Compact Global Attractors: Finite Number of Determining Modes and Regularity

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Dissipative systems

Many mathematical models in physics, chemistry or biology lead to systems of partial differential equations (PDE's) or retarded functional differential equations (RFDE's) with dissipative properties.

Very often, they generate an infinite-dimensional continuous semigroup $S(t)$ on a Banach space X and have a compact global attractor $\mathcal{A}$.

Two categories of systems

- Systems with smoothing or compactness properties in finite time, i.e. $S(t) : X \to X$ is a compact map, for $t \geq t_0 > 0$.
  Heat or 2D Navier-Stokes equations.

- Systems with only asymptotic smoothness or asymptotic compactness properties.
  Damped wave or damped Schrödinger equations.

Compact global attractors

X is a Banach space, $S(t) : X \to X$, $t \geq 0$, is a nonlinear semigroup (semiflow).
$S(t)$ has a compact global attractor $\mathcal{A}$ if

(a) $\mathcal{A}$ is a compact subset of X,

(b) $\mathcal{A}$ is invariant under $S(t)$, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for $t \geq 0$,

(c) $\mathcal{A}$ attracts every bounded set B of X, i.e. $\forall\, \varepsilon > 0$, $\forall$ bounded subset B of X, there exists $\tau = \tau(B, \varepsilon) \geq 0$ s.t.

$$S(t)B \subset \mathcal{N}_X(\mathcal{A}, \varepsilon),\ \forall t \geq \tau,$$

where $\mathcal{N}_X(\mathcal{A}, \varepsilon)$ is the ε-neighbourhood of $\mathcal{A}$ in X,

i.e. $\mathrm{dist}_X(S(t)B, \mathcal{A}) \to_{t \to +\infty} 0$, where $\mathrm{dist}_X(B_1, B_2) = \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \|b_1 - b_2\|_X$.

(a) and (b) imply $\mathcal{A} = \{u(t) \in \mathcal{C}_b^0(\mathbf{R}, X) \,|\, u(t)$ is a complete bounded orbit of $S(t)\}$.

Notation

X is a Hilbert space (for sake of simplicity), $S(t) : X \to X$, $t \geq 0$, denotes the continuous semigroup defined by the equation

$$\begin{aligned} \frac{du}{dt} &= Au + f(u) \equiv \mathcal{F}(u), \ t > 0, \\ u(0) &= u_0 \in X, \end{aligned}$$

where A is the generator of a (linear) $\mathcal{C}_0$ semigroup on X, $f \in \mathcal{C}^k(X, X)$, $k \geq 1$, or analytic.

We assume that this equation defines a non linear semigroup by $S(t)u_0 = u(t)$, where $u(t) \in \mathcal{C}^0([0, +\infty), X)$ is the mild solution

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(u(s))\, ds.$$

Part I

Properties of compact global attractors

The ultimate goal is the precise description of the flow on the global attractor. For the moment, this seems to be in general out of reach.

Three linked, modest questions:

- Does the semigroup $S(t)|_{\mathcal{A}}$ exhibit smoothness properties that are not shared by the semigroup $S(t)$ in general?
  Regularity in time and spatial variables?

- Can the asymptotic behaviour of the solutions be described by a finite number of degrees of freedom? Galerkin methods.

- Are compact global attractors robust objects with respect to perturbations? Structural stability.

Regularity on compact invariant sets

Importance of regularity in time or space variables:
-local study of periodic orbits, etc...
-structural stability properties,
-reduction to finite-dimensional problems.

ODE's on $\mathbf{R}^n$: every solution is as smooth in t as the vector field.
Parabolic type equations: same property is true for $t > 0$ and the solutions also enjoy regularity properties in the spatial variables.

[Time analyticity and Gevrey class regularity in space :
2D Navier-Stokes: Foias and Temam (1979, 1989),
General systems: Promislow (1991), Ferrari and Titi (1998)]

In our examples: we can expect smoothness in time only for solutions defined for all $t \in \mathbf{R}$ and contained in compact invariant sets.
To generalize regularity results of Hale and Scheurle (1985).

Finite-dimensional structure and Galerkin methods

Can we reduce the study of the flow on a compact invariant set $\mathcal{A}$ to the discussion of the flow of some system on a finite-dimensional space?

1. Finite number of determining modes

[Cesari (1964), Foias and Prodi for 2D Navier-Stokes (1967)]

Let P_n be the projection onto the space $V_n = P_n X$ generated by the first n eigenfunctions of A. There is n_0 so that, if $u_1(t)$, $u_2(t)$ are any two solutions satisfying

$$\|P_{n_0} u_1(t) - P_{n_0} u_2(t)\|_X \rightarrow_{t \rightarrow +\infty} 0,$$

then

$$\|u_1(t) - u_2(t)\|_X \rightarrow_{t \rightarrow +\infty} 0;$$

[Ladyzhenskaya (1972), Foias, Manley, Temam and Treve (1983), Jones and Titi (1993) for NS and parabolic-type equations Cockburn, Jones and Titi (1997), Oliver and Titi (1998) for the Schrödinger eq., Chueshov (1998)]

2. Ideal situation

To find a projection P onto a finite-dimensional subspace $PX \subset X$ s.t. $P|_{\mathcal{N}(\mathcal{A})}$ is invertible, where $\mathcal{N}(\mathcal{A})$ is a neighbourhood of $\mathcal{A}$ and to reduce the equation

$$u_t = \mathcal{F}(u),\ t > 0,\ u(0) = u_0,$$

to the finite-dimensional system for $v = Pu$

$$v_t = P\mathcal{F}((P)^{-1}v),\ v(0) = Pu_0.$$

If $\dim_F(\mathcal{A}) < +\infty$, such a projection P exists [Mañe (1981), Foias and Olson (1996)].
Unfortunately, $(P)^{-1}$ is only Hölder continuous. It may not define a flow.
[Eden, Foias, Nicolaenko and Temam (1994)]

Another approach:
To construct an inertial manifold $\mathcal{M}$ of $S(t)$, i.e. a finite-dimensional, smooth (at least $\mathcal{C}^1$), positively invariant (i.e. $S(t)\mathcal{M} \subset \mathcal{M}$) manifold $\mathcal{M} \subset X$, that contains $\mathcal{A}$.

Basic way for constructing inertial manifolds:
To obtain $\mathcal{M}$ as a smooth graph ($\mathcal{C}^1$) over the finite-dimensional space $V_n = P_n X$ and to apply the classical methods of center manifold theory.
$\mathcal{M} = \{u = v_n + \Phi(v_n), v_n \in V_n\}$, where $\Phi \in \mathcal{C}^1(V_n, Q_n X)$, $Q_n = Id - P_n$.

$\Rightarrow$ Inertial form

$$\frac{dv_n}{dt} = Av_n + P_n f(v_n + \Phi(v_n)).$$

$\Longrightarrow$ One encounters the same obstructions as in center manifold theory:
gap condition, cone condition.
These conditions are satisfied for some parabolic equations in 1D or special 2D domains.

[Foias, Sell and Temam (1988), Mallet-Paret and Sell (1988), etc...]

Here, we shall reduce the evolution equation to a finite-dimensional system of equations with delay.

3. New idea of low-dimensional reduction using rigorous computations and topological invariants (e.g. Conley index, degree theory) Cheap and accurate way to effectively compute fixed points, connecting orbits, periodic orbits etc...

[K. Mischaikow and P. Zgliczyński (2000) with applications to the Kuramoto-Sivashinsky equation]

Heuristically, one does not reduce the system

$$u_t = Au + f(u),$$

to an equivalent system, but rather replace it by the differential inclusion

$$\frac{dv_n}{dt} \in Av_n + P_n f(v_n + W_n^*),$$

$v_n \in V_n^* = P_n K$, $W_n^* = Q_n K$, $K \subset X$
Regularity condition: $\text{diam}(W_n^*) \to 0$ as $n \to +\infty$.

Main argument: the topological invariants are the same for any Galerkin system of the form $v_{j,t} \in Av_j + P_j f(v_j + W_j^*)$, where $v_j \in V_j^*$.

Structural stability

A basic problem in dynamics is to compare the flows defined by different semigroups.

Applications:

PDE's depending on several physical parameters, numerical approximations of systems....

Case of a finite-dimensional compact manifold $\mathcal{M}$:

two systems $S_1(t)$ and $S_2(t)$ are topologically equivalent on $\mathcal{M}$ if there exists a homeomorphism $h : \mathcal{M} \to \mathcal{M}$, which preserves orbits and the sense of direction in time.

$S_0(t)$ is stable (or structurally stable) if there exists a "neighbourhood" $\mathcal{N}_0$ of $S_0(t)$ s.t. any $S_1(t) \in \mathcal{N}_0$ is topologically equivalent to $S_0(t)$.

Infinite-dimensional case:

the strongest expected comparison of the dynamics of two semigroups $S_1(t)$ and $S_2(t)$ is the topological equivalence restricted to the compact attractors $\mathcal{A}_1$ and $\mathcal{A}_2$ ($h : \mathcal{A}_1 \to \mathcal{A}_2$).

Morse-Smale property:
compactness of the global attractor $\mathcal{A}$ (or of the maximal bounded invariant set)
hyperbolicity of equilibrium points and periodic orbits
transversality of the stable and unstable manifolds of them.

Morse-Smale property $\Rightarrow$ Structural stability.
[Palis (1969), Palis and Smale (1970),
Oliva (1982), Hale, Magalhães and Oliva (1984), Oliva (2000) in the infinite-dimensional case].

Transversality properties are very difficult to show in the infinite-dimensional case. It is easier in the context of gradient systems.

The semigroup $S(t)$ on X is a gradient system if it admits a Lyapunov functional $\Phi \in \mathcal{C}^0(X, \mathbf{R})$ s.t.
$\Phi(S(t)u_0) \leq \Phi(S(s)u_0)$, $t \geq s$,
and $\Phi(S(t)u_0) = \Phi(u_0)$ for any $t \geq 0$ implies that u_0 is an equilibrium point.

We consider the damped wave equation in a bounded domain $\Omega \subset \mathbf{R}^n$, $n = 1, 2, 3$, $t > 0$, where $\gamma > 0$,

$$U_{tt} + \gamma U_t - \Delta U + U = F(x, U)$$

with either homogeneous Dirichlet or homogeneous Neumann boundary conditions.

Theorem (Brunovsky and G.R. (2001)).
Generically in $\gamma \in \mathbf{R}^+$ and $F \in \mathcal{C}^k(\overline{\Omega} \times \mathbf{R})$, the semigroup generated by the damped wave equation has the Morse-Smale property and thus is structurally stable.

Main ingredients:
Generalized Sard theorem,
Analyticity in time of $S(t)|_{\mathcal{A}}$ when F is replaced by an analytic function

Genericity in $F(x, .)$ of the Morse-Smale property for the semilinear heat equation was proved by Brunovsky and Polacik in 1997.
The Morse-Smale property is always true for the semilinear heat equation in the one-dimensional case (n=1). [Henry (1985), Angenent (1986)]

Part II

Galerkin method and regularity

[Hale and G.R.]

We go back to the equation

$$\frac{du}{dt} = Au + f(u),\ t > 0,\ u(0) = u_0 \in X, \qquad (1)$$

where A is the generator of a (linear) $\mathcal{C}_0$ semi-group on X, $f \in \mathcal{C}^k(X, X)$, $k \geq 1$, or analytic.

We want to perform a Galerkin-type method to reduce the problem to a finite-dimensional system with delay and to prove regularity in time.

We do the following hypotheses:

(H1) $S(t)$ has a compact invariant set $\mathcal{A}$.

(H2) There exists an orthogonal decomposition of X into generalized eigenspaces of A, i.e. there is an orthogonal projection P_n s.t.

- $P_n A = A P_n$ on $\mathcal{D}(A)$,
- $P_n \to Id$ strongly as $n \to +\infty$,
- $\|P_n\|_{L(X,X)} \leq K_0\,,\ \forall\, n \in \mathbf{N}$.

(H3) $r(\sigma_{\mathrm{ess}}(e^{At})) \leq e^{-\delta_1 t}\,,\ t \geq 0\,,$
i.e. there exist an integer n_1 and $\delta_1 > 0$, $K_1 > 0$ s.t., for $t > 0$,

$$\|e^{At} Q_{n_1} u\|_X \leq K_1 e^{-\delta_1 t} \|Q_{n_1} u\|_X,\ \forall\, u \in X,$$

where $Q_n = Id - P_n$.

(H4) The set $\{Df(u_1)u_2 \,|\, u_1 \in \mathcal{A},\ \|u_2\|_X \leq 1\}$ is relatively compact in X.

Results of Hale and Scheurle [1985]:
If (H1), (H3) hold and if

$$\|Df(u)\|_{L(X,X)} \leq \eta$$

for u in some neighbourhood of $\mathcal{A}$, where $\eta > 0$ is small enough, then, for $u_0 \in \mathcal{A}$, the mapping $t \in \mathbf{R} \to S(t)u_0$ is as smooth as f.

Generalization:
Let $n \geq n_1$.
If $u(t) = P_n u(t) + Q_n u(t) \equiv v(t) + w(t)$ is a solution of (1), then (v, w) is a solution of the system

$$\begin{aligned} \frac{dv}{dt} &= Av + P_n f(v + w)\,, \\ \frac{dw}{dt} &= Aw + Q_n f(v + w)\,. \end{aligned}$$

If $u(\mathbf{R}) \subset \mathcal{A}$, (H3) implies that

$$w(t) = \int_{-\infty}^{t} e^{A(t-s)} Q_n f(v(s) + w(s))\, ds\,.$$

Let $d > 0$, we introduce the neighbourhoods

$$\mathcal{V}_n(d) = \mathcal{N}_{P_n X}(P_n \mathcal{A}, d)\,, \; \mathcal{W}_n(d) = \mathcal{N}_{Q_n X}(0, d)\,.$$

Theorem 1. Assume that (H1), (H2), (H3), (H4) hold.
For each d, $0 < d \le d_0$, (d_0 small enough), there exist an integer $N_0(d)$ and, for $n \ge N_0(d)$, a unique Lipschitz-continuous function

$$\mathcal{C}_b^0(\mathbf{R}, \mathcal{V}_n(d)) \to \mathcal{C}_b^0(\mathbf{R}, \mathcal{W}_n(d))\,, v \mapsto w^*(v)\,,$$

solution of

$$\frac{dw^*(v)}{dt} = Aw^* + Q_n f(v + w^*(v))\,.$$

The mapping $w^*(v)(t)$ depends only upon $v(s)$, $s \le t$ and $w^*(v)(t)$ is as smooth in v and t as f.

Given $v \in \mathcal{C}_b^0(\mathbf{R}, \mathcal{V}_n(d))$, $w^*(v)$ is the unique fixed point of the map T_v (strict contraction) from $\mathcal{C}_b^0(\mathbf{R}, \mathcal{W}_n(d))$ into itself, defined by

$$T_v(w)(t) = \int_{-\infty}^{t} e^{A(t-s)} Q_n f(v(s) + w(s))\, ds\,.$$

Consequences

We can choose $N_1 \geq N_0$, s.t., for any solution $u(t) \in \mathcal{A}$ of (1),

$$w(t) = Q_n u(t) \text{ is in } \mathcal{W}_n(d_0)\,, \,\forall n \geq N_1\,,$$

and thus, by uniqueness of the solutions of (1),

$$u(t) = v(t) + w^*(v)(t)\,,$$

where $v(t) = P_n u(t)$ satisfies the system of RFDE's (infinite delay)

$$v_t = Av + P_n f(v + w^*(v))\,. \qquad (2)$$

Thus, the flow on $\mathcal{A}$ is determined by the first N_1 modes.

Theorem 2. Suppose that (H1), (H2), (H3), (H4) hold and that $\mathcal{A}$ is the compact global attractor of (1). If $u_1(t)$ and $u_2(t)$ are two solutions of (1), not necessarily in $\mathcal{A}$, satisfying

$$\|P_{N_1}u_1(t) - P_{N_1}u_2(t)\|_X \to_{t\to+\infty} 0\,,$$

then,

$$\|u_1(t) - u_2(t)\|_X \to_{t\to+\infty} 0\,.$$

Theorem 3. Assume that (H1), (H2), (H3), (H4) hold.
Suppose that $AP_n \in L(X, X)$ and that f is in $\mathcal{C}^k_{bu}(X, X)$, $k \geq 1$, (resp. analytic), then, for any $u_0 \in \mathcal{A}$, $t \to S(t)u_0$ is in $\mathcal{C}^k_{bu}(\mathbf{R}, X)$ (resp. is analytic).

Regularity in the spatial variables

Under additional non restrictive hypotheses on f, one shows that the elements $u_0 \in \mathcal{A}$ have the same regularity in the spatial variables as the elements of P_nX (generalized eigenfunctions). Gevrey regularity, Analyticity.

Application: damped wave equation

Generalizations: The conditions $AP_n = P_nA$ and P_n being a projection can be weakened.

Case of PDE's in unbounded domains:
[P. Collet, G.R. and E. Titi (2001)]

An extension

The compactness assumption (H4) is rather strong. It does not hold for the Schrödinger equation.
$\Rightarrow$ To try to relax (H4) by a weaker condition involving a non-autonomous evolution system.

Remark:
If $u(t) \in \mathcal{A}$ is a (classical) solution of (1), then $w(t) = Q_n u(t)$ satisfies the equation

$$\frac{dw}{dt} = (A + Q_n Df(v))w + Q_n H(v, w)\,,$$

where $v = P_n u$ and, by Taylor's formula,

$$\begin{aligned} H(v,w) &= f(v+w) - Df(v)w \\ &= f(v) + \int_0^1 (Df(v+\sigma w) - Df(v))w d\sigma. \end{aligned}$$

Assume that $A + Q_n Df(v(t))Q_n$ generates a linear evolutionary operator $S_n(v,t,s)$ on $Q_n X$, with appropriate decay properties, then

$$w(t) = \int_{-\infty}^{t} S_n(v,t,s) Q_n H(v(s), w(s))\, ds\,.$$

New Hypotheses:

(H5) $Df : X \to L(X;X)$ is Lipschitz-continuous on the bounded sets of X.

(H6) $AP_n \in L(X, X)$.

(H7) There exist positive numbers d_2, δ_2, K_2 and an integer $n_2 \geq n_1$ s.t., for $n \geq n_2$, for $v(t) \in \mathcal{C}^0_{bu}(\mathbf{R}, \mathcal{V}_n(d)) \cap \mathcal{C}^1_{bu}(\mathbf{R}, P_n X)$, for $u \in X$ and $t > s$,

$$\|S_n(v, t, s)Q_n u\|_X \leq K_2 e^{-\delta_2(t-s)} \|Q_n u\|_X .$$

Theorem 4. The statements of Theorems 1, 2 and 3 still hold if Hypothesis (H4) is replaced by the Hypotheses (H5),(H6) and (H7).

As in Theorem 1, for $v \in \mathcal{C}^0_{bu}(\mathbf{R}, \mathcal{V}_n(d))$, $w^*(v)$ is the unique fixed point of the strict contraction $\mathcal{T}_v$ from $\mathcal{C}^0_b(\mathbf{R}, \mathcal{W}_n(d))$ into itself, defined by

$$\mathcal{T}_v(w)(t) = \int_{-\infty}^{t} S_n(v,t,s) Q_n H(v(s), w(s))\, ds\,.$$

Application to the Schrödinger equation:
$\mathcal{C}^k$-regularity (resp. analyticity) in time on the attractor $\mathcal{A}$ if f is $\mathcal{C}^k$ (resp. analytic).
Finite number of determining modes [Oliver and Titi (1998)]
$\mathcal{C}^k$-regularity in the spatial variables [Goubet (1996),(1998)]
Gevrey class regularity [Oliver and Titi (1998)]