Upper large deviations for the maximal flow through a domain of $\mathbb{R}^d$ in first passage percolation

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Abstract: We consider the standard first passage percolation model in the rescaled graph $\mathbb{Z}^d/n$ for $d \geq 2$, and a domain Ω of boundary Γ in $\mathbb{R}^d$. Let Γ^1 and Γ^2 be two disjoint open subsets of Γ, representing the parts of Γ through which some water can enter and escape from Ω. We investigate the asymptotic behaviour of the flow ϕ_n through a discrete version Ω_n of Ω between the corresponding discrete sets Γ_n^1 and Γ_n^2. We prove that under some conditions on the regularity of the domain and on the law of the capacity of the edges, the upper large deviations of ϕ_n/n^{d-1} above a certain constant are of volume order.

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1 First definitions and main result

We use many notations introduced in [5] and [6]. Let $d \geq 2$. We consider the graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$ having for vertices $\mathbb{Z}_n^d = \mathbb{Z}^d/n$ and for edges $\mathbb{E}_n^d$, the set of pairs of nearest neighbours for the standard L^1 norm. With each edge e in $\mathbb{E}_n^d$ we associate a random variable $t(e)$ with values in $\mathbb{R}^+$. We suppose that the family $(t(e), e \in \mathbb{E}_n^d)$ is independent and identically distributed, with a common law Λ: this is the standard model of first passage percolation on the graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$. We interpret $t(e)$ as the capacity of the edge e; it means that $t(e)$ is the maximal amount of fluid that can go through the edge e per unit of time.

We consider an open bounded connected subset Ω of $\mathbb{R}^d$ such that the boundary $\Gamma = \partial\Omega$ of Ω is piecewise of class $\mathcal{C}^1$ (in particular Γ has finite area: $\mathcal{H}^{d-1}(\Gamma) < \infty$). It means that Γ is included in the union of a finite number of hypersurfaces of class $\mathcal{C}^1$, i.e., in the union of a finite number of C^1 submanifolds of $\mathbb{R}^d$ of codimension 1. Let Γ^1, Γ^2 be two disjoint subsets of Γ that are open in

Γ. We want to define the maximal flow from Γ^1 to Γ^2 through Ω for the capacities $(t(e), e \in \mathbb{E}_n^d)$. We consider a discrete version $(\Omega_n, \Gamma_n, \Gamma_n^1, \Gamma_n^2)$ of $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$ defined by:

$$\begin{cases} \Omega_n \,=\, \{x \in \mathbb{Z}_n^d \,|\, d_\infty(x, \Omega) < 1/n\}\,, \\ \Gamma_n \,=\, \{x \in \Omega_n \,|\, \exists y \notin \Omega_n\,,\ \langle x, y\rangle \in \mathbb{E}_n^d\}\,, \\ \Gamma_n^i \,=\, \{x \in \Gamma_n \,|\, d_\infty(x, \Gamma^i) < 1/n\,,\ d_\infty(x, \Gamma^{3-i}) \geq 1/n\} \text{ for } i = 1, 2\,, \end{cases}$$

where d_∞ is the L^∞-distance, the notation $\langle x, y\rangle$ corresponds to the edge of endpoints x and y (see figure 1).

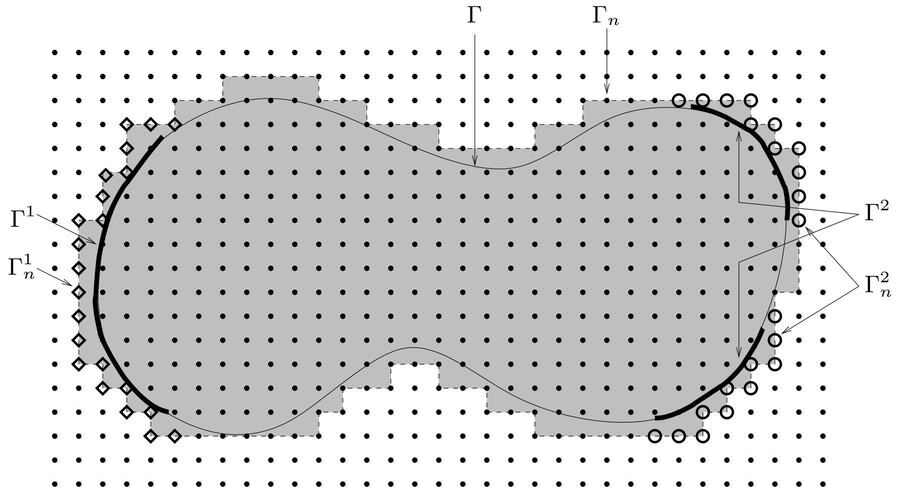

Figure 1: Domain Ω.

We shall study the maximal flow from Γ_n^1 to Γ_n^2 in Ω_n. Let us define properly the maximal flow $\phi(F_1 \rightarrow F_2 \text{ in } C)$ from F_1 to F_2 in C, for $C \subset \mathbb{R}^d$ (or by commodity the corresponding graph $C \cap \mathbb{Z}^d/n$). We will say that an edge $e = \langle x, y\rangle$ belongs to a subset A of $\mathbb{R}^d$, which we denote by $e \in A$, if the interior of the segment joining x to y is included in A. We define $\widetilde{\mathbb{E}}_n^d$ as the set of all the oriented edges, i.e., an element $\widetilde{e}$ in $\widetilde{\mathbb{E}}_n^d$ is an ordered pair of vertices which are nearest neighbours. We denote an element $\widetilde{e} \in \widetilde{\mathbb{E}}_n^d$ by $\langle\langle x, y\rangle\rangle$, where x, $y \in \mathbb{Z}_n^d$ are the endpoints of $\widetilde{e}$ and the edge is oriented from x towards y. We consider the set $\mathcal{S}$ of all pairs of functions (g, o), with $g : \mathbb{E}_n^d \rightarrow \mathbb{R}^+$ and $o : \mathbb{E}_n^d \rightarrow \widetilde{\mathbb{E}}_n^d$ such that $o(\langle x, y\rangle) \in \{\langle\langle x, y\rangle\rangle, \langle\langle y, x\rangle\rangle\}$, satisfying:

- for each edge e in C we have
$$0 \,\leq\, g(e) \,\leq\, t(e)\,,$$
- for each vertex v in $C \smallsetminus (F_1 \cup F_2)$ we have
$$\sum_{e \in C\,:\,o(e) = \langle\langle v, \cdot\rangle\rangle} g(e) \,=\, \sum_{e \in C\,:\,o(e) = \langle\langle \cdot, v\rangle\rangle} g(e)\,,$$

where the notation $o(e) = \langle\langle v, .\rangle\rangle$ (respectively $o(e) = \langle\langle ., v\rangle\rangle$) means that there exists $y \in \mathbb{Z}_n^d$ such that $e = \langle v, y\rangle$ and $o(e) = \langle\langle v, y\rangle\rangle$ (respectively $o(e) = \langle\langle y, v\rangle\rangle$). A couple $(g, o) \in \mathcal{S}$ is a possible stream in C from F_1 to F_2: $g(e)$ is the amount of fluid that goes through the edge e, and $o(e)$ gives the direction in which the fluid goes through e. The two conditions on (g, o) express only the fact that the amount of fluid that can go through an edge is bounded by its capacity, and that there is no loss of fluid in the graph. With each possible stream we associate the corresponding flow

$$\mathrm{flow}(g,o) \;=\; \sum_{u\in F_2\,,\, v\notin C\,:\,\langle u,v\rangle\in\mathbb{E}_n^d} g(\langle u,v\rangle)\mathbb{1}_{o(\langle u,v\rangle)=\langle\langle u,v\rangle\rangle} - g(\langle u,v\rangle)\mathbb{1}_{o(\langle u,v\rangle)=\langle\langle v,u\rangle\rangle}\,.$$

This is the amount of fluid that crosses C from F_1 to F_2 if the fluid respects the stream (g,o). The maximal flow through C from F_1 to F_2 is the supremum of this quantity over all possible choices of streams

$$\phi(F_1 \to F_2 \text{ in } C) \;=\; \sup\{\mathrm{flow}(g,o)\,|\,(g,o)\in\mathcal{S}\}\,.$$

We recall that we consider an open bounded connected subset Ω of $\mathbb{R}^d$ whose boundary Γ is piecewise of class $\mathcal{C}^1$, and two disjoint open subsets Γ^1 and Γ^2 of Γ. We denote by

$$\phi_n \;=\; \phi(\Gamma_n^1 \to \Gamma_n^2 \text{ in } \Omega_n)$$

the maximal flow from Γ_n^1 to Γ_n^2 in Ω_n. We will investigate the asymptotic behaviour of ϕ_n/n^{d-1} when n goes to infinity. More precisely, we will show that the upper large deviations of ϕ_n above a certain constant $\widetilde{\phi_\Omega}$ are of volume order. The description of $\widetilde{\phi_\Omega}$ will be given in section 2. Here we state the precise theorem:

Theorem 1. *We suppose that $d(\Gamma^1,\Gamma^2)>0$. If the law Λ of the capacity of an edge admits an exponential moment:*

$$\exists\theta>0 \qquad \int_{\mathbb{R}^+} e^{\theta x}d\Lambda(x) \;<\; +\infty\,,$$

then there exists a finite constant $\widetilde{\phi_\Omega}$ such that for all $\lambda>\widetilde{\phi_\Omega}$,

$$\limsup_{n\to\infty}\frac{1}{n^d}\log\mathbb{P}[\phi_n\geq\lambda n^{d-1}] \;<\; 0\,.$$

Remark 1. In the theorem 1 we need to impose that $d(\Gamma^1,\Gamma^2)>0$ because otherwise we cannot be sure that $\widetilde{\phi_\Omega}<\infty$, as we will see in section 4. Moreover, if $d(\Gamma^1,\Gamma^2)=0$, there exists a set of edges of constant cardinality (not depending on n) containing paths from Γ_n^1 to Γ_n^2 through Ω_n for all n along the common boundary of Γ^1 and Γ^2, and so it may be sufficient for these edges to have a huge capacity to obtain that ϕ_n is abnormally big too. Thus, we cannot hope to obtain upper large deviations of volume order (see [9] for a counter-example).

Remark 2. The large deviations we obtain are of the relevant order. Indeed, if all the edges in Ω_n have a capacity which is abnormally big, then the maximal flow ϕ_n will be abnormally big too. The probability for these edges to have an abnormally large capacity is of order $\exp -Cn^d$ for a constant C, because the number of edges in Ω_n is $C'n^d$ for a constant C'.

Remark 3. In the two companion papers [2] and [3], we prove in fact that $\widetilde{\phi_\Omega}$ is the almost sure limit of ϕ_n/n^{d-1} when n goes to infinity, and that the lower large deviations of ϕ_n/n^{d-1} below $\widetilde{\phi_\Omega}$ are of surface order.

2 Computation of $\widetilde{\phi_\Omega}$

2.1 Geometric notations

We start with some geometric definitions. For a subset X of $\mathbb{R}^d$, we denote by $\mathcal{H}^s(X)$ the s-dimensional Hausdorff measure of X (we will use $s = d-1$ and $s = d-2$). The r-neighbourhood $\mathcal{V}_i(X,r)$ of X for the distance d_i, that can be the Euclidean distance if $i=2$ or the L^∞-distance if $i=\infty$, is defined by

$$\mathcal{V}_i(X,r) \,=\, \{y \in \mathbb{R}^d \,|\, d_i(y,X) < r\}\,.$$

If X is a subset of $\mathbb{R}^d$ included in an hyperplane of $\mathbb{R}^d$ and of codimension 1 (for example a non degenerate hyperrectangle), we denote by $\mathrm{hyp}(X)$ the hyperplane spanned by X, and we denote by $\mathrm{cyl}(X,h)$ the cylinder of basis X and of height $2h$ defined by

$$\mathrm{cyl}(X,h) \,=\, \{x + tv \,|\, x \in X\,,\, t \in [-h,h]\}\,,$$

where v is one of the two unit vectors orthogonal to $\mathrm{hyp}(X)$ (see figure 2). For $x \in \mathbb{R}^d$, $r \geq 0$ and

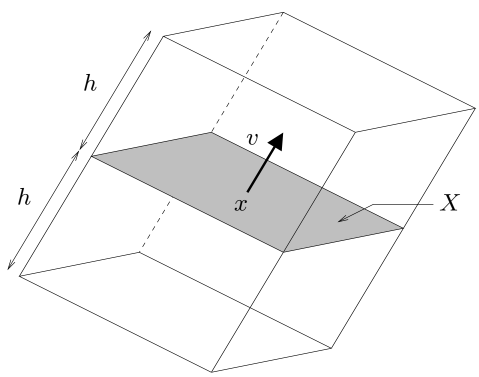

Figure 2: Cylinder $\mathrm{cyl}(X,h)$.

a unit vector v, we denote by $B(x,r)$ the closed ball centered at x of radius r.

2.2 Flow in a cylinder

Here are some particular definitions of flows through a box. It is important to know them, because all our work consists in comparing the maximal flow ϕ_n in Ω_n with the maximal flows in small cylinders. Let A be a non degenerate hyperrectangle, i.e., a box of dimension $d-1$ in $\mathbb{R}^d$. All hyperrectangles will be supposed to be closed in $\mathbb{R}^d$. We denote by v one of the two unit vectors orthogonal to $\mathrm{hyp}(A)$. For h a positive real number, we consider the cylinder $\mathrm{cyl}(A,h)$. The set $\mathrm{cyl}(A,h) \smallsetminus \mathrm{hyp}(A)$ has two connected components, which we denote by $\mathcal{C}_1(A,h)$ and $\mathcal{C}_2(A,h)$. For $i=1,2$, let A_i^h be the set of the points in $\mathcal{C}_i(A,h) \cap \mathbb{Z}_n^d$ which have a nearest neighbour in $\mathbb{Z}_n^d \smallsetminus \mathrm{cyl}(A,h)$:

$$A_i^h \,=\, \{x \in \mathcal{C}_i(A,h) \cap \mathbb{Z}_n^d \,|\, \exists y \in \mathbb{Z}_n^d \smallsetminus \mathrm{cyl}(A,h)\,,\, \langle x,y \rangle \in \mathbb{E}_n^d\}\,.$$

Let $T(A,h)$ (respectively $B(A,h)$) be the top (respectively the bottom) of $\mathrm{cyl}(A,h)$, i.e.,

$$T(A,h) \,=\, \{x \in \mathrm{cyl}(A,h) \,|\, \exists y \notin \mathrm{cyl}(A,h)\,,\, \langle x,y \rangle \in \mathbb{E}_n^d \text{ and } \langle x,y \rangle \text{ intersects } A + hv\}$$

and

$$B(A,h) \,=\, \{x\in \mathrm{cyl}(A,h)\,|\, \exists y \notin \mathrm{cyl}(A,h)\,,\ \langle x,y\rangle \in \mathbb{E}_n^d \text{ and } \langle x,y\rangle \text{ intersects } A-hv\}\,.$$

For a given realisation $(t(e), e\in \mathbb{E}_n^d)$ we define the variable $\tau(A,h) = \tau(\mathrm{cyl}(A,h),v)$ by

$$\tau(A,h) \,=\, \tau(\mathrm{cyl}(A,h),v) \,=\, \phi(A_1^h \rightarrow A_2^h \text{ in } \mathrm{cyl}(A,h))\,,$$

and the variable $\phi(A,h) = \phi(\mathrm{cyl}(A,h),v)$ by

$$\phi(A,h) \,=\, \phi(\mathrm{cyl}(A,h),v) \,=\, \phi(B(A,h) \rightarrow T(A,h) \text{ in } \mathrm{cyl}(A,h))\,,$$

where $\phi(F_1 \rightarrow F_2 \text{ in } C)$ is the maximal flow from F_1 to F_2 in C, for $C\subset \mathbb{R}^d$ (or by commodity the corresponding graph $C\cap \mathbb{Z}^d/n$) defined previously. The dependence in n is implicit here, in fact we can also write $\tau_n(A,h)$ and $\phi_n(A,h)$ if we want to emphasize this dependence on the mesh of the graph.

2.3 Max-flow min-cut theorem

The maximal flow $\phi(F_1 \rightarrow F_2 \text{ in } C)$ can be expressed differently thanks to the max-flow min-cut theorem (see [1]). We need some definitions to state this result. A path on the graph $\mathbb{Z}_n^d$ from v_0 to v_m is a sequence $(v_0,e_1,v_1,...,e_m,v_m)$ of vertices $v_0,...,v_m$ alternating with edges $e_1,...,e_m$ such that v_{i-1} and v_i are neighbours in the graph, joined by the edge e_i, for i in $\{1,...,m\}$. A set E of edges in C is said to cut F_1 from F_2 in C if there is no path from F_1 to F_2 in $C\smallsetminus E$. We call E an (F_1,F_2)-cut if E cuts F_1 from F_2 in C and if no proper subset of E does. With each set E of edges we associate its capacity which is the variable

$$V(E) \,=\, \sum_{e\in E} t(e)\,.$$

The max-flow min-cut theorem states that

$$\phi(F_1\rightarrow F_2 \text{ in } C) \,=\, \min\{\, V(E)\,|\, E \text{ is a } (F_1,F_2)\text{-cut}\,\}\,.$$

2.4 Definition of ν

The asymptotic behaviour of the rescaled expectation of $\tau_n(A,h)$ for large n is well known, thanks to the almost subadditivity of this variable. We recall the following result:

Theorem 2. *We suppose that*

$$\int_{[0,+\infty[} x\, d\Lambda(x) \,<\, \infty\,.$$

Then for each unit vector v there exists a constant $\nu(d,\Lambda,v)=\nu(v)$ (the dependence on d and Λ is implicit) such that for every non degenerate hyperrectangle A orthogonal to v and for every strictly positive constant h, we have

$$\lim_{n\rightarrow\infty} \frac{\mathbb{E}[\tau_n(A,h)]}{n^{d-1}\mathcal{H}^{d-1}(A)} \,=\, \nu(v)\,.$$

For a proof of this proposition, see [8]. We emphasize the fact that the limit depends on the direction of v, but not on h nor on the hyperrectangle A itself.

In fact, Rossignol and Théret proved in [8] that under some moment conditions and/or some condition on A, $\nu(v)$ is the limit of the rescaled variable $\tau_n(A,h)/(n^{d-1}\mathcal{H}^{d-1}(A))$ almost surely and in L^1. We also know, thanks to the works of Kesten [6], Zhang [11] and Rossignol and Théret [8] that the variable $\phi_n(A,h)/(n^{d-1}\mathcal{H}^{d-1}(A))$ satisfies the same law of large numbers in the particular case where A is a straight hyperrectangle, i.e., a hyperrectangle of the form $\prod_{i=1}^{d-1}[0,k_i]\times\{0\}$ for some $k_i>0$.

We recall some geometric properties of the map $\nu : v\in S^{d-1}\mapsto \nu(v)$, under the only condition on Λ that $\mathbb{E}(t(e))<\infty$. They have been stated in section 4.4 of [8]. There exists a unit vector v_0 such that $\nu(v_0)=0$ if and only if for all unit vector v, $\nu(v)=0$, and it happens if and only if $\Lambda(0)\geq 1-p_c(d)$, where $p_c(d)$ is the critical parameter of the bond percolation on $\mathbb{Z}^d$. This property has been proved by Zhang in [10]. Moreover, ν satisfies the weak triangle inequality, i.e., if (ABC) is a non degenerate triangle in $\mathbb{R}^d$ and v_A, v_B and v_C are the exterior normal unit vectors to the sides $[BC]$, $[AC]$, $[AB]$ in the plane spanned by A, B, C, then

$$\mathcal{H}^1([AB])\nu(v_C)\,\leq\,\mathcal{H}^1([AC])\nu(v_B)+\mathcal{H}^1([BC])\nu(v_A)\,.$$

This implies that the homogeneous extension ν_0 of ν to $\mathbb{R}^d$, defined by $\nu_0(0)=0$ and for all w in $\mathbb{R}^d$,

$$\nu_0(w)\;=\;|w|_2\nu(w/|w|_2)\,,$$

is a convex function; in particular, since ν_0 is finite, it is continuous on $\mathbb{R}^d$. We denote by $\nu_{\min}$ (respectively $\nu_{\max}$) the infimum (respectively supremum) of ν on S^{d-1}.

The last result we recall is Theorem 4 in [9] concerning the upper large deviations of the variable $\phi_n(A,h)$ above $\nu(v)$:

Theorem 3. *We suppose that*

$$\exists\gamma>0\qquad\int_{[0,+\infty[}e^{\gamma x}\,d\Lambda(x)\;<\;\infty\,.$$

Then for every unit vector v and every non degenerate hyperrectangle A orthogonal to v, for every strictly positive constant h and for every $\lambda>\nu(v)$ we have

$$\liminf_{n\to\infty}\frac{-1}{n^d\mathcal{H}^{d-1}(A)h}\log\mathbb{P}\left[\frac{\phi_n(A,h)}{n^{d-1}\mathcal{H}^{d-1}(A)}\geq\lambda\right]\;>\;0\,.$$

We shall rely on this result for proving Theorem 1. Moreover, Theorem 1 is a generalisation of Theorem 3, where we work in the domain Ω instead of a parallelepiped.

2.5 Continuous min-cut

We give here a definition of $\widetilde{\phi_\Omega}$ in terms of the map ν. When a hypersurface $\mathcal{S}$ is piecewise of class $\mathcal{C}^1$, we say that $\mathcal{S}$ is transverse to Γ if for all $x\in\mathcal{S}\cap\Gamma$, the normal unit vectors to $\mathcal{S}$ and Γ at x are not collinear; if the normal vector to $\mathcal{S}$ (respectively to Γ) at x is not well defined, this property must be satisfied by all the vectors which are limits of normal unit vectors to $\mathcal{S}$ (respectively Γ) at $y\in\mathcal{S}$ (respectively $y\in\Gamma$) when we send y to x - there is at most a finite number of such limits. We say that a subset P of $\mathbb{R}^d$ is polyhedral if its boundary ∂P is included in the union of a finite number of hyperplanes. For each point x of such a set P which is on the interior of one face of ∂P, we denote by $v_P(x)$ the exterior unit vector orthogonal to P at x. For $A \subset \mathbb{R}^d$, we denote by $\overset{\circ}{A}$ the interior of A. We define $\widetilde{\phi_\Omega}$ by

$$\widetilde{\phi_\Omega} = \inf \left\{ \mathcal{I}_\Omega(P) \,\middle|\, \begin{array}{c} P \subset \mathbb{R}^d \,,\, \overline{\Gamma^1} \subset \overset{\circ}{P} \,,\, \overline{\Gamma^2} \subset \mathbb{R}^d \smallsetminus \overset{\circ}{P} \\ P \text{ is polyhedral} \,,\, \partial P \text{ is transverse to } \Gamma \end{array} \right\},$$

where

$$\mathcal{I}_\Omega(P) = \int_{\partial P \cap \Omega} \nu(v_P(x))\, d\mathcal{H}^{d-1}(x)\,.$$

See figure 3 to have an example of such a polyhedral set P.

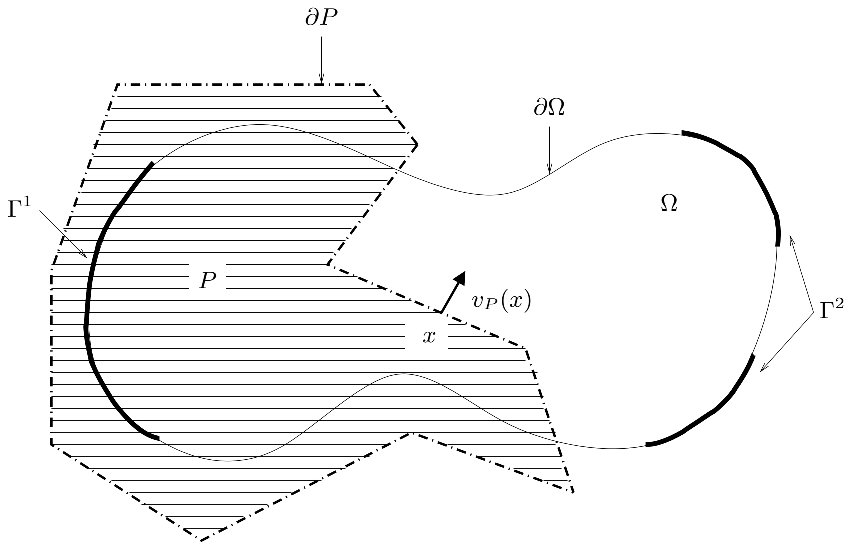

Figure 3: A polyhedral set P as in the definition of $\widetilde{\phi_\Omega}$.

The definition of the constant $\widetilde{\phi_\Omega}$ is not very intuitive. We propose to define the notion of a continuous cutset to have a better understanding of this constant. We say that $\mathcal{S} \subset \mathbb{R}^d$ cuts Γ^1 from Γ^2 in $\overline{\Omega}$ if every continuous path from Γ^1 to Γ^2 in $\overline{\Omega}$ intersects $\mathcal{S}$. In fact, if P is a polyhedral set of $\mathbb{R}^d$ such that

$$\overline{\Gamma^1} \subset \overset{\circ}{P} \quad \text{and} \quad \overline{\Gamma^2} \subset \mathbb{R}^d \smallsetminus \overset{\circ}{P}\,,$$

then $\partial P \cap \overline{\Omega}$ is a continuous cutset from Γ^1 to Γ^2 in $\overline{\Omega}$. Since $\nu(v)$ is the average amount of fluid that can cross a hypersurface of area one in the direction v per unit of time, it can be interpreted as the capacity of a unitary hypersurface orthogonal to v. Thus $\mathcal{I}_\Omega(P)$ can be interpreted as the capacity of the continuous cutset $\partial P \cap \overline{\Omega}$ defined by P. The constant $\widetilde{\phi_\Omega}$ is the solution of a min cut problem, because it is equal to the infimum of the capacity of a continuous cutset that satisfies some specific properties.

3 Sketch of the proof

We first prove that $\widetilde{\phi_\Omega}$ is finite, i.e., that there exists a polyhedral set $P \subset \mathbb{R}^d$ such that ∂P is transverse to Γ and

$$\overline{\Gamma^1} \subset \overset{\circ}{P}\,,\ \overline{\Gamma^2} \subset \mathbb{R}^d \smallsetminus \overset{\circ}{P}\,.$$

Then, we consider such a polyhedral set P whose capacity $\mathcal{I}_\Omega(P)$ is close to $\widetilde{\phi_\Omega}$. We construct a set Ω' that contains a small neighbourhood of Ω, thus Ω' contains Ω_n for all large n, and such that $\mathcal{H}^{d-1}(\partial P \cap (\Omega' \smallsetminus \Omega))$ is very small. We need the property that ∂P is transverse to Γ to obtain this control on $\mathcal{H}^{d-1}(\partial P \cap (\Omega' \smallsetminus \Omega))$. We want to construct a (Γ_n^1, Γ_n^2)-cut in Ω_n that is close to $\partial P \cap \Omega'$. We cover $\partial P \cap \Omega'$ with cylinders of arbitrarily small height; this is the reason why we need to consider a polyhedral set P. A part of $\partial P \cap \Omega'$ of very small area is missing in this covering. We construct then a (Γ_n^1, Γ_n^2)-cut in Ω_n with the help of cutsets in the cylinders constructed on $\partial P \cap \Omega'$. To achieve this, we have to add edges to cover the part of $\partial P \cap \Omega'$ missing in the covering by the cylinders, and to glue together the cutsets in the different cylinders. Thanks to the study of the upper large deviations for the maximal flow through cylinders made in [9], we obtain that the probability that the flow ϕ_n is greater than $\mathcal{I}_\Omega(P) n^{d-1}$ goes to zero. We want to prove that this probability decays exponentially fast in n^d. For that purpose, we have to consider a collection of cardinality of order n of possible sets of edges we can add to construct the cutset in Ω_n, and to choose the set that has the minimal capacity.

4 The constant $\widetilde{\phi_\Omega}$ is finite

To prove that $\widetilde{\phi_\Omega} < \infty$, it is sufficient to exhibit a set P satisfying all the conditions given in the definition of $\widetilde{\phi_\Omega}$. Indeed, if such a set P exists, then

$$\widetilde{\phi_\Omega} \,\leq\, \nu_{\max} \mathcal{H}^{d-1}(\partial P \cap \Omega) \,<\, \infty$$

since a polyhedral set has finite perimeter in Ω. We will construct such a set P. The idea of the proof is the following. We will cover $\overline{\Gamma^1}$ with small hypercubes which are transverse to Γ^1 and at positive distance of $\overline{\Gamma^2}$. Then, by compactness, we will extract a finite covering. We will denote by P the union of the hypercubes of this finite covering. Then P satisfies the desired properties.

We prove a geometric lemma:

Lemma 1. *Let Γ be an hypersurface (that is a C^1 submanifold of $\mathbb{R}^d$ of codimension 1) and let K be a compact subset of Γ. There exists a positive $M = M(\Gamma, K)$ such that:*

$$\forall \varepsilon > 0 \quad \exists\, r > 0 \quad \forall x, y \in K \qquad |x-y|_2 \leq r \quad \Rightarrow \quad d_2(y, \tan(\Gamma, x)) \leq M\, \varepsilon\, |x-y|_2\,.$$

($\tan(\Gamma, x)$ is the tangent hyperplane of Γ at x).

Proof :
By a standard compactness argument, it is enough to prove the following local property:

$$\forall x \in \Gamma \quad \exists\, M(x) > 0 \quad \forall \varepsilon > 0 \quad \exists\, r(x,\varepsilon) > 0 \quad \forall y, z \in \Gamma \cap B(x, r(x,\varepsilon))$$
$$d_2(y, \tan(\Gamma, z)) \leq M(x)\, \varepsilon\, |y-z|_2\,.$$

Indeed, if this property holds, we cover K by the open balls $\overset{o}{B}(x, r(x,\varepsilon)/2)$, $x \in K$, we extract a finite subcover $\overset{o}{B}(x_i, r(x_i,\varepsilon)/2)$, $1 \leq i \leq k$, and we set

$$M = \max\{\, M(x_i) : 1 \leq i \leq k\,\}, \quad r = \min\{\, r(x_i,\varepsilon)/2 : 1 \leq i \leq k\,\}.$$

Let now y, z belong to K with $|y - z|_2 \leq r$. Let i be such that y belongs to $B(x_i, r(x_i,\varepsilon)/2)$. Since $r \leq r(x_i,\varepsilon)/2$, then both y, z belong to the ball $B(x_i, r(x_i,\varepsilon))$ and it follows that

$$d_2(y, \tan(\Gamma, z)) \;\leq\; M(x_i)\,\varepsilon\,|y - z|_2 \leq M\,\varepsilon\,|y - z|_2\,.$$

We turn now to the proof of the above local property. Since Γ is an hypersurface, for any x in Γ there exists a neighbourhood V of x in $\mathbb{R}^d$, a diffeomorphism $f : V \mapsto \mathbb{R}^d$ of class C^1 and a $(d-1)$ dimensional vector space Z of $\mathbb{R}^d$ such that $Z \cap f(V) = f(\Gamma \cap V)$ (see for instance [4], 3.1.19). Let A be a compact neighbourhood of x included in V. Since f is a diffeomorphism, the maps $y \in A \mapsto df(y) \in \mathrm{End}(\mathbb{R}^d)$, $u \in f(A) \mapsto df^{-1}(u) \in \mathrm{End}(\mathbb{R}^d)$ are continuous. Therefore they are bounded:

$$\exists M > 0 \quad \forall y \in A \quad ||df(y)|| \leq M\,, \quad \forall u \in f(A) \quad ||df^{-1}(u)|| \leq M$$

(here $||df(x)|| = \sup\{\, |df(x)(y)|_2 : |y|_2 \leq 1\,\}$ is the standard operator norm in $\mathrm{End}(\mathbb{R}^d)$). Since $f(A)$ is compact, the differential map df^{-1} is uniformly continuous on $f(A)$:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall u, v \in f(A) \qquad |u - v|_2 \leq \delta \quad \Rightarrow \quad ||df^{-1}(u) - df^{-1}(v)|| \leq \varepsilon\,.$$

Let ε be positive and let δ be associated to ε as above. Let ρ be positive and small enough so that $\rho < \delta/2$ and $B(f(x), \rho) \subset f(A)$ (since f is a C^1 diffeomorphism, $f(A)$ is a neighbourhood of $f(x)$). Let r be such that $0 < r < \rho/M$ and $B(x, r) \subset A$. We claim that M associated to x and r associated to ε, x answer the problem. Let y, z belong to $\Gamma \cap B(x, r)$. Since $[y, z] \subset B(x, r) \subset A$, and $||df(\zeta)|| \leq M$ on A, then

$$|f(y) - f(x)|_2 \leq M|y - x|_2 \leq Mr < \rho\,, \quad |f(z) - f(x)|_2 < \rho\,,$$
$$|f(y) - f(z)|_2 < \delta\,, \quad |f(y) - f(z)|_2 < M|y - z|_2\,.$$

We apply next a classical lemma of differential calculus (see [7], I, 4, Corollary 2) to the map f^{-1} and the interval $[f(z), f(y)]$ (which is included in $B(f(x), \rho) \subset f(A)$) and the point $f(z)$:

$$|y - z - df^{-1}(f(z))(f(y) - f(z))|_2 \leq$$
$$|f(y) - f(z)|_2 \sup\{\, ||df^{-1}(\zeta) - df^{-1}(f(z))|| : \zeta \in [f(z), f(y)]\,\}\,.$$

The right–hand member is less than $M|y - z|_2\,\varepsilon$. Since $z + df^{-1}(f(z))(f(y) - f(z))$ belongs to $\tan(\Gamma, z)$, we are done.

■

We come back to our case. The boundary Γ of Ω is piecewise of class $\mathcal{C}^1$, i.e., it is included in a finite union of $\mathcal{C}^1$ hypersurfaces, which we denote by $(S_1, ..., S_p)$. The hypersurfaces $S_1, \ldots, S_p$ being $\mathcal{C}^1$ and the set Γ compact, the maps $x \in \Gamma \mapsto v_{S_k}(x)$, $1 \leq k \leq p$ (where $v_{S_k}(x)$ is the unit normal vector to S_k at x) are uniformly continuous:

$$\forall \delta > 0 \quad \exists \eta > 0 \quad \forall k \in \{\,1, \ldots, p\,\} \quad \forall x, y \in S_k \cap \Gamma \quad |x - y|_2 \leq \eta \Rightarrow \left|v_{S_k}(x) - v_{S_k}(y)\right|_2 < \delta\,.$$

Let η^* be associated to $\delta = 1$ by this property. Let $k \in \{1, \dots, p\}$. The set $S_k \cap \Gamma$ is a compact subset of the hypersurface S_k. Applying the previous lemma, we get:

$$\exists M_k \ \forall \delta_0 > 0 \ \exists\, \eta_k > 0 \ \forall x, y \in S_k \cap \Gamma \quad |x - y|_2 \leq \eta_k \ \Rightarrow \ d_2\big(y, \tan(S_k, x)\big) \leq M_k \delta_0 |x - y|_2 \,.$$

Let $M_0 = \max_{1 \leq k \leq p} M_k$ and let δ_0 in $]0, 1/2[$ be such that $M_0 \delta_0 < 1/2$. For each k in $\{1, \dots, p\}$, let η_k be associated to δ_0 as in the above property and let

$$\eta_0 \, = \, \min\Big(\min_{1 \leq k \leq p} \eta_k, \, \eta^*, \, \frac{1}{8d} \operatorname{dist}(\Gamma^1, \Gamma^2) \Big) \,.$$

We build a family of cubes $Q(x, r)$, indexed by $x \in \Gamma$ and $r \in]0, r_\Gamma[$ such that $Q(x, r)$ is a cube centered at x of side length r which is transverse to Γ. For $x \in \mathbb{R}^d$ and $k \in \{1, \dots, p\}$, let $p_k(x)$ be a point of $S_k \cap \Gamma$ such that

$$|x - p_k(x)|_2 \, = \, \inf \big\{ \, |x - y|_2 : y \in S_k \cap \Gamma \, \big\} \,.$$

Such a point exists since $S_k \cap \Gamma$ is compact. We define then for $k \in \{1, \dots, p\}$

$$\forall x \in \mathbb{R}^d \qquad v_k(x) \, = \, v_{S_k}(p_k(x)) \,.$$

We define also

$$d_r \, = \inf_{v_1, \dots, v_p \in S^{d-1}} \ \max_{b \in \mathcal{B}_d} \ \min_{\substack{1 \leq k \leq r \\ e \in b}} \ \big(|e - v_i|_2, | - e - v_i|_2 \big)$$

where $\mathcal{B}_d$ is the collection of the orthonormal basis of $\mathbb{R}^d$ and S^{d-1} is the unit sphere of $\mathbb{R}^d$. Let η be associated to $d_r/4$ as in the above continuity property. We set

$$r_\Gamma = \frac{\eta}{2d} \,.$$

Let $x \in \Gamma$. By the definition of d_r, there exists an orthonormal basis b_x of $\mathbb{R}^d$ such that

$$\forall e \in b_x \quad \forall k \in \{1, \dots, p\} \quad \min\big(|e - v_k(x)|_2, | - e - v_k(x)|_2 \big) \, > \, \frac{d_r}{2} \,.$$

Let $Q(x, r)$ be the cube centered at x of sidelength r whose sides are parallel to the vectors of b_x. We claim that $Q(x, r)$ is transverse to Γ for $r < r_\Gamma$. Indeed, let $y \in Q(x, r) \cap \Gamma$. Suppose that $y \in S_k$ for some $k \in \{1, \dots, p\}$, so that $v_k(y) = v_{S_k}(y)$ and $|x - p_k(x)|_2 < d r_\Gamma$. In particular, we have $|y - p_k(x)|_2 < 2 d r_\Gamma < \eta$ and $|v_{S_k}(y) - v_k(x)|_2 < d_r/4$. For $e \in b_x$,

$$\frac{d_r}{2} \, \leq \, |e - v_k(x)|_2 \, \leq \, |e - v_{S_k}(y)|_2 + |v_{S_k}(y) - v_k(x)|_2$$

whence

$$|e - v_{S_k}(y)|_2 \, \geq \, \frac{d_r}{2} - \frac{d_r}{4} \, = \, \frac{d_r}{4} \,.$$

This is also true for $-e$, therefore the faces of the cube $Q(x, r)$ are transverse to S_k.

Now we consider the collection

$$(\mathring{Q}(x, r), x \in \overline{\Gamma^1}, r < r_\Gamma) \,.$$

It covers $\overline{\Gamma^1}$. By compactness of $\overline{\Gamma^1}$, we can extract a finite covering $(\mathring{Q}(x_i, r_i), i \in I)$ from this collection. We define

$$P = \cup_{i \in I} Q(x_i, r_i) ,$$

We claim that P satisfies all the hypotheses in the definition of $\widetilde{\phi_\Omega}$. Indeed, P is obviously polyhedral and transverse to Γ. Moreover, we know that

$$\overline{\Gamma^1} \subset \mathring{P} ,$$

and since $d(P, \overline{\Gamma^2}) > 0$ we also obtain that

$$\overline{\Gamma^2} \subset \mathbb{R}^d \smallsetminus \mathring{P} .$$

5 Definition of the set Ω'

Let λ be in $]\widetilde{\phi_\Omega}, +\infty[$. We are studying

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] .$$

Suppose first that $\widetilde{\phi_\Omega} > 0$. There exists a positive s such that $\lambda > \widetilde{\phi_\Omega}(1+s)^2$. By definition of $\widetilde{\phi_\Omega}$, for every positive s, there exists a polyhedral subset P of $\mathbb{R}^d$, such that ∂P is transverse to Γ,

$$\overline{\Gamma^1} \subset \mathring{P} ,\ \overline{\Gamma^2} \subset \mathbb{R}^d \smallsetminus \mathring{P}$$

and

$$\mathcal{I}_\Omega(P) \leq \widetilde{\phi_\Omega}(1+s) .$$

Then $\lambda > \mathcal{I}_\Omega(P)(1+s)$ and

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P}[\phi_n \geq \mathcal{I}_\Omega(P)(1+s) n^{d-1}] .$$

Since ∂P is transverse to Γ, we know that there exists $\delta_0 > 0$ (depending on λ, P and Γ) such that for all $\delta \leq \delta_0$,

$$\mathcal{H}^{d-1}(\partial P \cap (\mathcal{V}_2(\Omega, \delta) \smallsetminus \Omega)) \leq \frac{s \mathcal{I}_\Omega(P)}{2 \nu_{\max}} .$$

Thus, for any set Ω' satisfying $\Omega \subset \Omega' \subset \mathcal{V}_2(\Omega, \delta_0)$, we have

$$\int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x) \leq \mathcal{I}_\Omega(P)(1+s/2) ,$$

then $\lambda > (1+s/2)(\int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x))$ and

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P}\left[\phi_n \geq \left(\int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x)\right)(1+s/2) n^{d-1}\right] .$$

Suppose now that $\widetilde{\phi_\Omega} = 0$. Then for an arbitrarily fixed $s \in]0,1[$, there exists a polyhedral subset P of $\mathbb{R}^d$, such that ∂P is transverse to Γ,

$$\overline{\Gamma^1} \subset \mathring{P} ,\ \overline{\Gamma^2} \subset \mathbb{R}^d \smallsetminus \mathring{P}$$

and

$$\mathcal{I}_\Omega(P) \,\leq\, \frac{\lambda}{1+s}\,,$$

and thus $\lambda > \mathcal{I}_\Omega(P)(1+s)$. If $\mathcal{I}_\Omega(P)>0$, we can use exactly the same argument as previously. We suppose that $\mathcal{I}_\Omega(P)=0$. We know as previously that there exists $\delta_0>0$ (depending on λ, P and Γ) such that for all $\delta \leq \delta_0$,

$$\mathcal{H}^{d-1}(\partial P \cap (\mathcal{V}_2(\Omega,\delta) \smallsetminus \Omega)) \,<\, \frac{\lambda}{\nu_{\max}(1+s/2)}\,.$$

Thus, in any case, we obtain that there exists $\delta_0>0$ such that, for any set Ω' satisfying $\Omega \subset \Omega' \subset \mathcal{V}_2(\Omega,\delta_0)$, we have

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \,\leq\, \mathbb{P}\left[\phi_n \geq \left(\int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x)\right)(1+s/2)n^{d-1}\right].$$

We will construct a particular set Ω' satisfying $\Omega \subset \Omega' \subset \mathcal{V}_2(\Omega,\delta_0)$. In the previous section, we have associated to each couple (x,r) in $\Gamma \times]0, r_\Gamma[$ a hypercube $Q(x,r)$ centered at x, of sidelength r, and which is transverse to Γ. Using exactly the same method, we can build a family of hypercubes

$$(Q'(x,r), x \in \Gamma, r < r_{(\Gamma,P)})$$

such that $Q'(x,r)$ is centered at x, of sidelength r, and it is transverse to Γ and ∂P. The family

$$(\overset{\circ}{Q'}(x,r), x \in \Gamma, r < \min(r_{(\Gamma,P)}, \delta_0/(2d)))$$

is a covering of the compact set Γ, thus we can extract a finite covering from this collection, we denote it by $(\overset{\circ}{Q'}(x_i,r_i), i \in J)$. We define

$$\Omega' \,=\, \Omega \cup \bigcup_{i \in J} \overset{\circ}{Q'}(x_i,r_i)\,.$$

Since $r_i \leq \delta_0/(2d)$ for all $i \in J$, we have $\Omega' \subset \mathcal{V}_2(\Omega,\delta_0)$. Moreover, ∂P is transverse to the boundary Γ' of Ω'. Finally, if we define

$$\delta_1 \,=\, \min_{i \in J} r_i/2\,,$$

we know that $\mathcal{V}_2(\Omega,\delta_1) \subset \Omega'$, and thus for all $n \geq 2d/\delta_1$, we have $\Omega_n \subset \Omega'$.

6 Existence of a family of (Γ^1_n, Γ^2_n)-cuts

In this section we prove that we can construct a family of disjoint (Γ^1_n, Γ^2_n)-cuts in Ω_n. Let ζ be a fixed constant larger than $2d$. We consider a parameter $h < h_0 = d(\partial P, \Gamma^1 \cup \Gamma^2)$. For $k \in \{0,..., \lfloor hn/\zeta \rfloor\}$ we define

$$P(k) \,=\, \{x \in \mathbb{R}^d \,|\, d(x,P) \leq k\zeta/n\}\,,$$

and for $k \in \{0,..., \lfloor hn/\zeta \rfloor - 1\}$ we define

$$\begin{aligned}\mathcal{U}(k) \,&=\, (\mathbb{R}^d \smallsetminus \overset{\circ}{P}_{k+1}) \smallsetminus \overset{\circ}{P}_k \\ &=\, \{x \in \mathbb{R}^d \,|\, k\zeta/n \leq d(x,P) < (k+1)\zeta/n\}\,,\end{aligned}$$

and $\mathcal{M}'(k) = \mathcal{U}(k) \cap \Omega'$ (see figure 4). We will prove the following lemma:

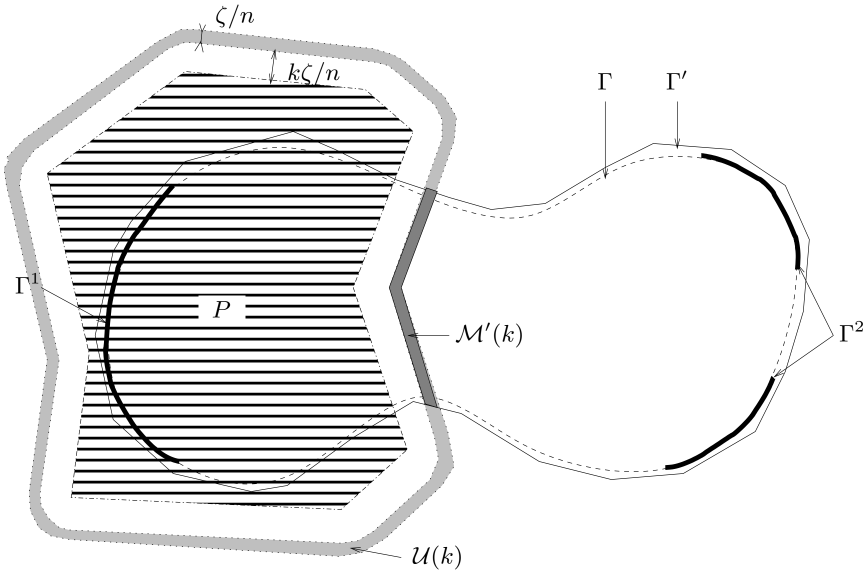

Figure 4: The sets P, $\mathcal{U}(k)$ and $\mathcal{M}'(k)$.

Lemma 2. *There exists N large enough such that for all $n \geq N$, every path on the graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$ from Γ_n^1 to Γ_n^2 in Ω_n contains at least one edge which is included in the set $\mathcal{M}'(k)$ for $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$.*

This lemma states precisely that for all $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$, $\mathcal{M}'(k)$ contains a (Γ_n^1, Γ_n^2)-cut in Ω_n.

Proof :
Let $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$. Let γ be a discrete path from Γ_n^1 to Γ_n^2 in Ω_n. In particular, γ is continuous, so we can parametrise it : $\gamma = (\gamma_t)_{0 \leq t \leq 1}$. There exists N large enough such that for all $n \geq N$, we have

$$\Omega_n \subset \Omega', \quad \Gamma_n^1 \subset \mathcal{V}_2(\Gamma^1, 2d/n) \subset \overset{\circ}{P}_k, \quad \text{and} \quad \Gamma_n^2 \subset \mathcal{V}_2(\Gamma^2, 2d/n) \subset \mathbb{R}^d \smallsetminus \overset{\circ}{P}_{k+1}.$$

Since γ is continuous, we know that there exists $t_1, t_2 \in]0,1[$ such that

$$t_1 = \sup\{t \in [0,1] \,|\, \gamma_t \in \overset{\circ}{P}_k\},$$

$$t_2 = \inf\{t \geq t_1 \,|\, \gamma_t \in \mathbb{R}^d \smallsetminus \overset{\circ}{P}_{k+1}\}.$$

Since

$$\overset{\circ}{P}_k \cup \mathcal{U}(k) \cup \mathbb{R}^d \smallsetminus \overset{\circ}{P}_{k+1}$$

is a partition of $\mathbb{R}^d$, we know that $(\gamma_t)_{t_1 \leq t < t_2}$, which is a continuous path, is included in $\mathcal{U}(k)$. The length of $(\gamma_t)_{t_1 \leq t < t_2}$ is larger than $d(\gamma_{t_1}, \gamma_{t_2})$. The segment $[\gamma_{t_1}, \gamma_{t_2}]$ intersects

$$\{x \in \mathbb{R}^d \,|\, d(x, P) = (k + 1/2)\zeta/n\}$$

at a point z, and we know that

$$\mathcal{V}_2(z, \zeta/(2n)) \subset \overset{\circ}{V}(k).$$

Thus $d(\gamma_{t_1}, \gamma_{t_2}) \geq \zeta/n$, and then the length of $(\gamma_t)_{t_1 \leq t < t_2}$ is larger than ζ/n. Finally, γ is composed of edges of length $1/n$, and $\zeta \geq 2d$, so $(\gamma_t)_{t_1 \leq t < t_2}$, and thus γ, contains at least one edge which is included in $\mathcal{U}(k)$. Noticing that for all $n \geq N$,

$$\gamma \subset \Omega_n \subset \Omega' ,$$

we obtain that this edge belongs to $\mathcal{U}(k) \cap \Omega' = \mathcal{M}'(k)$.

∎

7 Covering of $\partial P \cap \Omega'$ by cylinders

From now on we only consider $n \geq N$. According to lemma 2, we know that each set $\mathcal{M}'(k)$ for $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$ contains a (Γ_n^1, Γ_n^2)-cut in Ω_n, thus if we denote by $M'(k)$ the set of the edges included in $\mathcal{M}'(k)$, we obtain

$$\phi_n \leq \min\{V(M'(k)) \,,\, k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}\} \,.$$

However, we do not have estimates on $V(M'(k))$ that allow us to control ϕ_n using only the previous inequality. The estimates we can use are the one of the upper large deviations for the maximal flow from the top to the bottom of a cylinder (Theorem 3). In this section, we will transform our family of cuts $(M'(k))$ by replacing a huge part of the edges in each $\mathcal{M}'(k)$ by the edges of minimal cutsets in cylinders.

We denote by $H_i, i = 1, ..., \mathcal{N}$ the intersection of the faces of ∂P with Ω'. For each $i = 1, ..., \mathcal{N}$, we denote by v_i the exterior normal unit vector to P along H_i. We will cover $\partial P \cap \Omega'$ by cylinders, except a surface of $\mathcal{H}^{d-1}$ measure controlled by a parameter ε. To explain the construction of a cutset we will do with a huge number of cylinders, we present first the simpler construction of a cutset using one cylinder. Let R be a hyperrectangle that is included in H_j for a $j \in \{1, ..., \mathcal{N}\}$, and let B be the cylinder defined by

$$B = \{x + tv_j \,|\, x \in R \,,\, t \in [0, h]\} \,,$$

where $h \leq h_0$ is the same parameter as previously. The cylinder B is built on $\partial P \cap \Omega'$, in $\mathbb{R}^d \smallsetminus \mathring{P}$. We recall that $h_0 = d(\partial P, \Gamma^1 \cup \Gamma^2) > 0$, so we know that $d(B, \Gamma^1 \cup \Gamma^2) > 0$. We denote by E_a the set of the edges included in

$$\mathcal{E}_a = \{x + tv_j \,|\, x \in R \,,\, d(x, \partial R) < \zeta/n \,,\, t \in [0, h]\} \,.$$

The set $\mathcal{E}_a$ is a neighbourhood in B of the "vertical" faces of B, i.e., the faces of B that are collinear to v_j. We denote by E_b a set of edges in B that cuts the top $R + hv_j$ from the bottom R of B. Let $M'(k)$ be the set of the edges included in $\mathcal{M}'(k)$, for a $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$. Let B' be the thinner cylinder

$$B' = \{x + tv_j \,|\, x \in R \,,\, d(x, \partial R) \geq \zeta/n \,,\, t \in [0, h]\} \,.$$

Thus for all $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$, the set of edges

$$(M'(k) \cap (\mathbb{R}^d \smallsetminus B')) \cup E_a \cup E_b$$

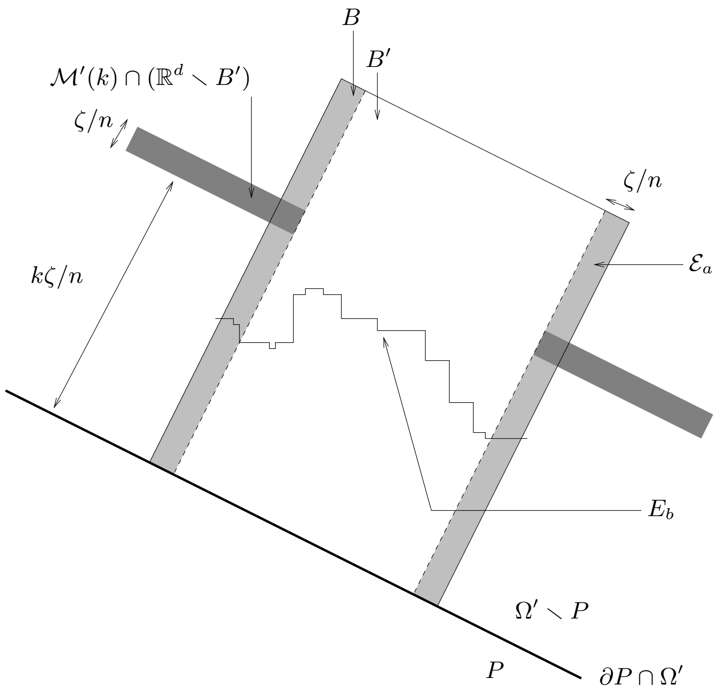

Figure 5: Construction of a (Γ_n^1, Γ_n^2)-cut in Ω_n using a cutset in a cylinder.

cuts Γ_n^1 from Γ_n^2 in Ω_n. Indeed, the set of edges $M'(k)$ is already a cut between Γ_n^1 and Γ_n^2 in Ω_n. We remove from it the edges that are inside B' which is in the interior of B, and we add to it a cutset E_b from the top to the bottom of B, and the set of edges E_a that glue together E_b and $M'(k) \cap (\mathbb{R}^d \smallsetminus B')$. This property is illustrated in the figure 5.

Remark 4. In this figure, we have represented E_b as a surface (so a path in dimension 2) that separates the top from the bottom of the cylinder to illustrate the fact that E_b cuts all discrete paths from the bottom to the top of B. Actually, we can mention that it is possible to define an object which could be the dual of an edge in dimension $d \geq 2$ (as a generalization of the dual of a planar graph). This object is a plaquette, i.e., a hypersquare of sidelength $1/n$ that is orthogonal to the edge and cuts it in its middle, and whose sides are parallel to the hyperplanes of the axis. Then the dual of a cutset is a hypersurface of plaquettes, thus the figure 5 is somehow intuitive.

We do exactly the same construction, but with a large number of cylinders, that will almost cover $\partial P \cap \Omega'$. We consider a fixed $\varepsilon > 0$. There exists a l sufficiently small (depending on F, P and ε) such that there exists a finite collection $(R_{i,j}, i = 1, ..., \mathcal{N}, j = 1, ..., N_i)$ of hypersquares of side l of disjoint interiors satisfying $R_{i,j} \subset H_i$ for all $i \in \{1, ..., \mathcal{N}\}$ and $j \in \{1, ..., N_i\}$, and for all $i \in \{1, ..., \mathcal{N}\}$,

$$\{x \in H_i \,|\, d(x, \partial H_i) \geq \varepsilon \mathcal{H}^{d-2}(\partial H_i)^{-1} \mathcal{N}^{-1}\} \subset \bigcup_{j=1}^{N_i} R_{i,j} \subset$$
$$\subset \{x \in H_i \,|\, d(x, \partial H_i) \geq \varepsilon \mathcal{H}^{d-2}(\partial H_i)^{-1} \mathcal{N}^{-1} 2^{-1}\}.$$

We immediately obtain that

$$\mathcal{H}^{d-1}\left((\partial P \cap \Omega') \smallsetminus \bigcup_{i=1}^{\mathcal{N}} \bigcup_{j=1}^{N_i} R_{i,j}\right) \leq \varepsilon .$$

We remark that

$$\int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x) \geq \sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i) ,$$

so that

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P}\left[\phi_n \geq (1+s/2) n^{d-1} \sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i)\right] .$$

Let $h < h_0$. For all $i \in \{1, ..., \mathcal{N}\}$ and $j \in \{1, ..., N_i\}$, we define

$$B_{i,j} = \{x + t v_i \,|\, x \in R_{i,j} , t \in [0, h]\} .$$

Since all the $B_{i,j}$ are at strictly positive distance of ∂H_i, there exists a positive h_1 such that for all $h < h_1$, the cylinders $B_{i,j}$ have pairwise disjoint interiors. We thus consider $h < \min(h_0, h_1)$ (see figure 6 for example). At this point, we could define a neighbourhood of the vertical faces of each

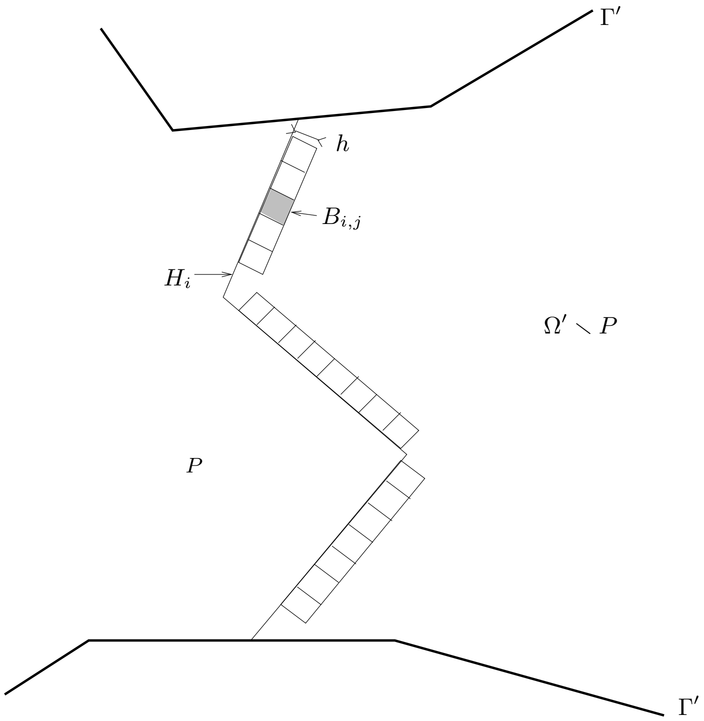

Figure 6: Covering of $\partial P \cap \Omega'$ by cylinders.

cylinder $B_{i,j}$, and do the same construction as in the previous example with one cylinder. Actually, we need to choose a little bit more carefully the sets of edges we define along the vertical faces of the cylinders. We will not consider only each cylinder $B_{i,j}$, but also thinner versions of these cylinders of the type

$$B_{i,j}(k) = \{x + tv_j \,|\, x \in R_{i,j}\,,\, d(x, \partial R_{i,j}) > k\zeta/n\,,\, t \in [0,h]\}$$

for different values of k. We will then consider the edges included in a neighbourhood of the vertical faces of each $B_{i,j}(k)$ (see the set $\mathcal{W}_{i,j}(k)$ above), and choose k to minimize the capacity of the union over i and j of these edges. The reason why we need this optimization is also the reason why we built a family $(M'(k))$ of cutsets and not only one cutset from Γ_n^1 to Γ_n^2 in Ω_n, we will try to explain it in remark 5.

Here are the precise definitions of the sets of edges. We still consider the same constants ζ bigger than $2d$ and $h < \min(h_0, h_1)$. We define another positive constant η that we will choose later (depending on P, s and Ω). For i in $\{1, ..., \mathcal{N}\}$ and j in $\{1, ..., N_i\}$ we recall the definition of $B_{i,j}$:

$$B_{i,j} = \{x + tv_i \,|\, x \in R_{i,j}\,,\ t \in [0,h]\}\,,$$

and we define the following subsets of $\mathbb{R}^d$:

$$B'_{i,j} = \{x + tv_i \,|\, x \in R_{i,j}\,,\ d(x, \partial R_{i,j}) > \eta\,,\ t \in [0,h]\}\,,$$

$$\forall k \in \{0, ..., \lfloor \eta n/\zeta - 1 \rfloor\}\,,\ \mathcal{W}_{i,j}(k) = \{x \in B_{i,j} \,|\, k\zeta/n \leq d_2(x, \partial R_{i,j} + \mathbb{R}v_i) < (k+1)\zeta/n\}\,,$$

$$\forall k \in \{0, ..., \lfloor hn\kappa/\zeta - 1 \rfloor\}\,,\ \mathcal{M}(k) = \mathcal{M}'(k) \smallsetminus \left(\bigcup_{i,j} B'_{i,j} \right)\,,$$

(see figures 7 and 8). We denote by $W_{i,j}(k)$ the set of the edges included in $\mathcal{W}_{i,j}(k)$ and we

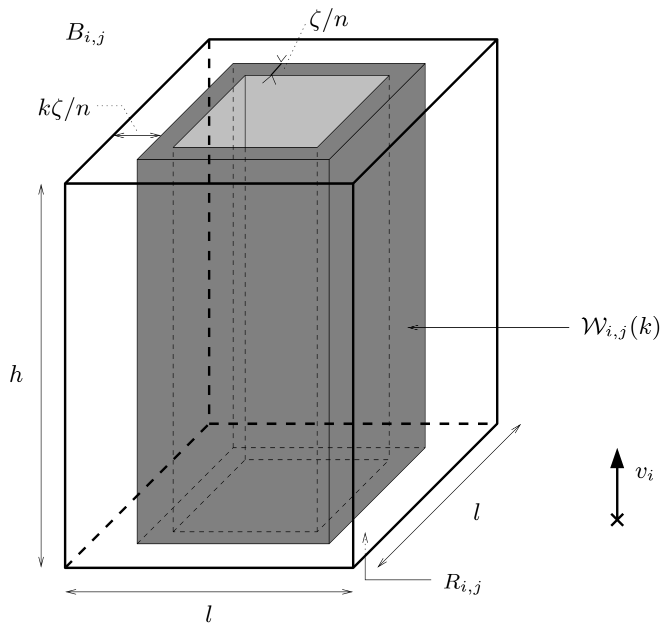

Figure 7: The set $\mathcal{W}_{i,j}(k)$.

define $W(k) = \cup_{i,j} W_{i,j}(k)$. We also denote by $M(k)$ the edges included in $\mathcal{M}(k)$. Exactly as in

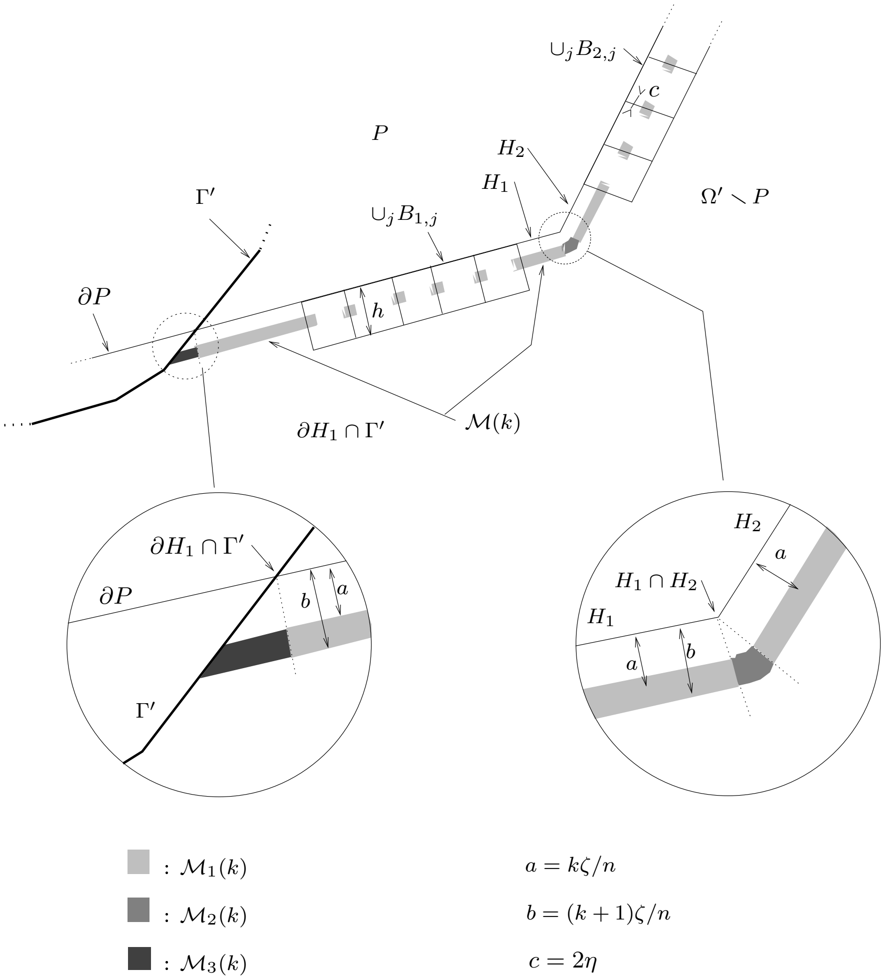

Figure 8: The set $\mathcal{M}(k)$.

the construction of a cutset with one cylinder, we obtain a cutset that is built with cutsets in each cylinders $B_{i,j}$. Indeed, if we denote by $E_{i,j}$ a set of edges that is a cutset from the top to the bottom of $B_{i,j}$ (oriented towards the direction given by v_i), then for each $k_1 \in \{0, ..., \lfloor \eta n/\zeta - 1 \rfloor\}$ and $k_2 \in \{0, ..., \lfloor hn/\zeta - 1 \rfloor\}$, the set of edges:

$$\bigcup_{\substack{i=1,...,\mathcal{N} \\ j=1,...,N_i}} E_{i,j} \cup W(k_1) \cup M(k_2)$$

contains a cutset from Γ_n^1 to Γ_n^2 in Ω_n. We deduce that

$$\phi_n \le \sum_{i,j} \phi_{B_{i,j}} + \min_{k_1} V(W(k_1)) + \min_{k_2} V(M(k_2)) \,. \tag{1}$$

8 Control of the cardinality of the sets of edges W and M

For the sake of clarity, we do not recall the sets in which the parameters take its values, we always assume that they are the following: $i \in \{1, ..., \mathcal{N}\}$, $j \in \{1, ..., N_i\}$, $k_1 \in \{0, ..., , \lfloor \eta n/\zeta - 1 \rfloor\}$ and $k_2 \in \{0, ..., \lfloor hn/\zeta - 1 \rfloor\}$. We have to evaluate the number of edges in the sets $W(k_1)$ and $M(k_2)$ to control the terms $\min_{k_1} V(W(k_1))$ and $\min_{k_2} V(M(k_2))$ in (1). There exist constants $c_1(d,\Omega)$, $c_2(P,d,\Omega)$ such that

$$\mathrm{card}\, W(k_1) \,\leq\, c_1 \frac{\mathcal{H}^{d-1}(\partial P \cap \Omega')}{l^{d-1}} \zeta l^{d-2} h n^{d-1} \,\leq\, c_2 l^{-1} h n^{d-1}\,.$$

The cardinality of $M(k_2)$ is a little bit more complicated to control. We will divide $M(k)$ (respectively $\mathcal{M}(k)$) into three parts: $M(k) \subset M_1(k) \cup M_2(k) \cup M_3(k)$ (respectively $\mathcal{M}(k) \subset \mathcal{M}_1(k) \cup \mathcal{M}_2(k) \subset \mathcal{M}_3(k)$), that are represented in figure 8.

We define $R'_{i,j} = \{x \in R_{i,j} \,|\, d(x, \partial R_{i,j}) > \eta\}$ which is the basis of $B'_{i,j}$. The set $\mathcal{M}_1(k)$ is a translation of the sets $H_i \smallsetminus (\cup_{j=1}^{N_i} R'_{i,j})$ along the direction given by v_i enlarged with a thickness $\zeta/(n\kappa)$:

$$\mathcal{M}_1(k) \,\subset\, \bigcup_{i=1}^{\mathcal{N}} \{x + t v_i \,|\, x \in H_i \smallsetminus (\cup_{j=1}^{N_i} R'_{i,j})\,,\; t \in [k\zeta/n, (k+1)\zeta/n[\}\,.$$

Here we have an inclusion and not an equality because $\mathcal{M}_1(k)$ can be a truncated version of this set (truncated at the junction between the translates of two different faces). Since we know that

$$\mathcal{H}^{d-1}\left((\partial P \cap \Omega') \smallsetminus \bigcup_{i=1}^{\mathcal{N}} \bigcup_{j=1}^{N_i} R_{i,j} \right) \,\leq\, \varepsilon\,,$$

and

$$\mathcal{H}^{d-1}\left(\bigcup_{i=1}^{\mathcal{N}} \bigcup_{j=1}^{N_i} (R_{i,j} \smallsetminus R'_{i,j}) \right) \,\leq\, \frac{\mathcal{H}^{d-1}(\partial P \cap \Omega')}{l^{d-1}} l^{d-2} \eta \,=\, \mathcal{H}^{d-1}(\partial P \cap \Omega') l^{-1} \eta\,,$$

we have the following bound on the cardinality of $M_1(k)$:

$$\mathrm{card}(M_1(k)) \,\leq\, c_3(\varepsilon + l^{-1}\eta) n^{d-1}\,,$$

for a constant $c_3(d, P, \Omega, \Omega')$.

The part $M_2(k)$ corresponds to the edges included in the "bends" of the neighbourhood of ∂P located around the boundary of the faces of ∂P in Ω', denoted by $\mathcal{M}_2(k)$, i.e.:

$$\mathcal{M}_2(k) \,\subset\, \bigcup_{i,j} (\mathcal{V}_2(H_i \cap H_j, (k+1)\zeta/n) \smallsetminus \mathcal{V}_2(H_i \cap H_j, k\zeta/n))\,,$$

and there exists a constant $c_4(d, P, \Omega')$ such that

$$\mathrm{card}\, M_2(k) \,\leq\, c_4 |k\zeta/n|^{d-2} n^{d-1} \,\leq\, c_4 h^{d-2} n^{d-1}\,.$$

The last part $\mathcal{M}_3(k)$ corresponds to the part of $\mathcal{M}(k)$ that is near the boundary Γ' of Ω'. Indeed, Γ' is not orthogonal to ∂P, thus for some k, the set $\mathcal{M}(k)$ may contain edges that are not included in

$$\bigcup_{i=1}^{\mathcal{N}} \{x + t v_i \,|\, x \in H_i \smallsetminus (\cup_{j=1}^{N_i} R'_{i,j})\,,\; t \in [k\zeta/n, (k+1)\zeta/n[\}\,,$$

neither in

$$\bigcup_{i,j} (\mathcal{V}_2(H_i \cap H_j, (k+1)\zeta/n) \setminus \mathcal{V}_2(H_i \cap H_j, k\zeta/n)) \ ,$$

(see figure 8). However, $\mathcal{M}(k) \subset \mathcal{U}(k)$, the problem is to evaluate the difference of cardinality between the different $M(k)$ due to the intersection of $\mathcal{U}(k)$ with Ω'. We have constructed Ω' such that Γ' is transverse to ∂P precisely to obtain this control. The sets Γ' and ∂P are polyhedral surfaces which are transverse. We denote by $(\mathcal{H}_i, i \in I)$ (resp. $(\mathcal{H}'_j, j \in J)$) the hyperplanes that contain ∂P (resp. Γ'), and by v_i (resp. v'_j) the exterior normal unit vector to P along $\mathcal{H}_i$ (resp. Ω' along $\mathcal{H}'_j$). The set $\Gamma' \cap \partial P$ is included in the union of a finite number of intersections $\mathcal{H}_i \cap \mathcal{H}'_j$ of transverse hyperplanes. To each such intersection $\mathcal{H}_i \cap \mathcal{H}'_j$, we can associate the angles between v_i and v'_j, and between v_i and $-v'_j$, in the plane of dimension 2 spanned by v_i and v'_j. Each such angle is strictly positive because $\mathcal{H}_i$ is transverse to $\mathcal{H}'_j$, and so the minimum θ_0 over the finite number of defined angles is strictly positive. This θ_0 and the measure $\mathcal{H}^{d-2}(\partial P \cap \Gamma')$ give to us a control on the volume of $\mathcal{M}_3(k)$, and thus on $\text{card}(M_3(k))$, as soon as these sets belong to a neighbourhood of $\partial P \cap \Gamma'$ (see figure 9). Thus, there exist $h_2(\Omega', P) > 0$ and a constant $c_5(d, P, \Omega, \Omega')$ such that

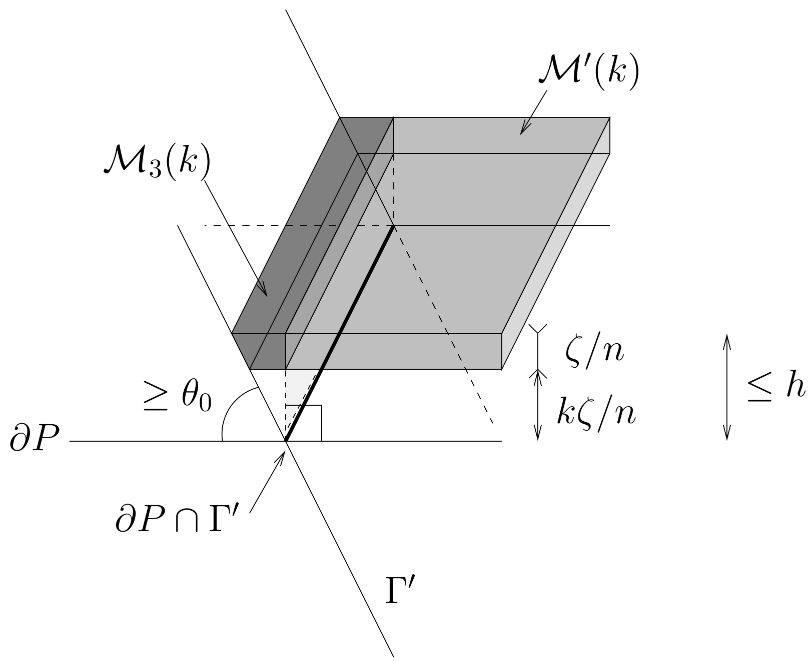

Figure 9: The set $\mathcal{M}_3(k)$.

for all $h \leq h_2$,

$$\text{card}(M_3)(k) \ = \ c_5 h n^{d-1} \, .$$

We conclude that there exists a positive constant $c_6(d, P, \Omega, \Omega')$ such that

$$\text{card}\, M(k) \ \leq \ c_6(\varepsilon + l^{-1}\eta + h^{d-2} + h) n^{d-1} \, .$$

9 Calibration of the constants

We remark that the sets $W(k)$ (resp., the sets $M(k)$) are pairwise disjoint for different k. Then we obtain that

$$\begin{aligned}
\mathbb{P}[\phi_n \geq \lambda n^{d-1}] &\leq \mathbb{P}\left[\phi_n \geq (1+s/2)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right] \\
&\leq \mathbb{P}\left[\sum_{i=1}^{\mathcal{N}}\sum_{j=1}^{N_i}\phi_{B_{i,j}} \geq (1+s/4)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right] \\
&\quad + \mathbb{P}\left[\min_{k_1} V(W(k_1)) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right] \\
&\quad + \mathbb{P}\left[\min_{k_2} V(M(k_2)) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right] \\
&\leq \sum_{i=1}^{\mathcal{N}}\sum_{j=1}^{N_i}\left(\max_{i,j}\mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1}\nu(v_i)(1+s/4)n^{d-1}]\right) \\
&\quad + \mathbb{P}\left[\sum_{i=1}^{c_2 l^{-1}hn^{d-1}} t(e_i) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right]^{\lfloor \eta n/\zeta\rfloor} \\
&\quad + \mathbb{P}\left[\sum_{i=1}^{c_6(\varepsilon+l^{-1}\eta+h^{d-2}+h)n^{d-1}} t(e_i) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right]^{2\lfloor hn/\zeta\rfloor}.
\end{aligned}$$

The terms

$$\mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1}\nu(v_i)(1+s/4)n^{d-1}]$$

have already been studied in [9] (we recalled it as Theorem 3 in this paper).

It remains to study two terms of the type

$$\mathcal{P}(n) \,=\, \mathbb{P}\left(\sum_{i=1}^{\alpha n^{d-1}} t(e_i) \geq \beta n^{d-1}\right).$$

As soon as $\beta > \alpha\mathbb{E}(t)$ and the law of the capacity of the edges admits an exponential moment, the Cramér theorem in $\mathbb{R}$ allows us to affirm that

$$\limsup_{n\to\infty}\frac{1}{n^{d-1}}\log\mathcal{P}(n) \,<\, 0\,.$$

Moreover, for all

$$\varepsilon \,\leq\, \varepsilon_0 \,=\, \frac{1}{2\nu_{\max}}\int_{\mathcal{P}\cap\Omega'}\nu(v_P(x))d\mathcal{H}^{d-1}(x)\,,$$

we have

$$\begin{aligned}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i) &\geq \int_{\partial P\cap\Omega'} \nu(v_P(x))d\mathcal{H}^{d-1}(x) - \varepsilon\nu_{max}\\ &\geq \frac{1}{2}\int_{\partial P\cap\Omega'} \nu(v_P(x))d\mathcal{H}^{d-1}(x)\\ &\geq \frac{\nu_{\min}}{2}\mathcal{H}^{d-1}(\partial P\cap\Omega')\,.\end{aligned}$$

Thus, for all $\varepsilon < \varepsilon_0$ and $h < \min(h_0, h_1, h_2)$, if the constants satisfy the two following conditions:

$$c_2 l^{-1} h \,<\, \mathcal{H}^{d-1}(\partial P\cap\Omega')\nu_{\min}\mathbb{E}(t(e))s/16\,, \tag{2}$$

and

$$c_6(\varepsilon + l^{-1}\eta + h^{d-2} + h) \,<\, \mathcal{H}^{d-1}(\partial P\cap\Omega')\nu_{\min}\mathbb{E}(t(e))s/16\,, \tag{3}$$

thanks Theorem 3 and the Cramér theorem in $\mathbb{R}$, we obtain that

$$\limsup_{n\to\infty} \frac{1}{n^d}\log\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \,<\, 0\,,$$

and theorem 1 is proved. We claim that it is possible to choose the constants such that conditions (2) and (3) are satisfied. Indeed, we first choose $\varepsilon < \varepsilon_0$ such that

$$\varepsilon \,<\, \frac{1}{4}\frac{\mathcal{H}^{d-1}(\partial P\cap\Omega)\nu_{\min}\mathbb{E}(t(e))s}{16c_6}\,.$$

To this fixed ε corresponds a l. Knowing ε and l, we choose $h \leq \min(h_0, h_1, h_2)$ and η such that

$$\max(h, h^{d-2}, l^{-1}h, l^{-1}\eta) \,<\, \frac{1}{4}\frac{\mathcal{H}^{d-1}(\partial P\cap\Omega')\nu_{\min}\mathbb{E}(t(e))s}{16\max(c_2, c_6)}\,.$$

This ends the proof of theorem 1.

Remark 5. We try here to explain why we built several sets $W(k_1)$ and $M(k_2)$, and not only one couple of such sets, that would have been sufficient to construct a cutset from Γ_n^1 to Γ_n^2 in Ω_n. To use estimates of upper large deviations of maximal flows in cylinder we already know, we want to compare ϕ_n with $\sum_{i,j}\phi_{B_{i,j}}$. Heuristically, to construct a (Γ_n^1,Γ_n^2)-cut in Ω_n from the union of cutsets in each cylinder $B_{i,j}$, we have to add edges to glue together the different cutsets at the common boundary of the small cylinders, and to extend these cutsets to $(\partial P\cap\Omega_n)\smallsetminus\bigcup_{i=1}^{\mathcal{N}}\bigcup_{j=1}^{N_i} R_{i,j}$. Yet we want to prove that the upper large deviations of ϕ_n are of volume order. If we only consider one possible set E of edges such that

$$\phi_n \,\leq\, \sum_{i,j}\phi_{B_{i,j}} + V(E)\,,$$

we will obtain that

$$\begin{aligned}\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq &\sum_{i,j}\mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1}\nu(v_i)(1+s/4)n^{d-1}]\\ &+\mathbb{P}\left[V(E) \geq n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)s/4\right]\,.\end{aligned}$$

We can choose such a set E so that it contains less than δn^{d-1} edges for a small δ (E is equal to $W(k_1) \cup M(k_2)$ for a fixed couple (k_1, k_2) for example), but the probability

$$\mathbb{P}\left[\sum_{i=1}^{\delta n^{d-1}} t(e_i) \geq C n^{d-1}\right]$$

does not decay exponentially fast with n^d in general. To obtain this speed of decay, we have to make an optimization over the possible choices of the set E, i.e., we choose E among a set of $C'n$ possible disjoint sets of edges $E_1, ..., E_{C'n}$; in this case, we obtain that

$$\phi_n \leq \sum_{i,j} \phi_{B_{i,j}} + \min_{k=1,\dots,C'n} V(E_k)\,,$$

and so

$$\begin{aligned}\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq & \sum_{i,j} \mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1}\nu(v_i)(1+s/4)n^{d-1}] \\ & + \prod_{k=1}^{C'n} \mathbb{P}\left[V(E_k) \geq n^{d-1} \sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i) s/4\right]\,. \end{aligned} \tag{4}$$

It is then sufficient to prove that for all k, $\mathbb{P}[V(E_k) \geq C'' n^{d-1}]$ decays exponentially fast with n^{d-1} to conclude that the last term in (4) decays exponentially fast with n^d. Theorem 3 gives a control on the terms

$$\mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1}\nu(v_i)(1+s/4)n^{d-1}]\,.$$

The conclusion is that to obtain the volume order of the upper large deviations, the optimization over the different possible values of k_1 and k_2 is really important, even if it is not needed if we only want to prove that $\mathbb{P}(\phi_n \geq \lambda n^{d-1})$ goes to zero when n goes to infinity.

References

[1] Béla Bollobás. *Graph theory*, volume 63 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. An introductory course.

[2] Raphaël Cerf and Marie Théret. Law of large numbers for the maximal flow through a domain of $\mathbb{R}^d$ in first passage percolation. Available from `arxiv.org/abs/0907.5504`.

[3] Raphaël Cerf and Marie Théret. Lower large deviations for the maximal flow through a domain of $\mathbb{R}^d$ in first passage percolation. Available from `arxiv.org/abs/0907.5501`.

[4] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[5] Harry Kesten. Aspects of first passage percolation. In *École d'Été de Probabilités de Saint Flour XIV*, number 1180 in Lecture Notes in Mathematics. Springer-Verlag, 1984.

[6] Harry Kesten. Surfaces with minimal random weights and maximal flows: a higher dimensional version of first-passage percolation. *Illinois Journal of Mathematics*, 31(1):99–166, 1987.

[7] Serge Lang. *Differential manifolds*. Springer-Verlag, New York, second edition, 1985.

[8] R. Rossignol and M. Théret. Lower large deviations and laws of large numbers for maximal flows through a box in first passage percolation. Available from `arxiv.org/abs/0801.0967v2`, 2009.

[9] Marie Théret. Upper large deviations for maximal flows through a tilted cylinder. Available from `arxiv.org/abs/0907.0614`, 2009.

[10] Yu Zhang. Critical behavior for maximal flows on the cubic lattice. *Journal of Statistical Physics*, 98(3-4):799–811, 2000.

[11] Yu Zhang. Limit theorems for maximum flows on a lattice. Available from `arxiv.org/abs/0710.4589`, 2007.