# A $d$–dimensional nucleation and growth model

Raphaël Cerf * and Francesco Manzo †

January 22, 2010

**Abstract**

We analyze the relaxation time of a ferromagnetic $d$–dimensional growth model on the lattice. The model is characterized by $d$ parameters which represent the activation energies of a site, depending on the number of occupied nearest neighbours. This model is a natural generalisation of the model studied by Dehghanpour and Schonmann [DS97a], where the activation energy of a site with more than two occupied neighbours is zero.

## 1 Introduction

Growth models have been extensively studied in many cases of physical relevance. Our model can be obtained with a particular choice of the parameters for Richardson's model on the lattice [Ric73] and it is closely related to the models studied by Eden [Ede61], Kesten and Schonmann [KS95], and specifically Dehghanpour and Schonmann [DS97a], with which it shares the same physical motivation, i.e., the study of the relaxation from a metastable state to the stable phase of a thermodynamic ferromagnetic system. In many physical cases, this event is triggered by the formation, growth and coalescence of many droplets of the stable phase in the midst of the metastable one. The model we study in this paper is inspired by the metastable behavior of the kinetic Ising model in the infinite-volume regime for small magnetic field and vanishing temperature. This regime was studied by Dehghanpour and Schonmann in the two dimensional case [DS97b]. The main ideas were presented in a simplified model in [DS97a]. We study here the model corresponding to the $d$–dimensional case. There are several problems to extend the approach of Dehghanpour and Schonmann when

*Université de Paris-Sud, Mathématique, Bât. 425, 91405 Orsay Cedex, France
E-mail: rcerf@math.u-psud.fr

†Università di Roma "Tor Vergata", dipartimento di matematica, via della ricerca scientifica 00133 Roma Italy, E-mail: manzo.fra@gmail.com

there are more than two activation energies. One of them is to control the speed of growth of large supercritical droplets. In the model with two activation energies, this was achieved with the technology of the "chronological paths" introduced by Kesten and Schonmann [KS95]. We did not manage to adapt this technology to deal with the three dimensional Ising model. In this paper, we present an alternative new strategy to control the speed of growth. This strategy relies on coupling arguments, where we consider specific boundary conditions called sandwich boundary conditions, as was done to analyze the bootstrap percolation model [CC99, CM02]. We hope to apply this strategy to control the growth of the supercritical droplets in the context of the three dimensional Ising model in the regime of low temperatures.

The model is an irreversible gas model on the lattice $\mathbb{Z}^d$. Sites are occupied at exponential times with rates that depend on the number of occupied neighbors. More precisely, our model is characterized by a set of parameters $\Gamma_n$, $n = 0, \dots, d$ that represent the activation energy of a "critical droplet" in dimension $n$. When a site has $i \le d$ occupied neighbors, its occupation rate is $\exp(-\beta\Gamma_{d-i})$. When a site has $d$ or more occupied neighbors, its occupation rate is 1. A natural choice for ferromagnetic systems is to assume

$$\Gamma_0 \le \Gamma_1 \le \cdots \le \Gamma_d \,.$$

We start from the void configuration in infinite volume or in a finite cube and look at the time $\tau_d$ when a given site, for instance the origin, is occupied. The scaling behavior of $\tau_d$ as $\beta$ goes to $\infty$ can be obtained with the help of the following simplified heuristics. The rate of creation of nuclei (namely, isolated occupied sites) is $\exp(-\beta\Gamma_d)$. Once a nucleus has appeared, it starts to grow, yet its speed of growth increases with its size. Let $l(\tau)$ be the typical diameter of a droplet grown from a nucleus after a time $\tau$. At time $2\tau$, the origin is likely to have been reached by any nucleus created at distance $l(\tau)$ before time $\tau$. The relaxation time $\tau_d$ should be such that the rate of creation of a nucleus within the space time cone $l(\tau_d)^d \times \tau_d$ is of order one. It turns out that $l(\tau)$ behaves as follows when $\beta$ goes to $\infty$:

$$l\big(\exp(\beta K)\big) \sim \begin{cases} 1 & \text{if } K < \Gamma_{d-1} \\ \exp\big(\beta(K - \kappa_{d-1})\big) & \text{if } K \ge \Gamma_{d-1} \end{cases}$$

Suppose that $\tau_d$ scales as $\exp(\beta\kappa_d)$ when $\beta$ goes to $\infty$. The value $\kappa_d$ will be the smallest value $K$ such that

$$l\big(\exp(\beta K)\big)^d \exp(\beta K) \exp(-\beta\Gamma_d)$$

is of order 1. Since $\Gamma_d \ge \Gamma_{d-1}$, then $K$ has to be larger than $\Gamma_{d-1}$, and it

satisfies therefore

$$\exp\big(d\beta(K-\kappa_{d-1})\big)\exp(\beta K)\exp(-\beta\Gamma_d) \;=\; 1\,.$$

This equation yields

$$K \;=\; \frac{\Gamma_d + d\kappa_{d-1}}{d+1}\,.$$

We conclude finally that

$$\kappa_d \;=\; \max\Big(\Gamma_{d-1}, \frac{\Gamma_d + d\kappa_{d-1}}{d+1}\Big)\,.$$

# 2 Main result

Our configuration space is $\{0,1\}^\Lambda$, where $\Lambda$ is a subset of $\mathbb{Z}^d$ (possibly equal to $\mathbb{Z}^d$ itself). A configuration is thus a map $\sigma : \Lambda \to \{\,0,1\,\}$, and a site $x \in \Lambda$ is empty (respectively occupied) in the configuration $\sigma$ if $\sigma(x) = 0$ (respectively $\sigma(x) = 1$). Sites which are occupied remain occupied forever. To define the dynamics, we consider a family of i.i.d. Poisson processes with rate one, associated with the sites in $\mathbb{Z}^d$. For $x \in \mathbb{Z}^d$, $i \geq 1$, we denote by $\tau(x,i)$ the $i$–th arrival time of the Poisson process associated with $x$. With each arrival time, we associate a uniform random variable $U(x,i)$ in $[0,1]$, independent of the Poisson processes and of the other uniform variables. We build a Markov process $(\sigma_{\Lambda,t})_{t\geq 0}$ with the help of these random objects. At time 0, we start from the empty configuration:

$$\forall x \in \mathbb{Z}^d \qquad \sigma_{\Lambda,0}(x) = 0\,.$$

We describe now the updating procedure of our process. Let $N(x,\sigma)$ be the number of occupied neighbors of the site $x$ in the configuration $\sigma$, i.e.,

$$N(x,\sigma) \;=\; \sum_{y\in\Lambda:|x-y|=1} \sigma(y)\,.$$

The rate at which a site becomes occupied depends only on the number of its occupied neighbors. These rates are given by a non–decreasing sequence

$$c(0) \;\leq\; c(1) \;\leq\; \cdots \;\leq\; c(2d)\,.$$

A site $x$ can become occupied only at a time corresponding to an arrival of its associated Poisson process. Suppose that $t = \tau(x,i)$ for some $i \geq 1$ and that $x$ was not occupied before time $t$. With probability one, all the arrival times are distinct and only the state of the site $x$ can change at time $t$. If

$$U(x,i) \;\leq\; c\big(N(x,\sigma_{\Lambda,t}(x))\big)$$

then $x$ becomes occupied at time $t$, otherwise it stays vacant. If the set $\Lambda$ is finite, the above rules define a Markov process $(\sigma_{\Lambda,t})_{t\geq 0}$. Whenever $\Lambda$ is infinite, one has to be more careful, because there is an infinite number of arrival times in any finite time interval and it is not possible to order them in an increasing sequence. However, because the rates are bounded, changes in the system propagate at a finite speed, and a Markov process can still be defined by taking the limit of finite volume processes (see [Lig05] for more details). Whenever $\Lambda = \mathbb{Z}^d$, we drop it from the notation, and we write $(\sigma_t)_{t\geq 0}$ for the infinite volume process in $\mathbb{Z}^d$. We will deal with exponentially small rates. However we need to have a sufficiently loose asymptotic condition in order to perform our inductive proof, so that we can compare the process in dimension $d$ with a $d-1$ dimensional process satisfying the same condition.

**Hypothesis on the rates.** We suppose that the occupation rates $c(n)$, $0 \leq n \leq 2d$, depend on a parameter $\beta > 0$ and that the following limits exist:

$$\forall n \in \{\, 0, \dots, d \,\} \qquad \lim_{\beta\to\infty} \frac{1}{\beta} \ln c_\beta(n) \,=\, -\Gamma_{d-n}\,,$$
$$\forall n \in \{\, d, \dots, 2d \,\} \qquad \lim_{\beta\to\infty} \frac{1}{\beta} \ln c_\beta(n) \,=\, 0\,.$$

Moreover, we suppose that

$$\Gamma_0 \,\leq\, \Gamma_1 \,\leq\, \cdots \,\leq\, \Gamma_d\,.$$

For $0 \leq n \leq d$, the parameter $\Gamma_n$ represents the activation energy of a critical droplet in dimension $n$. The conditions imposed on the sequence $\Gamma_n$, $0 \leq n \leq d$, simplify substantially the analysis and they are satisfied by the growth model associated to the metastability problem for the low–temperature Ising model. We define a sequence of critical constants $\kappa_i$ for $0 \leq i \leq d$ by setting $\kappa_0 = \Gamma_0$ and

$$\forall i \in \{\, 1, \dots, d \,\} \qquad \kappa_i \,=\, \max\Big(\Gamma_{i-1}, \frac{\Gamma_i + i\kappa_{i-1}}{i+1}\Big)\,.$$

Thus we have

$$\kappa_d \,=\, \max\Big(\Gamma_{d-1}, \frac{\Gamma_d + d\Gamma_{d-2}}{d+1}, \dots, \frac{\Gamma_d + \cdots + \Gamma_{d-i} + (d-i)\Gamma_{d-i-2}}{d+1}, \dots,$$
$$\frac{\Gamma_d + \cdots + \Gamma_3 + 3\Gamma_1}{d+1}, \frac{\Gamma_d + \cdots + \Gamma_2 + 2\Gamma_0}{d+1}, \frac{\Gamma_d + \cdots + \Gamma_1 + \Gamma_0}{d+1}\Big)\,.$$

Our main result states that, in infinite volume, the relaxation time of the system scales as $\exp(\beta\kappa_d)$.

**Theorem 2.1** (**Infinite volume.**) Let $\kappa > 0$ and let $\tau_\beta = \exp(\beta\kappa)$.
• If $\kappa < \kappa_d$, then

$$\lim_{\beta\to\infty} \mathbb{P}\left(\sigma_{\tau_\beta}(0) = 1\right) = 0\,.$$

• If $\kappa > \kappa_d$, then

$$\lim_{\beta\to\infty} \mathbb{P}\left(\sigma_{\tau_\beta}(0) = 0\right) = 0\,.$$

The first step of the proof consists in reducing the problem to some growth processes in a finite volume. Indeed, if $\kappa < K$ and we set

$$\tau_\beta = \exp(\beta\kappa)\,, \qquad \Lambda_\beta = \Lambda(\exp\beta K)\,,$$

then

$$\lim_{\beta\to\infty} \mathbb{P}\left(\sigma_{\tau_\beta}(0) = \sigma_{\Lambda_\beta,\tau_\beta}(0)\right) = 1\,.$$

This follows from a simple large-deviation estimate based on the fact that the maximum rate in the model is 1, see lemma 1 of [DS97b] for the complete proof. Let us shift next our attention to finite volumes. We have two possible scenarios for the growth process in order to fill completely a cube. If the cube is small, the system relaxes via the formation of a single nucleus that grows until filling the entire volume. If the cube is large, a more efficient mechanism consists in creating many droplets that grow and eventually coalesce. The critical side length of the cubes separating these two mechanisms scales exponentially with $\beta$ as $\exp(\beta L_d)$, where

$$L_d = \frac{\Gamma_d - \kappa_d}{d}\,.$$

There are three main factors controlling the relaxation time:

**Nucleation.** Within a box of sidelength $\exp(\beta L)$, the typical time when the first nucleus appears is of order $\exp(\beta(\Gamma_d - dL))$.
**Initial growth.** The typical time to grow a nucleus into a droplet travelling at the asymptotic speed is $\exp(\beta\Gamma_{d-1})$.
**Asymptotic growth.** A droplet travelling at the asymptotic speed covers a region of diameter $\exp(\beta L)$ in a time $\exp(\beta(L + \kappa_{d-1}))$.

The statement concerning the nucleation time contains no mystery. Let us try to explain the statements on the growth of the droplets. Once a nucleus is born, it starts to grow at speed $\exp(-\beta\Gamma_{d-1})$. As the droplet grows, the speed of growth increases, because the number of choices for the creation of a new protuberance attached to the droplet is of order the surface of the droplet. Thus the speed of growth of a droplet of size $\exp(\beta K)$ is

$$\exp(\beta(K(d-1) - \Gamma_{d-1}))\,.$$

When $K$ reaches the value $L_{d-1}$, the speed of growth is limited by the time needed for the protuberance to cover an entire face of the droplet. This time corresponds to the $d-1$ relaxation time and the droplet reaches its asymptotic speed, of order $\exp(-\beta\kappa_{d-1})$. The time needed to grow a nucleus into a droplet travelling at the asymptotic speed is

$$\sum_{1\le i\le \exp(\beta L_{d-1})} \exp\beta\Big(\Gamma_{d-1} - \frac{d-1}{\beta}\ln i\Big)$$

and it is still of order $\exp(\beta\Gamma_{d-1})$. With the help of the above facts, we can obtain easily an upper bound on the relaxation time in a box $\Lambda_\beta$ of sidelength $\exp(\beta L)$. Indeed, the relaxation time is smaller than the sum

$$\begin{pmatrix}\text{time for nucleation}\\ \text{in the box } \Lambda_\beta\end{pmatrix} + \begin{pmatrix}\text{time to grow a nucleus}\\ \text{into a droplet travelling}\\ \text{at the asymptotic speed}\end{pmatrix} + \begin{pmatrix}\text{time to cover}\\ \text{the box } \Lambda_\beta\end{pmatrix}$$
$$\sim \exp(\beta(\Gamma_d - dL)) + \exp(\beta\Gamma_{d-1}) + \exp(\beta(L+\kappa_{d-1}))$$

which is of order

$$\exp\Big(\beta\max\big(\Gamma_d - dL, \Gamma_{d-1}, L+\kappa_{d-1}\big)\Big).$$

Optimizing over the size of the box $\Lambda_\beta$, we conclude that the relaxation time in infinite volume satisfies

$$\tau_d \le \exp\Big(\beta\inf_L \max\big(\Gamma_d - dL, \Gamma_{d-1}, L+\kappa_{d-1}\big)\Big).$$

Let us now try to obtain a lower bound on the relaxation time. Suppose that we examine the state of the origin at a time $\exp(\beta\kappa)$. The origin becomes occupied when it is covered by a droplet. This droplet can result either from the growth of a single nucleus or from the coalescence of several droplets. Since the speed of propagation of the effects is finite, the state of the origin at time $\exp(\beta\kappa)$ is unlikely to have been influenced by any event occurring outside the box of sidelength $\exp(2\beta\kappa)$. Thus all the subsequent computations can be restricted to this box. In particular, a droplet which covers the origin before time $\exp(\beta\kappa)$ has to be born inside this box, meaning that the oldest site of the droplet belongs to this box. Let us consider the box $\Lambda_\beta$ of sidelength $\exp(\beta L)$. We can envisage two scenarios. If the droplet which covers the origin is born inside the box $\Lambda_\beta$, then nucleation has occurred inside this box. If the droplet which covers the origin is born outside the box $\Lambda_\beta$, then it has grown into a droplet of diameter at least $\frac{1}{2}\exp(\beta L)$ in order to reach the origin. Thus the relaxation time is larger

than

$$\min\left(\begin{pmatrix}\text{time for nucleation}\\ \text{in the box } \Lambda_\beta\end{pmatrix}, \begin{pmatrix}\text{time to grow a nucleus into}\\ \text{a droplet of diameter } \frac{1}{2}\exp(\beta L)\end{pmatrix}\right)$$
$$\sim \min\Big(\exp(\beta(\Gamma_d - dL))\,, \exp(\beta\Gamma_{d-1}) + \frac{1}{2}\exp(\beta(L+\kappa_{d-1}))\Big)$$

which is of order

$$\exp\Big(\beta \min\big(\Gamma_d - dL, \max(\Gamma_{d-1}, L+\kappa_{d-1})\big)\Big)\,.$$

By optimizing over the size of the box $\Lambda_\beta$, we conclude that the relaxation time in infinite volume satisfies

$$\tau_d \geq \exp\Big(\beta \sup_L \min\big(\Gamma_d - dL, \max(\Gamma_{d-1}, L+\kappa_{d-1})\big)\Big)\,.$$

Since the optimal value of $L$ solves $\Gamma_d - dL = L + \kappa_{d-1}$, the two constants appearing in the exponential in the lower and upper bounds for the relaxation time coincide, they are equal to

$$\kappa_d = \max\Big(\Gamma_{d-1}, \frac{\Gamma_d + d\kappa_{d-1}}{d+1}\Big)\,.$$

We state next precisely the finite volume results that we will prove.

**Terminology.** We say that a probability $\mathbb{P}(\cdot)$ is exponentially small in $\beta$ (written in short ES) if it satisfies

$$\limsup_{\beta\to\infty} \frac{1}{\beta}\ln\mathbb{P}(\cdot) < 0\,.$$

We say that a probability $\mathbb{P}(\cdot)$ is super–exponentially small in $\beta$ (written in short SES) if it satisfies

$$\lim_{\beta\to\infty} \frac{1}{\beta}\ln\mathbb{P}(\cdot) = -\infty\,.$$

**Theorem 2.2** (**Exponential volume.**) Let $L > 0$ and let $\Lambda_\beta = \Lambda(\exp(\beta L))$ be a cubic box of sidelength $\exp(\beta L)$. Let $\kappa > 0$ and let $\tau_\beta = \exp(\beta\kappa)$.
• If $\kappa < \max(\Gamma_d - dL, \kappa_d)$, then

$$\lim_{\beta\to\infty} \mathbb{P}\left(\sigma_{\Lambda_\beta;\tau_\beta}(0) = 1\right) = 0$$

and this probability is exponentially small in $\beta$.
• If $\kappa > \max(\Gamma_d - dL, \kappa_d)$, then

$$\lim_{\beta\to\infty} \mathbb{P}\left(\exists\, x \in \Lambda_\beta \quad \sigma_{\Lambda_\beta;\tau_\beta}(x) = 0\right) = 0$$

and this probability is super–exponentially small in $\beta$.

The hardest part of theorem 2.2 is the upper bound on the relaxation time, i.e., the first case where $\kappa < \max(\Gamma_d - dL, \kappa_d)$. The first ingredient in the proof is a lower bound on the time needed to create a large droplet.

**Proposition 2.3** Let $L > 0$ and let $\Lambda_\beta = \Lambda(\exp(\beta L))$ be a cubic box of sidelength $\exp(\beta L)$. Let $\kappa < \Gamma_{d-1}$ and let $\tau_\beta = \exp(\beta\kappa)$. The probability that an occupied cluster in $\sigma_{\Lambda_\beta;\tau_\beta}$ has diameter larger than $\beta$ is super–exponentially small in $\beta$.

The key result for the inductive proof is the following control on the size of the clusters in the configuration. We set

$$L_d \;=\; \frac{\Gamma_d - \kappa_d}{d}\,.$$

**Theorem 2.4** Let $L > 0$ and let $\Lambda_\beta = \Lambda(\exp(\beta L))$ be a cubic box of sidelength $\exp(\beta L)$. Let $\kappa < \kappa_d$ and let $\tau_\beta = \exp(\beta\kappa)$. The probability that an occupied cluster in $\sigma_{\Lambda_\beta;\tau_\beta}$ has diameter larger than $\exp(\beta L_d)$ is super–exponentially small in $\beta$.

By using theorem 2.4 inductively, we are able to show that the asymptotic speed of the droplets inside the box $\Lambda_\beta$ is of order $\exp(-\beta\kappa_{d-1})$. The proofs of proposition 2.3 and of theorem 2.4 involve both a bootstrap argument to control the coalescence of the droplets. In fact, one could make a general statement to control the maximal size of an occupied cluster at a given time. Yet it turns out that only the initial growth and the asymptotic speed of the droplets are relevant to compute the relaxation time, the intermediate stage of growth of the droplets is not a limiting factor.

# 3 Graphical construction

Throughout the paper, we use the standard graphical construction [DS97b]. All our processes are defined on the same probability space and they are built with the help of the arrival times of independent Poisson processes and the associated uniform random variables

$$\tau(x,i)\,,\quad U(x,i)\,,\quad i \geq 1\,,\quad x \in \mathbb{Z}^d\,.$$

This provides a natural coupling between the different growth processes. The process in a set $\Lambda$ with boundary conditions $\rho$ is denoted by

$$(\sigma^\rho_{\Lambda,t})_{t\geq 0}\,.$$

This coupling preserves the natural order on the configurations. A configuration $\alpha$ is included in a configuration $\rho$, which we denote by $\alpha \leq \rho$, if every site occupied in $\alpha$ is also occupied in $\rho$. The growth process in a box $\Lambda$ starting from the configuration $\alpha$ will always remain smaller than the growth process in $\Lambda$ starting from a larger configuration $\rho$. The growth processes in a box $\Lambda$ associated to different boundary conditions are also coupled in the same way, and the coupling respects the order on the boundary conditions, meaning that larger boundary conditions lead to larger growth processes. We rely repeatedly on this coupling in order to compare our model with simpler or lower-dimensional processes.

# 4 Bootstrap

Following [DS97b], we control the effect of the coalescence of the droplets with a bootstrap-percolation argument. We recall next the standard bootstrap procedure. Let $A$ be a finite subset of $\mathbb{Z}^d$. We start with a configuration $\eta \in \{0,1\}^A$ and we occupy iteratively all the sites which have at least two occupied neighbors, until exhaustion. Since the procedure is monotonic and the volume is finite, the algorithm will stop after a finite number of steps. We denote by bootstrap($\eta$) the final configuration obtained by bootstraping $\eta$. This final configuration is an union of occupied parallelepipeds, which are pairwise at distance larger than or equal to two. Following [AL88], we say that a set $E \subset \mathbb{Z}^d$ is internally spanned in the configuration $\eta$ if it is entirely covered in the final configuration of the dynamics restricted to $E$. More precisely, the initial configuration is the restriction of $\eta$ to $E$ and the dynamics runs on the sites of $E$ without taking into account sites outside $E$.

We will use the supremum norm, given by

$$\forall x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \qquad |x|_\infty \;=\; \max_{1 \leq i \leq d} |x_i| \,.$$

We denote by $\mathrm{d}_\infty$ the distance associated to the supremum norm and we define the $\mathrm{d}_\infty$ diameter $\operatorname{diam}_\infty C$ of a subset $C$ of $\mathbb{Z}^d$ by

$$\operatorname{diam}_\infty C \;=\; \sup \left\{ \, |x-y|_\infty : \; x, y \in C \, \right\}.$$

Thus $\operatorname{diam}_\infty C$ is the sidelength of the minimal cube surrounding $C$. The following lemma is a key observation of Aizenman and Lebowitz [AL88].

**Lemma 4.1** If a set $C$ is internally spanned in a configuration $\eta$ then for all integer $k \geq 1$ such that $2k+1 < \operatorname{diam}_\infty C$ there exists a subset $D$ of $C$ which is internally spanned in $\eta$ and such that $k \leq \operatorname{diam}_\infty D \leq 2k+1$.

We give the sketch of the proof, which can be found in [AL88]. It relies on the fact that if $\eta \leq \xi \leq \text{bootstrap}(\eta)$, then $\text{bootstrap}(\xi) = \text{bootstrap}(\eta)$. For this reason, we are free to change the updating order without affecting the final configuration. The idea is then to realize the bootstrap percolation by occupying a single site at each step. If the maximal diameter of the clusters present in the configuration is $k$ before one step of the algorithm, then right after occupying one site, the new maximal diameter is between $k$ and $2k+1$. Looking at the evolution of the maximal diameter of the occupied clusters, we get the thesis.

# 5 Proof of proposition 2.3

Let $L > 0$ and let $\Lambda_\beta = \Lambda(\exp(\beta L))$ be a cubic box of sidelength $\exp(\beta L)$. Let $\kappa < \Gamma_{d-1}$ and let $\tau_\beta = \exp(\beta\kappa)$. Let $\alpha$ be the random configuration defined as follows. For $x \in \Lambda_\beta$, we set $\alpha(x) = 1$ if there exists $i \geq 1$ such that $\tau(x,i) \leq \tau_\beta$ and $U(x,i) \leq c_\beta(1)$, otherwise we set $\alpha(x) = 0$. The law of the configuration $\alpha$ is the Bernoulli product law with parameter $p_\beta$ given by

$$p_\beta \ = 1 - \exp\big(-c_\beta(1)\tau_\beta\big)\,.$$

Taking logarithm, we see that

$$\lim_{\beta\to\infty} \frac{1}{\beta} \ln p_\beta \ = \ -\Gamma_{d-1} + \kappa \ < 0.$$

Let $\text{bootstrap}(\alpha)$ be the configuration obtained by bootstraping $\alpha$. The configuration $\sigma_{\Lambda_\beta;\tau_\beta}$ is smaller than or equal to $\text{bootstrap}(\alpha)$. Indeed, in order to grow beyond $\text{bootstrap}(\alpha)$, the process would have to occupy a site outside $\text{bootstrap}(\alpha)$ having 0 or 1 occupied neighbors, but all these events until time $\tau_\beta$ were already recorded in the initial configuration $\alpha$. Proposition 2.3 is therefore implied by the following lemma.

**Lemma 5.1** The probability that there exists an occupied cluster in the configuration $\text{bootstrap}(\alpha)$ whose $\mathrm{d}_\infty$ diameter is larger than $\beta$ is super–exponentially small in $\beta$.

**Proof.** We say that a box is crossed if, after applying the bootstrap operator restricted to the box, there is an occupied connected set joining two opposite faces of the box. By lemma 4.1, if there is an occupied cluster in $\text{bootstrap}(\alpha)$ whose $\mathrm{d}_\infty$ diameter is larger than $\beta$, then there exists an internally–spanned cluster in $\text{bootstrap}(\alpha)$ with diameter between $\beta$ and $2\beta+1$. Let $Q_\beta$ be a cube of minimal side length containing such a cluster. The cube $Q_\beta$ has to be crossed in one of the $d$ directions parallel to the axis, say for instance the vertical one. If there is an horizontal strip in $Q_\beta$

of height 2 which is void in the configuration $\alpha$ then the box $Q_\beta$ cannot be crossed vertically. Thus

$$\begin{aligned}
&\mathbb{P}\big(Q_\beta \text{ is crossed vertically}\big) \\
&\qquad \leq \mathbb{P}\begin{pmatrix}\text{each horizontal strip in } Q_\beta \text{ of height 2 is} \\ \text{non void in the initial configuration } \alpha\end{pmatrix} \\
&\qquad \leq \mathbb{P}\begin{pmatrix}\text{one fixed horizontal strip in } Q_\beta \text{ of height 2} \\ \text{is non void in the initial configuration } \alpha\end{pmatrix}^{\beta/2-1} \\
&\qquad \leq \big(1-(1-p_\beta)^{2(2\beta+1)^{d-1}}\big)^{\beta/2-1}.
\end{aligned}$$

To complete the estimate, we count the number of possible choices for the box $Q_\beta$:

$$\begin{aligned}
&\mathbb{P}\begin{pmatrix}\text{there is an occupied cluster in bootstrap}(\alpha) \\ \text{whose } \mathrm{d}_\infty \text{ diameter is larger than } \beta\end{pmatrix} \\
&\qquad \leq |\Lambda_\beta| \times 3\beta \times d\,\mathbb{P}\big(Q_\beta \text{ is crossed vertically}\big) \\
&\qquad \leq 3d\beta\exp(\beta dL)\big(1-(1-p_\beta)^{2(2\beta+1)^{d-1}}\big)^{\beta/2-1}
\end{aligned}$$

and this last bound is SES. □

# 6 Proof of theorem 2.4

In this section we prove theorem 2.4 with the help of an induction over the dimension $d$. The main point here is the bound on the asymptotic speed of growth of a droplet. Our approach gives a bound on the probability of a "too fast" growth. Since this bound is super-exponential, while both the volume and the time we are considering are exponential, we end up with a deterministic computation rather than a large-deviation estimate as in [DS97b]. This fact allows to avoid all combinatorial problems like counting the number of "chronological paths" and it is the main technical difference with the method used in [DS97b]. Heuristically, the process evolves as if the droplets were growing one shell after the other, filling the sites on one face before passing to the next. Since all the sites on a face are neighbors of an occupied site in the droplet, this growth mechanism is analogous to a nucleation and growth mechanism in dimension $d-1$. We use the $d-1$ dimensional bound on the size of the clusters to show that, up to SES events, a too–fast growth has to take place into a parallelepiped with "small" base. This is a SES bound, and the result holds in any exponential volume. Throughout the section, we let

$$\Lambda_\beta = \Lambda(\exp(\beta L))$$

be a cubic box of sidelength $\exp(\beta L)$, where $L > 0$. Let $\kappa < \kappa_d$ and let $\tau_\beta = \exp(\beta\kappa)$. Coalescence is a nontrivial effect only if $L \geq L_d$, since otherwise the number of droplets formed in $\Lambda_\beta$ before time $\tau_\beta$ is finite. Theorem 2.4 needs to be proved only for $L \geq L_d$.

## 6.1 Dilation, bootstrap and erosion

The procedure we are going to define is a modified version of standard bootstrap percolation and is specifically suited to our setting. The same results can be obtained by rescaling the lattice as in [DS97b] and using the standard bootstrap percolation arguments developed in [AL88, CM02]. We denote by $\mathrm{d}_\infty$ the distance associated to the supremum norm, given by

$$\forall x, y \in \mathbb{Z}^d \qquad \mathrm{d}_\infty(x, y) = |x - y|_\infty = \max_{1 \leq i \leq d} |x_i - y_i| .$$

Let $\Lambda$ be a subset of $\mathbb{Z}^d$, let $\eta$ be a configuration in $\{0,1\}^\Lambda$ and let $l \geq 0$. We define the dilated configuration $\mathrm{dilate}(\eta, l)$ by occupying all the sites of $\Lambda$ which are at a $\mathrm{d}_\infty$ distance strictly less than $l$ from a site occupied in $\eta$:

$$\forall x \in \Lambda \quad \mathrm{dilate}(\eta, l)(x) = \begin{cases} 1 & \text{if} \quad \exists\, y \in \Lambda \quad \mathrm{d}_\infty(x, y) < l\,, \quad \eta(y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

We define the eroded configuration $\mathrm{erode}(\eta, l)$ by emptying all the sites of $\Lambda$ which are at a $\mathrm{d}_\infty$ distance strictly less than $l$ from an empty site in $\eta$:

$$\forall x \in \Lambda \quad \mathrm{erode}(\eta, l)(x) = \begin{cases} 0 & \text{if} \quad \exists\, y \in \Lambda \quad \mathrm{d}_\infty(x, y) < l\,, \quad \eta(y) = 0 \\ 1 & \text{otherwise} \end{cases}$$

Dilation and erosion are classical operations in mathematical morphology.

Let $\eta$ be the random configuration defined as follows. For $x \in \Lambda_\beta$, we set $\eta(x) = 1$ if there exists $i \geq 1$ such that $\tau(x, i) \leq \tau_\beta$ and $U(x, i) \leq c_\beta(0)$, otherwise we set $\eta(x) = 0$. The law of the configuration $\eta$ is the Bernoulli product law with parameter $p_\beta$ given by

$$p_\beta = 1 - \exp\big(-c_\beta(0)\tau_\beta\big) .$$

Taking logarithm, we see that

$$\lim_{\beta \to \infty} \frac{1}{\beta} \ln p_\beta = -\Gamma_d + \kappa .$$

Let $\mathrm{bootstrap}(\eta)$ be the configuration obtained by bootstraping $\eta$.

**Proposition 6.1** Let $\rho$ be the configuration obtained by dilating $\eta$ with a distance $\beta^{-1}\exp(\beta L_d)$ and then bootstraping it:

$$\rho \,=\, \text{bootstrap}(\text{dilate}(\eta, \beta^{-1}\exp(\beta L_d)))\,.$$

The probability that there is an occupied cluster in $\rho$ whose $\mathrm{d}_\infty$ diameter is larger than $\exp(\beta L_d)$ is super–exponentially small in $\beta$.

**Proof.** By lemma 4.1, if there is an occupied cluster in $\rho$ whose $\mathrm{d}_\infty$ diameter is larger than $\exp(\beta L_d)$, then there exists an internally–spanned cluster in $\rho$ with diameter between $\exp(\beta L_d)$ and $2\exp(\beta L_d)+1$. Let $Q_\beta$ be a cube of minimal side length containing such a cluster. The cube $Q_\beta$ has to be crossed in one of the $d$ directions parallel to the axis, say for instance the vertical one. Let $Q'_\beta$ be the parallelepiped having the same center and the same height as $Q_\beta$ and whose sidelengths in the other directions are three times the sidelength of $Q_\beta$. If there is an horizontal strip in $Q'_\beta$ of height $3\beta^{-1}\exp(\beta L_d)$ which is void in the initial configuration $\eta$, then there is an horizontal strip in $Q_\beta$ of height 2 which is void in the intermediate configuration

$$\text{dilate}(\eta, \beta^{-1}\exp(\beta L_d))$$

and the box $Q_\beta$ cannot be crossed vertically after the bootstraping. Thus

$$\begin{aligned}
&\mathbb{P}\big(Q_\beta \text{ is crossed vertically in } \rho\big) \,\leq \\
&\qquad \mathbb{P}\begin{pmatrix}\text{each horizontal strip in } Q'_\beta \text{ of height } 3\beta^{-1}\exp(\beta L_d)\\ \text{is non void in the initial configuration } \eta\end{pmatrix} \\
&\leq\, \mathbb{P}\begin{pmatrix}\text{one fixed horizontal strip in } Q'_\beta \text{ of height } 3\beta^{-1}\exp(\beta L_d)\\ \text{is non void in the initial configuration } \eta\end{pmatrix}^{\beta/3} \\
&\leq\, \Big(1-\exp\Big(9^{d-1}\exp((d-1)\beta L_d)\times 3\beta^{-1}\exp(\beta L_d)\times\ln(1-p_\beta)\Big)\Big)^{\beta/3} \\
&\qquad \leq\, \big(-9^d\beta^{-1}\exp(d\beta L_d)\times\ln(1-p_\beta)\big)^{\beta/3}
\end{aligned}$$

and this last bound is SES because $dL_d+\Gamma_d = \kappa_d$. To complete the estimate, we count the number of possible choices for the box $Q_\beta$:

$$\begin{aligned}
&\mathbb{P}\begin{pmatrix}\text{there is an occupied cluster in } \rho \text{ whose}\\ \mathrm{d}_\infty \text{ diameter is larger than } \exp\beta L_d\end{pmatrix} \\
&\qquad \leq\, |\Lambda_\beta|\times 3\exp(\beta L_d)\times d\,\mathbb{P}\big(Q_\beta \text{ is crossed vertically in } \rho\big)
\end{aligned}$$

and the last term is SES. □

Let $\xi$ be the erosion of $\rho$ with a distance $(2\beta)^{-1}\exp(\beta L_d)$, i.e.,

$$\xi \,=\, \text{erode}(\rho, (2\beta)^{-1}\exp(\beta L_d))\,. \tag{6.2}$$

Since $\rho$ was obtained after applying the bootstrap procedure, it is a union of occupied parallelepipeds, which are pairwise at distance larger than two. After applying the erosion operator, we obtain again an union of occupied parallelepipeds, which are pairwise at distance larger than or equal to $(2\beta)^{-1}\exp(\beta L_d)$. Moreover we dilated $\eta$ with a distance $\beta^{-1}\exp(\beta L_d)$ before the bootstrap, thus the configuration $\eta$ is still included in $\xi$, so that all the sites where nucleation has occurred before time $\tau_\beta$ are occupied in the configuration $\eta$. By attractivity of the process, we have

$$\sigma_{\Lambda_\beta,\tau_\beta} \,\leq\, \sigma^{\eta}_{\Lambda_\beta,\tau_\beta} \,\leq\, \sigma^{\xi}_{\Lambda_\beta,\tau_\beta}$$

and because of the definition of $\eta$, no nucleation occurs in the growth process starting from $\eta$ until the time $\tau_\beta$. We are thus able to compare $\sigma_{\Lambda_\beta,\tau_\beta}$ with a process where nucleation events are cancelled, that we define in the next section. The crucial problem is then to control the speed of growth of the droplets and to show that, up to a SES event, the non–nucleating process starting from $\xi$ is still included in $\rho$ at time $\tau_\beta$.

## 6.2 Control of the speed of growth

In this section, we study the growth process where the nucleation is cancelled and we prove our key estimate to control the speed of growth of the droplets. The initial speed of growth of a nucleus is $\exp(-\beta\Gamma_{d-1})$. For a droplet of size $\exp(\beta K)$, the speed is

$$\exp\big(\beta((d-1)K-\Gamma_{d-1})\big)$$

for $K < L_{d-1}$ and $\exp(-\beta\kappa_{d-1})$ for $K \geq L_{d-1}$. It turns out that the time needed to create a droplet travelling at the asymptotic speed is $\exp(\beta\Gamma_{d-1})$, which is of the same order as the time needed to grow the initial nucleus into a droplet of diameter $\beta$. Hence we need only to control the speed of droplets having a diameter larger than $\exp(\beta L_{d-1})$, which travel at the asymptotic speed.

**Non–nucleating processes.** We define a *non-nucleating* process

$$(\widetilde{\sigma}_{\Lambda_\beta,t})_{t\geq 0}$$

associated to the rates

$$\widetilde{c}(0) \,=\, 0\,, \qquad \widetilde{c}(n) \,=\, c(n)\,, \quad 1\leq n\leq 2d\,.$$

In this process, a site cannot become occupied unless one of its neighbors is occupied. The activation energies for this process are given by

$$\widetilde{\Gamma}(d) \,=\, \infty\,, \qquad \widetilde{\Gamma}(n) \,=\, \Gamma(n)\,, \quad 0\leq n< d\,.$$

In the sequel, the various processes where nucleation is suppressed are denoted with a tilde above the symbol of the process.

**Floor boundary conditions.** Let $R$ be a cylinder with basis a $d-1$ dimensional cubic box $\Lambda^{d-1}$ and height $H$, i.e., of the form

$$R \,=\, \Lambda^{d-1} \times \{\, 0, \dots, H \,\}\,.$$

We call *floor* of $R$ its bottom face $\Lambda^{d-1} \times \{\, 0 \,\}$ and *ceiling* of $R$ its top face $\Lambda^{d-1} \times \{\, H \,\}$. We call floor boundary conditions on $R$ the boundary condition defined by the following configuration $\rho$:

$$\forall x \in \mathbb{Z}^d \qquad \rho(x) \,=\, \begin{cases} 1 \text{ if } x \in \Lambda^{d-1} \times \{\, -1 \,\} \\ 0 \text{ otherwise} \end{cases}$$

The process $(\widetilde{\sigma}^{\rho}_{R;t})_{t\geq 0}$ in $R$ with the floor boundary conditions is denoted by

$$(\widetilde{\sigma}^{-}_{R;t})_{t\geq 0}\,.$$

We say that a configuration *crosses* $R$ if it contains a cluster included in $R$ which connects the floor and the ceiling.

**Proposition 6.3** Let $d \geq 2$ and let $K > 0$. Let $R_\beta$ be the cylinder

$$R_\beta \,=\, \Lambda^{d-1}(\exp{(\beta K)}) \times \{\, 0, \dots, \beta \,\}\,.$$

Let $\kappa < \kappa_{d-1}$ and $\tau_\beta = \exp(\beta\kappa)$. Suppose that theorem 2.4 has been proved in dimension $d-1$. Then the probability that $\widetilde{\sigma}^{-}_{R_\beta;\tau_\beta}$ crosses $R_\beta$ is SES.

**Proof.** We start with the case $K > L_{d-1}$ and we set

$$\Lambda^{d-1}_\beta \,=\, \Lambda^{d-1}(\exp(\beta L_{d-1}))\,.$$

We use theorem 2.4 to show that, most likely, the cluster that crosses $R_\beta$ is contained in a smaller parallelepiped of basis $\Lambda^{d-1}_\beta$, i.e., a parallelepiped which is a translate of

$$T_\beta \,=\, \Lambda^{d-1}_\beta \times \{\, 0, \dots, \beta \,\}\,.$$

To this end, let us consider the process obtained from $(\widetilde{\sigma}^{-}_{R;t})_{t\geq 0}$ by occupying all the sites in each non empty column, and its projection $(\widehat{\sigma}_t)_{t\geq 0}$ on the floor $R_\beta$ defined for $t \geq 0$ by

$$\forall \widehat{x} \in \Lambda^{d-1}_\beta \quad \widehat{\sigma}_t(\widehat{x}) \,=\, \begin{cases} 0 & \text{if} \quad \widetilde{\sigma}^{-}_{R_\beta;t}(\widehat{x}, i) \,=\, 0 \text{ for all } i \in \{\, 0, \dots, \beta \,\} \\ 1 & \text{if} \quad \widetilde{\sigma}^{-}_{R_\beta;t}(\widehat{x}, i) \,=\, 1 \text{ for some } i \in \{\, 0, \dots, \beta \,\} \end{cases}$$

The process $(\widehat{\sigma}_t)_{t\geq 0}$ is a $(d-1)$-dimensional process with rates satisfying

$$c_\beta(n+1) \,\leq\, \widehat{c}_\beta(n) \,\leq\, 2c_\beta(n+1) + (\beta-2)c_\beta(n)\,, \quad 0 \leq n \leq d-1\,.$$

In terms of activation energies,

$$\widehat{\Gamma}(n) \,=\, \Gamma(n)\,, \quad 0 \leq n \leq d-1\,.$$

The idea is to use the $(d-1)$-dimensional bounds on the size of $(\widehat{\sigma}_t)_{t\geq 0}$ and attractivity to bound the size of the clusters of $(\widetilde{\sigma}^-_t)_{t\geq 0}$. Let Large be the event

$$\text{Large} \,=\, \left\{ \begin{array}{c} \text{there is an occupied cluster in } \widetilde{\sigma}^-_{R_\beta;\tau_\beta} \\ \text{whose projection on the floor of } R_\beta \\ \text{has a diameter larger than } \exp\left(\beta L_{d-1}\right) \end{array} \right\}.$$

By theorem 2.4 in dimension $d-1$, since $\kappa < \kappa_{d-1}$, the probability that an occupied cluster in $\widehat{\sigma}_{\tau_\beta}$ has diameter larger than $\exp(\beta L_{d-1})$ is SES. Since the volume of $R_\beta$ is exponential, the probability of the event Large is SES. We write then

$$\begin{aligned} \mathbb{P}(\widetilde{\sigma}^-_{R_\beta;\tau_\beta} \text{ crosses } R_\beta) \,&\leq\, \mathbb{P}(\text{Large}) \,+\, \mathbb{P}\big(\{\,\widetilde{\sigma}^-_{R_\beta;\tau_\beta} \text{ crosses } R_\beta\} \setminus \{\,\text{Large}\,\}\big) \\ &\leq\, SES \,+\, \mathbb{P}\left(\begin{array}{c}\text{there is a translate } y+T_\beta \text{ of } T_\beta \text{ included} \\ \text{in } R_\beta \text{ such that } \widetilde{\sigma}^-_{y+T_\beta;\tau_\beta} \text{ crosses } y+T_\beta\end{array}\right) \\ &\leq\, SES \,+\, |R_\beta|\;\mathbb{P}(\widetilde{\sigma}^-_{T_\beta;\tau_\beta} \text{ crosses } T_\beta)\,. \end{aligned}$$

In the last step, we used the fact that the model is translation invariant. We conclude by showing that

$$\mathbb{P}(\widetilde{\sigma}^-_{T_\beta;\tau_\beta} \text{ crosses } T_\beta)$$

is also SES, reducing ourselves to the case where $K \leq L_{d-1}$. We shall couple the process $(\widetilde{\sigma}^-_{T_\beta;t})_{t\geq 0}$ with floor boundary conditions in $T_\beta$ with another simpler process.

**Sandwich boundary conditions.** We call *slice* a parallelepiped with height 2 and basis $\Lambda_\beta^{d-1}$, which is a translate of

$$\Sigma \,=\, \Lambda_\beta^{d-1} \times \{\,0,1\,\}\,.$$

We call sandwich boundary conditions on $\Sigma$ the boundary condition defined by the following configuration $\rho$:

$$\forall x \in \mathbb{Z}^d \qquad \rho(x) \,=\, \begin{cases} 1 \text{ if } x \in \Lambda_\beta^{d-1} \times \{\,-1,2\,\} \\ 0 \text{ otherwise} \end{cases}$$

We denote by $(\widetilde{\sigma}^{=}_{\Sigma;t})_{t\geq 0}$ the process in $\Sigma$ evolving with the sandwich boundary conditions $\rho$.

**Multilayer process.** Let us partition the cylinder $T_\beta$ into translated slices as

$$T_\beta \,=\, \bigcup_{i=0}^{\beta/2} \Sigma_i \,,$$

where

$$\Sigma_i \,=\, \Lambda_\beta^{d-1} \times \{\, 2i, 2i+1 \,\} \,=\, \Sigma + (0,\dots,0,2i)\,.$$

We define the multilayer process $(\widetilde{\sigma}^{\equiv}_{T_\beta;t})_{t\geq 0}$ using the same graphical construction as $(\widetilde{\sigma}_{T_\beta;t})_{t\geq 0}$ but we use sandwich boundary conditions in each slice. More precisely, we set

$$\forall i \in \{\, 1,\dots,\beta/2 \,\} \quad \forall x \in \Sigma_i \quad \forall t \geq 0 \qquad \widetilde{\sigma}^{\equiv}_{T_\beta;t}(x) \,=\, \widetilde{\sigma}^{=}_{\Sigma_i;t}(x)\,.$$

A key point is that, once we put sandwich boundary conditions around each slice, the processes in the slices become independent of each other. Thanks to the coupling, the process $(\widetilde{\sigma}^{\equiv}_{T_\beta;t})_{t\geq 0}$ is always above the process $(\widetilde{\sigma}^{-}_{T_\beta;t})_{t\geq 0}$. Therefore, if $\widetilde{\sigma}^{-}_{T_\beta;\tau_\beta}$ crosses $T_\beta$, so does $\widetilde{\sigma}^{\equiv}_{T_\beta;\tau_\beta}$ and at least a nucleus must appear in each slice. Thus

$$\begin{aligned}
\mathbb{P}\big(\widetilde{\sigma}^{-}_{T_\beta;\tau_\beta} \text{ crosses } T_\beta\big) \,&\leq\, \mathbb{P}\big(\widetilde{\sigma}^{\equiv}_{T_\beta;\tau_\beta} \text{ crosses } T_\beta\big) \\
&\leq\, \mathbb{P}\big(\widetilde{\sigma}^{=}_{\Sigma_i;\tau_\beta} \text{ is not void } \text{ for } 1\leq i\leq \beta/2\big) \\
&\leq\, \mathbb{P}\big(\widetilde{\sigma}^{=}_{\Sigma;\tau_\beta} \text{ is not void }\big)^{\beta/2-1}.
\end{aligned}$$

Yet

$$\begin{aligned}
\mathbb{P}\big(\widetilde{\sigma}^{=}_{\Sigma;\tau_\beta} \text{ is void }\big) \,&=\, \mathbb{P}\begin{pmatrix}\text{for any } x\in\Sigma, \text{ there is no}\\ \text{nucleation at } x \text{ before } \tau_\beta\end{pmatrix} \\
&=\, \mathbb{P}\begin{pmatrix}\text{there is no nucleation}\\ \text{at the origin before } \tau_\beta\end{pmatrix}^{|\Sigma|} \\
&=\, \Big(\exp\Big(-c_\beta(1)\,\tau_\beta\Big)\Big)^{|\Sigma|} \\
&=\, \exp\Big(-2\,|\Lambda_\beta^{d-1}|\,c_\beta(1)\,\tau_\beta\Big) \\
&=\, \exp -\Big(2\exp\big(\beta(d-1)K + \ln c_\beta(1) + \beta\kappa\big)\Big)\,.
\end{aligned}$$

Since $K \leq L_{d-1}$ and $\kappa < \kappa_{d-1}$, we have

$$\lim_{\beta\to\infty}\, \frac{1}{\beta}\big(\beta(d-1)K + \ln c_\beta(1) + \beta\kappa\big) \,=\, (d-1)K - \Gamma_{d-1} + \kappa \,<\, 0$$

and there exists a positive constant $\delta$ such that, for $\beta$ large enough,

$$\mathbb{P}\big(\widetilde{\sigma}^{=}_{\Sigma;\tau_\beta} \text{ is void }\big) \;\geq\; \exp -\Big(2\exp(-\beta\delta)\Big)\,.$$

Reporting in the previous inequality, we get

$$\mathbb{P}\big(\widetilde{\sigma}^{=}_{T_\beta;\tau_\beta} \text{ crosses } T_\beta\big) \;\leq\; \Big(1-\exp -\Big(2\exp(-\beta\delta)\Big)\Big)^{\beta/2-1}\,.$$

Hence the above probability is also SES. □

**Corollary 6.4** Let $d \geq 2$ and let $K, L > 0$. Let $R_\beta$ be the cylinder

$$R_\beta \;=\; \Lambda^{d-1}(\exp{(\beta K)}) \times \{\,0, \ldots, \exp(\beta L)\,\}\,.$$

Let $\kappa > 0$ be such that $\kappa < L + \kappa_{d-1}$ and $\tau_\beta = \exp(\beta\kappa)$. Suppose that theorem 2.4 has been proved in dimension $d-1$. Then the probability that $\widetilde{\sigma}^{-}_{R_\beta;\tau_\beta}$ crosses $R_\beta$ is SES.

**Proof.** For $i \in \mathbb{N}$, let $\tau_i$ be the first time when a site of the layer

$$\Lambda^{d-1}(\exp{(\beta K)}) \times \{\,i\beta\,\}$$

becomes occupied in the process $(\widetilde{\sigma}^{=}_{R_\beta;t})_{t\geq 0}$. Let us set

$$l \;=\; \left\lfloor \frac{\exp(\beta L)}{\beta} \right\rfloor\,.$$

With these definitions, we see that if $\widetilde{\sigma}^{=}_{R_\beta;\tau_\beta}$ crosses $R_\beta$, then $\tau_l \leq \tau_\beta$. Yet

$$\tau_l \;=\; \sum_{0\leq i<l} \tau_{i+1} - \tau_i$$

and moreover, by using the Markov property and the attractivity of the process, we see that, for any $i \geq 0$, the time $\tau_{i+1} - \tau_i$ stochastically dominates the time $\tau_1$. Therefore

$$\begin{aligned}\mathbb{P}(\tau_l \leq \tau_\beta) \;&\leq\; \mathbb{P}\Big(\exists i < l \quad \tau_{i+1} - \tau_i \leq l^{-1}\exp(\beta\kappa)\Big)\\ &\leq\; l\,\mathbb{P}\Big(\tau_1 \leq l^{-1}\exp(\beta\kappa)\Big)\,.\end{aligned}$$

By hypothesis, we have $\kappa - L < \kappa_{d-1}$. Proposition 6.3 implies that this last bound is SES. □

### 6.3 Conclusion of the proof of theorem 2.4

We proceed now by induction over the dimension $d$. The case of dimension 0 is straightforward. In this case the lattice $\mathbb{Z}^0$ is reduced to the singleton $\{0\}$ and $\kappa_0 = \Gamma_0$, $L_0 = 0$. In particular, it is impossible to see an occupied cluster of diameter strictly larger than 0. Let $d \geq 1$. Suppose that the result has been proved in dimension $d-1$. Let $L > 0$ and let $\Lambda_\beta = \Lambda(\exp(\beta L))$ be a $d$–dimensional cubic box of sidelength $\exp(\beta L)$. Let $\kappa < \kappa_d$ and let $\tau_\beta = \exp(\beta\kappa)$.

We apply corollary 6.4 to show that, up to a SES event, $\sigma_{\Lambda_\beta,\tau_\beta}$ is included in the configuration $\rho$. Indeed, suppose that it is not the case. Then the configuration $\widetilde{\sigma}^{\xi}_{\Lambda_\beta,\tau_\beta}$ is also not included in $\rho$. Yet the configuration $\xi$ is an union of occupied parallelepipeds, which are pairwise at distance larger than or equal to $(2\beta)^{-1}\exp(\beta L_d)$ (see (6.2)), and the configuration $\rho$ is obtained from $\xi$ by dilating these parallelepipeds with a distance $(2\beta)^{-1}\exp(\beta L_d)$. We consider the first time and place when the process $(\widetilde{\sigma}^{\xi}_{\Lambda_\beta,t})_{t\geq 0}$ occupies a site not occupied in $\rho$. This happens close to the boundary of a face $F$ of one of the parallelepipeds $Q$ occupied in $\rho$. Let $R_\beta$ be the cylinder included in $Q$ having for basis this face $F$ and for height $(2\beta)^{-1}\exp(\beta L_d)$. By corollary 6.4, the probability that $\widetilde{\sigma}^{-}_{R_\beta;\tau_\beta}$ crosses $R_\beta$ is SES. Since the number of choices of times and places above is exponential in $\beta$, we conclude that, up to a SES event, the configuration $\sigma_{\Lambda_\beta,\tau_\beta}$ is included in $\rho$. This estimate, together with proposition 6.1, implies theorem 2.4.

# 7 Proof of the upper bound of theorem 2.2

Let $L > 0$ and let $\Lambda_\beta = \Lambda(\exp(\beta L))$ be a cubic box of sidelength $\exp(\beta L)$. Let $\kappa < \max(\Gamma_d - dL, \kappa_d)$ and let $\tau_\beta = \exp(\beta\kappa)$. We distinguish three different cases.

• First case: $\kappa < \Gamma_d - dL$. If the origin is occupied at time $\tau_\beta$ for the growth process in $\Lambda_\beta$, then a nucleation must have taken place in the box $\Lambda_\beta$ before the time $\tau_\beta$, thus

$$\mathbb{P}\left(\sigma_{\Lambda_\beta;\tau_\beta}(0) = 1\right) \leq |\Lambda_\beta| \left(1 - \exp\left(-c_\beta(0)\tau_\beta\right)\right).$$

Taking logarithm, we see that

$$\limsup_{\beta\to\infty} \frac{1}{\beta} \ln \mathbb{P}\left(\sigma_{\Lambda_\beta;\tau_\beta}(0) = 1\right) \leq dL - \Gamma_d + \kappa\,.$$

Yet $\kappa < \Gamma_d - dL$ and the probability that the origin is occupied at time $\tau_\beta$ for the growth process in $\Lambda_\beta$ is therefore ES in $\beta$.

• Second case: $\kappa < \Gamma_{d-1}$. Let $\Lambda'_\beta = \Lambda(3\beta)$ be a cubic box of sidelength $3\beta$. Suppose that the origin is occupied at time $\tau_\beta$ for the growth process in $\Lambda_\beta$. The droplet which has reached the origin is either born inside the box $\Lambda'_\beta$ or outside of it. In the first scenario, a nucleation event must have taken place in the box $\Lambda'_\beta$ before the time $\tau_\beta$. In the second scenario there is an occupied cluster in $\sigma_{\Lambda_\beta;\tau_\beta}$ with diameter larger than $\beta$. We have thus

$$\begin{aligned}\mathbb{P}\left(\sigma_{\Lambda_\beta;\tau_\beta}(0)=1\right) \;\le\; & \mathbb{P}\begin{pmatrix}\text{a nucleation event takes}\\ \text{place in } \Lambda'_\beta \text{ before } \tau_\beta\end{pmatrix}\\ & + \mathbb{P}\begin{pmatrix}\text{there is an occupied cluster in } \sigma_{\Lambda_\beta;\tau_\beta}\\ \text{whose } \mathrm{d}_\infty \text{ diameter is larger than } \beta\end{pmatrix}.\end{aligned}$$

Proceeding as in the first case, we bound the probability of a nucleation by

$$|\Lambda'_\beta|\left(1-\exp\big(-c_\beta(0)\tau_\beta\big)\right)$$

which is ES in $\beta$ since $\kappa < \Gamma_{d-1} \le \Gamma_d$. By proposition 2.3, the probability that an occupied cluster in $\sigma_{\Lambda_\beta;\tau_\beta}$ has diameter larger than $\beta$ is super–exponentially small in $\beta$.

• Third case: $\kappa < \kappa_d$. Let $\Lambda'_\beta = \Lambda\big(3\exp(\beta L_d)\big)$ be a cubic box of sidelength $3\exp(\beta L_d)$. Suppose that the origin is occupied at time $\tau_\beta$ for the growth process in $\Lambda_\beta$. The droplet which has reached the origin is either born inside the box $\Lambda'_\beta$ or outside of it. In the first scenario, a nucleation event must have taken place in the box $\Lambda'_\beta$ before the time $\tau_\beta$. In the second scenario there is an occupied cluster in $\sigma_{\Lambda_\beta;\tau_\beta}$ with diameter larger than $\exp(\beta L_d)$. We have thus

$$\begin{aligned}\mathbb{P}\left(\sigma_{\Lambda_\beta;\tau_\beta}(0)=1\right) \;\le\; & \mathbb{P}\begin{pmatrix}\text{a nucleation event takes}\\ \text{place in } \Lambda'_\beta \text{ before } \tau_\beta\end{pmatrix}\\ & + \mathbb{P}\begin{pmatrix}\text{there is an occupied cluster in } \sigma_{\Lambda_\beta;\tau_\beta}\\ \text{whose } \mathrm{d}_\infty \text{ diameter is larger than } \exp(\beta L_d)\end{pmatrix}.\end{aligned}$$

Proceeding as in the first case, we bound the probability of a nucleation by

$$|\Lambda'_\beta|\left(1-\exp\big(-c_\beta(0)\tau_\beta\big)\right).$$

Taking logarithm, we see that

$$\limsup_{\beta\to\infty}\ \frac{1}{\beta}\ln\mathbb{P}\begin{pmatrix}\text{a nucleation event takes}\\ \text{place in } \Lambda'_\beta \text{ before } \tau_\beta\end{pmatrix} \;\le\; dL_d - \Gamma_d + \kappa \;<0\,.$$

By proposition 2.3, the probability that an occupied cluster in $\sigma_{\Lambda_\beta;\tau_\beta}$ has diameter larger than $\exp(\beta L_d)$ is super–exponentially small in $\beta$.

In the three cases, the probability

$$\mathbb{P}\left(\sigma_{\Lambda_\beta;\tau_\beta}(0)=1\right)$$

is ES in $\beta$.

# 8 Proof of the lower bound of theorem 2.2

We prove here part 2 of theorem 2.2 by induction over the dimension $d$. Let us consider first the case $d=0$. We have then $\kappa_0=\Gamma_0$. The box $\Lambda_\beta$ is reduced to the singleton $\{\,0\,\}$. Let $\kappa>\kappa_0$ and let $\tau_\beta=\exp(\beta\kappa)$. We have

$$\mathbb{P}\left(\sigma_{\Lambda_\beta;\tau_\beta}(0)=0\right) \;=\; \exp-(c_\beta(0)\tau_\beta) \;=\; \exp-(c_\beta(0)\exp(\beta\kappa))\,.$$

Since by hypothesis,

$$\lim_{\beta\to\infty}\;\frac{1}{\beta}\ln c_\beta(0) \;=\; -\Gamma_0$$

we conclude that the above probability is SES. We suppose now that $d\geq 1$ and that the result has been proved in dimension $d-1$. Let $L>0$ and let $\Lambda_\beta=\Lambda(\exp(\beta L))$ be a cubic box of sidelength $\exp(\beta L)$. Let $\kappa>0$ and let $\tau_\beta=\exp(\beta\kappa)$. Let $\varepsilon>0$. We define the nucleation time $\tau_{\text{nucleation}}$ in $\Lambda_\beta$ as

$$\tau_{\text{nucleation}} \;=\; \inf\left\{\,t\geq 0:\exists\,x\in\Lambda_\beta\quad \sigma_{\Lambda_\beta;t}(x)=1\,\right\}.$$

We have

$$\forall t>0\qquad \mathbb{P}(\tau^N\;>\;t)=\exp\left(-|\Lambda_\beta|\,c_\beta(0)\,t\right).$$

Therefore, up to a SES event, the first nucleus in the box $\Lambda_\beta$ appeared before time

$$\exp\Big(\beta\big(\Gamma_d-dL+\varepsilon\big)\Big)\,.$$

For $i\geq 1$, we define the first time $\tau^i$ when there is an occupied parallelepiped of diameter larger than or equal to $i$ in $\Lambda_\beta$, i.e.,

$$\tau^i \;=\; \inf\left\{\,t\geq 0:\begin{array}{c}\text{there is an occupied parallelepiped included in }\Lambda_\beta\\ \text{whose }\mathrm{d}_\infty\text{ diameter is larger than or equal to }i\end{array}\right\}$$

The restriction of the process $(\sigma_{\Lambda_\beta;t})_{t\geq 0}$ to the sites which are the neighbors of a face of an occupied parallelepiped is a $d-1$ dimensional growth process whose rates satisfy the hypothesis of our model. From the induction hypothesis, we know that, up to a SES event, the $d-1$ dimensional process in a box of sidelength $\exp(\beta K)$ is fully occupied at a time

$$\exp\Big(\beta\big(\max(\Gamma_{d-1}-(d-1)K,\kappa_{d-1})+\varepsilon\big)\Big)\,.$$

This implies that, up to a SES event, the box $\Lambda_\beta$ is fully occupied at time

$$\tau^{\exp(\beta L)} \le \tau_{\text{nucleation}} + \sum_{1\le i<\exp(\beta L)} (\tau^{i+1}-\tau^i) \le \exp\Big(\beta\big(\Gamma_d - dL + \varepsilon\big)\Big)$$
$$+ \sum_{1\le i<\exp(\beta L)} 2d\exp\Big(\beta\big(\max(\Gamma_{d-1} - \frac{d-1}{\beta}\ln i, \kappa_{d-1}) + \varepsilon\big)\Big)$$

We consider two cases.
• First case: $L \le L_{d-1}$. Notice that $L_0 = 0$, hence this case can happen only whenever $d \ge 2$. In this case, we have

$$\forall i < \exp(\beta L) \qquad \kappa_{d-1} \le \Gamma_{d-1} - \frac{d-1}{\beta}\ln i$$

and

$$\sum_{1\le i<\exp(\beta L)} \exp\Big(\beta\max(\Gamma_{d-1} - \frac{d-1}{\beta}\ln i, \kappa_{d-1})\Big)$$
$$\le \exp(\beta\Gamma_{d-1}) \sum_{1\le i<\exp(\beta L)} \frac{1}{i^{d-1}}$$
$$\le \exp(\beta\Gamma_{d-1}) \sum_{1\le i<\exp(\beta L)} \frac{1}{i} \le \beta L\exp(\beta\Gamma_{d-1})\,.$$

• Second case: $L > L_{d-1}$. We have then

$$\sum_{\exp(\beta L_{d-1})\le i<\exp(\beta L)} \exp\Big(\beta\max(\Gamma_{d-1} - \frac{d-1}{\beta}\ln i, \kappa_{d-1})\Big)$$
$$\le \Big(\exp(\beta L) - \exp(\beta L_{d-1})\Big)\exp(\beta\kappa_{d-1})$$
$$\le \exp\Big(\beta(L+\kappa_{d-1})\Big)\,.$$

We conclude that, in both cases, for any $\varepsilon > 0$, up to a SES event, the box $\Lambda_\beta$ is fully occupied at a time

$$2d\beta L\exp(\beta\varepsilon)\left(\exp\Big(\beta(\Gamma_d - dL)\Big) + \exp(\beta\Gamma_{d-1}) + \exp\Big(\beta(L+\kappa_{d-1})\Big)\right).$$

Therefore, for any $\kappa$ such that

$$\kappa > \max\Big(\Gamma_d - dL, \Gamma_{d-1}, L + \kappa_{d-1}\Big)$$

the probability that the box $\Lambda_\beta$ is not fully occupied at a time $\exp(\beta\kappa)$ is SES. If $L \le L_d$ then

$$\max\Big(\Gamma_d - dL, \Gamma_{d-1}, L + \kappa_{d-1}\Big) = \Gamma_d - dL$$

and we have the desired estimate. Suppose next that $L > L_d$. By the previous result, we know that, for any $\kappa > \kappa_d$, up to a SES event, a box of sidelength $\exp(\beta L_d)$ is fully occupied at a time $\exp(\beta\kappa)$. We cover $\Lambda_\beta$ by boxes of sidelength $\exp(\beta L_d)$. Such a cover contains at most $\exp(\beta dL)$ boxes, thus

$$\begin{aligned}\mathbb{P}\left(\Lambda_\beta \text{ is not fully occupied at time } \tau_\beta\right) &\leq \mathbb{P}\begin{pmatrix}\text{there exists a box included in } \Lambda_\beta \text{ of sidelength}\\ \exp(\beta L_d) \text{ which is not fully occupied at time } \tau_\beta\end{pmatrix}\\ &\leq \exp(\beta dL)\,\mathbb{P}\begin{pmatrix}\text{the box } \Lambda(\exp(\beta L_d)) \text{ is not}\\ \text{fully occupied at time } \tau_\beta\end{pmatrix}.\end{aligned}$$

The last probability being SES, we are done.

**Acknowledgements:** Raphaël Cerf thanks Roberto Schonmann for discussions on this problem while he visited UCLA in 1995.

# References

[AL88] M. Aizenman and J. L. Lebowitz. Metastability effects in bootstrap percolation. *J. Phys. A*, 21(19):3801–3813, 1988.

[CC99] Raphaël Cerf and Emilio N. M. Cirillo. Finite size scaling in three-dimensional bootstrap percolation. *Ann. Probab.*, 27(4):1837–1850, 1999.

[CM02] R. Cerf and F. Manzo. The threshold regime of finite volume bootstrap percolation. *Stochastic Process. Appl.*, 101(1):69–82, 2002.

[DS97a] Pouria Dehghanpour and Roberto H. Schonmann. Metropolis dynamics relaxation via nucleation and growth. *Comm. Math. Phys.*, 188(1):89–119, 1997.

[DS97b] Pouria Dehghanpour and Roberto H. Schonmann. A nucleation-and-growth model. *Probab. Theory Related Fields*, 107(1):123–135, 1997.

[Ede61] Murray Eden. A two-dimensional growth process. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. IV*, pages 223–239. Univ. California Press, Berkeley, Calif., 1961.

[KS95] Harry Kesten and Roberto H. Schonmann. On some growth models with a small parameter. *Probab. Theory Related Fields*, 101(4):435–468, 1995.

[Lig05] Thomas M. Liggett. *Interacting particle systems*. Classics in Mathematics. Springer-Verlag, Berlin, 2005. Reprint of the 1985 original.

[Ric73] Daniel Richardson. Random growth in a tessellation. *Proc. Cambridge Philos. Soc.*, 74:515–528, 1973.