Dynamics of second grade fluids

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Dedicated to Jürgen Scheurle on the occasion of his 60th birthday
(joint works with J. Hale, M. Paicu, A. Rekalo)

In petroleum industry, in polymer technology, non-Newtonian (Rivlin-Ericksen) fluids of diffferential type often arise.
The constitutive law of incompressible homogeneous fluids of degree 2 is given by

$$\sigma = -pI + 2\nu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2,$$

where σ is the Cauchy tensor, A_1 and A_2 are the first two Rivlin-Ericksen tensors:

$$A_1(u) = \frac{1}{2}[\nabla u + \nabla u^T], \quad A_2 = \frac{DA_1}{Dt} + (\nabla u)^T A_1 + A_1(\nabla u)$$

and

$$\frac{D}{Dt} = \partial_t + u.\nabla.$$

is the material derivative.
Dunn and Fosdick (1974) established that

$$\alpha_1 + \alpha_2 = 0, \quad \alpha_1 \geq 0.$$

Writing then the equation $\frac{Du}{Dt} = u_t + u.\nabla u = \operatorname{div} \sigma$, one obtains the second grade fluid equations.

System of second grade fluid equations in $\mathbb{T}^2$

For $\alpha > 0$, we consider the following system in the two-dimensional torus $\mathbb{T}^2$:

$$\begin{aligned} &\partial_t(u - \alpha\Delta u) - \nu\Delta u + \mathrm{curl}\,(u - \alpha\Delta u) \times u + \nabla p = f, \\ &\mathrm{div}\, u = 0, \\ &u(0, x) = u_0(x). \end{aligned} \tag{1}$$

Here f is the force, $u = (u_1, u_2) \equiv (u_1, u_2, 0)$ is the velocity field and $\mathrm{curl}\, u \equiv \mathrm{rot}\, u \equiv (0, 0, \partial_1 u_2 - \partial_2 u_1)$.

There is a "bad" nonlinear term in (1)

If $\alpha = 0$, we recover the Navier-Stokes equations,

$$\begin{aligned} &\partial_t u - \nu\Delta u + (\mathrm{curl}\, u) \times u + \nabla q = f, \\ &\mathrm{div}\, u = 0, \\ &u(0, x) = u_0(x). \end{aligned} \tag{2}$$

Other form:

$$\partial_t u - \nu\Delta u + u \cdot \nabla u + \nabla p = f,$$

Comparaison with the 2D Navier-Stokes equations

Properties of the 2D Navier-Stokes equations

- NS Equations define a non-linear semigroup $S_0(t)$ on $L^2_{\mathrm{div}}(\mathbb{T}^2)^2$: global existence and uniqueness of a classical solution u.
- Smoothing in finite positive time (parabolic type)
- Existence of a compact global attractor $\mathcal{A}_0$.
- Finite dimensional properties (determining modes, finite fractal dimension of $\mathcal{A}_0$, etc..)

Features of the second grade fluid system, $\alpha > 0$

- System (1) defines a non-linear group $S_\alpha(t)$ on $H^3_{\mathrm{div}}(\mathbb{T}^2)^2$.
- No smoothing in finite time (asymptotic smoothness).
- (1) is a non-regular perturbation of 2D NS, for α small.
- $S_\alpha(t)$ is not Hölderian in t.
- But, good structure (cancellations).

Objectives of this talk

Question: Can we extend some properties of the Navier-Stokes equations to the second grade fluid equations ?

Part 1: Properties of the compact global attractor $\mathcal{A}_\alpha$ in $\mathbb{T}^2$ (Paicu, Rekalo, G.R.)

1. Regularity of $\mathcal{A}_\alpha$.
2. Finite-dimensional behaviour :
   Finite number of determining modes ?
   Reduction of (1) to a finite-dimensional system of ODE's with infinite delay?

Part 2: Perturbation results in $\mathbb{T}^2$ (Hale, G.R.)
Comparison of the dynamics of $S_\alpha(t)$ with the ones of $S_0(t)$ (periodic orbits, etc..)

Part 3: Fast rotating second grade fluids in $\mathbb{T}^3$

Classical properties of System (1)

Let V^m, $m \in \mathbb{N}$, be the closure in $H^m(\mathbb{T}^2)^2$ of the space

$$\{u \in C^\infty(\mathbb{T}^2)^2 \,|\, u \text{ is periodic}, \operatorname{div} u = 0, \int_{\mathbb{T}^2} u\,dx = 0\}.$$

We set $H = V^0$.
Likewise, let H^m_{per} be the closure in $H^m(\mathbb{T}^2)^2$ of the space

$$\{u \in C^\infty(\mathbb{T}^2)^2 \,|\, u \text{ is periodic}, \int_{\mathbb{T}^2} u\,dx = 0\}.$$

Proposition (Cioranescu,Ouazar (1983))

Let $\alpha > 0$, $T_1 > 0$ and $T_2 > 0$ be given. For any $f \in L^\infty((-T_1, T_2), H^1_{per})$ and any $u_0 \in V^3$, (1) has a unique solution $u(t) \in C^0([-T_1, T_2], V^3) \cap W^{1,\infty}((-T_1, T_2), V^2)$.

If f does not depend on t, $S_\alpha(t) : u_0 \in V^3 \mapsto u(t) \in V^3$ is a (continuous) group

Compact global attractor

We assume that f does not depend on t.

Theorem (Moise, Rosa and Wang (1998))

Let $f \in V^1$. For any $\alpha > 0$, $S_\alpha(t)$ has a compact global attractor $\mathcal{A}_\alpha$ in V^3, that is,
- $\mathcal{A}_\alpha$ is a compact set in V^3
- $\mathcal{A}_\alpha$ is invariant (i.e. $S_\alpha(t)\mathcal{A}_\alpha = \mathcal{A}_\alpha$, for any $t \geq 0$)
- $\mathcal{A}_\alpha$ attracts every bounded set of V^3.

Proof: method of functionals of J. Ball

Remark:
- $S_\alpha(t)$ is a group $\rightsquigarrow$ there is no smoothing in finite time
($S_\alpha(t)$ is asymptotically smooth or asymptotically compact)

Part I: Regularity of the global attractor $\mathcal{A}_\alpha$ in V^s, $s > 3$

Theorem (1: Regularity of the global attractor, P. R. R.)

1) If $f \in H^2_{per}$ and $a_0 = \nu - \alpha(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$, then $\mathcal{A}_\alpha$ is bounded in V^4. Moreover, for any $u \in \mathcal{A}_\alpha$,

$$\|u\|^2_{V^3} + \inf(\alpha, 1)\|u\|^2_{V^4} \leq M_1 \ .$$

2) For any $\alpha > 0$, there exists $0 < \theta \leq 1$, depending only on α and $\|f\|_{H^1}$, s. t., if $f \in H^{1+\theta}_{per}$, then $\mathcal{A}_\alpha$ is bounded in $V^{3+\theta}$. And, for any $u \in \mathcal{A}_\alpha$,

$$\|u\|^2_{V^{2+\theta}} + \inf(\alpha, 1)\|u\|^2_{V^{3+\theta}} \leq M_\theta.$$

3) If $f \in H^{m+1}_{per}$ and $a_m = \nu - \alpha d_m(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$, then $\mathcal{A}_\alpha$ is bounded in V^{m+3}. Moreover, for any $u \in \mathcal{A}_\alpha$,

$$\|u\|^2_{V^{m+2}} + \inf(\alpha, 1)\|u\|^2_{V^{m+3}} \leq M_m \ .$$

Part I: Regularity of the global attractor (continued)

Open Problem: If f is analytical, can we show that $\mathcal{A}_\alpha$ is analytical?

Remarks:
1) The quantity $\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}$ is bounded by a constant depending only on $\|f\|_{V^1}$.
2) Other form of the second grade fluid system:

$$\partial_t \mathrm{curl}\,(u-\alpha\Delta u) - \nu\Delta \mathrm{curl}\, u + \mathrm{curl}\,\big(\mathrm{curl}\,(u-\alpha\Delta u)\times u\big) = \mathrm{curl}\, f.$$

Identity: If v and u^* are divergence-free and regular enough,

$$(\mathrm{curl}\,(\mathrm{curl}\, v\times u^*), \mathrm{curl}\, v) = 0 \qquad (3)$$

Part I: Regularity of the global attractor (continued)

Decomposition of the second grade fluid system:

Let $u(t) = S_\alpha(t)u_0 \subset \mathcal{A}_\alpha$. We set $u(t) = v_n(t) + w_n(t)$, where $v_n(t)$ and $w_n(t)$ are solutions of the non-autonomous equations

$$\partial_t(v_n - \alpha\Delta v_n) - \nu\Delta v_n + \mathrm{curl}\,(v_n - \alpha\Delta v_n) \times u + \nabla p_n = f,\ \ t > s_n,$$
$$v_n(s_n, x) = 0,$$

and

$$\partial_t(w_n - \alpha\Delta w_n) - \nu\Delta w_n + \mathrm{curl}\,(w_n - \Delta w_n) \times u + \nabla \tilde{p}_n = 0,\ \ t > s_n,$$
$$w_n(s_n, x) = u(s_n, x),$$

where $s_n \in \mathbb{R}$ will go to $-\infty$.

Properties:

1. $w_n(t) \to 0$ in V^3 and thus $v_n(t) \to u(t)$ in V^3 as $n \to +\infty$
2. $v_n(t)$ is uniformly bounded in V^4 with respect to n
3. A subsequence $v_{n_k}(t)$ converges weakly to $u(t)$ in V^4.
4. $u(t)$ is uniformly bounded in V^4

Part I: Finite-dimensional properties of $\mathcal{A}_\alpha$

As for the Navier-Stokes equations, we are not able to prove the existence of an inertial manifold containing $\mathcal{A}_\alpha$. But we have the result below.

Let P_n be the orthogonal projection in H onto the space generated by the eigenvectors corresponding to the first n eigenvalues of $A = -\mathbb{P}\Delta$.

Theorem (2: Retarded system)

If $\nu - 4\alpha(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$ and if $f \in H^{1+d}_{per}$, $d \in (0,1]$, there exists N_1 s. t., for $n \geq N_1$, any element $u(t) \in \mathcal{A}_\alpha$ writes as

$$u = v_n + q_n(v_n)\ , \quad v_n \in P_n\mathcal{A}_\alpha,$$

where $q_n \equiv q_{n,\alpha}$ maps $C^0(\mathbb{R}; N_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha, r_0))$ into $C^0(\mathbb{R}, B_{Q_nV^3}(0, r_1))$ and $q_n(v_n)(t)$ depends only on $v_n(s)$, $s \leq t$. Moreover on $\mathcal{A}_\alpha$, Equation (1) reduces to the RFDE

$$\partial_t(v_n - \alpha\Delta v_n) - \nu\Delta v_n$$
$$+ P_n\mathbb{P}\big(\mathrm{curl}\,(v_n + q_n(v_n) - \alpha\Delta(v_n + q_n(v_n))) \times (v_n + q_n(v_n))\big) = P_n\mathbb{P}f.$$

Part I: Finite number of determining modes

The property of "finite number of determining modes" was proved, by Foias and Prodi in 1967, for the 2D Navier-Stokes equations.

Theorem (3: Finite number of determining modes, PRR)

If $\nu - 4\alpha(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$ *and if* $f \in H^{1+d}_{per}$, $d > 0$, *then* (1) *has the property of finite number of determining modes, that is, there exists a positive integer* N_0 *such that, for any* u_0, u_1 *in* V^3, *the property*

$$\|P_{N_0}S_\alpha(t)u_0 - P_{N_0}S_\alpha(t)u_1\|_{V^3} \longrightarrow_{t\to+\infty} 0$$

implies that

$$\|S_\alpha(t)u_0 - S_\alpha(t)u_1\|_{V^3} \longrightarrow_{t\to+\infty} 0 \; .$$

Like in [Hale, GR 2003], we deduce Theorem 3 from Theorem 2.

Part II: Comparison with 2D Navier-Stokes

Theorem (4: Comparison with 2D Navier-Stokes, PRR)

There exists $\alpha_m > 0$ *such that, if* $f \in H^m_{per}$ *and* $u_0 \in V^{m+2}$*, we have, for* $0 \leq s \leq 2$ *and* $0 < \alpha \leq \alpha_m$*,*

$$\|S_\alpha(t)u_0 - S_0(t)u_0\|^2_{V^{s+m-1}} + \alpha\|S_\alpha(t)u_0 - S_0(t)u_0\|^2_{V^{s+m}} \leq \alpha^{2-s} \exp K(\|f\|^2_{H^m}, \|u_0\|^2_{V^{m+2}})$$

$m = 1$ (resp. $m = 2$) $\Rightarrow$ estimate in V^ℓ, $\ell < 2$ (resp. $\ell < 3$).

Corollary

Let $f \in H^m_{per}$, $m \geq 1$. *Then, the global attractors* $\mathcal{A}_\alpha$ *are upper-semicontinuous in* V^ℓ, $0 \leq \ell < m + 1$*, that is,*

$$\lim_{\alpha \to 0} \sup_{u_\alpha \in \mathcal{A}_\alpha} \inf_{u \in \mathcal{A}_0} \|u_\alpha - u\|_{V^\ell} = 0$$

Theorem 4 leads to compare the dynamics of $S_\alpha(t)$ and $S_0(t)$

Part II: Persistence of periodic orbits

Assume that $f \in H^m_{per}$ is chosen so that the 2D Navier-Stokes system $S_0(t)$ admits a periodic orbit

$$\Gamma_0 = \{p_0(t) = S_0(t)p_0(0) \,|\, 0 \leq t \leq \omega_0\}$$

where $p_0(t)$ is periodic of (minimal) period $\omega_0 > 0$. Suppose that Γ_0 is non-degenerate. (Existence of periodic orbits: Yudovich; Iooss; Chen and Price, Com. Math. Physics, 1999)

Question : Does $S_\alpha(t)$ admit a periodic orbit $\Gamma_\alpha = \{p_\alpha(t) = S_\alpha(t)p_\alpha(0)|0 \leq t \leq \omega_\alpha\}$ close to Γ_0 of minimal period ω_α close to ω_0? Is this periodic orbit unique?

Definition: $p_0(t)$ is a non-degenerate or simple periodic solution of period ω_0 if 1 is an isolated (algebraically) simple eigenvalue of the period map $\Pi_0(T_0, 0) \equiv D_u(S_0(\omega_0)p_0(0))$.

General classical Poincaré method

The classical method for showing persistence of non-degenerate orbits is the well-known Poincaré method.

Two difficulties:

- $S_\alpha(t) : V^3 \to V^3$ is not Hölderian in the time variable;
- $S_\alpha(t)$ is not a regular perturbation of $S_0(t)$.

But $S_\alpha(t)$ is asymptotically smoothing and the periodic orbits are smoother.

General method (J. Hale, G.R.),

Ingredients:

1. Reinterpret the Poincaré method as a Lyapunov-Schmidt method,
2. Use the fact that the periodic orbits are more regular
3. Work with two spaces $Z_\alpha \subset X_\alpha$ with compact injection.
4. For the existence of the periodic solution $p_\alpha(t)$, apply the Leray-Schauder fixed point theorem.

Modified Poincaré method

Let $X_\alpha = V^3$ and $Z_\alpha = V^5$ equipped with the norms

$$\|u\|_{X_\alpha} = \|u\|_{V^2} + \alpha^{1/2}\|u\|_{V^3}, \quad \|u\|_{Z_\alpha} = \|u\|_{V^4} + \alpha^{1/2}\|u\|_{V^5}$$

Theorem (5: Persistence of periodic orbits, Hale, R.)

Suppose that $f \in H^3_{per}$ and the Navier-Stokes system has a simple periodic orbit $\Gamma_0 = \{p_0(t) = S_0(t)p_0(0) | 0 \le t \le \omega_0\}$ of (minimal) period $\omega_0 > 0$ Then, there exist positive constants $\alpha_0 > 0$, R_0 and η_0, s. t., for $0 < \alpha \le \alpha_0$, $S_\alpha(t)$ has a unique periodic orbit $\Gamma_\alpha = \{p_\alpha(t) = S_\alpha(t)p_\alpha(0) | 0 \le t \le \omega_\alpha\}$ of minimal period ω_α s. t.

$$\Gamma_\alpha \subset \mathcal{N}_{X_\alpha}(\Gamma_0, \eta_0) \cap \mathcal{N}_{Z_\alpha}(0, R_0) \ , \quad |\omega_\alpha - \omega_0| \le \eta_0 \ .$$

And $(p_\alpha(0), \omega_\alpha)$ goes to $(p_0(0), \omega_0)$ as α goes to 0 (Estimates).

Corollary

There exist $f \in H^3_{per}$ and $\alpha_0 > 0$, s. t., for $0 \le \alpha \le \alpha_0$, (1) has at least a periodic solution of minimal period $\omega_\alpha \neq 0$.

Part II: Further comparison results

1. Similar results (simpler) for equilibria
2. comparaison of the local stable and unstable manifolds of equilibria (difficulties)
3. comparaison of the local stable and unstable manifolds of periodic orbits (in progress)

Part III : Rotating second grade fluids in $\mathbb{T}^3$

The system of rotating second grade fluid is given by

$$\begin{aligned} &\partial_t(u_\varepsilon - \alpha\Delta u_\varepsilon) - \nu\Delta u_\varepsilon + \mathrm{curl}\,(u_\varepsilon - \alpha\Delta u_\varepsilon) \times u_\varepsilon + \frac{e_3 \times u_\varepsilon}{\varepsilon} \\ &\qquad\qquad = \nabla p_\varepsilon + f, \qquad (4) \\ &\mathrm{div}\, u_\varepsilon = 0, \\ &u_\varepsilon(0,x) = u_0, \end{aligned}$$

where $\mathbb{T}^3 = \prod_{i=1}^{i=3}(0, 2\pi a_i)$ and e_3 is the unit vector in the vertical direction. We introduce the vertical average operator M.

Theorem (6: Global existence, B. Jaffal)

For a.e. (a_1, a_2, a_3) (non-resonant case), one has the following existence result. For any $f \in L^2(\mathbb{R}^+, H^1_{per}) \cap H^1(\mathbb{R}^+, L^2_{per})$, s.t. $Mf \in L^2(\mathbb{R}^+, H^2_{per})$, for any $u_0 \in V^3$, there exist α_0 and ε_0, s. t., for $\alpha \leq \alpha_0$, $\varepsilon \leq \varepsilon_0$, (4) has a unique global solution $u_\varepsilon \in L^\infty(\mathbb{R}^+, V^3) \cap L^2(\mathbb{R}^+, V^3)$ (Estimates).

Part III : Rotating fluid (continued)

1) When α is large, one obtains a global existence result under a smallness condition on the vertical components of Mf and of Mu_0.
2) In the proof of Theorem 6, one uses the filtered vector field $v_\varepsilon = L_\alpha(\frac{-t}{\varepsilon})u_0$, where $u = L_\alpha(t)u_0$ is the solution of

$$\partial_t(u - \alpha\Delta u) + \mathbb{P}(e_3 \times u) = 0 \ , \quad u(0) = u_0.$$

As ε goes to 0, $(v_\varepsilon)_\varepsilon$ strongly converges to a vector field v and Mv satisfies the system of 3 equations defined on $\mathbb{T}^2$:

$$\partial_t(Mv - \alpha\Delta_h Mv) - \nu\Delta_h Mv + \mathbb{P}(\mathrm{curl}\,(Mv - \alpha\Delta_h Mv) \times Mv) = \mathbb{P}(Mf), \tag{5}$$

and

$$\mathrm{div_h}\, Mv = 0 \ , \quad Mv(0) = Mu_0, \tag{6}$$

where Δ_h and $\mathrm{div_h}$ are the horizontal Laplacian and divergence.

Part III : Rotating fluid (continued)

3) In the case of rotating Navier-Stokes equations, one shows global existence of solutions for any size of initial data and forcing terms, provided that ε is small enough (Babin, Mahalov and Nicolaenko, 1997 - Gallagher, 1998).

4) Open Problem : For α large, does the limiting system (5), (6) admit a (unique) global solution for initial data and forcing terms of any size? (This is true for the corresponding limiting system in the Navier-Stokes case)

**Alles Gute
zum Geburtstag!**

Happy Birthday!