Persistence of periodic orbits in perturbed dissipative evolutionary equations

(Joint work with Jack Hale)

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Dedicated to Jack Hale on the occasion of his 80th birthday

Introduction

Let X be a Banach space and $S_0(t) : X \mapsto X$ be a dynamical system.
We assume that $S_0(t)$ admits a periodic orbit $p_0(t)$ of (least) period T_0 and that this periodic orbit is non-degenerate.

For $\varepsilon > 0$ small, let $S_\varepsilon(t) : X \mapsto X$ be a perturbed dynamical system, such that $S_\varepsilon(t)$ "converges" in some sense to $S_0(t)$ when $\varepsilon \to 0$.

Question Does $S_\varepsilon(t)$ admit a periodic orbit $p_\varepsilon(t)$ close to $p_0(t)$ of period T_ε close to T_0? Is this periodic orbit unique?

An abstract setting

$$S_0(t): \qquad u_t = A_0 u(t) + G_0(u(t)) \,, \;\; u(0) = u_0 \in X, \quad (1)$$

and

$$S_\varepsilon(t): \qquad u_t = A_\varepsilon u(t) + G_\varepsilon(u(t)) \,, \;\; u(0) = u_\varepsilon \in X, \quad (2)$$

where, for $\varepsilon \geq 0$, A_ε is the generator of a C^0-semigroup and either $G_\varepsilon : X \to Y$, where $Y \hookrightarrow X$ or $G_\varepsilon : X \to Z$ where $X \hookrightarrow Z$. We assume that $A_\varepsilon \to A_0$ and $G_\varepsilon \to G_0$.

Definition: $p_0(t)$ is a non-degenerate periodic solution of (least) period T_0 if 1 is an isolated (algebraically) simple eigenvalue of the period map $U_0(T_0, 0)$ where $U_0(\sigma, s)w^s = w(\sigma)$ is the solution of

$$w_t = A_0 w + DG_0(p_0(t))w \,, \;\; w(s) = w^s \in X. \qquad (3)$$

Classical result

Classical theorem If (1) and (2) are finite-dimensional systems of ODE's, if A_ε and G_ε are continuous in ε and if $p_0(t)$ is a non-degenerate periodic orbit of (1) of period T_0, then there exist $\varepsilon_0 > 0$ and $r > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, (2) has a unique periodic solution $p_\varepsilon(t)$ of period T_ε with $|T_\varepsilon - T_0| \leq r$ and $\|p_0(0) - p_\varepsilon(0)\| \leq r$. And $p_\varepsilon(t)$ is continuous in ε.

Proof: Poincaré method or Lyapunov-Schmidt method ....

The same theorem holds in the case of parabolic equations.

Persistence of periodic orbits in equations of retarded type and neutral type (J. Hale and M. Weedermann, JDE, 2004)

Problems arising in infinite dimensions

- In general, given $u_0 \in X$, $t \in \mathbb{R} \mapsto S_\varepsilon(t)u_0 \in X$ is not a Hölder-continuous map.

-The interesting perturbations are not regular (not continuous in ε)

Goals:

-To generalize the previous theorem to dissipative systems (2 methods).

-To describe the local unstable (stable) manifolds of $\Gamma = \{p_0(t) | t \in [0, T_0)\}$

Part I: Examples

Example 1: A system of damped wave equations

$\Omega \subset \mathbf{R}^n$ is a bounded smooth domain.
$X = (H^1(\Omega) \times L^2(\Omega))^2$, $f_i : (u,v) \in H^1(\Omega) \times H^1(\Omega) \mapsto H^s(\Omega)$, $0 < s < 1$, $i = 1, 2$.
The dynamical system $S_0(t)$ is defined by

$$\begin{aligned} u_{tt} + \beta u_t - \Delta u + \alpha u &= f_1(u,v) \\ v_{tt} + \beta v_t - \Delta v + \alpha u &= f_2(u,v) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \\ (u, u_t, v, v_t)(0,x) &= (u_0, u_1, v_0, v_1) \in X \ . \end{aligned} \tag{4}$$

where $\alpha > 0, \beta > 0$. It can be written as a first order system

$$w_t = A_0 w(t) + G_0(w)\,,\ \ w(0) = w_0 \in X,\ \text{ where}$$

$G_0 : w \in X \mapsto G_0(w) = (0, f_1, 0, f_2) \in Y = (H^{s+1}(\Omega) \times H^s(\Omega))^2$
The embedding of Y in X is compact.

Properties of Example 1

1. $S_0(t)$ is asymptotically smooth (a.s.): $\Rightarrow$ Every bounded invariant set B is relatively compact
2. Regularity in space and time of bounded invariant sets
   If $w(t)$ is a bounded solution for $t \in (-\infty, T)$, then, for $t \leq T$,

   $$w(t) = \int_{-\infty}^{t} e^{A_0(t-s)} G_0(w(s)) ds \in (H^{s+1} \times H^s)^2,$$

   Same regularity properties for linearized equations along smooth global bounded solutions and thus for the map $U(T_0, 0)$ if we have a periodic orbit.
   Same regularity for the eigenfunctions of $U(T_0, 0)$ associated with eigenvalues λ, $|\lambda| \geq 1$.

Examples of perturbations of $S_0(t)$

1. Regular perturbations: perturbations $f_i(\varepsilon, u, v)$ or $\gamma(\varepsilon)$
2. Non regular perturbations: perturbations of the domain Ω
   Thin domains: $Q_\varepsilon = \{(x, y) \,|\, x \in \Omega, 0 < y < \varepsilon h(x)\}$

   $h(x) = 1$: R. Johnson, M. Kamenski, P. Nistri, JDDE 1998
   General h, $n = 1$: B. Abdelhedi, 2005

   The thin domain perturbation is singular since we can only estimate $\|S_\varepsilon(t)w_0 - S_0(t)w_0\|_{\tilde{X}}$ for $w_0 \in \tilde{Y}$ where $\tilde{Y} \subsetneq \tilde{X}$.

Example 2: Hyperbolic perturbation of the Navier-Stokes equations

We consider the equations defined on $\Omega = \mathbf{T}^2$:

$$\begin{aligned} \varepsilon u^\varepsilon_{tt} + u^\varepsilon_t - \nu \Delta u^\varepsilon &= -\nabla p^\varepsilon - u^\varepsilon \cdot \nabla u^\varepsilon + f \ , \\ \mathrm{div} u^\varepsilon &= 0, \\ (u^\varepsilon, u^\varepsilon_\tau)(0,y) &= (u_0(y), u_1(y)) \ . \end{aligned}$$

$S_\varepsilon(t)$ is a local dynamical system on $X = V^{s+1}(\Omega) \times V^s(\Omega)$, for $s \geq 0$, where $V^s = \{w \in H^s(\Omega)^2 \,|\, \text{div } w = 0, \ w \text{ is } 2\pi\text{-periodic}\}$.
When ε goes to 0, we obtain the system $S_0(t)$ generated by the Navier-Stokes equations

$$\begin{aligned} v_\tau - \nu \Delta v &= -\nabla p - v \cdot \nabla v + f \ , \\ \mathrm{div} v &= 0, \\ v(0,y) &= u_0 \ . \end{aligned}$$

Existence of periodic orbits: Chen and Price, Com. Math. Physics, 1999.
Paicu, R.

Other examples

Example 3: Second grade fluids Let $\Omega = \mathbf{T}^2$. For $\alpha > 0$, $S_\alpha(t)$ is a global dynamical system on V^3,

$$\partial_t(u - \alpha\Delta u) - \nu\Delta u + \mathrm{curl}(u - \alpha\Delta u) \times u + \nabla p = -\nabla p + f,$$

$$\mathrm{div} u = 0,$$

$$u(0,x) = u_0(x),$$

where $\mathrm{curl}\, u = (0, 0, \partial_1 u_2 - \partial_2 u_1)$ if $u = (u_1, u_2)$.

1. $S_\alpha(t)$ is not smoothing in finite time.
2. $S_\alpha(t)$ is a non regular perturbation of $S_0(t)$.
3. $S_\alpha(t)$ is a.s. and the bounded invariant sets are more regular.

Example 4: A system of weakly damped Schrödinger equations

Summary

In these examples, $S_\varepsilon(t)$ has no Hölder-regularity in time and the convergence of $S_\varepsilon(t)$ to $S_0(t)$ is not good. BUT

1. $S_\varepsilon(t)$ is a.s. due to dissipation (second grad fluids or Schrödinger equations) or to compactness of the non-linearity (damped wave equation).
2. The bounded invariant sets of $S_\varepsilon(t)$ are more regular.
3. $S_\varepsilon(t)$ can be compared to $S_0(t)$ for smoother data,

$$\|S_\varepsilon(t)u - S_0(t)u\|_X \leq C(t, \|u\|_Y)\varepsilon^d$$

Part II. A first method : An integral or functional method

$$S_\varepsilon(t): \qquad u_t = A_\varepsilon u(t) + G(u)\,,\ u(0) = u_\varepsilon \in X. \qquad (5)$$

We assume that

- (A.1) $G \in C^2(X, Y)$, where $Y \subsetneq X$ and X, Y are Hilbert spaces.
- (A.2) A_ε is the generator of a C^0-semigroup on X and Y and

$$\|e^{A_\varepsilon t}\|_{L(X,X)} + \|e^{A_\varepsilon t}\|_{L(Y,Y)} \le Ce^{-\alpha t}\ ,\ \alpha > 0$$

- (H.1) $S_0(t)$ admits a non-degenerate periodic solution $p_0(t)$ of period T_0 and, for $t \ge 0$,

$$\|p_0(t)\|_{\tilde{Y}} + \|\dot{p}_0(t)\|_{\tilde{Y}} \le R_0\ ,$$

  where $\tilde{Y} \subset D((A_0)^a)$, $a > 0$.
- (H.2) If $U(T_0, 0)^*\varphi_0^* = \varphi_0^*$ and $\langle \dot{p}_0, \varphi_0^* \rangle = 1$, then

$$\|\varphi_0^*\|_{\tilde{Y}} \le C_0.$$

- (H.3) One has the estimate

$$\|e^{A_\varepsilon t} - e^{A_0 t}\|_{L(Y,X)} \leq C\varepsilon^d,$$

where $d > 0$.

- (H.4) $t \mapsto e^{A_0 t} D_u G(p_0)$ is locally Hölder-continuous from $\mathbf{R}^+$ into $L(X, X)$ and $t \mapsto e^{A_0 t} G(p_0)$ is locally Lipschitz-continuous from $\mathbf{R}^+$ into X.
- (H.5) $|\langle A_\varepsilon e^{A_\varepsilon t} G(w), \varphi_0^* \rangle| \leq C\|w\|_X$, for any $w \in X$.

Theorem

Under the hypotheses (A.1), (A.2), (H.1) to (H.5), there exist $\varepsilon_0 > 0$ and $r > 0$, such that, for $0 < \varepsilon \leq \varepsilon_0$, there exists a unique periodic solution $p_\varepsilon(t)$ of Eq.(5) with period T_ε such that $|T_\varepsilon - T_0| \leq r$ and $\|p_0(t) - p_\varepsilon(t)\|_X \leq r$. Moreover, p_ε converges to p_0 and T_ε converges to T_0 when $\varepsilon \to 0$ (estimates).

Consequence: Existence (and uniqueness) of perturbed periodic orbits of period close to T_0 for systems of damped wave equations on thin domains.

Proof

Ingredients:

- Use of the variation of constants formula (Krasnoselskii, Zabreiko, Pustylnik and Sobolevskii, 1966; Gurova and Kamenskii, 1996, R. Johnson, M. Kamenski, P. Nistri, JDDE 1998, etc..)
- Lyapunov-Schmidt method
- Strict contraction fixed point theorem

Idea: If $p_\varepsilon(t)$ is a periodic orbit of (5) of period T_ε, then,

$$p_\varepsilon(0) = p_\varepsilon(T_\varepsilon) = e^{A_\varepsilon T_\varepsilon} p_\varepsilon(0) + \int_0^{T_\varepsilon} e^{A_\varepsilon (T_\varepsilon - s)} G(p_\varepsilon(s)) ds$$

or

$$\begin{aligned} p_\varepsilon(0) =& (I - e^{A_\varepsilon T_\varepsilon})^{-1} \int_0^{T_\varepsilon} e^{A_\varepsilon (T_\varepsilon - s)} G(p_\varepsilon(s)) ds \\ p_\varepsilon(t) =& e^{A_\varepsilon t} (I - e^{A_\varepsilon T_\varepsilon})^{-1} \int_0^{T_\varepsilon} e^{A_\varepsilon (T_\varepsilon - s)} G(p_\varepsilon(s)) ds + \int_0^{t} e^{A_\varepsilon (t - s)} G(p_\varepsilon(s)) ds \end{aligned}$$

Change of time scaling $t \to \frac{T}{T_0} t$ leads the equation

$$\tilde{u}_t = \frac{T}{T_0} A_\varepsilon \tilde{u}(t) + \frac{T}{T_0} G(\tilde{u}) \,, \;\; \tilde{u}(0) = \tilde{u}_\varepsilon \in X. \qquad (6)$$

We set:

$\Sigma_{\varepsilon,T}(t) = e^{\frac{T}{T_0} A_\varepsilon t}$, $C^{per}_{T_0} = \{w \in C^0(X,X) | w \text{ is } T_0\text{-periodic}\}$.

$$(J_\varepsilon(T)w)(t) = \Sigma_{\varepsilon,T}(t)(I - \Sigma_{\varepsilon,T}(T_0))^{-1} \int_0^{T_0} \Sigma_{\varepsilon,T}(T_0 - s) w(s) ds + \int_0^t \Sigma_{\varepsilon,T}(t-s) w(s) ds$$

$$F_\varepsilon(T, w) = J_\varepsilon(T)(\frac{T}{T_0} G(w)) \; .$$

- *$\varphi \in C^{per}_{T_0}$ is a fixed point of $F_\varepsilon(T,.)$ iff φ is a periodic solution of (6).*
- *$D_u F_0(T_0, p_0)\dot{p}_0 = \dot{p}_0$, 1 is a (algebraically) simple eigenvalue, and $D_u F_0(T_0, p_0) - I \in L(C^{per}_{T_0}(X))$ is Fredholm of index 0.*
- *Good estimates of $F_\varepsilon - F_0$ and of $D_u F_\varepsilon - D_u F_0$.*

Let $v_0^* \in (C^{per}_{T_0}(X))^*$ such that $(D_u F_0(T_0, p_0) - I)^* v_0^* = 0$ and $\langle \dot{p}_0, v_0^* \rangle = 1$, then

$$C^{per}_{T_0}(X) = \{\dot{p}_0\} + Z \ , \quad Z = \{w \in C^{per}_{T_0}(X) \, / \, \langle w, v_0^* \rangle = 0\}$$

Goal: For $r > 0$ small, find $(T, \varphi) \in B_{\mathbf{R}}(T_0, r) \times B_{C^{per}_{T_0}(X)}(0, r)$, such that

$$F_\varepsilon(T, p_0 + \varphi) - (p_0 + \varphi) = 0 \ .$$

Use a Lyapunov-Schmidt method and solve

$$L_\varepsilon(\varphi, T) = F_\varepsilon(T, p_0 + \varphi) - (p_0 + \varphi) - \langle F_\varepsilon(T, p_0 + \varphi) - (p_0 + \varphi), v_0^* \rangle = 0$$
$$\langle F_\varepsilon(T, p_0 + \varphi) - (p_0 + \varphi), v_0^* \rangle = 0 \ .$$

Step 1: One shows that

$$\mathcal{L}_\varepsilon(\varphi, T) = \varphi - ((D_u F_0(T_0, p_0) - I)_{/Z})^{-1} L_\varepsilon(\varphi, T)$$

is a strict contraction in $B_Z(0, r)$, for $\varepsilon > 0$ and r small.
$\Rightarrow$ For ε small and T close to T_0, there exists a unique fixed point $\varphi(\varepsilon, T) \in B_Z(0, r)$ of $\mathcal{L}_\varepsilon(\varphi, T)$.

Step 2: One solves the equation

$$M_\varepsilon(T) = \langle F_\varepsilon(T, p_0 + \varphi(\varepsilon, T)) - (p_0 + \varphi(\varepsilon, T)), v_0^* \rangle = 0. \quad (7)$$

Under the hypothesis (H.5), M_ε is a strict contraction from $B_{\mathbf{R}}(0, r)$ into itself. Thus there exists a unique $T_\varepsilon \in B_{\mathbf{R}}(0, r)$ such that (7) holds. And $\varphi(\varepsilon, T_\varepsilon)$ is the T_0-periodic solution of (6).

Remarks:

1. If $p_0(t)$ is of class C^2, we show, without the hypothesis (H.5) that (7) has a solution in $B_{\mathbf{R}}(0, r)$ by the Schauder fixed point theorem. If we assume that all periodic orbits of (5) are of class $C^{1+\delta}$, $\delta > 0$, then one can prove the uniqueness.
2. In the case of thin product domains, R. Johnson, M. Kamenski and P. Nistri proved the existence of the perturbed periodic solution by using a topological degree argument.

Part III. A modified Poincaré method

Appropriate method if the non-linearity is non-compact, but dissipative.

Let $\tilde{Y} \hookrightarrow Y \hookrightarrow_{\text{compact}} X$ be Hilbert spaces. Let $S_\varepsilon(t)$, $\varepsilon \geq 0$, be a dynamical system on X, Y and $\tilde{Y}$. We assume (H.1), (H.2) and

- (H.3) $t \mapsto p_0(t)$ is of class $C^2(\mathbf{R}, X)$.
- (H.4) $r(\sigma_{\text{ess}}(U(T_0, 0))) < 1$.
- (H.5) There exist $R_1 \geq 2R_0$ and $0 < k_1 < 1$ s.t., for $T_0/2 \leq t \leq 2T_0$, $\|u\|_Y \leq R_1$, for $\varepsilon \geq 0$,

$$\|S_\varepsilon(t)u\|_Y \leq k_1 R_1.$$

- (H.6) There exists $\beta > 0$ s.t., for $R > 0$, $0 < t_0 < t \leq 2T_0$, for $w \in B_Y(0, R)$,

$$\|S_\varepsilon(t)w - S_0(t)w\|_X \leq \varepsilon^\beta K_0(R)$$
$$\|(DS_\varepsilon(t)w)u - (DS_0(t)w)u\|_X \leq \varepsilon^\beta K_0(R)\|u\|_Y.$$

- (H.7) Besides $\lambda_0 = 1$, there exist m distinct eigenvalues $\lambda_i \neq 1$ of $U(T_0, 0)$ with algebraic multiplicity d_i such that $|\lambda_i| \geq 1$. The corresponding (generalized) eigenvectors are in $\tilde{Y}$.

  There exists $0 < k_2 < 1$ s.t., if P_i is the spectral projection onto the eigenspace associated with λ_i, $i = 0, 1..., m$ and $Q = I - \sum_{i=0}^{m} P_i$, we have

$$\|U(T_0, 0)\varphi\|_X \leq k_2 \|\varphi\|_X \ , \quad \forall \varphi \in QX.$$

- (H.8) There exists $K > 0$ s.t., if $u_\varepsilon(t)$ is a bounded orbit of $S_\varepsilon(t)$, for $t \in \mathbf{R}$, then $u_\varepsilon(t)$ is bounded in $\tilde{Y}$ and

$$\sup_t \|u_\varepsilon(t)\|_{\tilde{Y}} \leq K \sup_t \|u_\varepsilon(t)\|_X.$$

  If $u_\varepsilon^1(t)$ and $u_\varepsilon^2(t)$ are bounded orbits of $S_\varepsilon(t)$, for $t \in \mathbf{R}$, then

$$\sup_{t \in \mathbf{R}} \|u_\varepsilon^1(t) - u_\varepsilon^2(t)\|_Y \leq K \sup_{t \in \mathbf{R}} \|u_\varepsilon^1(t) - u_\varepsilon^2(t)\|_X.$$

- (H.9) For any $\varphi \in \tilde{Y}$, the map $t \mapsto S_\varepsilon(t)\varphi$ is locally Lipschitz-continuous from $\mathbf{R}$ into X.

Second method: Results

Theorem

Under the hypotheses (H.1) to (H.9), there are $\varepsilon_0 > 0$ *and* $r > 0$*, such that, for* $0 < \varepsilon \leq \varepsilon_0$*, there exists a unique periodic orbit* $p_\varepsilon(t)$ *of* $S_\varepsilon(t)$ *with period* T_ε *such that* $|T_\varepsilon - T_0| \leq r$ *and* $\|p_0(0) - p_\varepsilon(0)\|_X \leq r$*. Moreover,* p_ε *converges to* p_0 *and* T_ε *converges to* T_0 *when* $\varepsilon \to 0$ *(estimates).*

Consequence: Existence (and uniqueness) of perturbed periodic orbits of period close to T_0 in the examples 2 to 4 (hyperbolic Navier-Stokes; second grade fluids, weakly damped Schrödinger equations).

Ingredients of the proof:

1. Lyapunov-Schmidt method (modified Poincaré method)
2. Schauder fixed point theorem (topological degree also?)
3. Uniqueness

Proof

We write $X = \{\dot{p}_0(0)\} \oplus Z_1 \oplus Z_2$ where $Z_1 = \oplus_{i=1}^m P_i X$, $Z_2 = QX$.
We recall that $P_0 w = \langle w, \varphi_0^* \rangle \dot{p}_0(0)$ and $\langle v, \varphi_0^* \rangle = 0$ if $v \in Z_1 \oplus Z_2$.
Goal: For $r > 0$ small, find $(T_\varepsilon, p_\varepsilon(0)) \in B_{\mathbf{R}}(T_0, r) \times B_X(p_0(0), r)$, such that

$$S_\varepsilon(T_\varepsilon)p_\varepsilon(0) = p_\varepsilon(0).$$

Step 1: We write: $p_\varepsilon(0) = p_0(0) + \varphi + \psi$, where $\psi = \sum_{j=1}^d \alpha_j \psi_j \in Z_1$, $\varphi \in Z_2$. For $r_2 > 0$ small, we set

$$\mathcal{B} = \{\varphi \in B_{Z_2}(0, r_2) | \varphi + p_0(0) \in B_Y(0, R_1)\},$$

and, for $\psi \in B_{Z_1}(0, r_1)$, $|T - T_0| < \eta$, $0 < \varepsilon \leq \varepsilon_0$, where ε_0, r_1, η are small, we define the map

$$\mathcal{L}_\varepsilon(T, \varphi, \psi) = (I - P_0 - \sum_{i=1}^m P_i)\big(S_\varepsilon(T)(p_0(0) + \varphi + \psi) - p_0(0)\big).$$

$\mathcal{L}_\varepsilon(T, \varphi, \psi) \in \mathcal{B}$. By Schauder fixed point theorem, there exists a fixed point $\varphi_\varepsilon(T, \psi) \in \mathcal{B}$ of the map $\mathcal{L}_\varepsilon(T, \cdot, \psi)$.

Proof (continued)

Step 2: For $|T - T_0| < \eta$, $0 < \varepsilon \le \varepsilon_0$, to find $\psi \in B_{Z_1}(0, r_1)$ s. t.

$$\mathcal{M}_\varepsilon(T, \psi) \equiv \sum_{i=1}^{m} P_i\big(S_\varepsilon(T)(p_0(0) + \varphi_\varepsilon(T, \psi) + \psi) - p_0(0)\big) = \psi$$

If $r_1 > 0$, $\eta > 0$ and $\varepsilon_0 > 0$ are small, $\mathcal{M}_\varepsilon(T, \psi) \in B_{Z_1}(0, r_1)$.
By Schauder fixed point theorem, the map $\mathcal{M}_\varepsilon(T, \cdot)$ has a fixed point $\psi_\varepsilon(T) \in B_{Z_1}(0, r_1)$.

Step 3: For $0 < \varepsilon \le \varepsilon_0$, to find T with $|T - T_0| < \eta$, s.t.

$$\langle S_\varepsilon(T_\varepsilon)\big(p_0(0) + \varphi_\varepsilon(T, \psi_\varepsilon(T)) + \psi_\varepsilon(T)) - p_0(0)\big), \varphi_0^*\rangle = 0,$$

or to find a fixed point τ, $|\tau| < \eta$ of the map

$$F_\varepsilon(\tau) = \tau \dot{p}_0(0) - \langle S_\varepsilon(T_0 + \tau)\big(p_0(0) + \varphi_\varepsilon(T_0 + \tau, \psi_\varepsilon(T_0 + \tau)) \\ + \psi_\varepsilon(T_0 + \tau)\big) - p_0(0), \varphi_0^*\rangle.$$

For $\varepsilon_0 > 0$ small enough, F_ε has a fixed point τ_ε, with $|\tau| < \eta$.

Proof (end)

Set $T_\varepsilon = T_0 + \tau_\varepsilon$. Thus, $S_\varepsilon(\cdot)$ has a periodic orbit

$$p_\varepsilon(t) = S_\varepsilon(t)\big(p_0(0) + \varphi_\varepsilon(T_\varepsilon, \psi_\varepsilon(T_\varepsilon)) + \psi_\varepsilon(T_\varepsilon)\big).$$

of period T_ε. Since $p_\varepsilon(t)$ is a periodic orbit of $S_\varepsilon(\cdot)$, $p_\varepsilon(t)$ is uniformly bounded in $\tilde{Y}$.

Step 4: Since $p_\varepsilon(t)$ is uniformly bounded in $\tilde{Y}$, we can use the hypotheses (H.8) and (H.9) to show the uniqueness of the periodic solution with period T_ε s.t. $|T_\varepsilon - T_0| < r$ and $\|p_0(0) - p_\varepsilon(0)\|_X \leq r$ (Strict contraction argument).

Part IV. Local unstable and stable manifolds

We take $\varepsilon = 0$ in Equation (5),

$$u_t = A_0 u + G(u) \ , \quad u(0) = u_0 \in X$$

and suppose:

- (A.1) and (A.2) hold,
- $p_0(t) \in C^2(\mathbf{R}, X)$,
- $r(\sigma_{\text{ess}}(U(T_0, 0))) < 1$
- the periodic orbit $\Gamma_0 = \{p_0(t) | t \in [0, T_0)\}$ is hyperbolic, i.e.

$$(\sigma(U(T_0, 0)) - \{1\}) \cap S^1 = \emptyset.$$

- $D_u G(p_0(t)) \in L(X, X)$ is a compact map, for $t \in [0, T_0)$.

Thus the index $i(\Gamma_0) \equiv i(U(T_0, 0))$, which is the number of eigenvalues λ of $U(T_0, 0)$ with $|\lambda| > 1$, is finite.

Let V be a neighbourhood of Γ_0, we define the local stable and unstable sets by

$$W^s_{loc}(\Gamma_0) \equiv W^s(\Gamma_0, V) = \{u \in X | S_0(t)u \in V, \forall t \geq 0; \lim_{t \to +\infty} \delta_X(S_0(t)u, \Gamma_0) = 0\},$$

$$W^u_{loc}(\Gamma_0) \equiv W^u(\Gamma_0, V) = \{u \in X | S_0(t)u \in V, \forall t \leq 0; \lim_{t \to -\infty} \delta_X(S_0(t)u, \Gamma_0) = 0\}$$

Theorem (J. Hale, R)

$W^s_{loc}(\Gamma_0)$ (resp. $W^u_{loc}(\Gamma_0)$) is a C^1-submanifold of codimension $i(\Gamma_0)$ (respectively of dimension $i(\Gamma_0) + 1$). Moreover, there exist positive constants α and β such that, for any u_0 in $W^s_{loc}(\Gamma_0)$ (resp. in $W^u_{loc}(\Gamma_0)$), there is a number τ_{u_0} (resp. θ_{u_0}) s.t.

$$\|S_0(t)u_0 - p_0(t + \tau_{u_0})\|_X \leq Ce^{-\alpha t}, \quad t \geq 0,$$

$$(\text{ resp. } \|S_0(t)u_0 - p_0(t + \theta_{u_0})\|_X \leq Ce^{\beta t}, \quad t \leq 0).$$

Corollary *Under the above hypotheses and the hypotheses of Part II, if* Γ_0 *is the periodic orbit of period* T_0 *of* $S_0(t)$ *and* Γ_ε *the periodic orbit of period* T_ε *of* $S_\varepsilon(t)$*, then*

$$dist_X(W^u_{loc}(\Gamma_0), W^u_{loc}(\Gamma_\varepsilon)) \leq C\varepsilon^d,$$

where $d > 0$*.*
(Particular case of thin domains has been proved by B. Abdelhedi (2005))

Ingredients of the proof:

- If V is a neighbourhood of Γ_0, we define the local synchronized stable and local synchronized unstable sets of $p_0(\theta_0) \in \Gamma_0$ as

$$\mathcal{W}^s_{\theta_0,loc}(\Gamma_0) \equiv \mathcal{W}^s_{\theta_0}(\Gamma_0, V) = \{u \in X | S_0(t)u \in V, \forall t \geq 0; \lim_{t\to+\infty} S_0(t)u - p_0(t+\theta_0) = 0\},$$

$$\mathcal{W}^u_{\theta_0,loc}(\Gamma_0) \equiv \mathcal{W}^u_{\theta_0}(\Gamma_0, V) = \{u \in X | S_0(t)u \in V, \forall t \leq 0; \lim_{t\to-\infty} S_0(t)u - p_0(t+\theta_0) = 0\}$$

We define the local synchronized stable and local synchronized unstable sets of Γ_0 as

$$\mathcal{W}^s_{loc}(\Gamma_0) = \cup_{\theta_0 \in [0,T_0)} \mathcal{W}^s_{\theta_0}(\Gamma_0, V)$$
$$\mathcal{W}^u_{loc}(\Gamma_0) = \cup_{\theta_0 \in [0,T_0)} \mathcal{W}^u_{\theta_0}(\Gamma_0, V).$$

and we show that

$$\mathcal{W}^s_{loc}(\Gamma_0) = W^s_{loc}(\Gamma_0) \ , \quad \mathcal{W}^u_{loc}(\Gamma_0) = W^u_{loc}(\Gamma_0).$$

- We introduce a new coordinates system around Γ_0. Every u in a small tubular neighbourhood of Γ_0 is written as

$$u = p_0(\theta) + Q(\theta)w \ , \|w\|_X \leq \delta \ ,$$

where $\delta > 0$ is small enough, $\theta \in R$, $w \in W$, W is a linear subspace of X of codimension 1, $Q(\theta)$ is a continuous T_0-periodic map of W into X and $Q(\theta)W$ is transversal to $\dot{p}_0(\theta)$ for all θ.

We obtain the new system

$$\dot{\theta} = 1 + g(w)$$
$$\dot{w} = \tilde{A}w + h(\theta, w)$$