu}\mathcal{C}(nU)$$

are exponentially contiguous with respect to the metric D, that is, for any positive δ,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n^2} \ln P\Big(D\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU),\, n^{-1}\operatorname{fifa}\mathcal{C}(nU)\big) \geq \delta \Big) \;&=\; -\infty\,,\\
\limsup_{n\to\infty} \frac{1}{n^2} \ln P\Big(D\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU),\, n^{-1}\operatorname{aglu}\mathcal{C}(nU)\big) \geq \delta \Big) \;&=\; -\infty\,.
\end{aligned}$$

Proof. We write, for any positive a, and for **** being either fifa or aglu,

$$\begin{aligned}
&P\Big(D\Big(\mathcal{V}_\infty\mathcal{C}(nU), {}^{****}\mathcal{C}(nU) \Big) \geq \delta n^3 \Big) \;\leq\\
&P\Big(\exists\, C \in \mathcal{C}(nU, f(n)) \quad an^2 \leq \operatorname{diam}_\infty C < \infty \Big) +\\
&P\Big(\mathcal{P}(\operatorname{fifa}\mathcal{C}(nU)) \geq an^2, \text{ each set of } \operatorname{fifa}\mathcal{C}(nU)_0 \text{ is included in } \mathcal{V}_\infty(nU, an^2 + f(n)) \Big)\\
&\qquad\qquad + P\Big(D\Big(\mathcal{V}_\infty\mathcal{C}(nU), {}^{****}\mathcal{C}(nU) \Big) \geq \delta n^3,\, \mathcal{P}(\operatorname{fifa}\mathcal{C}(nU)) < an^2 \Big)\,.
\end{aligned}$$

By Lemma 12.2, the first term is bounded by $(n\mathrm{diam}_\infty U+f(n))^3 b(k)\exp(-c(k)an^2)$. The second term is bounded by $P\big(\mathcal{P}\big(\mathrm{fifa}\,\mathcal{C}(nU),\mathcal{V}_\infty(nU,an^2+f(n))\big)\geq an^2\big)$ which, in turn, by Proposition 12.1, is bounded by $b(k)\exp(-c(k)an^2)$. We now examine the third term. We suppose that the event $\{\,\mathcal{P}(\mathrm{fifa}\,\mathcal{C}(nU))<an^2\,\}$ occurs and we distinguish the case of fifa and aglu to bound the third term.

• Case of fifa. Proceeding as in Lemma 12.4, we have

$$\begin{aligned}D\Big(\mathcal{V}_\infty\mathcal{C}(nU),\mathrm{fifa}\,\mathcal{C}(nU)\Big)\ \leq\ \max\Big(&f(n)^3\mathrm{card}\,\{\,y\in\Lambda(0,2an^2)\cap f(n)\mathbb{Z}^3:Y(y)=1\,\}\\&+280an^2f(n)^2,\mathcal{L}^3\big(\mathcal{V}_\infty\mathcal{C}(nU)_\infty\Delta\mathrm{fifa}\,\mathcal{C}(nU)_\infty\big)\Big)\,.\end{aligned}$$

The first term in the max is controlled by Lemma 12.7. For the second term,

$$\begin{aligned}\mathcal{L}^3\big(\mathcal{V}_\infty\mathcal{C}(nU)_\infty\Delta\mathrm{fifa}\,\mathcal{C}(nU)_\infty\big)\ &=\ \mathcal{L}^3\big(\mathrm{cup}\,\mathcal{V}_\infty\mathcal{C}(nU)_0\Delta\mathrm{cup}\,\mathrm{fifa}\,\mathcal{C}(nU)_0\big)\\&\leq\sum_{C\in\mathcal{C}(nU,f(n))}\mathcal{L}^3\big(\mathcal{V}_\infty(C,f(n))\setminus\mathrm{fifa}\,C\big)\\&\leq\sum_{C\in\mathcal{C}(nU,f(n))}140f(n)^2\mathcal{P}(\mathrm{fifa}\,C)\ \leq\ 140an^2f(n)^2\,.\end{aligned}$$

• Case of aglu. Proceeding as in Lemma 12.6, we have

$$\begin{aligned}&D\Big(\mathcal{V}_\infty\mathcal{C}(nU),\mathrm{aglu}\,\mathcal{C}(nU)\Big)\ \leq\\&\max\Big(27f(n)^3\mathrm{card}\,\{\,y\in\Lambda(0,2an^2)\cap f(n)\mathbb{Z}^3:Y(y)=1\,\}\\&\qquad+17920an^2f(n)^2,\mathcal{L}^3\big(\mathcal{V}_\infty\mathcal{C}(nU)_\infty\Delta\mathrm{aglu}\,\mathcal{C}(nU)_\infty\big)\Big)\,.\end{aligned}$$

The first term in the max is controlled by Lemma 12.7. For the second term, using the same argument as in Lemma 9.15, we get

$$\begin{aligned}\mathcal{L}^3\big(\mathcal{V}_\infty\mathcal{C}(nU)_\infty\Delta\mathrm{aglu}\,\mathcal{C}(nU)_\infty\big)\ &=\ \mathcal{L}^3\big(\mathrm{cup}\,\mathcal{V}_\infty\mathcal{C}(nU)_0\Delta\mathrm{cup}\,\mathrm{aglu}\,\mathcal{C}(nU)_0\big)\\&\leq\ 3^3f(n)^3\mathrm{card}\,\big\{\,y\in\Lambda(0,2an^2):Y(y)=1\,\big\}+17920an^2f(n)^2\,.\end{aligned}$$

In both cases, we conclude by sending successively n and a to ∞. ☐

Lemma 12.12. *Let U be a bounded Borel subset of $\mathbb{R}^3$. For any positive δ, we have*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\Big(\mathcal{L}^3\big(\mathrm{overlap}\,\mathrm{fifa}\,\mathcal{C}(nU)\big)\geq\delta n^3\Big)\ =\ -\infty\,,$$
$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P\Big(\mathcal{L}^3\big(\mathrm{overlap}\,\mathrm{aglu}\,\mathcal{C}(nU)\big)\geq\delta n^3\Big)\ =\ -\infty\,.$$

Proof. For any open cluster C, $\mathrm{fifa}\,C$ is included in $\mathcal{V}_\infty(C,f(n))$ and $\mathrm{aglu}\,C$ is included in $\mathcal{V}_\infty(C,5f(n))$. This implies that

$$\mathrm{overlap}\,\mathrm{fifa}\,\mathcal{C}(nU)\subset\mathrm{overlap}\,\mathcal{V}_\infty\mathcal{C}(nU)\,.$$

Therefore, Proposition 2.8 yields the result concerning $\mathrm{fifa}\,\mathcal{C}(nU)$. The result on $\mathrm{aglu}\,\mathcal{C}(nU)$ follows from a slight variation of the argument. ☐

Lemma 12.13. *For any closed subset $\mathbb{F}^m$ of $BC(B_1)\times\cdots\times BC(B_m)$,*

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\sup_{(x_1,\dots,x_m)\in T(B_1,\dots,B_m)}P\Big(\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)\big)\in\mathbb{F}^m\Big)$$
$$\leq-\inf\big\{\,\mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m):(\mathcal{A}_1,\dots,\mathcal{A}_m)\in\mathbb{F}^m\,\big\}\,.$$

Proof. We prove the large deviations upper bound in a standard fashion, by proving the $\mathcal{I}^m$–tightness together with a local estimate. Moreover the estimates are uniform with respect to $(x_1,\dots,x_m)$ in $T(B_1,\dots,B_m)$.

• *$\mathcal{I}^m$–tightness.* Let λ,δ be positive. For any $(x_1,\dots,x_m)$ in $T(B_1,\dots,B_m)$, we have

$$\begin{aligned}P\Big(\inf\Big\{&D_m\big(\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}\big(n^{-1}\mathrm{fifa}\,\mathcal{C}(nU)\big),(\mathcal{A}_1,\dots,\mathcal{A}_m)\big):\\&(\mathcal{A}_1,\dots,\mathcal{A}_m)\in\mathrm{BC}(B_1)\times\cdots\times\mathrm{BC}(B_m),\,\mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m)\leq\lambda\Big\}\geq\delta\Big)\\&\leq P\Big(\mathcal{I}^m\Big(\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}\big(n^{-1}\mathrm{fifa}\,\mathcal{C}(nU)\big)\Big)\geq\lambda\Big)\\&\leq P\Big(\mathcal{I}(n^{-1}\mathrm{fifa}\,\mathcal{C}(nU),x_1+\mathrm{int}\,B_1)+\cdots+\mathcal{I}(n^{-1}\mathrm{fifa}\,\mathcal{C}(nU),x_m+\mathrm{int}\,B_m)\geq\lambda\Big)\\&\leq P\big(\mathcal{I}(n^{-1}\mathrm{fifa}\,\mathcal{C}(nU))\geq\lambda\big)\;\leq\;P\big(\mathcal{P}(\mathrm{fifa}\,\mathcal{C}(nU))\geq n^2\lambda/||\tau||_\infty\big)\,.\end{aligned}$$

In the last step, we have used Lemma 11.1. Next, letting $\lambda'=\lambda/||\tau||_\infty$,

$$\begin{aligned}&P\big(\mathcal{P}(\mathrm{fifa}\,\mathcal{C}(nU))\geq n^2\lambda'\big)\;\leq\;P\Big(\exists\,C\in\mathcal{C}(nU,f(n))\quad\lambda'n^2\leq\mathrm{diam}_\infty C<\infty\Big)+\\&P\Big(\mathcal{P}(\mathrm{fifa}\,\mathcal{C}(nU))\geq\lambda'n^2,\text{each set of }\mathrm{fifa}\,\mathcal{C}(nU)_0\text{ is included in }\mathcal{V}_\infty(nU,\lambda'n^2+f(n))\Big).\end{aligned}$$

By Lemma 12.2, the first term is less than $(n\mathrm{diam}_\infty U+f(n))^3b(k)\exp(-c(k)\lambda'n^2)$. The second term is bounded by

$$P\Big(\mathcal{P}\big(\mathrm{fifa}\,\mathcal{C}(nU),\mathcal{V}_\infty(nU,an^2+f(n))\big)\geq\lambda'n^2\Big)\,,$$

which, in turn, by Proposition 12.1, is bounded by $b(k)\exp(-c(k)\lambda'n^2)$. Moreover we have, using Lemma 12.10,

$$\begin{aligned}&P\Big(D_m\big(\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathcal{V}_\infty\mathcal{C}(nU)\big),\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathrm{fifa}\,\mathcal{C}(nU)\big)\big)\geq\delta\Big)\;\leq\\&P\Big(\mathcal{L}^3\big(\mathrm{overlap}\,\frac{1}{n}\mathcal{V}_\infty\mathcal{C}(nU)\big)+D\big(\frac{1}{n}\mathcal{V}_\infty\mathcal{C}(nU),\frac{1}{n}\mathrm{fifa}\,\mathcal{C}(nU)\big)\\&\qquad\qquad+\mathcal{L}^3\big(\mathrm{overlap}\,\frac{1}{n}\mathrm{fifa}\,\mathcal{C}(nU)\big)\geq\delta\Big)\,.\end{aligned}$$

These bounds are independent of $(x_1,\dots,x_m)$ in $T(B_1,\dots,B_m)$. Using Propositions 2.8, 12.11 and Lemma 12.12, we deduce from the preceding bounds that there exists a positive constant $c'=c(k)/||\tau||_\infty$ such that for any positive λ,δ,

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\sup_{x_1,\dots,x_m}P\Big(\inf\Big\{D_m\big(\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathcal{V}_\infty\mathcal{C}(nU)\big),(\mathcal{A}_1,\dots,\mathcal{A}_m)\big):$$
$$(\mathcal{A}_1,\dots,\mathcal{A}_m)\in \mathrm{BC}(B_1)\times\cdots\times\mathrm{BC}(B_m),\,\mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m)\le\lambda\Big\}\ge\delta\Big)\;\le\;-c'\lambda$$

where the supremum is taken over $(x_1,\dots,x_m)$ in $T(B_1,\dots,B_m)$.

• *Local estimate.* Let $(\mathcal{A}_1,\dots,\mathcal{A}_m)$ be an element of $\mathrm{BC}(B_1)\times\cdots\times\mathrm{BC}(B_m)$ such that $\mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m)<\infty$. We have to show that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n^2}\ln\sup_{x_1,\dots,x_m}P\Big(D_m\big(\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathcal{V}_\infty\mathcal{C}(nU)\big),(\mathcal{A}_1,\dots,\mathcal{A}_m)\big)<\delta\Big)$$
$$\le\;-\mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m)$$

where the supremum is taken over $(x_1,\dots,x_m)$ in $T(B_1,\dots,B_m)$. Let ε be positive. Let δ_0 be associated to ε as in Lemma 8.1. For each l in $\{1,\dots,m\}$, by Lemma 11.5 applied to the Caccioppoli partition $\mathcal{A}_l$ and the ball B_l, there exists a finite collection of disjoint balls $B(x_i^l,r_i^l)$, $i\in I(l)$, such that: for any i in $I(l)$, x_i^l belongs to $\partial^*\mathcal{A}_l\cap B_l$, r_i^l belongs to $]0,1[$, $B(x_i^l,r_i^l)$ is included in B_l and

$$\mathcal{L}^3\big((A_{l,1}(x_i)\cap B(x_i^l,r_i^l))\Delta B_-(x_i^l,r_i^l,\nu_{\mathcal{A}_l}(x_i^l))\big)\;\le\;\delta_0\,(r_i^l)^3/3\,,$$
$$\mathcal{L}^3\big((A_{l,2}(x_i)\cap B(x_i^l,r_i^l))\Delta B_+(x_i^l,r_i^l,\nu_{\mathcal{A}_l}(x_i^l))\big)\;\le\;\delta_0\,(r_i^l)^3/3\,,$$
$$\Big|\mathcal{I}(\mathcal{A}_l,\mathrm{int}\,B_l)-\sum_{i\in I(l)}\pi(r_i^l)^2\tau(\nu_{\mathcal{A}_l}(x_i^l))\Big|\;\le\;\varepsilon\,.$$

For the notation $A_{l,1}$, $A_{l,2}$, see the beginning of chapter 11. Let δ be a positive real number strictly smaller than $\min\{\,\delta_0(r_i^l)^3/6:1\le l\le m,\,i\in I(l)\,\}$. Let $\mathcal{E}(\delta)$ be the event

$$\mathcal{E}(\delta)\;=\;\Big\{\,D_m\big(\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}\big(n^{-1}\mathrm{aglu}\,\mathcal{C}(nU)\big),(\mathcal{A}_1,\dots,\mathcal{A}_m)\big)<2\delta\,\Big\}\,.$$

For $(x_1,\dots,x_m)$ in $T(B_1,\dots,B_m)$, we write, using Lemma 12.10,

$$\begin{aligned}
&P\Big(D_m\big(\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathcal{V}_\infty\mathcal{C}(nU)\big),(\mathcal{A}_1,\dots,\mathcal{A}_m)\big)<\delta\Big)\\
&\quad\le P\Big(D_m\big(\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathcal{V}_\infty\mathcal{C}(nU)\big),\phi^{B_1,\dots,B_m}_{x_1,\dots,x_m}\big(\frac{1}{n}\mathrm{aglu}\,\mathcal{C}(nU)\big)\big)\ge\delta\Big)\;+\;P(\mathcal{E}(\delta))\\
&\qquad\le P\Big(\mathcal{L}^3\big(\mathrm{overlap}\,\frac{1}{n}\mathcal{V}_\infty\mathcal{C}(nU)\big)+D\big(\frac{1}{n}\mathcal{V}_\infty\mathcal{C}(nU),\frac{1}{n}\mathrm{aglu}\,\mathcal{C}(nU)\big)+\\
&\qquad\qquad\mathcal{L}^3\big(\mathrm{overlap}\,\frac{1}{n}\mathrm{aglu}\,\mathcal{C}(nU)\big)\ge\delta\Big)\;+\;P(\mathcal{E}(\delta))\,.
\end{aligned}$$

Propositions 2.8, 12.11 and Lemma 12.12 give a sufficient control on the first term. Hence we need only to estimate $P(\mathcal{E}(\delta))$. Suppose that $\mathcal{E}(\delta)$ occurs. Suppose also that n is large enough to have

$$\forall\, l \in \{1,\dots,m\} \quad \forall\, i \in I(l) \qquad 4\pi\sqrt{3}(r_i^l)^2 n^2 + \pi\sqrt{3} < \delta_0 (r_i^l)^3 n^3/3\,.$$

Let l belong to $\{1,\dots,m\}$ and let i belong to $I(l)$. There exists a set $C_n(x_i^l)$ belonging to $\operatorname{aglu}\mathcal{C}(nU)$ such that

$$\mathcal{L}^3\big((nA_{l,1}(x_i^l)\cap nB_l)\Delta\big(C_n(x_i^l)\cap(nB_l+nx_l)-nx_l\big)\big) \;<\; 2\delta\, n^3\,.$$

Let $\mathcal{C}(i,l)$ be the collection of the open clusters of the configuration restricted to the set $C_n(x_i^l)\cap B(nx_l+nx_i^l,nr_i^l)$. Because of the definition of $\operatorname{aglu}\mathcal{C}(nU)$, the clusters of $\mathcal{C}(l,i)$ are open clusters of the configuration restricted to $B(nx_l+nx_i^l,nr_i^l)$. Moreover

$$\begin{aligned}
&\mathcal{L}^3\Big(\mathcal{V}_\infty(\operatorname{cup}\mathcal{C}(l,i),1/2)\Delta B_-(nx_l+nx_i^l,nr_i^l,\nu_{\mathcal{A}_l}(x_i^l))\Big)\\
&\quad\le \mathcal{L}^3\Big(\mathcal{V}_\infty(\operatorname{cup}\mathcal{C}(l,i),1/2)\Delta(C_n(x_i^l)\cap B(nx_l+nx_i^l,nr_i^l))\Big)\\
&\qquad\qquad + \mathcal{L}^3\Big((C_n(x_i^l)\cap B(nx_l+nx_i^l,nr_i^l))\Delta B_-(nx_l+nx_i^l,nr_i^l,\nu_{\mathcal{A}_l}(x_i^l))\Big)\\
&\quad\le \mathcal{L}^3\big(B(nx_i^l,nr_i^l+\sqrt{3}/2)\setminus B(nx_i^l,nr_i^l-\sqrt{3}/2)\big)\\
&\qquad + \mathcal{L}^3\big(\big((nB_l+nx_l)\cap C_n(x_i^l)-nx_l\big)\Delta(nA_{l,1}(x_i^l)\cap nB_l)\big)\\
&\qquad\qquad + \mathcal{L}^3\Big((nA_{l,1}(x_i^l)\cap B(nx_i^l,nr_i^l))\Delta B_-(nx_i^l,nr_i^l,\nu_{\mathcal{A}_l}(x_i^l))\Big)\\
&\quad< 4\pi\sqrt{3}(r_i^l)^2n^2+\pi\sqrt{3}+2\delta n^3+\delta_0(r_i^l)^3n^3/3<\delta_0(r_i^l)^3n^3\,.
\end{aligned}$$

Therefore the collection $\mathcal{C}(l,i)$ realizes the event $\operatorname{sep}(n,x_l+x_i^l,r_i^l,\nu_{\mathcal{A}_l}(x_i^l),\delta_0)$ and

$$\mathcal{E}(\delta)\;\subset\;\bigcap_{1\le l\le m}\bigcap_{i\in I(l)}\operatorname{sep}(n,x_l+x_i^l,r_i^l,\nu_{\mathcal{A}_l}(x_i^l),\delta_0)\,.$$

Since the balls $B(x_i^l,r_i^l)$, $i\in I(l)$, are disjoint for any l in $\{1,\dots,m\}$ and since $(x_1,\dots,x_m)$ belongs to $T(B_1,\dots,B_m)$, then the balls $B(nx_l+nx_i^l,nr_i^l)$, $i\in I(l)$, $1\le l\le m$, are still disjoint and the events $\operatorname{sep}(n,x_l+x_i^l,r_i^l,\nu_{\mathcal{A}_l}(x_i^l),\delta_0)$, $i\in I(l)$, $1\le l\le m$, are independent, whence

$$P(\mathcal{E}(\delta))\;\le\;\prod_{1\le l\le m}\prod_{i\in I(l)}P\big(\operatorname{sep}(n,x_l+x_i^l,r_i^l,\nu_{\mathcal{A}_l}(x_i^l),\delta_0)\big)\,.$$

Since the model is invariant under integer translations, then for l in $\{1,\dots,m\}$ and i in $I(l)$

$$P\big(\operatorname{sep}(n,x_l+x_i^l,r_i^l,\nu_{\mathcal{A}_l}(x_i^l),\delta_0)\big)\;=\;P\big(\operatorname{sep}(n,x_i^l,r_i^l,\nu_{\mathcal{A}_l}(x_i^l),\delta_0)\big)$$

and

$$P(\mathcal{E}(\delta)) \le \prod_{1\le l\le m}\prod_{i\in I(l)} P\big(\mathrm{sep}\,(n, x_i^l, r_i^l, \nu_{\mathcal{A}_l}(x_i^l), \delta_0)\big)\,.$$

This last bound is independent of $(x_1,\dots,x_m)$ in $T(B_1,\dots,B_m)$. By the choice of δ_0 (see Lemma 8.1) and of the balls $B(x_i^l, r_i^l)$ (see Lemma 11.5)

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n^2}\ln\ \sup\Big\{P(\mathcal{E}(\delta)) : (x_1,\dots,x_m)\in T(B_1,\dots,B_m)\Big\}&\\ \le -\sum_{1\le l\le m}\sum_{i\in I(l)}\tau(\nu_{\mathcal{A}_l}(x_i^l))\pi(r_i^l)^2(1-\varepsilon)&\\ \le -\sum_{1\le l\le m}\Big(\mathcal{I}(\mathcal{A}_l,\mathrm{int}\,B_l)\,(1-\varepsilon)-\varepsilon\Big)&\\ = -\mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m)\,(1-\varepsilon)+m\varepsilon\,.&\end{aligned}$$

Letting δ and then ε go to zero, we get the upper bound we were seeking for. $\square$

We turn now to the proof of the enhanced upper bound of Theorem 2.11. Let $\mathbb{F}$ be a closed subset of $(\mathrm{BC}(\mathbb{R}^3), D)$. Let a,δ be positive with $\delta\le 1\le a$. We write

$$\begin{aligned}&P\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)\in\mathbb{F}\big) \le P\Big(\exists\, C\in\mathcal{C}(nU,f(n))\quad an^2\le \mathrm{diam}_\infty C<\infty\Big)+\\ &P\Big(\mathcal{P}(\mathrm{fifa}\,\mathcal{C}(nU)_0)\ge an^2,\ \text{each set of fifa}\,\mathcal{C}(nU)_0\text{ is included in }\mathcal{V}_\infty(nU,an^2+f(n))\Big)\\ &\qquad\qquad + P\big(D(\mathcal{V}_\infty\mathcal{C}(nU),\mathrm{fifa}\,\mathcal{C}(nU))\ge\delta n^3\big) + P(\mathcal{E})\,,\end{aligned}$$

where $\mathcal{E}$ is the event

$$\Big\{\, n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)\in\mathbb{F},\ \mathcal{P}(\mathrm{fifa}\,\mathcal{C}(nU)_0)<an^2,\ D(\mathcal{V}_\infty\mathcal{C}(nU),\mathrm{fifa}\,\mathcal{C}(nU))<\delta n^3\,\Big\}\,.$$

We examine successively each term of the right–hand side. By Lemma 12.2, the first term is bounded by $(n\mathrm{diam}_\infty U+f(n))^3 b(k)\exp(-c(k)an^2)$. The second term is bounded by

$$P\Big(\mathcal{P}\big(\mathrm{fifa}\,\mathcal{C}(nU),\mathcal{V}_\infty(nU,an^2+f(n))\big)\ge an^2\Big)\,,$$

which, in turn, by Proposition 12.1, is bounded by $b(k)\exp(-c(k)an^2)$. The third term is controlled by Proposition 12.11. We examine now the last term $P(\mathcal{E})$ of the inequality. Suppose that the event

$$\{\,\mathcal{P}(\mathrm{fifa}\,\mathcal{C}(nU)_0)<an^2\,\}$$

occurs. By the isoperimetric inequality, we have

$$\mathcal{L}^3\big(\mathrm{cup\,fifa}\,\mathcal{C}(nU)_0\big)\le c_{\mathrm{iso}}a^{3/2}n^3\,.$$

Let X be the random subset of $\mathbb{Z}^3$ defined by

$$X \;=\; \{\, x \in \mathbb{Z}^3 : \mathcal{L}^3(B(nx,n) \cap \operatorname{cup} \operatorname{fifa} \mathcal{C}(nU)_0) \geq \delta n^3 \,\}\,.$$

Since

$$\forall\, C \in \mathcal{C}(nU, f(n)) \qquad \operatorname{diam}_\infty \operatorname{fifa} C \leq \mathcal{P}(\operatorname{fifa} \mathcal{C}(nU)_0) < an^2\,,$$

then X is included in $B(0, n\operatorname{diam} U + 3an^2)$. Since a point of $\mathbb{R}^3$ belongs to at most eight balls among the balls $B(x,1)$, $x \in \mathbb{Z}^3$, then

$$\begin{aligned}
\delta n^3 \operatorname{card} X \;&\leq\; \sum_{x\in X} \mathcal{L}^3\big(B(nx,n) \cap \operatorname{cup} \operatorname{fifa} \mathcal{C}(nU)_0\big)\\
&\leq\; 8\mathcal{L}^3\big(\operatorname{cup} \operatorname{fifa} \mathcal{C}(nU)_0\big) \;\leq\; 8c_{\mathrm{iso}} a^{3/2} n^3
\end{aligned}$$

and therefore $\operatorname{card} X \leq 8c_{\mathrm{iso}} a^{3/2}/\delta$. For x in $\mathbb{Z}^3 \setminus X$, by the isoperimetric inequality relative to the ball $B(nx,n)$, taking into account that

$$\mathcal{L}^3\big(B(nx,n) \cap \operatorname{cup} \operatorname{fifa} \mathcal{C}(nU)_0\big) < \delta n^3\,,$$

we have, for any A in $\operatorname{fifa} \mathcal{C}(nU)_0$,

$$\mathcal{L}^3\big(A \cap B(nx,n)\big) \;\leq\; \delta^{1/3} n\, b_{\mathrm{iso}}^{2/3} \mathcal{P}\big(A, \operatorname{int} B(nx,n)\big)\,.$$

Summing this inequality over A in $\operatorname{fifa} \mathcal{C}(nU)_0$ and over x in $\mathbb{Z}^3 \setminus X$,

$$\begin{aligned}
&\mathcal{L}^3\big(\operatorname{cup} \operatorname{fifa} \mathcal{C}(nU)_0 \setminus \operatorname{cup} \{\, B(nx,n) : x \in X \,\}\big)\\
&\qquad\leq\; \sum_{x \in \mathbb{Z}^3 \setminus X} \mathcal{L}^3\big(B(nx,n) \cap \operatorname{cup} \operatorname{fifa} \mathcal{C}(nU)_0\big)\\
&\qquad\leq\; \sum_{x \in \mathbb{Z}^3 \setminus X} \; \sum_{A \in \operatorname{fifa} \mathcal{C}(nU)_0} \mathcal{L}^3\big(B(nx,n) \cap A\big)\\
&\qquad\leq\; \delta^{1/3} n b_{\mathrm{iso}}^{2/3} \sum_{x \in \mathbb{Z}^3 \setminus X} \mathcal{P}\big(\operatorname{fifa} \mathcal{C}(nU)_0, \operatorname{int} B(nx,n)\big)\\
&\qquad\leq\; 8\delta^{1/3} n b_{\mathrm{iso}}^{2/3} \mathcal{P}\big(\operatorname{fifa} \mathcal{C}(nU)_0\big) \;\leq\; 8a\delta^{1/3} b_{\mathrm{iso}}^{2/3} n^3\,.
\end{aligned}$$

We set $M = 8c_{\mathrm{iso}} a^{3/2}/\delta$ and $\eta = \delta + 8a\delta^{1/3} b_{\mathrm{iso}}^{2/3}$. Whenever $\mathcal{E}$ occurs, we have therefore

$$X \subset B(0, n\operatorname{diam} U + 3an^2)\,, \quad \operatorname{card} X \leq M\,,$$
$$\mathcal{L}^3\big(\operatorname{cup} \operatorname{fifa} \mathcal{C}(nU)_0 \setminus \operatorname{cup} \{\, B(nx,n) : x \in X \,\}\big) \;\leq\; 8a\delta^{1/3} b_{\mathrm{iso}}^{2/3} n^3 \;=\; (\eta - \delta) n^3\,.$$

Let $E(X)$ be a subset of $X \times \{1, \dots, 3^{\operatorname{card} X}\}$ associated to X as in Lemma 9.20. We decompose the event $\mathcal{E}$ according to the possible values of the set $E(X)$:

$$P(\mathcal{E}) \;=\; \sum_{1 \leq m \leq M} \; \sum_{r_1, \dots, r_m} \; \sum_{y_1, \dots, y_m} P\big(\mathcal{E},\, E(X) = \{\, (y_1, r_1), \dots, (y_m, r_m) \,\}\big)$$

where the second summation extends over the integers $r_1,\dots,r_m$ in $\{1,\dots,3^M\}$ and the third summation extends over $y_1,\dots,y_m$ in $\mathbb{Z}^3\cap B(0,n\mathrm{diam}\,U+3an^2)$. The number of possible choices for $y_1,\dots,y_m$ is less than $\mathcal{L}^3(B(0,n\mathrm{diam}\,U+3an^2+2))^m$, which is a polynomial function in n. The number of possible choices for the first two sums is less than $3^{M(M+1)}$. We estimate now the term inside the sums. Let $\{(y_1,r_1),\dots,(y_m,r_m)\}$ be a value for the random set $E(X)$ compatible with $\mathcal{E}$ (that is a value which occurs with positive probability). By the construction of $E(X)$, the balls $B(y_l,r_l+1)$, $1\leq l\leq m$, are pairwise disjoint so that $(y_1,\dots,y_m)$ belongs to $T(B(0,r_1+1),\dots,B(0,r_m+1))$; moreover the balls $B(y_i,r_i)$, $1\leq i\leq m$, cover the balls $B(x,1)$, $x\in X$, whence

$$\mathcal{L}^3\big(\mathrm{cup}\,n^{-1}\mathrm{fifa}\,\mathcal{C}(nU)_0\setminus B(y_1,r_1)\setminus\cdots\setminus B(y_m,r_m)\big)\ \leq\ \eta-\delta\,.$$

Since $D(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU),n^{-1}\mathrm{fifa}\,\mathcal{C}(nU))<\delta$, then

$$\mathcal{L}^3\big(n^{-1}\mathrm{cup}\,\mathcal{V}_\infty\mathcal{C}(nU)_0\Delta n^{-1}\mathrm{cup}\,\mathrm{fifa}\,\mathcal{C}(nU)_0\big)=\\ \mathcal{L}^3\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)_\infty\Delta n^{-1}\mathrm{fifa}\,\mathcal{C}(nU)_\infty\big)\leq\delta$$

and it follows that

$$\mathcal{L}^3\big(\mathrm{cup}\,n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)_0\setminus B(y_1,r_1)\setminus\cdots\setminus B(y_m,r_m)\big)\ \leq\ \eta\,.$$

Let $\mathbb{F}^m_\eta(r_1,\dots,r_m)$ be the subset of $\mathrm{BC}(B(0,r_1+1))\times\cdots\times\mathrm{BC}(B(0,r_m+1))$ defined by

$$\mathbb{F}^m_\eta(r_1,\dots,r_m)\ =\ \Big\{\,\phi^{B(0,r_1+1),\dots,B(0,r_m+1)}_{z_1,\dots,z_m}(\mathcal{A}):\\ \mathcal{A}\in\mathbb{F},\ (z_1,\dots,z_m)\in(\mathbb{Z}^3)^m\text{ satisfying the conditions below}\Big\}$$

The conditions imposed on $\mathcal{A}$ and $(z_1,\dots,z_m)$ are:
- $\mathcal{A}$ contains exactly one set A_∞ such that $\mathcal{L}^3(A_\infty)=\infty$,
- $\mathbb{R}^3$ is the disjoint union of A_∞ and $\mathrm{cup}\,\{A\in\mathcal{A}:\mathcal{L}^3(A)<\infty\}$,
- $(z_1,\dots,z_m)$ belongs to $T(B(0,r_1+1),\dots,B(0,r_m+1))$,
- $\mathcal{L}^3\big(\mathrm{cup}\,\{A\in\mathcal{A}:\mathcal{L}^3(A)<\infty\}\setminus B(z_1,r_1)\setminus\cdots\setminus B(z_m,r_m)\big)\leq\eta\,.$

We write then

$$\begin{aligned}
&P\big(\mathcal{E},\,E(X)=\{(y_1,r_1),\dots,(y_m,r_m)\}\big)\\
&\leq\ P\Big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)\in\mathbb{F},\ \mathcal{L}^3\big(\mathrm{cup}\,n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)_0\setminus B(y_1,r_1)\setminus\cdots\setminus B(y_m,r_m)\big)\leq\eta\Big)\\
&\leq\ P\Big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)\in\big\{\mathcal{A}\in\mathbb{F},\ \mathcal{L}^3\big(\mathrm{cup}\,\{A\in\mathcal{A}:\mathcal{L}^3(A)<\infty\}\setminus\bigcup_{1\leq i\leq m}B(y_i,r_i)\big)\leq\eta\big\}\Big)\\
&\qquad\leq\ P\Big(\phi^{B(0,r_1+1),\dots,B(0,r_m+1)}_{y_1,\dots,y_m}\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)\big)\in\mathbb{F}^m_\eta(r_1,\dots,r_m)\Big)\,.
\end{aligned}$$

But $\mathbb{F}_\eta^m(r_1,\dots,r_m)$ depends on $(r_1,\dots,r_m)$ and η only and not on $(y_1,\dots,y_m)$ in $T(B(0,r_1+1),\dots,B(0,r_m+1))$. Coming back to the innermost summation,

$$\sum_{y_1,\dots,y_m} P\big(\mathcal{E},\, E(X)=\{\,(y_1,r_1),\dots,(y_m,r_m)\,\}\big) \;\le\; \mathcal{L}^3(B(0,n\mathrm{diam}\,U+3an^2+2))^m$$
$$\times \sup_{y_1,\dots,y_m} P\Big(\phi_{y_1,\dots,y_m}^{B(0,r_1+1),\dots,B(0,r_m+1)}\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)\big)\in\mathbb{F}_\eta^m(r_1,\dots,r_m)\Big)$$

where the supremum is taken over $(y_1,\dots,y_m)$ in $T(B(0,r_1+1),\dots,B(0,r_m+1))$. This inequality and the upper bound of Lemma 12.13 yield

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln\sum_{y_1,\dots,y_m} P\big(\mathcal{E},\, E(X)=\{\,(y_1,r_1),\dots,(y_m,r_m)\,\}\big) \;\le$$
$$-\inf\,\big\{\,\mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m):(\mathcal{A}_1,\dots,\mathcal{A}_m)\in\mathrm{clo}\,\mathbb{F}_\eta^m(r_1,\dots,r_m)\,\big\}\,.$$

Since the number of terms involved in the first two sums is bounded by $3^{M(M+1)}$ which is independent of n, we conclude that

$$\limsup_{n\to\infty}\frac{1}{n^2}\ln P(\mathcal{E}) \;\le\; -\inf\Big\{\,\mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m):1\le m\le M,$$
$$r_1,\dots,r_m\in\{\,1,\dots,3^M\,\},\,(\mathcal{A}_1,\dots,\mathcal{A}_m)\in\mathrm{clo}\,\mathbb{F}_\eta^m(r_1,\dots,r_m)\Big\}\,.$$

It remains to evaluate this infimum. Let m belong to $\{1,\dots,M\}$, let $r_1,\dots,r_m$ belong to $\{\,1,\dots,3^M\,\}$ and let $(\mathcal{A}_1,\dots,\mathcal{A}_m)$ belong to $\mathrm{clo}\,\mathbb{F}_\eta^m(r_1,\dots,r_m)$. We suppose in addition that $\mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m)$ is finite. By the very definition of the set $\mathrm{clo}\,\mathbb{F}_\eta^m(r_1,\dots,r_m)$, there exist a Borel collection $\mathcal{A}$ in $\mathbb{F}$ and a m–uple $(z_1,\dots,z_m)$ in $T(B(0,r_1+1),\dots,B(0,r_m+1))$ such that: $\mathcal{A}$ contains exactly one set A_∞ having infinite volume, $\mathbb{R}^3$ is the disjoint union of A_∞ and $\mathrm{cup}\,\{\,A\in\mathcal{A}:\mathcal{L}^3(A)<\infty\,\}$, and

$$\mathcal{L}^3\Big(\mathrm{cup}\,\{\,A\in\mathcal{A}:\mathcal{L}^3(A)<\infty\,\}\setminus B(y_1,r_1)\setminus\cdots\setminus B(y_m,r_m)\Big)\le\eta\,,$$
$$\max_{1\le i\le m} D\big((\mathcal{A}\cap B(z_i,r_i+1)),(z_i+\mathcal{A}_i)\big)\;<\;\eta/m\,.$$

Therefore each Caccioppoli partition $\mathcal{A}_i$ contains exactly one set $A_{i,\infty}$ having infinite volume and

$$\max_{1\le i\le m}\mathcal{L}^3_{B(z_i,r_i+1)}\big(A_\infty\Delta(z_i+A_{i,\infty})\big)\;<\;\eta/m$$

whence

$$\sum_{1\le i\le m}\mathcal{L}^3\big(B(0,r_i+1)\setminus B(0,r_i)\setminus A_{i,\infty}\big)\;\le\;\eta+\mathcal{L}^3\Big(\bigcup_{1\le i\le m}\big(B(z_i,r_i+1)\setminus B(z_i,r_i)\big)\setminus A_\infty\Big)$$
$$\le\;\eta+\mathcal{L}^3\Big(\mathrm{cup}\,\{\,A\in\mathcal{A}:\mathcal{L}^3(A)<\infty\,\}\setminus B(y_1,r_1)\setminus\cdots\setminus B(y_m,r_m)\Big)\le 2\eta\,.$$

By Lemma 6.7, for i in $\{1,\dots,m\}$, for $\mathcal{H}^1$ almost all t in $]0,1[$, we have

$$\mathcal{I}(B(0,r_i+t)\setminus A_{i,\infty}) \le \mathcal{I}(\mathbb{R}^3\setminus A_{i,\infty},\operatorname{int}B(0,r_i+t)) + ||\tau||_\infty \mathcal{H}^2(\partial B(0,r_i+t)\setminus A_{i,\infty})$$

and for all A in $\mathcal{A}_i\setminus\{A_{i,\infty}\}$,

$$\mathcal{I}(A\cap B(0,r_i+t)) \le \mathcal{I}(A,\operatorname{int}B(0,r_i+t)) + ||\tau||_\infty \mathcal{H}^2(A\cap\partial B(0,r_i+t)),$$

and also, because we are dealing with Caccioppoli partitions of $\mathbb{R}^3$,

$$\sum_{A\in\mathcal{A}_i\setminus\{A_{i,\infty}\}} \mathcal{H}^2\big(A\cap\partial B(0,r_i+t)\big) = \mathcal{H}^2\big(\partial B(0,r_i+t)\setminus A_{i,\infty}\big).$$

Let T be the subset of $]0,1[$ where all the above inequalities hold simultaneously. Certainly $\mathcal{H}^1(T)=1$ and by integrating in polar coordinates

$$\sum_{1\le i\le m} \mathcal{L}^3\big(B(0,r_i+1)\setminus B(0,r_i)\setminus A_{i,\infty}\big) = \int_T \sum_{1\le i\le m} \mathcal{H}^2\big(\partial B(0,r_i+t)\setminus A_{i,\infty}\big)\,dt,$$

so that there exists t in T such that

$$\sum_{1\le i\le m}\ \sum_{A\in\mathcal{A}_i\setminus\{A_{i,\infty}\}} \mathcal{H}^2\big(A\cap\partial B(0,r_i+t)\big) = \sum_{1\le i\le m} \mathcal{H}^2\big(\partial B(0,r_i+t)\setminus A_{i,\infty}\big) \le 2\eta.$$

Let $(A(j),j\in\mathbb{N})$ be an arrangement of $\mathcal{A}$ such that $A(0)$ is the unique element of $\mathcal{A}$ having infinite volume. By Corollary 10.3, and the inequalities satisfied by $\mathcal{A},\mathcal{A}_1,\dots,\mathcal{A}_m$, there exist arrangements $(A_i(j),j\in\mathbb{N})$, $1\le i\le m$, of $\mathcal{A}_1,\dots,\mathcal{A}_m$ such that

$$\max_{1\le i\le m} \mathcal{L}^3_{B(z_i,r_i+1)}\big(A(0)\Delta(z_i+A_i(0))\big) < \eta/m,$$
$$\max_{1\le i\le m}\sup_{j\ge1} \mathcal{L}^3\big((A(j)\cap B(z_i,r_i+1))\Delta(z_i+A_i(j))\big) < \eta/m.$$

We set

$$A'(0) = \Big(\mathbb{R}^3\setminus\bigcup_{1\le i\le m} B(z_i,r_i+t)\Big)\cup\bigcup_{1\le i\le m}\big(z_i+A_i(0)\cap B(0,r_i+t)\big)$$

and for $j\ge1$,

$$A'(j) = \bigcup_{1\le i\le m}\big(z_i+A_i(j)\cap B(0,r_i+t)\big).$$

Let $\mathcal{A}'$ be the Borel collection of the non–negligible sets of the sequence $(A'(j))_{j\in\mathbb{N}}$. Then $\mathcal{A}'$ is Caccioppoli partition of $\mathbb{R}^3$, $(A'(j))_{j\in\mathbb{N}}$ is an arrangement of $\mathcal{A}'$ and

$$\begin{aligned}
&\mathcal{L}^3\big(A(0)\Delta A'(0)\big)\\
&\quad\le \mathcal{L}^3\Big(\mathbb{R}^3\setminus A(0)\setminus\bigcup_{1\le i\le m} B(z_i,r_i+t)\Big) + \sum_{1\le i\le m}\mathcal{L}^3_{B(z_i,r_i+t)}\big(A(0)\Delta(z_i+A_i(0))\big)\\
&\quad\le \mathcal{L}^3\Big(\mathbb{R}^3\setminus A(0)\setminus\bigcup_{1\le i\le m} B(z_i,r_i)\Big) + \sum_{1\le i\le m}\mathcal{L}^3_{B(z_i,r_i+1)}\big(A(0)\Delta(z_i+A_i(0))\big) \le 2\eta.
\end{aligned}$$

For $j \geq 1$, we have

$$
\begin{aligned}
&\mathcal{L}^3\big(A(j)\Delta A'(j)\big)\\
&\quad\leq \mathcal{L}^3\big(A(j)\setminus \bigcup_{1\leq i\leq m} B(z_i, r_i+t)\big) + \sum_{1\leq i\leq m} \mathcal{L}^3_{B(z_i,r_i+t)}\big(A(j)\Delta(z_i+A_i(j))\big)\\
&\leq \mathcal{L}^3\big(A(j)\setminus \bigcup_{1\leq i\leq m} B(z_i, r_i)\big) + \sum_{1\leq i\leq m} \mathcal{L}^3\big((A(j)\cap B(z_i,r_i+1))\Delta(z_i+A_i(j))\big) \leq 2\eta\,.
\end{aligned}
$$

Using the previous inequalities, we obtain that

$$D(\mathcal{A},\mathcal{A}') \leq \sup_{j\in\mathbb{N}} \mathcal{L}^3\big(A(j)\Delta A'(j)\big) \leq 2\eta\,.$$

Furthermore, because of the choice of t

$$
\begin{aligned}
\mathcal{I}(\mathcal{A}') &= \frac{1}{2}\sum_{j\geq 0}\mathcal{I}(A'(j)) = \sum_{1\leq i\leq m} \mathcal{I}\big(\mathcal{A}_i\cap B(0,r_i+t)\big)\\
&\leq \sum_{1\leq i\leq m} \mathcal{I}(\mathcal{A}_i, \operatorname{int} B(0,r_i+t)) + ||\tau||_\infty \sum_{1\leq i\leq m}\ \sum_{A\in\mathcal{A}_i\setminus\{A_{i,\infty}\}} \mathcal{H}^2(A\cap\partial B(0,r_i+t))\\
&\leq \mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m) + 2\eta||\tau||_\infty\,.
\end{aligned}
$$

It follows that

$$\inf\big\{\,\mathcal{I}(\mathcal{A}) : \mathcal{A}\in\mathcal{V}_D(\mathbb{F},2\eta)\,\big\} \leq \mathcal{I}^m(\mathcal{A}_1,\dots,\mathcal{A}_m) + 2\eta||\tau||_\infty\,.$$

Passing to the infimum over $(\mathcal{A}_1,\dots,\mathcal{A}_m)$, $(r_1,\dots,r_m)$ and m,

$$\limsup_{n\to\infty} \frac{1}{n^2}\ln P\big(\mathcal{E}\big) \leq -\inf\big\{\,\mathcal{I}(\mathcal{A}) : \mathcal{A}\in\mathcal{V}_D(\mathbb{F},2\eta)\,\big\} + 2\eta||\tau||_\infty\,.$$

Coming back to the initial inequality,

$$
\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^2}\ln P\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)\in\mathbb{F}\big) \leq\\
&\qquad\qquad -\min\big(c(k)a,\, \inf\big\{\,\mathcal{I}(\mathcal{A}) : \mathcal{A}\in\mathcal{V}_D(\mathbb{F},2\eta)\,\big\} - 2\eta||\tau||_\infty\big)\,.
\end{aligned}
$$

Recalling that $\eta = \delta + 8a\delta^{1/3}b_{\text{iso}}^{2/3}$, sending first δ to 0 and then a to ∞, we obtain

$$\limsup_{n\to\infty} \frac{1}{n^2}\ln P\big(n^{-1}\mathcal{V}_\infty\mathcal{C}(nU)\in\mathbb{F}\big) \leq -\sup_{\varepsilon>0}\inf\big\{\,\mathcal{I}(\mathcal{A}) : \mathcal{A}\in\mathcal{V}_D(\mathbb{F},\varepsilon)\,\big\}\,.$$

This is the enhanced upper bound we were seeking for. □

APPENDIX

For the convenience of the reader, we include here some basic and classical material of large deviations theory (see [28] for a detailed exposition).

Let $(F, \mathcal{O})$ be a topological space. A rate function on $(F, \mathcal{O})$ is a lower semicontinuous map from $(F, \mathcal{O})$ to $\mathbb{R}^+ \cup \{\infty\}$. A rate function I on $(F, \mathcal{O})$ is good if its level sets $\{\, x \in F : I(x) \leq \lambda \,\}$, $\lambda > 0$, are compact. We endow $(F, \mathcal{O})$ with the Borel σ–field $\mathcal{B}(F)$ generated by the open sets $\mathcal{O}$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables defined on Ω with values in F. Let I be a rate function on $(F, \mathcal{O})$ and let ϕ be a map from $\mathbb{N}$ to $\mathbb{R}^+$. The sequence of random variables $(X_n)_{n \in \mathbb{N}}$ satisfies the large deviation principle with speed $\phi(n)$ and rate function I if for any A in $\mathcal{B}(F)$

$$
\begin{aligned}
-\inf \{\, I(x) : x \in \operatorname{int} A \,\} \;&\leq\; \liminf_{n \to \infty} \;\frac{1}{\phi(n)} \ln P(X_n \in A) \\
&\leq\; \limsup_{n \to \infty} \;\frac{1}{\phi(n)} \ln P(X_n \in A) \;\leq\; -\inf \{\, I(x) : x \in \operatorname{clo} A \,\},
\end{aligned}
$$

where $\operatorname{int} A$ is the interior of A and $\operatorname{clo} A$ is the closure of A with respect to the topology $\mathcal{O}$. The sequence of random variables $(X_n)_{n \in \mathbb{N}}$ satisfies a weak large deviation principle if the upper bound holds only for compact sets.

We suppose next that the topology $\mathcal{O}$ is associated to a metric d on F. The large deviations lower bound is equivalent to the following statement:

$$
\forall\, x \in F \quad \forall\, \delta > 0 \qquad \liminf_{n \to \infty} \;\frac{1}{\phi(n)} \ln P\big(d(X_n, x) < \delta\big) \;\geq\; -I(x)\,.
$$

We always use the above statement to prove large deviations lower bound.

Definition. We say that the sequence of random variables $(X_n)_{n \in \mathbb{N}}$ is I–tight if

$$
\forall\, \delta > 0 \qquad \lim_{\lambda \to \infty} \limsup_{n \to \infty} \;\frac{1}{\phi(n)} \ln P\big(d(X_n, I^{-1}([0, \lambda])) > \delta\big) \;=\; -\infty\,.
$$

Our proofs of the large deviations upper bound rely on the following formulation.

Lemma. *Suppose that the rate function I is good. The sequence of random variables $(X_n)_{n\in\mathbb{N}}$ satisfies the large deviations upper bound if and only if it is I–tight and it satisfies the local estimate*

$$\forall\, x \in F, \quad I(x) < \infty, \quad \forall\, \varepsilon > 0 \quad \exists\, \delta = \delta(x,\varepsilon) > 0$$
$$\limsup_{n\to\infty} \frac{1}{\phi(n)} \ln P\big(d(X_n, x) < \delta\big) \;\le\; -I(x)(1-\varepsilon)\,.$$

Remark. Notice that the above property is equivalent to

$$\forall\, x \in F, \quad I(x) < \infty, \qquad \limsup_{\delta\to 0}\, \limsup_{n\to\infty} \frac{1}{\phi(n)} \ln P\big(d(X_n, x) < \delta\big) \;\le\; -I(x)\,.$$

Remark. In the situation where the rate function is not good (that is, it is only lower semicontinuous), we have the following result: If the sequence of random variables $(X_n)_{n\in\mathbb{N}}$ is I–tight and satisfies the local estimate, then the weak large deviations upper bound holds.

Proof. Let A be a closed subset of F. Let λ, δ, η be positive with $\delta \le \eta$. Let $A^\delta = \{\, x : d(x, A) \le \delta \,\}$. We write

$$\begin{aligned} P(X_n \in A) \;&\le\; P\big(X_n \in A,\, d(X_n, I^{-1}([0,\lambda])) \le \delta\big) \;+\; P\big(d(X_n, I^{-1}([0,\lambda])) > \delta\big) \\ &\le\; P\big(d(X_n,\, A^\eta \cap I^{-1}([0,\lambda])) \le \delta\big) \;+\; P\big(d(X_n, I^{-1}([0,\lambda])) > \delta\big)\,. \end{aligned}$$

Since A is closed and I is a good rate function, the set $A^\eta \cap I^{-1}([0,\lambda])$ is compact. Let ε be positive. To each x in $A^\eta \cap I^{-1}([0,\lambda])$ we associate a positive $\delta(x,\varepsilon)$ as given by the local estimate. The collection of sets

$$\{\, y : d(x,y) < \delta(x,\varepsilon) \,\}\,, \quad x \in A^\eta \cap I^{-1}([0,\lambda])$$

is an open covering of $A^\eta \cap I^{-1}([0,\lambda])$, from which we can extract a finite subcover associated to a finite number of points x_i, $1 \le i \le r$. The open set

$$\bigcup_{1\le i\le r} \{\, y : d(x_i, y) < \delta(x_i,\varepsilon) \,\}$$

contains a neighbourhood of $A^\eta \cap I^{-1}([0,\lambda])$. Hence, for δ sufficiently small,

$$P\big(d(X_n,\, A^\eta \cap I^{-1}([0,\lambda])) \le \delta\big) \;\le\; \sum_{1\le i\le r} P\big(d(X_n, x_i) < \delta(x_i,\varepsilon)\big)\,.$$

By the definition of the $\delta(x_i,\varepsilon)$, $1 \le i \le r$, using the local estimate and the previous inequalities, we get

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{\phi(n)} \ln P\big(d(X_n,\, A^\eta \cap I^{-1}([0,\lambda])) \le \delta\big) \;&\le\; -\min_{1\le i\le r} I(x_i)(1-\varepsilon) \\ &\le\; -(1-\varepsilon)\, \inf\{\, I(x) : x \in A^\eta \cap I^{-1}([0,\lambda]) \,\}\,. \end{aligned}$$

Using the fact that the sequence is I–tight, with the help of the previous inequalities, we can choose λ large enough to obtain

$$\limsup_{n\to\infty}\ \frac{1}{\phi(n)}\ln P\big(X_n\in A\big)\ \le\ -(1-\varepsilon)\ \inf\{\,I(x): x\in A^\eta\,\}\,.$$

Since I is lower semicontinuous, the large deviations upper bound follows from this inequality by sending ε and η to 0. $\square$

Two sequences of random variables $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are exponentially contiguous if

$$\forall\,\delta>0\qquad \limsup_{n\to\infty}\ \frac{1}{\phi(n)}\ln P\big(d(X_n,Y_n)>\delta\big)\ =\ -\infty\,.$$

If $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are exponentially contiguous and if $(X_n)_{n\in\mathbb{N}}$ satisfies the large deviation principle with speed $\phi(n)$ and rate function I then so does $(Y_n)_{n\in\mathbb{N}}$.

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