TWO DIMENSIONAL ZONOIDS AND CHEBYSHEV MEASURES

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Abstract. We give an alternative proof to the well known fact that each convex compact centrally symmetric subset of $\mathbb{R}^2$ containing the origin is a zonoid i.e. the range of a two dimensional vector measure and we show that a two dimensional zonoid whose boundary contains the origin is strictly convex if and only if it is the range of a Chebyshev measure. We give a condition under which a two dimensional vector measure admits a decomposition as the difference of two Chebyshev measures, a necessary condition on the density function for the strict convexity of the range of a measure and a characterization of two dimensional Chebyshev measures.

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1. Introduction

A well known Theorem of Lyapunov [11, 13] states that the range of a non–atomic vector measure is compact and convex. Conversely (e.g. [1]) each compact convex centrally symmetric subset of $\mathbb{R}^2$ containing the origin is the range of a two dimensional measure (such a set is called a zonoid).

Some problems related to the bang–bang principle in control theory led us to work with the class of the Chebyshev measures. Our definition of a Chebyshev measure is essentially a linear independence condition on some vectors of its range. In [5,6] we proved that the range of a $n-$dimensional Chebyshev measure is strictly convex and its boundary contains the origin. Recently Schneider showed in [14] that the range of a $n-$dimensional measure is strictly convex if and only if for every set A with $\mu(A) \neq 0$ there exist n measurable subsets $A_1, \dots, A_n$ of A such that $\mu(A_1), \dots, \mu(A_n)$ are linearly independent. A result by Neyman [9] states that if the origin is an extreme point of the boundary of a zonoid Z and μ is a vector measure such that $\mathcal{R}(\mu) = Z$ then Z determines the $m-$range of μ i.e. the set of $m-$uples $(\mu(A_1), \dots, \mu(A_m))$ where $A_1, \dots, A_m$ are a measurable partition of the space. A $n-$dimensional strictly convex zonoid whose boundary contains the origin is then naturally expected to be the range of a Chebyshev measure.

Here we prove that a strictly convex, centrally symmetric, compact subset of $\mathbb{R}^2$ whose boundary contains the origin is actually the range of a two dimensional Chebyshev measure. We give two different proofs: the first one involves the representation theorem for Chebyshev measures proved in [6]; the second one is based on a new simple representation result for convex sets in $\mathbb{R}^2$. Our technique allows also, given an arbitrary convex centrally symmetric compact set, to build explicitly a measure whose range coincides with it. The method of defining the measure through its density with respect to a reference measure was used in [2] where the authors characterize the range of a couple of positive (quasi–) measures. Moreover, we give a condition under which a two dimensional vector measure admits a decomposition as the difference of two Chebyshev measures.

Further, for two dimensional measures, we state a necessary condition on the density function of μ with respect to its total variation for the strict convexity of the range $\mathcal{R}(\mu)$ of μ; as an application we show that μ is a Chebyshev measure on $[0, 1]$ if and only if the map θ defined by $\theta(\alpha, \beta) = \mu([\alpha, \beta])$ for $0 < \alpha < \beta < 1$ is a homeomorphism onto $\operatorname{int} \mathcal{R}(\mu)$.

2. Notation and preliminary results

Let $\mu = (\mu_1, \mu_2)$ be a two dimensional vector measure defined on the interval $[0, 1]$ equipped with a $\sigma-$ field $\mathcal{M}$ and $|\mu|$ be its total variation. The determinant measure $\det \mu$ associated to μ is the two dimensional measure on $[0, 1]^2$ defined by

$$\det \mu = \mu_1 \otimes \mu_2 - \mu_2 \otimes \mu_1;$$

we point out that if A, B are measurable then $\det \mu(A \times B) = \det(\mu(A), \mu(B))$.

We assume that $\mathcal{M}$ contains the Borelians and we set $\Gamma = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}$.

Definition 1. *The measure μ is a Chebyshev measure (or simply T–measure) with respect to the intervals $([0,\alpha])_{0\leq\alpha\leq 1}$ if it is non–atomic and each $|\mu|\otimes|\mu|$– non negligible measurable subset of Γ has a positive (or negative) $\det\mu$ measure.*

Remark. In what follows we will always assume $\det\mu$ to be positive whenever μ is a Chebyshev measure; its properties do not change in the other case.
In particular if μ is a Chebyshev measure and A, B are $|\mu|$ – non negligible subsets of $[0,1]$ such that $\sup A \leq \inf B$ then $\det(\mu(A),\mu(B)) > 0$.

For Φ being an endomorphism of $\mathbb{R}^2$ and μ a two dimensional measure on $[0,1]$ we define the two dimensional measure $\Phi\mu$ by $\Phi\mu(A) = \Phi(\mu(A))$ for every measurable $A \subset [0,1]$. The next proposition is a straightforward consequence of the definitions.

Proposition 1. *Let μ be a T–measure and Φ be a rotation; then $\Phi\mu$ is a T–measure.*

Definition 2. *A function f in $L^1_\nu([0,1],\mathbb{R}^2)$ is a Chebyshev system (or simply T–system) with respect to a prescribed measure ν whenever the determinant $\det(f(t_1), f(t_2))$ is positive for $\nu\otimes\nu$– almost every (t_1,t_2) in Γ.*

Proposition 2. [6, Th. 3.4] *A measure μ is a Chebyshev measure if and only if the density of μ with respect to its total variation is a T–system.*

For $\mu = (\mu_1,\mu_2)$ being a two dimensional measure on $[0,1]$ we denote by $\mathcal{R}(\mu)$ the range of μ defined by $\mathcal{R}(\mu) = \{\mu(E) = (\mu_1(E),\mu_2(E)) \,:\, E \in \mathcal{M}\}$ and by $\theta : \Gamma \to \mathcal{R}(\mu)$ the map defined by $\theta(\alpha,\beta) = \mu([\alpha,\beta])$ for every (α,β) in Γ.
We denote by $\operatorname{int} A$ the interior of a set A, by $\operatorname{cl} A$ its closure, by ∂A its boundary and by $\operatorname{co} A$ its convex hull; for L being a convex set in $\mathbb{R}^n$ we denote by $\operatorname{ri} L$ its relative interior. We refer to [12] for the definitions of these sets.
The peculiar properties of a Chebyshev measure rely on the following result.

Theorem 1. [6] *Let μ be a Chebyshev measure on $[0,1]$. Then the restriction of θ to $int\,\Gamma$ induces a homeomorphism onto $int\,\mathcal{R}(\mu)$; in particular $\mathcal{R}(\mu) = \{\mu([\alpha,\beta]) : 0\leq\alpha\leq\beta\leq 1\}$ and $\partial\mathcal{R}(\mu) = \{\mu([0,\alpha]) : 0\leq\alpha\leq 1\} \cup \{\mu([\beta,1]) : 0\leq\beta\leq 1\}$.*

3. A characterization of planar strictly convex zonoids

Theorem 2. *Let K be a subset of $\mathbb{R}^2$. We have the following equivalence:*

i) *the set K is strictly convex, compact, centrally symmetric and $(0,0)\in\partial K$;*
ii) *there exists a two–dimensional Chebyshev measure μ such that $K = \mathcal{R}(\mu)$.*

Proof. Assume that i) holds. Let Φ be a rotation such that $\Phi(K) \subset \{(x,y)\in\mathbb{R}^2 \,:\, x\geq 0\}$; let for simplicity $[0,1]$ be the projection of $\Phi(K)$ on the $x-$axis. For each x in $[0,1]$ let

$$y(x) = \inf\{y\in\mathbb{R} \,:\, (x,y)\in\Phi(K)\}.$$

Clearly $p = \frac{1}{2}(1, y(1))$ is the center of $\Phi(K)$ and the boundary of $\Phi(K)$ is the union of the graph of y and its symmetric with respect to p. Since y is strictly convex and $y(0) = 0$ there exists a strictly increasing function g such that

$$\forall x \in [0,1] \qquad y(x) = \int_0^x g(t)\,dt.$$

Let μ be the two dimensional vector measure on $[0,1]$ whose density function with respect to the Lebesgue measure is $f(t) = (1, g(t))$. Since g is strictly increasing, f is a T–system with respect to the Lebesgue measure: Proposition 2 then implies that μ is a T–measure. By Theorem 1 the boundary points of the range of μ are exactly the points $\mu([0,x])$ where x varies in $[0,1]$ together with their symmetric with respect to $\frac{1}{2}\mu([0,1])$. By the definition of μ we have $\mu([0,x]) = (x, y(x))$; it follows that the boundaries of $\mathcal{R}(\mu)$ and of $\Phi(K)$ coincide: these sets being convex and closed we obtain $\mathcal{R}(\mu) = \Phi(K)$ so that $K = \mathcal{R}(\Phi^{-1}\mu)$. By Proposition 1, $\Phi^{-1}\mu$ is a T–measure, proving ii). The converse is a trivial consequence of the Lyapunov Theorem and of the results stated in [5,6]. □

4. Bidimensional zonoids

A point P of a convex set C is said to be exposed (see [12]) if there exists an hyperplane whose intersection with C is reduced to P. It is well–known (Straszewicz–Klee Theorem, [10, 15]) that a compact subset of a normed space has at least an exposed point.
Let C be a compact, convex, centrally symmetric subset of $\mathbb{R}^2$. We assume here that O is an exposed point of C and that

$$C \subset \{(x,y) \in \mathbb{R}^2 : x \geq 0\}, \qquad C \cap \{(0,y) : y \in \mathbb{R}\} = \{O\}.$$

Let $L > 0, M$ in $\mathbb{R}$ be such that $(L/2, M/2)$ are the coordinates of the center I of C; clearly C is contained in $[0,L] \times \mathbb{R}$. Let y be the function defined by

$$\forall x \in [0,L]: \qquad y(x) = \min\{y \in \mathbb{R} : (x,y) \in C\}.$$

Clearly y is convex, bounded and (thus) continuous on its domain and its graph coincides with $\partial^- C = \partial C \cap \{(x,y) \in \mathbb{R}^2 : y \leq \frac{M}{L}x\}$. Let $G : [0,L] \to \partial^- C$ be the map defined by

$$\forall x \in [0,L]: \qquad G(x) = (x, y(x)).$$

Remark that for x in $[0,L]$ the symmetric point of $G(L-x)$ with respect to I is the point $(x, M - y(L-x))$ of the boundary of C. It follows that

$$\forall (x,y) \in [0,L] \times \mathbb{R}: \qquad (x,y) \in C \iff y(x) \leq y \leq M - y(L-x). \tag{$\circ$}$$

We will widely use the next representation result.

Proposition 3. *The following identity holds:*

$$C = \{G(x_2) - G(x_1) : x_1, x_2 \in [0, L],\ x_1 \leq x_2\}.$$

Proof. Let $x_1 \leq x_2$; if $x_1 = x_2$ then $O = G(x_1) - G(x_1) \in C$.
Assume that $x_1 < x_2$; since $0 \leq x_1$ and $x_2 - x_1 \leq x_2$ then by convexity we have

$$\frac{y(x_2 - x_1)}{x_2 - x_1} \leq \frac{y(x_2) - y(x_1)}{x_2 - x_1};$$

similarly since $x_1 \leq L - (x_2 - x_1)$ and $x_2 \leq L$ then

$$\frac{y(x_2) - y(x_1)}{x_2 - x_1} \leq \frac{y(L) - y(L - (x_2 - x_1))}{L - (L - (x_2 - x_1))}.$$

It follows that $y(x_2 - x_1) \leq y(x_2) - y(x_1) \leq M - y(L - (x_2 - x_1))$; thus by ($\circ$) the point $(x_2 - x_1, y(x_2) - y(x_1)) = G(x_2) - G(x_1)$ belongs to C.
Conversely let $z = (a, b) \in C$. Let $\varphi : [0, L - a] \to \mathbb{R}$ be the map defined by

$$\forall x \in [0, L - a] : \qquad \varphi(x) = y(x + a) - y(x) - b.$$

Clearly φ is continuous; moreover by ($\circ$) we have $y(a) \leq b \leq M - y(L - a)$. Therefore $\varphi(0) = y(a) - b \leq 0$ and $\varphi(L - a) = M - y(L - a) - b \geq 0$: it follows that there exists x_1 such that $\varphi(x_1) = 0$. Then if we put $x_2 = x_1 + a$ we obtain $y(x_2) = b + y(x_1)$ implying that $G(x_2) = z + G(x_1)$, which is the desired conclusion. $\square$

The construction in Theorem 2 suggests an alternative proof (and an improvement) to the well–known fact that C is the range of a measure (see [1]).
For I, J being intervals in $\mathbb{R}$ we write that $I < J$ if both I and J are not trivial and $\sup I \leq \inf J$; we shall denote by λ the Lebesgue measure in $\mathbb{R}$.

Theorem 3. *Let K be a non empty, compact, centrally symmetric, convex subset of $\mathbb{R}^2$ containing the origin. Then there exists a non–atomic measure μ on the Borelians of $[0, 1]$ such that $K = \mathcal{R}(\mu)$ and for every x in K there exist $\alpha, \beta, \gamma, \delta$ in $[0, 1]$ such that $x = \mu([\alpha, \beta]) - \mu([\gamma, \delta])$. Moreover if the origin is an exposed point of K then*

$$\mathcal{R}(\mu) = \{\mu([\alpha, \beta]) : 0 \leq \alpha \leq \beta \leq 1\}.$$

Proof. Let e be an exposed point of K; such a point exists by the Straszewicz–Klee Theorem. Then O is an exposed point of $-e + K$. Let T be a rotation such that

$$T(-e + K) \subset \{(x, y) \in \mathbb{R}^2 : x \geq 0\}, \qquad T(-e + K) \cap \{(0, y) : y \in \mathbb{R}\} = \{O\}$$

and let $I = (L/2, M/2)$ be the center of $T(-e+K)$; we will assume that $L = 1$ and set $C = T(-e+K)$. Correspondingly let y and G be the functions defined above.
By [12, Corollary 24.2.1] there exists an increasing function $g : [0,1] \to \mathbb{R}$ such that

$$\forall x_1, x_2 \in [0,1] : \qquad y(x_2) - y(x_1) = \int_{x_1}^{x_2} g(t)\, dt.$$

Let ν be the measure whose density function with respect to the Lebesgue measure is $(1, g)$. Proposition 3 then yields

$$C = \{\nu([x_1, x_2]) : x_1, x_2 \in [0,1],\, x_1 \leq x_2\} \tag{$*$}$$

so that, in particular, $C \subset \mathcal{R}(\nu)$. To prove the opposite inclusion let $I_1 < \cdots < I_m$ be m disjoint non trivial open intervals and set $V = I_1 \cup \cdots \cup I_m$. Let

$$0 = x_0 < x_1 < \cdots < x_m \leq 1 \quad \text{and} \quad 1 = y_0 > y_1 > \cdots > y_m \geq 0$$

be such that, for i in $\{1, \ldots, m\}$, the intervals $J_i =]x_{i-1}, x_i[$ and $L_i =]y_{m-i+1}, y_{m-i}[$ are translates of I_i, so that for each i there exist two positive real numbers a_i and b_i satisfying $I_i = a_i + J_i$ and $L_i = b_i + I_i$. Then

$$J_1 < \cdots < J_m, \quad L_1 < \cdots < L_m, \quad x_m = \lambda(V), \quad y_m = 1 - x_m.$$

The function g being increasing we obtain

$$\forall i \in \{1, \ldots, m\} : \quad \int_{J_i} g(t)\, dt \leq \int_{J_i} g(a_i + t)\, dt = \int_{I_i} g(t)\, dt \leq \int_{I_i} g(b_i + t)\, dt = \int_{L_i} g(t)\, dt$$

and thus

$$\int_0^{x_m} g(t)\, dt = \sum_{i=1}^m \int_{J_i} g(t)\, dt \leq \int_V g(t)\, dt \leq \sum_{i=1}^m \int_{L_i} g(t)\, dt = \int_{y_m}^1 g(t)\, dt.$$

Now by Proposition 3 we have

$$p = \left(x_m, \int_0^{x_m} g(t)\, dt\right) = G(x_m) - G(0) \in C, \quad q = \left(x_m, \int_{y_m}^1 g(t)\, dt\right) = G(1) - G(y_m) \in C;$$

therefore by convexity we obtain

$$\nu(V) = \left(x_m, \int_V g(t)\, dt\right) \in \operatorname{co}\{p, q\} \subset C.$$

Let A be a measurable subset of $[0, 1]$; the measure ν being regular there exists a $\mathcal{G}_\delta$ subset E such that $\nu(A) = \nu(E)$. We may write $E = \cap_m V_m$ where $(V_m)_m$ is a decreasing sequence of countable unions of disjoint open intervals. Since $\nu(E) = \lim_m \nu(V_m)$ then the previous remarks and the closure of C imply that $\nu(A) = \nu(E)$ is in C. It follows that

$$C = \mathcal{R}(\nu) \tag{$**$}$$

and therefore $K = e + T^{-1}\mathcal{R}(\nu) = e + \mathcal{R}(T^{-1}\nu)$. If O is an exposed point of K we may take $e = O$, proving the claim. Otherwise since $O \in TK$ there exists a set E such that $\nu(E) = -Te$; let ν' be the measure on the Borelians of $[0, 1]$ defined by

$$\nu'(B) = \nu(B \setminus E) - \nu(B \cap E).$$

It is well known [1, Lemma 1.3] (and easy to check) that the range of ν' is given by

$$\mathcal{R}(\nu') = \mathcal{R}(\nu) - \nu(E)$$

so that $\mathcal{R}(\nu') = TK$ and therefore $K = T^{-1}(\mathcal{R}(\nu')) = \mathcal{R}(\mu)$ where $\mu = T^{-1}\nu'$.
Now let A be a measurable subset of $[0, 1]$. Then

$$\mu(A) = T^{-1}\nu(A \setminus E) - T^{-1}\nu(A \cap E);$$

$(*)$ and $(**)$ yield the conclusion. $\square$

Remark. A generalized version of the integral inequalities that we use to show that $\mathcal{R}(\nu)$ is contained in C was stated in [3]; their proof in this less general context is simpler and it is given here for the convenience of the reader.

The above arguments yield an alternative proof of Theorem 2.

Corollary. *Let K be a non empty, compact, centrally symmetric, strictly convex subset of $\mathbb{R}^2$ such that O belongs to ∂K. Then there exists a Chebyshev measure μ on the Borelians of $[0, 1]$ such that $K = \mathcal{R}(\mu)$.*

Proof. Since K is strictly convex and O belongs to ∂K then O is exposed: with the notation of the proof of Theorem 3 we may take $e = O$ and thus no translation is needed. Then $C = T(K)$ so that by $(**)$ we obtain $K = \mathcal{R}(T^{-1}\nu)$ where ν is the measure whose density with respect to λ is the vector $(1, g)$. Since the function y is strictly convex then g is strictly monotonic and therefore $(1, g)$ is a T–system. Proposition 2 then shows that ν is a Chebyshev measure; Proposition 1 yields the result. $\square$

Remark. The main difference between the two proofs is that, in Theorem 3, the representation result for convex sets (Proposition 3) is used as a substitute of the representation Theorem 1 for Chebyshev measures.

5. Decomposition of measures

Let $(X, \mathcal{M})$ be a measurable space and μ be a non–atomic positive measure on X. There exists a family $(M_i)_{i\in[0,1]}$ of sets of $\mathcal{M}$ such that μ is a Chebyshev measure with respect to μ and to $(M_i)_{i\in[0,1]}$ (we refer to [6] for the definition of T–measure in this more general setting). In fact Lyapunov Theorem on the range of measures yields the existence of an increasing family $(M_i)_{i\in[0,1]}$ such that $\mu(M_i) = i\mu(X)$ for each i in $[0, 1]$.
More generally, if μ is a signed measure on X, by the Hahn decomposition theorem we may decompose X into a disjoint union $X^- \cup X^+$ such that if we set

$$\mu^+(\cdot) = \mu(X^+ \cap \cdot), \qquad \mu^-(\cdot) = -\mu(X^- \cap \cdot)$$

then $\mu = \mu^+ - \mu^-$ and μ^+, μ^- are positive measures. The latter property together with the non–atomicity yield the existence of two increasing families $(M_i^+)_{i\in[0,1]}$ and $(M_i^-)_{i\in[0,1]}$ such that μ^+ (resp. μ^-) is a Chebyshev measure with respect to $(M_i^+)_{i\in[0,1]}$ (resp. $(M_i^-)_{i\in[0,1]}$). Thus μ is the difference of two Chebyshev measures.
We give now a condition under which the same conclusion holds for two dimensional vector measures. For a vector v of $\mathbb{R}^2 \setminus \{(0,0)\}$ we denote by $\arg v$ its principal argument in $]-\pi, \pi]$.
Let f be a measurable function with values in $\mathbb{R}^2$.

Theorem 4. *Let μ be a two dimensional non–atomic vector measure on $(X, \mathcal{M})$ and let $f = (f_1, f_2)$ be its density function with respect to $|\mu|$. If $|\mu|(\{x : \arg f(x) = \theta\}) = 0$ for each θ in $]-\pi, \pi]$ then there exist two T–measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$.*

Proof. We define $X^+ = \{x \in X : \arg f(x) \geq 0\}$, $X^- = \{x \in X : \arg f(x) < 0\}$ and

$$\forall i \in [0,1] : \quad M_i^+ = \{x \in X^+ : \arg f(x) \leq i\pi\}, \quad M_i^- = \{x \in X^- : \arg f(x) \geq -i\pi\}.$$

Let f^+ and f^- be the functions $f^+ = f\mathbf{1}_{X^+}$ and $f^- = f\mathbf{1}_{X^-}$. Then f^+ (resp. f^-) is a T–system on X^+ (resp. X^-) with respect to $|\mu|$ and $(M_i^+)_{i\in[0,1]}$ (resp. $(M_i^-)_{i\in[0,1]}$). Then setting $d\mu^+ = f^+ d\,|\mu|$ and $d\mu^- = f^- d\,|\mu|$ we obtain a decomposition of μ as the difference of two Chebyshev measures. □

Remark. Under the above assumptions Theorem 5.1 in [6] then implies that for every A in $\mathcal{M}$ there exist i_1, i_2, j_1, j_2 in $[0, 1]$ such that $\mu(A) = \mu^+(M_{i_2}^+ \setminus M_{i_1}^+) - \mu^-(M_{j_2}^- \setminus M_{j_1}^-)$. This result looks similar to the one stated in Theorem 3; however here the measure μ is imposed whereas in Theorem 3, given a zonoid, the measure is built.

6. A characterization of two dimensional Chebyshev measures

Let μ be a two dimensional non–atomic vector measure on $([0, 1], \mathcal{M})$ and let f be its density with respect to the total variation $|\mu|$. We denote by $< u : u \in E >$ the vector subspace of $\mathbb{R}^2$ spanned by the vectors u belonging to a set E and by "$\cdot$" the usual scalar product in $\mathbb{R}^2$.
The next result will be applied later and has an interest in itself.

Theorem 5. *If $\mathcal{R}(\mu)$ is strictly convex then the determinant* $\det(f(x), f(y))$ *of the vectors* $f(x), f(y)$ *is not zero* $|\mu| \otimes |\mu|$*– a.e. on* $[0,1]^2$.

Proof. Let A, Z, A_1 be the sets defined by

$$A = \{(x,y) : \det(f(x), f(y)) = 0\}, \quad Z = \{x : f(x) = 0\}$$
$$A_1 = \{(x,y) : f(x) \neq 0,\ f(y) \in < f(x) > \};$$

clearly we have $A = (Z \times [0,1]) \cup A_1$ and $(Z \times [0,1]) \cap A_1 = \emptyset$. Let τ be the map defined by $\tau(a,b) = (-b,a)$; then $A_1 = \{(x,y) : f(x) \neq 0, f(y) \cdot \tau(f(x)) = 0\}$ so that A_1 is measurable. Moreover Fubini's Theorem gives

$$|\mu| \otimes |\mu|(A_1) = \int_{[0,1]\setminus Z} \left\{ \int_{D_x} d|\mu|(y) \right\} d|\mu|(x)$$

where, for x in $[0,1]$, $D_x = \{y : f(y) \cdot \tau(f(x)) = 0\}$.
If $|\mu| \otimes |\mu|(A_1) \neq 0$ there exists x in $[0,1] \setminus Z$ such that $|\mu|(D_x) \neq 0$. The very definition of D_x implies that for every measurable subset B of D_x we have

$$\mu(B) \cdot \tau(f(x)) = \int_B f(y) \cdot \tau(f(x))\, d|\mu|(y) = 0$$

and thus the vector space $< \mu(B) : B \in \mathcal{M}, B \subset D_x >$ is at most one dimensional: Theorem 3.1.2 in [14] then implies that $\mathcal{R}(\mu)$ is not strictly convex, a contradiction. Obviously $|\mu|(Z) = 0$; thus $|\mu| \otimes |\mu|(A) = |\mu| \otimes |\mu|(Z \times [0,1]) + |\mu| \otimes |\mu|(A_1) = 0$, proving the claim. □

We will use the following results.

Lemma 1. *Let A be a non–empty open convex bounded subset of $\mathbb{R}^2$ and assume that ∂A contains a non trivial segment L. Then $ri\, L$ is open in ∂A.*

For the convenience of the reader, a proof of a more general result is given in the appendix.

Lemma 2. *Let A, B be open bounded subsets of $\mathbb{R}^n$ and $\psi : cl\, A \to \mathbb{R}^n$ be a continuous map inducing a homeomorphism from A onto B. Then $\psi(\partial A) = \partial B$.*

We recall that we denote by λ the Lebesgue measure on $[0,1]$; in what follows we assume that there exists a strictly positive function h in $L^1_\lambda([0,1])$ such that $d|\mu| = h\, d\lambda$; in particular $|\mu|$ is absolutely continuous with respect to λ.
We prove here that Theorem 1 characterizes the Chebyshev measures.

Theorem 6. *Let θ be the map defined in §2. If θ induces a homeomorphism from intΓ onto int$\mathcal{R}(\mu)$ then μ is a Chebyshev measure.*

Proof. Remark first that $\operatorname{int}\mathcal{R}(\mu)$, being isomorphic to $\operatorname{int}\Gamma$, is non–empty.
Since Γ and $\mathcal{R}(\mu)$ are convex and closed then Theorem 6.3 in [12] yields $\Gamma = \operatorname{cl}(\operatorname{int}\Gamma)$ and $\mathcal{R}(\mu) = \operatorname{cl}(\operatorname{int}\mathcal{R}(\mu))$: applying Lemma 2 with $\psi = \theta$, $A = \operatorname{int}\Gamma$, $B = \operatorname{int}\mathcal{R}(\mu)$ we obtain $\theta(\partial\Gamma) = \partial\mathcal{R}(\mu)$; in particular

$$\partial\mathcal{R}(\mu) = \{\mu([0,\alpha]) \,:\, 0 \le \alpha \le 1\} \cup \{\mu([\beta,1]) \,:\, 0 \le \beta \le 1\}.$$

Assume that the boundary of $\mathcal{R}(\mu)$ contains a non trivial segment L; let (for instance) α in $[0,1]$ be such that $x = \mu([0,\alpha])$ belongs to the relative interior of L. By Lemma 1 there exists an open neighbourhood V of x such that $V \cap \partial\mathcal{R}(\mu) = V \cap \operatorname{ri} L$. By continuity there exist α_1, α_2 in $]0,1[$ such that $\alpha_1 < \alpha < \alpha_2$ and

$$\{\mu([0,t]) : t \in [\alpha_1,\alpha_2]\} \subset V;$$

Lemma 2 then implies that $\mu([0,t]) \in V \cap \operatorname{ri} L$ for every $t \in]\alpha_1,\alpha_2[$. Therefore, if $p \in \mathbb{R}^2 \setminus \{O\}, c \in \mathbb{R}$ are such that $L \subset \{x \in \mathbb{R}^2 : p \cdot x = c\}$ we have

$$\forall t \in]\alpha_1,\alpha_2[: \qquad p \cdot \mu([0,t]) = c.$$

Let U be the open subset of $\operatorname{int}\Gamma$ defined by

$$U = \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha_1 < \alpha < \beta < \alpha_2\}.$$

Our assumption implies that $\theta(U)$ is an open subset of $\mathbb{R}^2$; however we have

$$\forall (\alpha,\beta) \in U \qquad p \cdot \theta(\alpha,\beta) = p \cdot \mu([\alpha,\beta]) = p \cdot \mu([0,\beta]) - p \cdot \mu([0,\alpha]) = 0,$$

a contradiction; it follows that $\mathcal{R}(\mu)$ is strictly convex. Theorem 5 then implies that

$$\det(f(\alpha), f(\beta)) \neq 0 \quad |\mu| \otimes |\mu| - \text{ a.e. in } [0,1]^2.$$

By [16, Corollary 10.50] we have

$$\lim_{x\to 0} \frac{\mu([\alpha,\alpha+x])}{|\mu|([\alpha,\alpha+x])} = f(\alpha) \quad |\mu| - \text{a.e.} \quad \text{and} \quad \lim_{x\to 0} \frac{|\mu|([\alpha,\alpha+x])}{\lambda([\alpha,\alpha+x])} = h(\alpha) \quad \lambda - \text{a.e.}$$

so that

$$\lim_{x\to 0} \frac{\mu([\alpha,\alpha+x])}{x} = f(\alpha)h(\alpha) \quad |\mu| - \text{a.e.}$$

Therefore the map θ is differentiable $|\mu| \otimes |\mu|$–a.e. on $[0,1]^2$ and its Jacobian is given by

$$\operatorname{Jac}(\theta)(\alpha,\beta) = (-f(\alpha)h(\alpha), f(\beta)h(\beta)) \quad |\mu| \otimes |\mu| - \text{ a.e.}$$

so that in particular the determinant of the Jacobian vanishes only on a negligible set.
The map θ is a homeomorphism on $\operatorname{int}\Gamma$ and Γ is connected; as a consequence the degree $\deg(\operatorname{int}\Gamma, \theta, p)$ is constantly equal to 1 or -1 for every p in $\operatorname{int}\mathcal{R}(\mu)$ [8, Theorem 3.35], assume for instance that it equals -1. Then by [8, Lemma 5.9] we have

$$\operatorname{sgn} \det(-f(\alpha), f(\beta)) = \operatorname{sgn} \det \operatorname{Jac}(\theta)(\alpha,\beta) = \deg(\operatorname{int}\Gamma, \theta, p) = -1 \quad |\mu| \otimes |\mu| - \text{a.e. in } \Gamma$$

and therefore f is a T–system; Proposition 2 yields the conclusion. □

Appendix: faces of codimension 1

The above Lemma 1 can be formulated in a more general setting.

Theorem 7. *Let A be an open convex bounded subset of $\mathbb{R}^n$ and assume that ∂A contains a relatively open subset L of an hyperplane. Then L is open in ∂A.*

Proof (G. De Marco). It is not restrictive to assume that $O \in A$ and that $L \subset H$ where

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = \lambda\} \qquad \text{for some } \lambda > 0.$$

Clearly H is a supporting hyperplane so that $x_n \le \lambda$ for every $x = (x_1, \dots, x_n) \in \operatorname{cl} A$. We denote by $\|\cdot\|$ the norm of $\mathbb{R}^n$ defined by $\|(x_1, \dots, x_n)\| = \max_i |x_i|$; we recall that the map $p : x \mapsto \frac{x}{\|x\|}$ is a homeomorphism from ∂A onto the unit sphere S (in the $\|\cdot\|$–norm) of $\mathbb{R}^n$ (see for instance [7]). It is not restrictive to assume that

$$L \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > \max\{|x_1|, \dots, |x_{n-1}|\}\};$$

in fact it is enough to transform A and H with the map $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, rx_n)$ for a sufficiently large r. Then in particular we have $\|x\| = \lambda$ for every x in L. It follows that $K = p(L) \subset S \cap Q$ where $Q = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 1\} = \frac{1}{\lambda} H$ and that $p(x) = \frac{x}{\lambda}$ for every x in L so that K is homothetic to L and is thus open in Q. Moreover K is contained in the open set $B = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0,\ 1 > \max\{|x_1|, \dots, |x_{n-1}|\}\}$ and $Q \cap B = S \cap B$: therefore K is open in S. $\square$

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