Dynamics of second grade fluids: the Lagrangian approach

M. Paicu and G. Raugel

This paper is dedicated to Professor Jürgen Scheurle on the occasion of his sixtieth birthday.

Abstract This article is devoted to the mathematical analysis of the second grade fluid equations in the two-dimensional case. We first begin with a short review of the existence and uniqueness results, which have been previously proved by several authors. Afterwards, we show that, for any size of the material coefficient $\alpha > 0$, the second grade fluid equations are globally well posed in the space $V^{3,p}$ of divergence-free vectors fields, which belong to the Sobolev space $W^{3,p}(\mathbb{T}^2)^2$, $1 < p < +\infty$, where $\mathbb{T}^2$ is the two-dimensional torus. Like previous authors, we introduce an auxiliary transport equation in the course of the proof of this existence result. Since the second grade fluid equations are globally well posed, their solutions define a dynamical system $S_\alpha(t)$. We prove that $S_\alpha(t)$ admits a compact global attractor $\mathscr{A}_\alpha$ in $V^{3,p}$. We show that, for any $\alpha > 0$, there exists $\beta(\alpha) > 0$, such that $\mathscr{A}_\alpha$ belongs to $V^{3+\beta(\alpha),p}$ if the forcing term is in $W^{1+\beta(\alpha)}(\mathbb{T}^2)^2$. We also show that this attractor is contained in any Sobolev space $V^{3+m,p}$ provided that α is small enough and the forcing term is regular enough. The method of proof of the existence and regularity of the compact global attractor is new and rests on a Lagrangian method. The use of Lagrangian coordinates makes the proofs much simpler and clearer.

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1 Introduction

In petroleum industry, in polymer technology, in problems of liquid crystals suspensions, non-Newtonian (also called Rivlin-Ericksen) fluids of diffferential type often arise. The constitutive law of incompressible homogeneous fluids of grade 2 is given by

$$\sigma = -pI + 2\nu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2,$$

where σ is the Cauchy tensor, A_1 and A_2 are the first two Rivlin-Ericksen tensors:

$$A_1(u) = \frac{1}{2}[\nabla u + \nabla u^T], \quad A_2(u) = \frac{DA_1}{Dt} + (\nabla u)^T A_1 + A_1(\nabla u)$$

and

$$\frac{D}{Dt} = \partial_t + u.\nabla.$$

is the material derivative.
In 1974, Dunn and Fosdick ([16]) established that a fluid modelled by the above relations is compatible with thermodynamics (that is, the Clausius-Duhem inequality and the assumption that the Helmholtz free energy is a minimum, when the fluid is at rest) if the following conditions

$$\alpha_1 + \alpha_2 = 0, \quad \alpha_1 \geq 0\ ,$$

are imposed.
Writing then the equation $\frac{Du}{Dt} = u_t + u.\nabla u = \operatorname{div}\sigma$, one obtains the second grade fluid equations (2) below.

If $\alpha_1 \geq 0$, the fluid has asymptotic stability properties. In [18], it was showed that if $\alpha_1 + \alpha_2$ is arbitrary and $\alpha_1 < 0$, then the second grade fluid has an anomalous behaviour (unstable behaviour). There has been an extensive discussion on the modelling of the second grade fluids and on the restrictions, which have to be imposed on the coefficients α_1 and α_2 (see [16], [18], [17], for example).

If one does not impose the condition $\alpha_1 + \alpha_2 = 0$, the system of second grade can be written as

$$\begin{aligned}
&\partial_t(u - \alpha_1 \Delta u) - \nu \Delta u + \operatorname{rot}(u - (2\alpha_1 + \alpha_2)\Delta u) \times u \\
&\qquad\qquad + (\alpha_1 + \alpha_2)(-\Delta(u \cdot \nabla u) + 2u \cdot \nabla(\Delta u)) + \nabla p = f, \quad t > 0,\ x \in \Omega, \\
&\operatorname{div} u = 0, \quad t > 0,\ x \in \Omega, \\
&u(0,x) = u_0(x), \quad x \in \Omega\ ,
\end{aligned} \tag{1}$$

where $\alpha_1 \geq 0$. When $\alpha_1 + \alpha_2 = 0$, setting $\alpha = \alpha_1$, we obtain the system of second grade fluids in the simplified form,

$$\begin{aligned}&\partial_t(u-\alpha\Delta u)-\nu\Delta u+\operatorname{rot}(u-\alpha\Delta u)\times u+\nabla p=f,\quad t>0,\ x\in\Omega,\\&\operatorname{div}u=0,\quad t>0,\ x\in\Omega,\\&u(0,x)=u_0(x),\quad x\in\Omega\ ,\end{aligned}\tag{2}$$

where Ω is either a bounded simply connected regular enough domain in $\mathbb{R}^d$, or the d-dimensional torus $\mathbb{T}^d$, $d=2,3$. In the two-dimensional case, we use the convention that $\operatorname{rot}u\equiv\operatorname{curl}u=(0,0,\partial_1u_2-\partial_2u_1)$ and we identify each 2-component vector-field $u=(u_1,u_2)$ with the 3-component vector field $u=(u_1,u_2,0)$ and each scalar m with the 3-component vector field $w=(0,0,m)$. If Ω is a bounded domain in $\mathbb{R}^d$, the equations (2) are completed with boundary conditions. In most of the papers, one assumes that the fluid adheres to the boundary $\partial\Omega$, that is, one requires homogeneous Dirichlet boundary conditions

$$u(x,t)=0\ ,\quad t>0,\ x\in\partial\Omega\ .\tag{3}$$

The condition (3) is sufficient to determine a unique (local) solution of the system (2) despite the fact that the non-linearity in (2) contains derivatives of higher order than 2. One can also consider the system (2) with non-homogeneous Dirichlet boundary conditions

$$u(x,t)=g(x,t)\ ,\quad t>0,\ x\in\partial\Omega\ ,\tag{4}$$

where g must satisfy the compatibility condition $\int_{\partial\Omega}g\cdot n ds=0$, n being the outward normal to the boundary $\partial\Omega$. For such boundary conditions, in the case of three-dimensional bounded domains Ω, Galdi et al. [26] proved the existence of local solutions of the system (2). They showed the uniqueness of the solutions, when the boundary is impermeable, that is, when $g\cdot n\equiv 0$. In the case where the boundary is impermeable, Girault and Scott [27] proved the existence of stationary solutions, when Ω is a two-dimensional domain. Under additional smallness conditions on the data, Girault and Scott obtained the uniqueness of the stationary solutions. The second grade fluid model with fully non-homogeneous Dirichlet boundary conditions is actually not well posed. For example, Gupta and Rajagopal [29] have given examples in which the stationary problem has multiple solutions. For this reason, it is important to require that $g\cdot n\equiv 0$.
C. Le Roux [45] has studied the system (2) subject to non-linear partial slip boundary conditions in a bounded simply-connected domain in $\mathbb{R}^3$. Under appropriate growth restrictions on the data, he has proved the existence and uniqueness of a classical solution.

Before describing the contents of this paper, we briefly recall the main known existence and uniqueness results of the solutions of (2) in the case of the homogeneous Dirichlet boundary conditions (3). Since there are many papers devoted to this case, we cannot quote all of them. In particular, we will not recall the results concerning the stationary solutions (see for example, [7], [4], [21], [24], [27]).
The first general existence and uniqueness results of solutions of (2) are due to Cioranescu and Ouazar in 1984 (see [13] and [14]). Assuming that the initial

data u_0 belong to the space $W = \{v \in H^3(\Omega)^d \,|\, \mathrm{div}\, v = 0, v_{|\partial\Omega} = 0\}$ and that the force f belongs to $L^2((0,T),H^1(\Omega)^d)$ and using a Galerkin method with a special basis, Cioranescu and Ouazar proved that (2) has a unique (weak) solution $u \in L^\infty((0,T^*),W) \cap W^{1,\infty}((0,T^*),W')$, where $T^* = T$ in the case $d = 2$ and $0 < T^* \le T$ in the case $d = 3$. Later, in 1997, Ciorasnecu and Girault [12] completed these results by showing the global existence of weak solutions in the three-dimensional case under the assumption that the data are small enough, and showed that these solutions are more regular if the data are smoother. In 1993, Galdi, Grobbelaarvandalsen and Sauer ([25]) have shown the local existence and uniqueness of classical solutions of (1) and also the global existence of solutions of (1) under a smallness condition of the data, when α_1 is large enough. These local and global existence of classical solutions results have been improved in 1994 by Galdi and Sequeira ([23]) and, in particular, the requirement that α be large enough has been removed. In [25] (resp. [23]), the local (or global) existence of classical solutions has been proved by writing an equation for the auxiliary variable $v = u - \alpha\Delta u$ (resp. $v = curl(u - \alpha\Delta u)$) and by applying the Leray-Schauder fixed point theorem. For instance, in [23], assuming that the forcing term f vanishes and that $v_0 = curl(u_0 - \alpha\Delta u_0)$ belongs to $X_m = \{v \in H^m(\Omega)^3 \,|\, \mathrm{div}\, v = 0\}$, $m \ge 1$, the authors have proved the local existence and uniqueness of the solution u of (2) in $C^0((0,T),X_{m+2}) \cap L^\infty((0,T),H^{m+3}(\Omega)^3)$ with $\frac{du}{dt} \in L^\infty((0,T),H^{m+2}(\Omega)^3)$, where $T > 0$. Under a smallness condition on u_0, they proved that the solution is global.

Later in 1998, Bernard ([3]) has generalized the existence result of (weak) solutions of (2) to the system (1) by using a Galerkin method with the special basis as in [12] or [14]. Roughly speaking, assuming that f and u_0 belong to $L^1((0,+\infty),H^1(\Omega)^3) \cap L^\infty((0,+\infty),L^2(\Omega)^3)$ and to W respectively and are both small enough, Bernard has proved the global existence and uniqueness of a solution $u \in L^\infty((0,+\infty),W)$ and that $\frac{du}{dt} \in L^\infty((0,+\infty),H^1(\Omega)^3)$.

Also in 1998, in the three-dimensional case, Bresch and Lemoine ([8]) have obtained the existence and uniqueness of solutions of (2), when $f \in L^r((0,T),L^r(\Omega)^3)$ and u_0 is a divergence-free vector field in $X_1 \cap W^{2,r}(\Omega)^3$, where $r > 3$. More precisely, under these hypotheses, they showed that there exists a (unique) solution $u(t) \in C^0([0,T^*],W^{2,r}(\Omega)^3 \cap X_1)$, with $\frac{du}{dt} \in L^r((0,T^*),W^{1,r}(\Omega)^3)$ where $0 < T^* \le T$. If $f \in L^\infty((0,+\infty),W^{1,r}(\Omega)^3))$ and $u_0 \in X_1 \cap W^{2,r}(\Omega)^3$ are small enough and if α is larger than a constant depending only on r and Ω, then the solution $u(t)$ is global and belongs to $C^0_b([0,+\infty),W^{2,r}(\Omega)^3 \cap X_1)$, with $\frac{du}{dt} \in L^\infty((0,+\infty),W^{1,r}(\Omega)^3)$. In their proof, given u, the authors introduce the unique solution w of the linear equation $w_t + (\nu/\alpha)w + u \cdot \nabla w + \nabla u \cdot w = (\nu/\alpha)u + f$, with $w(0) = u(0) - \alpha\Delta u(0)$. Then, they consider the unique solution (z,π) of the "Stokes" problem $z - \alpha\Delta z + \nabla\pi = w$, where z is divergence-free and the mean value of π vanishes. Finally, applying the Leray-Schauder fixed point theorem, they show that the map $u \mapsto z$ has a fixed point. Of course, arguing in the same way, one can prove similar existence and uniqueness results when Ω is a bounded domain in $\mathbb{R}^2$ and $r > 2$. One notices that Bresch and Lemoine have used a similar strategy in [9] to prove the existence and uniqueness of a solution for third grade fluids. For other existence results in $W^{2,r}(\Omega)^3$, $r > 3$, see also [6].

In 2007, Girault and Saadouni ([28]) considered the equations (2) on a two-dimensional Lipschizian domain Ω. They proved the existence of a weak solution of (2) and obtained the uniqueness of the solution if Ω is a convex polygon. Introducing the auxiliary variable $z = \mathrm{rot}(u - \alpha\Delta u)$, they have replaced the system (2) by the equivalent system

$$\begin{aligned}
&\partial_t(u-\alpha\Delta u) - \nu\Delta u + z\times u + \nabla p = f, \quad t>0,\ x\in\Omega,\\
&\alpha\partial_t z + \nu z + \alpha u\cdot\nabla z = \alpha \mathrm{rot} f + \nu \mathrm{rot} u, \quad t>0,\ x\in\Omega,\\
&\mathrm{div}\, u = 0, \quad t>0,\ x\in\Omega,\\
&u(0,x) = u_0(x), \quad x\in\Omega\ .
\end{aligned} \tag{5}$$

They proved the existence of a (weak) solution by using a semi-discretization in time of the system (5).
For the asymptotic behaviour in time of the solutions of (2), when Ω is replaced by $\mathbb{R}^2$ (respectively $\mathbb{R}^3$), we refer the reader to [40], [41] and [15] (respectively to [15] and [51]). Additional interesting related results about non-Newtonian second grade or third grade fluids are contained in [5, 22, 39, 48, 57, 59].

We would like to notice that the equations (2) differ from the so-called α-Navier-Stokes system (see e.g. [20] and the references therein). Indeed, the α-Navier-Stokes model contains the strong regularizing term $-\nu\Delta(u-\alpha\Delta u)$ instead of $-\nu\Delta u$, and thus is a semilinear problem, which is much easier to solve than the second grade fluid equations where the dissipation is weaker.

In the inviscid case $\nu = 0$, the local existence and uniqueness of regular solutions still hold and, in the two-dimensional case, these solutions are global (see [10] for example). For the convergence of the solutions u_ν of (2) towards the solution u^* of the equation (2) for $\nu = 0$, when ν goes to zero, we refer the reader to [60]. For additional results in the inviscid case, we also refer to [47].

Until now, only few papers have been devoted to the dynamics of second grade fluids. In 1998, Moise, Rosa and Wang ([50]) have considered the second grade fluid equations (2) with time-independent forcing term $f \in H^1(\Omega)^2$, where Ω is a bounded simply-connected domain in $\mathbb{R}^2$. In this case, we can introduce the dynamical system $S_\alpha(t)$ on W, defined by $S_\alpha(t)u_0 = u(t)$, where $u(t)$ is the solution of (2). Moise, Rosa and Wang have shown that the map $u_0 \in W \mapsto S_\alpha(t)u_0 \equiv u(t) \in W$ is continuous and that every solution u of (2) belongs to $C^0([0,+\infty),W)$. Applying the method of functionals of J. Ball, they have proved that $S_\alpha(t)$ is asymptotically compact in W, which implies, since S_α has an absorbing set in W, that $S_\alpha(t)$ admits a compact global attractor $\mathscr{A}_\alpha$ in W (for the notions of asymptotic compactness and absorbing set, see Section 3 below).
In [55], Paicu, Raugel and Rekalo have proved that there exists a positive constant $\delta = \delta(\alpha, \|f\|_{H^1})$ such that the compact global attractor $\mathscr{A}_\alpha$ in W is actually bounded in $H^{3+\delta}(\Omega)^2$, when f belongs to $H^{1+\delta}(\Omega)^2$. Moreover, $\mathscr{A}_\alpha$ is bounded in $H^{3+m}(\Omega)^2$, $m \geq 0$, provided α is small enough and f belongs to $H^{1+m}(\Omega)^2$. They have also shown that, on the attractor, the second grade fluid equations (2) reduce

to a finite number of ordinary differential equations with an infinite delay term ([55, Section 5]). From these properties, they deduced that, as for the Navier-Stokes equations, the property of finite number of determining modes holds. Let us recall that the global attractor contains all the interesting asymptotic dynamics, in particular the equilibrium points and the periodic orbits. We would like to emphasize that the regularity property of the global attractor has important consequences. For example, they allow to prove persistence of non-degenerate equilibrium points or periodic orbits, when various parameters in the system (2) vary, such as the coefficient α or the domain Ω (see [35], [36], [40], [49]). In particular, if the Navier-Stokes equations admit a non-degenerate periodic orbit of minimal period $\omega > 0$, using these regularity properties, one obtains that, for $\alpha > 0$ small enough, (2) has a unique periodic orbit, which is close to the corresponding one of the Navier-Stokes equations ([35]) and has minimal period ω_α close to ω. If Ω is a three-dimensional bounded domain, there exists a compact attractor if f is small enough. But this attractor is a local one, since we do not know if the solutions exist globally for any size of the initial data. Thus, the study of this (local) attractor is less interesting. The above-mentioned regularity properties are certainly still true for the local attractor.

In this paper, we consider the equations of second grade, when the forcing term belongs to $L^\infty((0,+\infty), W^{1,p}(\Omega)^2)$ and the initial data are divergence-free and belong to $W^{3,p}(\Omega)^2$, where $p > 1$. First, we prove the existence and uniqueness of the weak solution of (2), give some a priori estimates and show that the equations (2) generate a dynamical system $S_\alpha(t)$ on the subspace of divergence free vector fields of $W^{3,p}(\Omega)^2$. In Section 3, we show that the dynamical system $S_\alpha(t)$ admits a compact global attractor $\mathscr{A}_\alpha$, which is bounded in a more regular space. We prove the existence and regularity of $\mathscr{A}_\alpha$ by using the Lagrangian coordinates. By adopting the Lagrangian approach, we simplify the previous proofs of the existence and the regularity of the compact global attractor.
For the sake of simplicity, we will only prove the results in the case where $\Omega = \mathbb{T}^2$.

Before we briefly describe these results, we introduce the needed notation. We denote $V^{m,p}$, $m \in \mathbb{N}$, $p \geq 1$, the closure of the space

$$\{u \in [C^\infty(\mathbb{T}^2)]^2 \,|\, u \text{ is periodic }, \operatorname{div} u = 0, \int_{\mathbb{T}^2} u\,dx = 0\},$$

in $W^{m,p}(\mathbb{T}^2)^2$. If $p = 2$, we set $V^m \equiv V^{m,2}$ and we simply write $H = V^0$. We equip the space $V^{m,p}$ with the classical $W^{m,p}(\mathbb{T}^2)^2$-norm, denoted $\|\cdot\|_{V^{m,p}} \equiv \|\cdot\|_{W^{m,p}}$. We will also use the usual $L^2(\mathbb{T}^2)^2$-scalar product $(\cdot,\cdot)$.
Finally, we denote $W^{m,p}_{per} \equiv W^{m,p}_{per}(\mathbb{T}^2)^2$ the space of vector fields $u \in W^{m,p}(\mathbb{T}^2)^2$, which are periodic and whose mean value vanishes.
If $m \in \mathbb{N}$, we define the spaces $W^{-m,p}_{per}$ as the dual space of W^{m,p^*}_{per}, where $\frac{1}{p} + \frac{1}{p^*} = 1$.

As several authors have already done it (see e.g. [12], [23],[27], [45], etc ..), we consider the auxiliary variable $\omega = curl(u - \alpha\Delta u) \equiv \operatorname{rot}(u - \alpha\Delta u)$. Applying the curl (also called rotational) operator to the first equation in (2), we formally obtain the equation

$$\partial_t \omega + \frac{\nu}{\alpha}\omega + u \cdot \nabla \omega = \mathrm{rot}\, f + \frac{\nu}{\alpha}\mathrm{rot}\, u, \quad t > 0,\ x \in \Omega. \tag{6}$$

We thus replace the system (2) by the following system

$$\begin{aligned}
&\partial_t \omega + \frac{\nu}{\alpha}\omega + u \cdot \nabla \omega = \mathrm{rot}\, f + \frac{\nu}{\alpha}\mathrm{rot}\, u, \quad t > 0,\ x \in \Omega,\\
&\omega(0,x) = \mathrm{rot}\,(u_0(x) - \alpha \Delta u_0(x)), \quad x \in \Omega\ ,\\
&\omega = \mathrm{rot}\,(u - \alpha \Delta u), \quad t > 0,\ x \in \Omega,\\
&\mathrm{div}\, u = 0, \quad t > 0,\ x \in \Omega,
\end{aligned} \tag{7}$$

where $\mathrm{rot}\, f \in L^\infty((0,+\infty), L^p_{per})$ and $u_0 \in V^{3,p}$.
In Section 2, we will prove that (2) (or (7)) has a solution u by showing that the map $J : u \in L^\infty((0,+\infty), V^{3,p}) \mapsto \omega \mapsto z \in L^\infty((0,+\infty), V^{3,p})$ has a fixed point, where, given u, ω is the solution of the affine equation (6) and z is the solution of the equation $\omega = \mathrm{rot}\,(z - \alpha \Delta z)$. The fixed point is obtained by applying the Leray-Schauder fixed point theorem and "adopting a Lagrangian point of view". As it was recalled in the above lines, the idea of applying the Leray-Schauder fixed point theorem is not new (however, the existence result below is new to our knowledge). Elementary a priori estimates will show that the solution u is unique in $L^\infty((0,+\infty), V^{2,2})$. This will lead us to the following theorem.

Theorem 1. *(i) Assume that $p > 1$ and that the forcing term f is in $L^\infty((0,+\infty), W^{1,p}_{per})$. Then, for every $u_0 \in V^{3,p}$, there exists a unique solution $u(t)$ of the equations* (2) *such that $u(t) \in C^0([0,+\infty), V^{3,p})$ and $\frac{d}{dt}u(t) \in L^\infty((0,+\infty), V^{2,p})$. Moreover, for any $t \geq 0$, the map $u_0 \in V^{3,p} \mapsto u(t) \in V^{3,p}$ is continuous.*
(ii) Likewise, if f belongs to $L^\infty(\mathbb{R}, W^{1,p}_{per})$, then, for every $u_0 \in V^{3,p}$, there exists a unique solution $u(t)$ of the equations (2) *such that $u(t) \in C^0(\mathbb{R}, V^{3,p})$ and $\frac{d}{dt}u(t) \in L^\infty((0,+\infty), V^{2,p}) \cap L^\infty_{loc}(\mathbb{R}, V^{2,p})$. Moreover, for any $t \in \mathbb{R}$, the map $u_0 \in V^{3,p} \mapsto u(t) \in V^{3,p}$ is continuous.*

More precise upper bounds of the solutions are given in Section 2.4.

Likewise, by adopting the Lagrangian point of view, one could also prove the existence of a unique solution $u(t)$ and the boundedness of it, when the viscosity ν vanishes. We will give the details in this case in a subsequent paper.

Assume now that $f \in W^{1,p}_{per}$ is time-independent, then (2) is an autonomous system and the map $S_\alpha(t) : u_0 \in V^{3,p} \mapsto S_\alpha(t)u_0 \equiv u(t) \in V^{3,p}$ (where $u(t)$ is the solution of (2)) is a dynamical system and even a non-linear continuous group, that is, $S_\alpha(t)$ has the following properties

1. $S_\alpha(t)S_\alpha(s) = S_\alpha(t+s)$, for any $t,s \in \mathbb{R}$,
2. $u_0 \in V^{3,p} \mapsto S_\alpha(t)u_0 \equiv u(t) \in V^{3,p}$ is continuous from $V^{3,p}$ into $V^{3,p}$, for any $t \in \mathbb{R}$,
3. $t \mapsto S_\alpha(t)u_0 \in V^{3,p}$ belongs to $C^0(\mathbb{R}, V^{3,p})$, for any $u_0 \in V^{3,p}$.

The proof of Theorem 1 implies that $S_\alpha(t)$ admits a bounded absorbing set, that is, there exists a bounded set $\mathscr{B}_\alpha$ in $V^{3,p}$, such that, for any bounded set $B \in V^{3,p}$, there exists a time $\tau(B)$ such that, for $t \geq \tau(B)$,

$$S_\alpha(t)B \subset \mathscr{B}_\alpha \ .$$

From the proof of Theorem 1, one deduces that, in the case where f is time-independent, $\frac{d}{dt}u$ belongs to $C^0(\mathbb{R}, V^{2,p})$, which allows to state the following corollary.

Corollary 1. *Assume that $p > 1$ and that the forcing term $f \in W^{1,p}_{per}$ is time-independent, then for every $u_0 \in V^{3,p}$, there exists a unique solution $u(t)$ of the equations* (2) *such that $u(t) \in C^0(\mathbb{R}, V^{3,p})$ and $\frac{d}{dt}u(t) \in C^0(\mathbb{R}, V^{2,p}) \cap L^\infty((0,+\infty), V^{2,p}) \cap L^\infty_{loc}(\mathbb{R}, V^{2,p})$. Moreover, the dynamical system $S_\alpha(t)$ admits a bounded absorbing set in $V^{3,p}$.*

A dynamical system which has a bounded absorbing set, is called *bounded dissipative* (for further details, see [31], [32] or [58] for example). If a dynamical system is bounded dissipative, one may wonder if it has also asymptotic compactness properties, which will imply that it admits a compact global attractor (see [32, Theorem 3.4.6] or [58, Theorem 2.26] for example). Before stating the existence theorem of a compact global attractor, we recall its definition.

Definition 1. Let X be a Banach space and $S(t)$ be a dynamical system on X. A compact set $\mathscr{A} \in X$ is a compact *global attractor* if

- $\mathscr{A}$ is invariant, that is, $S(t)\mathscr{A} = \mathscr{A}$, for any $t \geq 0$,
- $\mathscr{A}$ attracts all bounded sets of X, that is, for any $\varepsilon > 0$, for any bounded set B in X, there exists a time $T = T(\varepsilon, B)$ such that

$$S(t)B \subset \mathscr{N}_X(\mathscr{A};\varepsilon) \ , \quad \text{for any t } \geq T \ ,$$

  where $\mathscr{N}_X(\mathscr{A};\varepsilon)$ denotes the ε-neighbourhood of $\mathscr{A}$ in X.

The compact global attractor plays an important role, since all the asymptotic (and interesting) dynamics are contained in it. In Section 3, we are going to show that $S_\alpha(t)$ is *asymptotically smooth* or *asymptotically compact*.
We recall that a dynamical system $S(t)$ on a Banach space X is *asymptotically compact* (or *asymptotically smooth*; for an equivalent definition of asymptotic smoothness, see [32, Chapter 3.2] or [58, Definition 2.12 and Proposition 2.15] or [33]) if, for any bounded subset B of X such that $\cup_{t \geq 0} S(t+\tau)(B)$ is bounded for some $\tau \geq 0$, every set of the form $\{S(t_n)z_n\}$, with $z_n \in B$ and $t_n \geq \tau$, $t_n \to_{n\to+\infty} +\infty$ is relatively compact in X.

Since the equation (2) is fully non-linear (and not only semi-linear), the asymptotic compactness of $S_\alpha(t)$ is not straighrforward. In [50], for the case $p = 2$, Moise, Rosa and Wang had proved it by using the method of functionals of J. Ball. Here, using the Lagrangian point of view, we will be able to write $S_\alpha(t)$ as the sum $S_\alpha(t)u_0 = \Sigma_\alpha(t)u_0 + K_\alpha(t)u_0$, where $\Sigma_\alpha(t)u_0$ is a map, which is "asymptotically contracting" on $V^{3,p}$ and $K_\alpha(t)$ is a compact map from $V^{3,p}$ into itself (see Section 3 for more details). This property implies by [32, Lemma 3.2.6] or [58, Theorem

2.31]) that $S_\alpha(t)$ is asymptotically compact. Since $S_\alpha(t)$ is also bounded dissipative, [32, Theorem 3.4.6] or [58, Theorem 2.26] then imply that $S_\alpha(t)$ has a compact global attractor in $V^{3,p}$, which is also connected.

Theorem 2. *For $p > 1$, if the forcing term f is time-independent and belongs to $W^{1,p}_{per}$, then $S_\alpha(t)$ admits a compact global attractor $\mathscr{A}_\alpha$ in $V^{3,p}$ and $\mathscr{A}_\alpha$ is connected.*

The fact that $S_\alpha(t)$ is a non-linear group prevents smoothing properties in finite time. Thus, in view of the applications (persistence of equilibrium points, of periodic orbits, of local stable and unstable manifolds under perturbations of the equations (2), it is interesting to know if the elements or trajectories on the global attractor $\mathscr{A}_\alpha$ are more regular.
Numerous authors have shown regularity properties of the compact global attractor in the case of dynamical systems which are not smoothing in finite time. Such results were obtained already more than thirty years ago for retarded functional differential equations in $\mathbb{R}^n$ with finite delay or neutral functional differential equations by Hale [30] and Nussbaum [53]. For dissipative evolutionary equations, which admit a compact global attractor, regularity results have later been proved by several authors, using different methods (see [34] and [55] for references). We recall that one of the first regularity results applicable to partial differential evolutionary equations, has been shown by Hale and Scheurle [37] in 1985, who considered the equation

$$\dot{u} = Au + f(u) \ , \quad u(0) = u_0 \in X \ , \tag{8}$$

on a Banach space X, where A is the generator of a (linear) C^0 semi-group and $f(\cdot)$ is a smooth map on X. It is known that, for any $u_0 \in X$, there exists a unique local mild solution $u(t) \in C^0([0,T);X)$ of (8). Let us assume that all the solutions exist on $[0,+\infty)$. Then, (8) defines a dynamical system $S(t)$ on X, given by $S(t)u_0 = u(t)$ where $u(t)$ is the solution of (8). Hale and Scheurle have proved that, if $S(t)$ has a compact invariant set $\mathscr{J}$ in X, then there exists a positive number η such that, if $\|Df(v)\|_{L(X,X)} \leq \eta$ for any v in a small neighborhood of $\mathscr{J}$, the mapping $t \in \mathbb{R} \to S(t)u \in X$, for any $u \in \mathscr{J}$, is as smooth as f. The smoothness in the time variable implies smoothness in the spatial variable if (8) is the abstract version of a PDE. In particular, if the restriction of $S(t)$ to $\mathscr{J}$ is of class C^1, then $\mathscr{J}$ is bounded in the domain $D(A)$, which usually is a smoother space than X.

The system of second grade (2) is more complex than the abstract equation (8) and one cannot deduce spatial regularity properties from the time regularity results. In [60], using Lagrangian coordinates, Shkoller has proved time regularity properties of all the solutions of (2). However, from these time regularity results, one cannot deduce spatial regularity properties.

In [55, Section 2], in the special case where $p = 2$, we have proved the regularity of the attractor $\mathscr{A}_\alpha$ by establishing a series of appropriate a priori estimates for the solutions of the linear equation (which is the analogous of the transport equation (6))

$$\begin{aligned}
&\partial_t(w^* - \alpha\Delta w^*) - \nu\Delta w^* + \mathrm{rot}\,(w^* - \alpha\Delta w^*) \times u^* + \nabla p^* = f, \quad t > 0,\ x \in \mathbb{T}^2, \\
&\mathrm{div}\, w^* = 0, \quad t > 0,\ x \in \mathbb{T}^2, \\
&w^*(0,x) = u_0(x), \quad x \in \mathbb{T}^2\ ,
\end{aligned} \tag{9}$$

where $f \in H^{m+1}_{per}$ and $u^* \in L^\infty((0,+\infty), V^{m+2}) \cap C^0([0,+\infty), V^2)$ and by using the decomposition of $S_\alpha(t)u_0$ into $S_\alpha(t)u_0 = v_n(t) + (S_\alpha(t)u_0 - v_n(t))$, where $v_n(t)$ is the solution at time t of the equations (2) satisfying $v(s_n) = 0$ and where s_n is a sequence converging to $-\infty$. In the course of this proof, we have obtained "good" estimates of the size of the elements of $\mathscr{A}_\alpha$ in various norms. However, the proofs were long.
Here, using the system (7) for $p > 1$ and the Lagrangian coordinates, we are proving the regularity of $\mathscr{A}_\alpha$ in a more elegant way (see Section 3). Notice that, in the case $m = 1$ below, we recover the same condition as in [55, Theorem 1.1]. In the case $m > 1$, we obtain a better condition for the regularity than in [55, Theorem 1.1].

Theorem 3. *Let $p > 1$.*
1) Let $f \in W^{2,p}_{per}$. Assume that $\sup_{v \in \mathscr{A}_\alpha} \|\nabla v\|_{L^\infty} < \frac{\nu}{\alpha}$ and let $a_1 \equiv \frac{\nu}{\alpha} - \sup_{v \in \mathscr{A}_\alpha} \|\nabla v\|_{L^\infty} > 0$. Then, the following upper bound holds for any u belonging to the global attractor

$$\|\nabla(\mathrm{rot}\, u - \alpha\Delta \mathrm{rot}\, u)\|_{L^p} \leq a_1^{-1}(\|\mathrm{rot} f\|_{W^{1,p}} + \frac{\nu}{\alpha} M_\alpha(p))\ ,$$

where $M_\alpha(p)$ is given in (84) below.
2) There always exists $0 < \theta \leq 1$ such that $a_{1,\theta} \equiv \frac{\nu}{\alpha} - \theta \sup_{v \in \mathscr{A}_\alpha} \|\nabla v\|_{L^\infty} > 0$. If $f \in W^{1+\theta,p}_{per}$, then the following estimate is true for any u belonging to the global attractor

$$\|\mathrm{rot}\, u - \alpha\Delta \mathrm{rot}\, u\|_{W^{\theta,p}} \leq a_{1,\theta}^{-1}(\|\mathrm{rot} f\|_{W^{\theta,p}} + \frac{\nu}{\alpha} M_\alpha(p))\ .$$

3) More generally, if $f \in W^{1+m,p}_{per}$ and $a_m \equiv \frac{\nu}{\alpha} - (2m-1) \sup_{v \in \mathscr{A}_\alpha} \|\nabla v\|_{L^\infty} > 0$, then the following upper bound holds for any u belonging to the global attractor

$$\|\mathrm{rot}\, u - \alpha\Delta \mathrm{rot}\, u\|_{W^{m,p}} \leq a_m^{-1} M_{m,\alpha}(p)\ ,$$

where $M_{m,\alpha}(p)$ is a positive constant.

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1 and to several remarks about the solutions of (2) In Section 3, we first prove that $S_\alpha(t)$ is asymptotically smooth in $V^{3,p}$ and thus admits a compact global attractor $\mathscr{A}_\alpha$ in $V^{3,p}$. Afterwards, we prove Theorem 3, that is, the regularity properties of $\mathscr{A}_\alpha$ if the forcing term is smoother.

2 Existence results for the second grade fluid equations

Theorem 1 can be proved in different ways. For example, we could remark that the local existence result [8, Theorem 1] can be extended to the two-dimensional case and the periodic boundary conditions, when the initial data belong to $V^{2,q}$, $q > 2$ and the forcing term f is in $L^\infty((0,+\infty), L^q_{per})$. Since $W^{1,p}(\mathbb{T}^2)$, $p > 1$, is continuously embedded into the space $L^{q_0}(\mathbb{T}^2)$, where $q_0 > 2$, we could deduce from Theorem 1 of [8] that, for every $u_0 \in V^{3,p}$, there exists a unique local solution $u(t) \in C^0([0,T), V^{2,q_0})$ of (2), where $T > 0$. Afterwards, we could show that this solution is unique and is actually more regular.
However, since we want to emphasize the important role of the transport equation (6), we will give a complete direct proof of Theorem 1.

2.1 The transport equation

Since the existence of the solution of (2) will be proved by a fixed point argument involving the solution ω of the transport equation (6), we first study the following general transport equation (where $\nu > 0$ and $\alpha > 0$),

$$\begin{aligned} &\partial_t w + \frac{\nu}{\alpha} w + u \cdot \nabla w = g \ , \quad t > 0, \ x \in \mathbb{T}^2 \ , \\ &w(0,x) = w_0(x) \ , \quad x \in \mathbb{T}^2 \ , \end{aligned} \tag{10}$$

where, for the sake of simplicity (and in view of the applications), $u \in C^0([0,+\infty), V^{2,p}) \cap L^\infty((0,+\infty), V^{3,p})$, $p > 1$.

Before stating an existence and uniqueness result of solutions of (10), we introduce the "Lagrangian coordinates", that is, the following ordinary differential equation, for $t, \tau \in [0,+\infty)$, $x \in \mathbb{T}^2$,

$$\partial_t \varphi(t;\tau,x) = u(t, \varphi(t;\tau,x)) \ , \quad \varphi(\tau;\tau,x) = x \in \mathbb{T}^2 \ . \tag{11}$$

Since $u \in C^0([0,+\infty) \times \mathbb{T}^2, \mathbb{T}^2) \cap L^\infty((0,+\infty), V^{1,\infty})$, the classical Cauchy-Lipschitz theorem implies that, for every $x \in \mathbb{T}^2$, there exists a unique solution $\varphi(t;\tau,x) \in C^1([0,+\infty), \mathbb{T}^2)$ and the function $\varphi(t;\tau,x) : x \in \mathbb{T}^2 \mapsto \varphi(t;\tau,x) \in \mathbb{T}^2$ is Lipschitz-continuous with respect to x, where the Lipschitz constant may depend on t. Moreover, the function $\varphi(t;\tau,x) : (t,\tau,x) \mapsto \varphi(t;\tau,x)$ belongs to $C^1([0,+\infty)^2 \times \mathbb{T}^2, \mathbb{T}^2)$. The integral form of the equation (11) is as follows

$$\varphi(t;\tau,x) = x + \int_\tau^t u(s, \varphi(s;\tau,x)) ds \ . \tag{12}$$

Of course, the solution $\varphi(t;\tau,x)$ also depends on u. If we want to emphasize that $\varphi(t;\tau,x)$ also depends on u or when we let u vary, we will use the notation $\varphi_u(t;\tau,x)$ instead of $\varphi(t;\tau,x)$.

In what follows, we will often use the following estimates without further notice. Below $\mathrm{Jac}\,\varphi$ denotes the Jacobian matrix of φ.

Lemma 1.
Let $u \in C^0([0,+\infty),V^{2,p}) \cap L^\infty((0,+\infty),V^{3,p})$, $p > 1$.
1) Then,

$$(\det\, Jac\ \varphi)(t;\tau,x)) = 1\ ,\forall\tau\ ,\forall t\ ,\forall x\ , \tag{13}$$

2) The following estimate holds, for any $1 \leq q \leq +\infty$, any $t \geq \tau$ (resp. $\tau \geq t$)

$$\|\nabla\varphi(t;\tau,\cdot)\|_{L^q} \leq \exp(\int_\tau^t \|\nabla u(s)\|_{L^\infty} ds\ ) \tag{14}$$

(resp. $\|\nabla\varphi(t;\tau,\cdot)\|_{L^q} \leq \exp(\int_t^\tau \|\nabla u(s)\|_{L^\infty} ds)$).
3) Let u_i, $i = 1,2$ be two elements in $C^0([0,+\infty),V^{2,p}) \cap L^\infty((0,+\infty),V^{3,p})$, $p > 1$ and denote $\varphi_{u_i}(t;\tau,x)$ the corresponding solutions of (10). Then, for any $q \geq 1$, any $t \geq \tau$ (resp. $\tau \geq t$)

$$\|\varphi_{u_1}(t;\tau,\cdot) - \varphi_{u_2}(t;\tau,\cdot)\|_{L^q} \leq \|u_1 - u_2\|_{L^\infty((\tau,t),L^q)} \exp(\int_\tau^t \|\nabla u_1(s)\|_{L^\infty} ds\ ) \tag{15}$$

(resp. $\|\varphi_{u_1}(t;\tau,\cdot) - \varphi_{u_2}(t;\tau,\cdot)\|_{L^q} \leq \|u_1 - u_2\|_{L^\infty((t,\tau),L^q)} \exp(\int_t^\tau \|\nabla u_1(s)\|_{L^\infty} ds)$).

Proof.
1) The property (13) is well known. It is a consequence of the fact that $\mathrm{div}\, u = 0$ (see, for example, [11]).
2) Let $t \geq \tau$. We set

$$\psi_k(t;\tau,x) = \frac{\partial}{\partial x_k}\varphi(t;\tau,x)\ .$$

and notice that

$$\partial_t \psi_k(t;\tau,x) = \sum_{i=1}^{2} \frac{\partial u}{\partial x_i}(t,\varphi(t;\tau,x)) \frac{\partial \varphi_i}{\partial x_k}(t;\tau,x)\ ,$$

which implies that, for $t \geq \tau$,

$$\|\psi_k(t;\tau,\cdot)\|_{L^q} \leq \|\psi_k(\tau;\tau,\cdot)\|_{L^q} + \int_\tau^t \|\nabla u(s)\|_{L^\infty} \|\psi_k(s;\tau,\cdot)\|_{L^q} ds\ .$$

Noticing that $D_x\varphi(\tau;\tau,x) = I$ for any x and using the Gronwall inequality, we deduce the estimate (14) from the above inequality.
The statement of 3) is proved in the same way.

Theorem 4.
1) Let $p > 1$. *Let* $k = 0,1$, *for any* $w_0 \in W^{k,p}_{per}$ *and any* $g \in L^\infty((0,T), W^{k,p}_{per})$, *there exists a unique (mild) solution* $w(t) \in C^0([0,T], W^{k,p}_{per})$ *of* (10) *and* $\partial_t w$ *belongs to* $L^\infty((0,T), W^{k-1,p}_{per})$, *where* $T > 0$.
2) For $k \geq 2$, *assume that* u *belongs to* $C^0([0,+\infty), V^{k+1,p}) \cap L^\infty((0,+\infty), V^{k+2,p})$, *then, for any* $w_0 \in W^{k,p}_{per}$ *and any* $g \in L^\infty((0,T), W^{k,p}_{per})$, *there exists a unique (mild) solution* $w(t) \in C^0([0,T], W^{k,p}_{per})$ *of* (10) *and* $\partial_t w$ *belongs to* $L^\infty((0,T), W^{k-1,p}_{per})$, *where* $T > 0$.
3) Moreover, we have the following estimate, for any $0 \leq t \leq T$,

$$\begin{aligned}
\|w(t)\|_{L^p} \leq & e^{-\frac{\nu}{\alpha}t}\|w_0\|_{L^p} + \frac{\alpha}{\nu}(1 - e^{-\frac{\nu}{\alpha}t})\|g\|_{L^\infty(I,L^p)} \\
\|\partial_t w(t)\|_{W^{-1,p}} \leq & \big(\frac{\nu}{\alpha} + \|u\|_{L^\infty(I,L^\infty)}\big)\big(e^{-\frac{\nu}{\alpha}t}\|w_0\|_{L^p} + \frac{\alpha}{\nu}(1 - e^{-\frac{\nu}{\alpha}t})\|g\|_{L^\infty(I,L^p)}\big) \\
& + \|g\|_{L^\infty(I,L^p)} ,
\end{aligned} \tag{16}$$

where $I = (0,T)$. *These inequalities hold for any* $t \geq 0$, *if* $w_0 \in W^{0,p}_{per}$ *and* $g \in L^\infty((0,+\infty), W^{0,p}_{per})$.

Proof. To prove this theorem, we proceed as Beirão da Veiga ([2]), but replace the Dirichlet boundary conditions by the periodic ones. Let us consider the equation

$$\begin{aligned}
&\partial_t w + aw + u \cdot \nabla w = g , \quad t > 0,\ x \in \mathbb{T}^2 , \\
&w(0,x) = w_0(x) , \quad x \in \mathbb{T}^2 ,
\end{aligned} \tag{17}$$

where for simplicity a is a given constant. To solve this equation, Beirão da Veiga considered the differential operator

$$\tilde{A}_a(t)w \equiv aw + u \cdot \nabla w , \quad t \in [0,T] ,$$

acting in the distributional sense on the functions w on $\Omega \equiv \mathbb{T}^2$. For $k \geq 1$, he introduced the space

$$D^k(t) \equiv \{w \in W^{k,p}_{per} \,|\, u \cdot \nabla w \in W^{k,p}_{per}\} ,$$

and defined the operator

$$A^k_a(t)w \equiv \tilde{A}_a(t)w , \quad \forall w \in D^k(t) . \tag{18}$$

In the case $k = 0$, one defines the operator A^0_a as the closure in L^p of the operator $A^1_a : D^k(t) \to W^{1,p}_{per}$.
In [2], Beirão da Veiga proved that, under the above regularity hypotheses made on $u(t)$, the family $\{A^k_a(t)\}_{t\in I}$, where $I = [0,T]$, is $(1,\theta_k)$-stable in the sense of Kato ([42], [43] and also [56]), with $\theta_k \geq 0$. Thus, the evolution operator $U_a(t,s)$ associated with the family $\{A^k_a(t)\}_{t\in I}$ is strongly continuous in $W^{k,p}_{per}$, for $k \geq 0$ (see [2, Theorem 2.2 and Sections 3 and 4]) and, for any $w_0 \in W^{k,p}_{per}$ and any $g \in L^\infty((0,T), W^{k,p}_{per})$, there exists a unique (mild) solution $w(t) \in C^0([0,T], W^{k,p}_{per})$ of (17) given by

$$w(t) = U_a(t,0)w_0 + \int_0^t U_a(t,s)g(s)ds \ . \tag{19}$$

Moreover, $w(t)$ is a strong solution in $W_{per}^{k-1,p}$, that is, the equality (17) holds in $W_{per}^{k-1,p}$ a.e. in t.
One remarks that

$$U_a(t,s) = e^{-a(t-s)}U(t,s) \ , \tag{20}$$

where $U(t,s) \equiv U_0(t,s)$ and thus

$$w(t) = e^{-at}U(t,0)w_0 + \int_0^t e^{-a(t-s)}U(t,s)g(s)ds \ . \tag{21}$$

Theorem 2.2 of [2] implies that, for any $k \geq 0$, one has, for $0 \leq t \leq T$,

$$\|w(t)\|_{W^{k,p}} \leq \exp(\theta_k T)(\|w_0\|_{W^{k,p}} + \int_0^T \|g(s)\|_{W^{k,p}} ds) \ . \tag{22}$$

In [2], Beirão da Veiga also proved that the evolution operator $U_a(t,s)$ is strongly continuous from $W_{per}^{-k,p}$ into itself, for $k \geq 0$, which implies that the estimate (22) still holds if k is replaced by $-k$, that is, one has, for $0 \leq t \leq T$,

$$\|w(t)\|_{W^{-k,p}} \leq \exp(\theta_k T)(\|w_0\|_{W^{-k,p}} + \int_0^T \|g(s)\|_{W^{-k,p}} ds) \ . \tag{23}$$

We apply the above results with $a = \frac{\nu}{\alpha}$. In our case, we obtain a better estimate for $k = 0$. Indeed, assume first that $w_0 \in W_{per}^{1,p}$ and $g \in L^\infty((0,T), W_{per}^{1,p})$. We first take the inner product of the equality (10) with $(\delta + |w|^2)^{(p-2)/2}w$, where $\delta > 0$ is small, then integrate by parts by taking into account that $\operatorname{div} u = 0$ and that $u \in L^\infty((0,T), W^{1,\infty}(\mathbb{T}^2)^2)$, and finally let δ go to zero. Then, we obtain that, for $0 \leq t \leq T$,

$$\partial_t \|w(t)\|_{L^p} + \frac{\nu}{\alpha}\|w(t)\|_{L^p} \leq \|g\|_{L^p} \ . \tag{24}$$

Integrating (24) with respect to the time variable and applying the Gronwall lemma, we deduce from (24) that, for $0 \leq t \leq T$,

$$\|w(t)\|_{L^p} \leq e^{-\frac{\nu t}{\alpha}}\|w_0\|_{L^p} + \int_0^t e^{\frac{\nu}{\alpha}(s-t)}\|g(s)\|_{L^p} ds \leq e^{-\frac{\nu t}{\alpha}}\|w_0\|_{L^p} + \frac{\alpha}{\nu}\|g\|_{L^\infty(I,L^p)} \ . \tag{25}$$

This inequality is also valid in the case where the interval $I \equiv (0,T)$ is replaced by $(0,+\infty)$ in the statement of the theorem.
Arguing by density, one readily shows that the inequality (25) still holds if w_0 and g only belong to L_{per}^p and $L^\infty((0,T), L_{per}^p)$.

By the general theory developped in [43] or [56, Chapter 5], we also know that $\partial_t w$ belongs to $L^\infty((0,T), W^{k-1,p})$. If moreover g is in $C^0([0,T], W^{k,p})$, then $\partial_t w$ belongs to $C^0([0,T], W^{k-1,p})$. Using the equality (10) and the inequality (25), we also show by density as above that, for $t \geq 0$,

$$\|\partial_t w(t)\|_{W^{-1,p}} \leq \big(\frac{\nu}{\alpha} + \|u\|_{L^\infty(I,L^\infty)}\big)\big(e^{-\frac{\nu}{\alpha}t}\|w_0\|_{L^p} + \frac{\alpha}{\nu}(1-e^{-\frac{\nu}{\alpha}t})\|g\|_{L^\infty(I,L^p)}\big) + \|g\|_{L^\infty(I,L^p)} \ . \tag{26}$$

Finally, let $\tilde{w}$ be the solution of the equation (10), where u, g and w_0 are replaced by $\tilde{u}$, $\tilde{g}$ and $\tilde{w}_0$ respectively. Then, $W = \tilde{w} - w$ is a solution of the equation

$$\begin{aligned} &\partial_t W + \frac{\nu}{\alpha} W + \tilde{u}\cdot\nabla W = \tilde{g} - g + (u - \tilde{u})\cdot\nabla w \ , \quad t > 0, x \in \mathbb{T}^2 \ , \\ &W(0,x) = \tilde{w}_0(x) - w_0(x) \ , \quad x \in \mathbb{T}^2 \ , \end{aligned} \tag{27}$$

Assume that w_0, $\tilde{w}_0$ and g, $\tilde{g}$ belong to $W^{1,p}_{per}$ and $L^\infty((0,T),W^{1,p}_{per})$, respectively. Applying the estimate (24) to the equation (27), we obtain, that, for $0 \leq t \leq T$,

$$\begin{aligned} \partial_t\|W(t)\|_{L^p} + \frac{\nu}{\alpha}\|W(t)\|_{L^p} &\leq \big(\|(\tilde{u}-u)\nabla w\|_{L^p} + \|\tilde{g}-g\|_{L^p}\big) \\ &\leq \|(\tilde{u}-u)(t)\|_{L^\infty}\|w(t)\|_{W^{1,p}} + \|(\tilde{g}-g)(t)\|_{L^p} \ . \end{aligned} \tag{28}$$

Integrating with respect to t and taking into account the inequality (22), we finally get the following estimate, for $0 \leq t \leq T$,

$$\begin{aligned} \|W(t)\|_{L^p} \leq & e^{-\frac{\nu t}{\alpha}}\|w_0 - \tilde{w}_0\|_{L^p} + \frac{\alpha}{\nu}\|g - \tilde{g}\|_{L^\infty(I,L^p)} \\ & + \frac{\alpha}{\nu}\|(\tilde{u}-u)(t)\|_{L^\infty(I,L^\infty)}\big[\exp(\theta_1 T)(\|w_0\|_{W^{1,p}} + \int_0^T \|g(s)\|_{W^{1,p}}ds)\big] \ . \end{aligned} \tag{29}$$

Using the estimates (23) and (25) we also show that, for $0 \leq t \leq T$,

$$\begin{aligned} \|W(t)\|_{W^{-1,p}} \leq & e^{\theta_1 T}\Big[\|w_0 - \tilde{w}_0\|_{W^{-1,p}} + \|g - \tilde{g}\|_{L^1(I,W^{-1,p})} \\ & + \|(\tilde{u}-u)(t)\|_{L^1(I,L^\infty)}(e^{-\frac{\nu t}{\alpha}}\|w_0\|_{L^p} + \frac{\alpha}{\nu}\|g\|_{L^\infty(I,L^p)})\Big] \ . \end{aligned} \tag{30}$$

By density, the above inequality also holds if w_0, $\tilde{w}_0$ and g, $\tilde{g}$ only belong to L^p_{per} and $L^\infty((0,T),L^p_{per})$, respectively.

Remark 1. In [44], Ladyzenskaya and Solonnikov proved that, if the data w_0 and f are regular enough, the solution w of (17) with $a = 0$ is given by

$$w(t,x) = w_0(\varphi(0;t,x)) + \int_0^t g(s,\varphi(s;t,x))ds \ . \tag{31}$$

This implies by uniqueness of the solution that, if $w_0 \in W^{k,p}_{per}$ and $g \in L^\infty((0,T),W^{k,p}_{per})$, for $k \geq 0$, the solution w of the equation (10) is given by

$$w(t,x)=e^{-\frac{\nu}{\alpha}t}w_0(\varphi(0;t,x))+\int_0^t e^{-\frac{\nu}{\alpha}(t-s)}g(s,\varphi(s;t,x))ds\ . \tag{32}$$

The integral formula allows to prove the above estimates in another (elegant) way, without using the inequalities of [2]. Let W be the solution of (27). It satisfies the integral equation:

$$\begin{aligned} W(t,x)=&e^{-\frac{\nu}{\alpha}t}\big(\tilde{w}_0(\tilde{\varphi}(0;t,x))-w_0(\tilde{\varphi}(0;t,x))\big)\\ &+\int_0^t e^{-\frac{\nu}{\alpha}(t-s)}(\tilde{g}(s,\tilde{\varphi}(s;t,x))-g(s,\tilde{\varphi}(s;t,x)))ds\\ &+\int_0^t e^{-\frac{\nu}{\alpha}(t-s)}(u-\tilde{u})(s,\tilde{\varphi}(s;t,x))\cdot\nabla w(\tilde{\varphi}(s;t,x))ds\ , \end{aligned} \tag{33}$$

where φ and $\tilde{\varphi}$ are the solutions of the equation (11) associated with u and $\tilde{u}$ respectively. From the equality (33), we at once deduce, by applying Lemma 1 and (25), that, for $0\le t\le T$,

$$\begin{aligned} \|W(t)\|_{W^{-1,p}}\le&C^*(T,\tilde{u})\Big(e^{-\frac{\nu}{\alpha}t}\|w_0-\tilde{w_0}\|_{W^{-1,p}}+\frac{\alpha}{\nu}\|g-\tilde{g}\|_{L^\infty(I,W^{-1,p})}\\ &+\frac{\alpha}{\nu}\|(\tilde{u}-u)(t)\|_{L^\infty(I,L^\infty)}\|w\|_{L^\infty(L^p)}\Big)\\ \le&C^*(T,\tilde{u})\Big[e^{-\frac{\nu}{\alpha}t}\|w_0-\tilde{w_0}\|_{W^{-1,p}}+\frac{\alpha}{\nu}\|g-\tilde{g}\|_{L^\infty(I,W^{-1,p})}\\ &+\frac{\alpha}{\nu}\|(\tilde{u}-u)(t)\|_{L^\infty(I,L^\infty)}\big(\|w_0\|_{L^p}+\frac{\alpha}{\nu}\|g\|_{L^\infty(I,L^p)}\big)\Big]\ , \end{aligned} \tag{34}$$

where $C^*(T,\tilde{u})=\exp\int_0^T\|\nabla\tilde{u}(s)\|_{L^\infty}ds$.
We point out that the above inequality also holds if w_0, $\tilde{w}_0$ and g, $\tilde{g}$ only belong to L^p_{per} and $L^\infty((0,T),L^p_{per})$, respectively.

We are actually interested in the solution of the following transport equation

$$\begin{aligned} &\partial_t\omega+\frac{\nu}{\alpha}\omega+u\cdot\nabla\omega=\mathrm{rot}\,f+\frac{\nu}{\alpha}\mathrm{rot}\,u,\quad t>0,\ x\in\mathbb{T}^2\\ &\omega(0,x)=\omega_0(x)\ ,\quad x\in\mathbb{T}^2\ , \end{aligned} \tag{35}$$

where $u\in C^0([0,+\infty),V^{2,p})\cap L^\infty((0,+\infty),V^{3,p})$, $p>1$.
As an immediate consequence of Theorem 4, we obtain the following corollary.

Corollary 2. *Let $p>1$. For any $\omega_0\in L^p_{per}$ and $\mathrm{rot}\,f\in L^\infty((0,T),L^p_{per})$, there exists a unique (mild) solution $\omega\in C^0([0,T],L^p_{per})$ (with $\partial_t\omega\in L^\infty((0,T),W^{-1,p}_{per})$) of the equation* (35). *Moreover, the following estimate holds, for $t\ge 0$,*

$$\|\omega(t)\|_{L^p}\le e^{-\frac{\nu}{\alpha}t}\|\omega_0\|_{L^p}+\frac{\alpha}{\nu}\|\mathrm{rot}\,f\|_{L^\infty(I,L^p)}+(1-e^{-\frac{\nu}{\alpha}t})\|\mathrm{rot}\,u\|_{L^\infty(I,L^p)}\ . \tag{36}$$

This inequality holds for any $t\ge 0$, if $I\equiv(0,T)$ is replaced above by $I\equiv(0,+\infty)$.

The upper bound for $\|\partial_t\omega\|_{W^{-1,p}}$ follows from (26).

2.2 An auxiliary problem

In Corollary 2, we have obtained the solution ω of the equation (35). We next want to show that there exists a unique divergence free vector field z such that $\omega = z - \alpha\Delta z$. This will be an easy consequence of the following two lemmas.

Lemma 2. *1) For any $w \in W^{k,p}_{per}$, $k \geq 0$, there exists a unique vector field $\psi \in V^{k+1,p}$ such that*

$$\mathrm{rot}\,\psi(x) = w(x)\ , \quad \forall x \in \mathbb{T}^2\ . \tag{37}$$

Moreover, there exists a positive constant $C_0(k)$ such that,

$$\|\psi\|_{W^{k+1,p}} \leq C_0(k)\|u\|_{W^{k,p}}\ . \tag{38}$$

2) Likewise, for any $w \in W^{m,\infty}((0,T),W^{k,p}_{per})$ (resp. in $C^m([0,T],W^{k,p}_{per})$), $k \geq 0$, $m \geq 0$, there exists a unique vector field $\psi \in W^{m,\infty}((0,T),V^{k+1,p})$ (resp. in $C^m([0,T],V^{k+1,p})$) such that the equality (37) *holds.*

Proof.
1) For any $w \in W^{k,p}_{per}$, following [1, Lemma,2.3] for example, we construct the vector field

$$\psi = \nabla^{\perp} G(w)\ , \tag{39}$$

where $G(w)$ is the solution of the problem: to find $G(w) \in W^{1,p}_{per}$ such that,

$$\Delta G(w) = w\ . \tag{40}$$

The solution $G(w)$ is unique in $W^{1,p}_{per}$. The regularity of ψ is a consequence of the regularity properties of the solutions of the Laplace equation.
We remark that the vector field ψ is unique in $V^{0,p}_{per}$. Indeed, if ψ_1 and ψ_2 are two solutions of (37), then $\Delta(\psi_1 - \psi_2) = 0$, which has a unique solution in $V^{0,p}_{per}$. Statement 2) is proved in the same way.

We next show that $w \in W^{k,p}_{per}$ can be written in the form $w = z - \alpha\Delta z$, where $z \in V^{k+3,p}$.

Lemma 3. *1) For any $w \in W^{k,p}_{per}$, $k \geq 0$, there exists a unique vector field $z \in V^{k+3,p}$ such that*

$$\mathrm{rot}\,(z - \alpha\Delta z)(x) = w(x)\ , \quad \forall x \in \mathbb{T}^2\ . \tag{41}$$

Moreover, there exists a positive constant $C_1(k,\alpha)$ such that,

$$\|z\|_{W^{k+3,p}} \leq C_1(k,\alpha)\|u\|_{W^{k,p}}\ . \tag{42}$$

2) Likewise, for any $w \in W^{m,\infty}((0,T),W^{k,p}_{per})$ (resp. in $C^m([0,T],W^{k,p}_{per})$), $k \geq 0$, $m \geq 0$, there exists a unique vector field $z \in W^{m,\infty}((0,T),V^{k+3,p})$ (resp. in $C^m([0,T],V^{k+3,p})$) such that the equality (41) *holds.*

Proof.
1) By Lemma 2, we know that there exists a unique vector field $\psi \in V^{k+1,p}$ such that $\operatorname{rot}\psi = w$ (and ψ is unique in $V^{1,p}$). But, it is well known that the problem: to find $z \in V^{1,p}$ such that

$$z - \alpha\Delta z = \psi \tag{43}$$

has a unique solution. Moreover, the regularity properties of the Laplacian operator imply that $z \in V^{k+3,p}$ and that the inequality (42) holds. Statement 2) is proved in the same way.

From the Corollary 2 and the Lemmata 2 and 3, we at once deduce the following Corollary.

Corollary 3. *Let $p > 1$. For any $\omega_0 \in L^p_{per}$ and $\operatorname{rot} f \in L^\infty((0,T), L^p_{per})$, there exists a unique $z \in C^0([0,T], V^{3,p})$ (with $\partial_t z \in L^\infty((0,T), V^{2,p})$) such that*

$$\omega = \operatorname{rot}(z - \alpha\Delta z) \tag{44}$$

is the unique (mild) solution of the equation (35)*. Moreover, the following estimates hold, for $0 \le t \le T$,*

$$\begin{aligned}
\|z(t)\|_{W^{3,p}} \le & C_1(0,\alpha)\big[e^{-\frac{\nu}{\alpha}t}\|\omega_0\|_{L^p} + (1 - e^{-\frac{\nu}{\alpha}t})(\frac{\alpha}{\nu}\|\operatorname{rot} f\|_{L^\infty(I,L^p)} + \|\operatorname{rot} u\|_{L^\infty(I,L^p)})\big] \\
\|\partial_t z(t)\|_{W^{2,p}} \le & C_2(\alpha)\big[\big(\frac{\nu}{\alpha} + \|u\|_{L^\infty(I,L^p)}\big) \\
& \times \big(e^{-\frac{\nu}{\alpha}t}\|\omega_0\|_{L^p} + (1 - e^{-\frac{\nu}{\alpha}t})(\frac{\alpha}{\nu}\|\operatorname{rot} f\|_{L^\infty(I,L^p)} + \|\operatorname{rot} u\|_{L^\infty(I,L^p)})\big) \\
& + \|\operatorname{rot} f\|_{L^\infty(I,L^p)} + \frac{\nu}{\alpha}\|\operatorname{rot} u\|_{L^\infty(I,L^p)}\big] \ ,
\end{aligned} \tag{45}$$

where $C_2(\alpha)$ is a positive constant depending only on α.
These inequalities hold for any $t \ge 0$, if $I \equiv (0,T)$ is replaced above by $I \equiv (0,+\infty)$.

Proof.
The existence and uniqueness of $z(t) \in C^0([0,T], W^{3,p}_{per})$, such that $\omega = \operatorname{rot}(z - \alpha\Delta z)$ is the mild solution of (35), is a direct consequence of Corollary 2 and of Lemma 3. Taking the derivative of (44) with respect to t, we obtain the equality

$$\operatorname{rot}\partial_t z - \alpha\Delta \operatorname{rot}\partial_t z = \partial_t \omega \ .$$

Since $\partial_t\omega$ belongs to $W^{-1,p}_{per}$, the regularity properties of the above equation imply that $\operatorname{rot}\partial_t z$ is in $W^{2,p}_{per}$ and thus $\partial_t z$ belongs to $W^{3,p}_{per}$. The inequalities (45) are a direct consequence of the inequalities of Corollary 2 and of Lemma 3 and of (26).

2.3 Local existence and uniqueness of solutions in $V^{3,p}$, $p>1$

Let $u_0 \in V^{3,p}$ be given. We first remark that, if ω is a solution of the transport equation (35) with $\omega_0 = \mathrm{rot}\,(u_0 - \alpha\Delta u_0)$ and z is the solution of (44), then there exists a unique pressure $p \in W^{1,p}_{per}$ such that,

$$\begin{aligned}
&\partial_t(z-\alpha\Delta z) - \nu\Delta z + \mathrm{rot}\,(z-\alpha\Delta z)\times u + \nabla p = f, \quad t>0,\ x\in\Omega,\\
&\mathrm{div}\, u = 0, \quad t>0,\ x\in\Omega,\\
&z(0,x) = u_0(x), \quad x\in\Omega\ .
\end{aligned} \tag{46}$$

Local existence of the solution of (2)

Now we are ready to show the local existence of solutions of (2). Let $T>0$ be fixed (the choice of T will be made more precise below). Let $u_0 \in V^{3,p}$ and $f \in L^\infty((0,T),W^{1,p}_{per})$ be given.

As we have explained in the introduction, we define the following map J_T : $L^\infty((0,T),V^{3,p}) \cap C^0([0,T],V^{2,p})$ into itself as follows

$$u \in L^\infty((0,T),V^{3,p}) \cap C^0([0,T],V^{2,p}) \mapsto \omega \in C^0([0,T],L^p_{per}) \mapsto z \in C^0([0,T],V^{2,p})\ , \tag{47}$$

where ω is the solution of the equation (35) with $\omega_0 = \mathrm{rot}\,(u_0 - \alpha\Delta u_0)$ and z is the solution of the equation (44). We will show that J_T is a continous compact map from a closed convex subset E_T of $L^\infty((0,T),V^{3,p}) \cap C^0([0,T],V^{2,p})$ into E_T. Then applying the Leray-Schauder fixed point theorem, we deduce that J_T has a fixed point u^*. We notice that the idea of introducing such a type of map and of using the Leray-Schauder theorem goes back to [25], where the local existence of solutions has been proved (see also [23], [8], [45] for example). The map, constructed in these papers differs from the one here. Indeed, these authors considered the map $\mathscr{F} : w \mapsto u \mapsto \mathscr{F}(w) = \omega$, where u satisfies $\mathrm{div}\, u = 0$ and $w = \mathrm{rot}\,(u - \alpha\Delta u)$ and where ω is the solution of (35). In [25], and [23], the authors work in more regular spaces. Even if there are some differences, our proof follows the same main lines. First we introduce the positive constant K given by

$$K \equiv C_1(0,\alpha)\big(\|\omega_0\|_{L^p} + \frac{\alpha}{\nu}\|\mathrm{rot}\, f\|_{L^\infty(I,L^p)}\big)\ , \tag{48}$$

where $C_1(0,\alpha)$ is given in Corollary 3 and then choose $T>0$ such that

$$2KC_1(0,\alpha)(1-e^{-\frac{\nu}{\alpha}T}) < K\ . \tag{49}$$

Finally, we define the (non empty) set

$$E_T \equiv \{v \in L^\infty((0,T),V^{3,p}) \cap C^0([0,T],V^{2,p}) \,|\, v(0) = u_0\ ,\ \|v\|_{L^\infty(W^{3,p})} \le 2K\}\ . \tag{50}$$

We equip E_T with the classical topology of the space $X_T = C^0([0,T],V^{2,p})$.

First, one checks that E_T is a closed subset of X_T. As in [23] or in [9], one considers a sequence v_n, $n \in \mathbb{N}$, in X_T converging to v. Since the sequence v_n is bounded in $L^\infty((0,T),V^{3,p})$, it converges in $L^\infty((0,T),V^{3,p})$ weak $*$ to an element U in $L^\infty((0,T),V^{3,p})$ and

$$\|U\|_{L^\infty((0,T),V^{3,p})} \leq 2K \ .$$

Due to the uniqueness of the limit in the space of distributions in $(0,T)\times\mathbb{T}^2$, $U = v$ and thus v belongs to E_T.
With the above choice of K, Corollary 3 at once implies that $J(E_T) \subset E_T$.
Actually, $J(E_T)$ is relatively compact in E_T. Indeed, by Corollary 3, $J(E_T)$ is bounded in $W^{1,\infty}((0,T),W^{2,p}_{per}) \cap L^\infty((0,T),V^{3,p})$. Since the injection of $W^{3,p}$ into $W^{2,p}$ is compact, we deduce from [46, Assertion(12.10), page 142] that every bounded set in $W^{1,\infty}((0,T),W^{2,p}(\mathbb{T}^2)) \cap L^\infty((0,T),W^{3,p}(\mathbb{T}^2))$ is relatively compact in $C^0([0,T],W^{2,p}(\mathbb{T}^2))$. Thus, $J(E_T)$ is relatively compact in E_T.
It remains to verify that the map $J : u \in E_T \mapsto \omega \mapsto z \in E_T$ is continuous for the topology of X_T. Let u_1 and u_2 be two elements of E_T, let ω_1 and ω_2 be the two corresponding solutions of the equation (35) and finally let z_1, z_2 be the two corresponding solutions of (44). From the estimate (34) in Remark 1, we deduce that, for $0 \leq t \leq T$,

$$\begin{aligned}
&\|(\omega_1-\omega_2)(t)\|_{W^{-1,p}} \\
&\leq C^*(T,u_2)\big[\frac{\alpha}{\nu}\|u_1-u_2\|_{L^\infty(I,L^\infty)}\big(\|\omega_1(0)\|_{L^p} + \|\mathrm{rot} f\|_{L^\infty(I,L^p)} + \frac{\nu}{\alpha}\|\mathrm{rot}\, u_1\|_{L^\infty(I,L^p)}\big) \\
&\qquad + \|u_1-u_2\|_{L^\infty(I,L^p)}\big] \\
&\leq C(K)\|u_1-u_2\|_{L^\infty(I,W^{2,p})} \ ,
\end{aligned} \tag{51}$$

where $C(K)$ is a positive constant depending on K. Using the regularity properties of the Laplacian and arguing as in Corollary 3, one deduces from the inequality (51) that, for $0 \leq t \leq T$,

$$\|(z_1-z_2)(t)\|_{W^{2,p}} \leq C\|(\omega_1-\omega_2)(t)\|_{W^{-1,p}} \leq CC(K)\|u_1-u_2\|_{L^\infty(I,W^{2,p})} \ . \tag{52}$$

From the inequality (52), one at once deduces that the map J is continuous for the topology of X_T.
Now we may apply the Leray-Schauder fixed point theorem to the map J. Thus, there exists a fixed point u of J, that is, a function $u \in E_T$ satisfying the system (7). Moreover, by Theorem 4 and by Corollary 3, u belongs to $C^0([0,T],V^{3,p}) \cap W^{1,\infty}((0,+\infty),W^{2,p}_{per})$ and $u \equiv z$ satisfies the estimates (45). Morever, applying Theorem 2.2 of [2] in the “negative order Sobolev space” $W^{-1,p}_{per}$, we deduce that $\partial_t u$ actually belongs to $C^0([0,T],W^{2,p}_{per})$. Finally, introducing the pressure term as in (46), we have proved that the system (2) admits a solution (u,p) if $T > 0$ is small enough (T depending only on u_0 and f).

The propagation of the regularity of u is a direct consequence of Theorem 4. Assume that $u_0 \in V^{4,p}$ and $f \in L^\infty((0,T),W^{2,p}_{per})$, then, in the equation (35), ω_0 and $\mathrm{rot}\, f + \frac{\nu}{\alpha}\mathrm{rot}\, u$ belong to $V^{1,p}$ and $L^\infty((0,T),W^{1,p}_{per})$ respectively. Thus, by Theorem 4, the solution ω of (35) belongs to $C^0((0,T),W^{1,p}_{per})$. Since $\omega = \mathrm{rot}\,(u-\alpha\Delta u)$, it follows that u belongs to $C^0((0,T),V^{4,p}_{per})$. When $k \geq 2$, we proceed by recursion on k. Indeed, if $u_0 \in V^{k+3,p}$, $f \in L^\infty((0,T),W^{k+1,p}_{per})$, then, by Theorem 4, u belongs to $C^0([0,T],V^{k+3,p})$, provided that u is in $C^0([0,T],V^{k+1,p}) \cap L^\infty((0,T),W^{k+2,p}_{per})$. But this regularity property is known by application of Theorem 4 at the order $k-1$.

Uniqueness of the solution of (2)

The proof of the uniqueness of the solutions of (2) is well known and goes back to [14]. For the sake of completeness, we give a quick proof of it. Actually, we will prove a more general continuity result. Let $u_i(t) \in C^0([0,T],V^{3,p})$ (with $\partial_t u_i(t) \in L^\infty((0,T),V^{2,p})$), $i=0,1$, be two solutions of (2). Then, $U = u_1 - u_2$ satisfies the equation

$$\begin{aligned}
\partial_t(U-\alpha\Delta U) - \nu\Delta U + \mathrm{rot}\,(U-\alpha\Delta U)\times u_1 + \mathrm{rot}\,(u_2-\alpha\Delta u_2)\times U \\
= -\nabla(p_1-p_2)\,, \quad t>0,\ x\in\Omega\,, \\
U(0,x) = u_1(0)-u_2(0)\,, \quad x\in\Omega\,.
\end{aligned} \tag{53}$$

In [55, Theorem A.1], we have shown the following equality

$$\begin{aligned}
(\mathrm{rot}\,\Delta U\times u_1, U) \equiv \int_{\mathbb{T}^2}\mathrm{rot}\, U\left(\Delta u_1^1 U^2 - \Delta u_1^2 U^1\right)dx \\
+2\int_{\mathbb{T}^2}\mathrm{rot}\, U\left(\nabla u_1^1\cdot\nabla U^2 - \nabla u_1^2\cdot\nabla U^1\right)dx.
\end{aligned} \tag{54}$$

Taking the inner product of the first equation in (53) with U in L^2 and using the equality (54) together with classical Sobolev inequalities, we obtain, for $0\leq t\leq T$,

$$\begin{aligned}
&\partial_t(\|U(t)\|^2_{L^2} + \alpha\|\nabla U(t)\|^2_{L^2}) + \nu\|\nabla U(t)\|^2_{L^2} \\
&\leq 2\left|\int_{\mathbb{T}^2}(\mathrm{rot}\,(U-\alpha\Delta U)\times u_1)U\,dx\right| \\
&\leq C_1\Big(\|u_1(t)\|_{L^\infty}\|U\|_{L^2}\|\nabla U\|_{L^2} + \alpha\|\nabla u_1(t)\|_{L^\infty}\|\nabla U\|^2_{L^2} + \alpha\|\nabla U\|_{L^2}\|\Delta u_1\|_{W^{1,p}}\|U\|_{L^{\frac{p}{p-1}}}\Big) \\
&\leq C_2\Big(\|u_1(t)\|_{L^\infty}\|U\|_{L^2}\|\nabla U\|_{L^2} + \alpha\|\nabla u_1(t)\|_{L^\infty}\|\nabla U\|^2_{L^2} + \alpha\|\nabla U\|^2_{L^2}\|\Delta u_1\|_{W^{1,p}}\Big) \\
&\leq C_3\|u_1(t)\|_{W^{3,p}}\Big(\frac{1+\alpha}{\alpha}\Big)\big(\|U\|^2_{L^2} + \alpha\|\nabla U\|^2_{L^2}\big)\,.
\end{aligned} \tag{55}$$

Integrating with respect to t and aplying the Gronwall lemma, we obtain that, for $0\leq t\leq T$,

$$\|U(t)\|_{L^2}^2+\alpha\|\nabla U(t)\|_{L^2}^2 \le \big[\|u_1(0)-u_2(0)\|_{L^2}^2+\alpha\|\nabla(u_1-u_2)(0)\|_{L^2}^2\big] \times \exp\int_0^T C_3(\frac{1+\alpha}{\alpha})\|u_1(s)\|_{W^{3,p}}ds\,. \tag{56}$$

If $u_1(0)=u_2(0)$, then (56) implies that $U(t)\equiv 0$, that is, that the solution $u(t)$ of (2) is unique.

Continuity of the map $u_0\in V^{3,p}\mapsto u(t)\in V^{3,p}$

Let $f\in L^\infty((0,T),W^{1,p}_{per})$ (resp. $f\in L^\infty((0,+\infty),W^{1,p}_{per})$) be given. In the next section, we will show that the solution $u(t,x)\equiv u(t,x;u_0)$, with $u(0,x;u_0)=u_0(x)$ of (2) exists on $(0,T)$ (resp. $(0,+\infty)$) and is uniformly bounded in time for u_0 belonging to bounded sets of $V^{3,p}$. So we do not need to worry about blow-up in finite time. To simplify the notation, we will sometimes only write $u(t;u_0)$ instead of $u(t,x;u_0)$.

Assume that f belongs to $L^\infty((0,T),W^{1,p}_{per})$. The estimate (56) implies that the map $u_0\in V^{3,p}\mapsto u(t;u_0)\in V^{1,2}$ is continuous and even Lipschitzian on the bounded sets of $V^{3,p}$. Since, for any bounded set B_0 in $V^{3,p}$, there exists a bounded set $\gamma^+(B_0)\in V^{3,p}$ such that $u(t;u_0)\in\gamma^+(B_0)$ for any $0\le t\le T$ and any $u_0\in B_0$, we deduce, by interpolation, that, for every $0\le\theta<3$, the map $u_0\mapsto u(t;u_0)\in V^{\theta,p}$ is Hölder continuous on the bounded sets of $V^{3,p}$ and, in particular, $u_0\to u(t)$ belongs to $C^0(V^{3,p},V^{\theta,p})\cap L^\infty(V^{3,p},V^{3,p})$. We next prove that actually $u_0\to u(t)$ belongs to $C^0(V^{3,p},V^{3,p})$.

Below, we set $\omega(t,x;u_0)=\mathrm{rot}\,(u(t,x;u_0)-\alpha\Delta u(t,x;u_0))$, $\omega(x;u_0)=\mathrm{rot}\,(u_0(x)-\alpha\Delta u_0(x))$ and we denote $\varphi_{u_0}(t;\tau,x)$ the solution of the equation (11), where $u(t)$ is replaced by $u(t,x;u_0)$. We recall that $\omega(t,x;u_0)$ writes, for $0\le t\le T$,

$$\begin{aligned}\omega(t,x;u_0)=&e^{-\frac{\nu}{\alpha}t}\omega(\varphi_{u_0}(0;t,x);u_0)\\&+\int_0^t e^{-\frac{\nu}{\alpha}(t-s)}\big(\mathrm{rot}\,f(s,\varphi_{u_0}(s;t,x))+\frac{\nu}{\alpha}\mathrm{rot}\,u(s,\varphi_{u_0}(s;t,x);u_0)\big)ds\,.\end{aligned}\tag{57}$$

Let u_{0n} be a sequence converging to u_0 in $V^{3,p}$; we want to show that $\omega(t,\cdot;u_{0n})$ converges to $\omega(t,\cdot;u_0)$ in $L^\infty((0,T),L^p)$ when n goes to $+\infty$. We at once remark that, for $0\le t\le T$,

$$\begin{aligned}&\|\omega(\varphi_{u_0}(0;t,x);u_0)-\omega(\varphi_{u_{0n}}(0;t,x);u_{0n})\|_{L^p}\\&\quad\le\|\omega(\varphi_{u_0}(0;t,x);u_0)-\omega(\varphi_{u_{0n}}(0;t,x);u_0)\|_{L^p}+(1+\alpha)\|u_0-u_{0n}\|_{V^{3,p}}\\&\quad\le\|\omega_m(\varphi_{u_0}(0;t,x))-\omega_m(\varphi_{u_{0n}}(0;t,x))\|_{L^p}+(1+\alpha)\|u_0-u_{0n}\|_{V^{3,p}}\\&\qquad+2\|\omega(\cdot;u_0)-\omega_m(\cdot)\|_{L^p}\,,\end{aligned}\tag{58}$$

where ω_m is a sequence in $W^{3,p}_{per}$ converging to ω in L^p. Next we use the Taylor formula and apply Lemma 1 to obtain, for $0\le t\le T$,

$$\begin{aligned}\|\omega_m(\varphi_{u_0}(0;t,x)) - \omega_m(\varphi_{u_{0n}}(0;t,x))\|_{L^p} &\leq C(T)\|\nabla\omega_m\|_{L^\infty}\|\varphi_{u_0}(0;t,\cdot) - \varphi_{u_{0n}}(0;t,\cdot)\|_{L^p} \\ &\leq C(T,\|u_0\|_{V^{3,p}})C(T)\|\nabla\omega_m\|_{L^\infty}\|u_{0n}-u_0\|_{W^{2,p}}\end{aligned} \tag{59}$$

The inequalities (58) and (59) show that the map $u_0 \in V^{3,p} \mapsto \omega(t,\cdot;u_0) \in L^p$ is continuous. In the same way, we prove that the map $u_0 \in V^{2,p} \mapsto \mathrm{rot}\, f(s,\varphi_{u_0}(s;t,x))$ is continuous (uniformly with respect to s). To show the continuity (uniformly with respect to s) of $\mathrm{rot}\, u(s,\varphi_{u_0}(s;t,x);u_0)$, we argue in the same way and in addition we use the fact that there exists $0<\theta<3$ such that $\|\mathrm{rot}\, u(s,y;u_0) - \mathrm{rot}\, u(s,y;u_{0n})\|_{L^p} \leq C(\theta,u_0)\|u_0 - u_{0n}\|_{V^{\theta,p}}$.

Remark 2. We notice that the local existence of solutions as well as the continuity properties also hold for negative time, if the force f belongs to $L^\infty((-T,0),W^{k+1,p})$, $k \geq 0$, $1<p<+\infty$.

Remark 3. Mutadis mutandis, one can also use the above method of proof to show the corresponding local existence and continuity of the solutions of (2), when the periodic boundary conditions are replaced by homogeneous Dirichlet ones, provided the domain Ω is smooth enough (of class C^2) and simply connected. We emphasize that the proof of Bresch and Lemoine ([8]) of local existence of solutions u in the spaces $V^{2,q}$, $q>2$, requires less regularity of the domain Ω since they do not consider the transport equation satisfied by ω.

2.4 Global existence of solutions in $V^{3,p}$, $p>1$

Let u_0 be given in $V^{3,p}$. We assume here that f belongs to $L^\infty(\mathbb{R}^+,W^{1,p}_{per})$. We set

$$T^*(u_0) = \sup\{T>0 \,|\, (7) \text{ has a solution } \omega = (u-\alpha\Delta u) \in C^0([0,T],L^p_{per})\}\ .$$

The proof of the local existence implies that $T^*(u_0)>0$. If $T^*(u_0)<+\infty$, then $\|\omega(t)\|_{L^p}$ goes to infinity, when t goes to $T^*(u_0)$. Indeed, if this is not true, then there exist $r>0$ and a sequence t_n converging to $T^*(u_0)$, with $t_n<T^*(u_0)$ such that $\|\omega(t_n)\|_{L^p}\leq r$, for any n. Due to the proof of the local existence (in particular, see the choices of K and T in (48) and in (49)), there exists $\tilde{T}(r)>0$ such that, for any n, $\omega(t)$, which exists on $[0,t_n]$ extends to $[0,t_n+\tilde{T}(r)]$. But, for n large enough, $t_n+\tilde{T}(r)>T^*(u_0)$, which is a contradiction. Thus $\|\omega(t)\|_{L^p}$ goes to infinity, when t goes to $T^*(u_0)$.

By the inequality (16) in Theorem 4, $\omega = \mathrm{rot}\,(u-\alpha\Delta u)$ satisfies the following estimate for $0\leq t<T^*(u_0)$,

$$\begin{aligned}\|\mathrm{rot}\,(u-\alpha\Delta u)(t)\|_{L^p} \leq e^{-\frac{\nu}{\alpha}t}\|\mathrm{rot}\,(u_0-\alpha\Delta u_0)\|_{L^p} &+ \frac{\alpha}{\nu}\|\mathrm{rot}\, f\|_{L^\infty(\mathbb{R}^+,L^p)} \\ &+ \|\mathrm{rot}\, u\|_{L^\infty((0,t),L^p)}\ .\end{aligned} \tag{60}$$

It remains to bound the term $\|\mathrm{rot}\, u\|_{L^\infty(\mathbb{R}^+, L^p)}$. We will first estimate this term for $1 < p \leq 2$. We proceed as in the proof of the uniqueness (see also [55]). Taking the inner product of the first equation in (2) with u and using the Young inequality, we obtain, for $0 \leq t < T^*(u_0)$,

$$\partial_t(\|u(t)\|_{L^2}^2 + \alpha\|\nabla u(t)\|_{L^2}^2) + \nu\|\nabla u(t)\|_{L^2}^2 \leq \frac{1}{\nu\lambda_1}\|f(t)\|_{L^2}^2 ,$$

where $\lambda_1 > 0$ is the first eigenvalue of the Stokes operator in L^p. From the above inequality, we deduce that, for $0 \leq t < T^*(u_0)$,

$$\partial_t(\|u(t)\|_{L^2}^2 + \alpha\|\nabla u(t)\|_{L^2}^2) + \frac{\nu}{2(\lambda_1^{-1} + \alpha)}(\|u(t)\|_{L^2}^2 + \alpha\|\nabla u(t)\|_{L^2}^2) + \frac{\nu}{2}\|\nabla u(t)\|_{L^2}^2 \leq \frac{1}{\nu\lambda_1}\|f(t)\|_{L^2}^2 . \tag{61}$$

Integrating the inequality (61) and applying the Gronwall lemma, we obtain, for $0 \leq t < T^*(u_0)$,

$$\begin{aligned}\|u(t)\|_{L^2}^2 + \alpha\|\nabla u(t)\|_{L^2}^2 &+ \frac{\nu}{2}\int_0^t \exp(\frac{\nu}{2(\lambda_1^{-1} + \alpha)}(s-t))\|\nabla u(s)\|_{L^2}^2 ds \\ &\leq \exp(-\frac{\nu}{2(\lambda_1^{-1} + \alpha)}t)\left[\|u_0\|_{L^2}^2 + \alpha\|\nabla u_0\|_{L^2}^2\right] \\ &\quad + \frac{2(1+\lambda_1\alpha)}{\lambda_1^2\nu^2}\|f\|_{L^\infty(\mathbb{R}^+, L^2)}^2 .\end{aligned} \tag{62}$$

From the estimates (60) and (62), we at once deduce that, for $0 \leq t < T^*(u_0)$, for $1 < p \leq 2$,

$$\begin{aligned}\|\mathrm{rot}\,(u - \alpha\Delta u)(t)\|_{L^p} \leq &\exp(-\frac{\nu}{\alpha}t)\|\mathrm{rot}\,(u_0 - \alpha\Delta u_0)\|_{L^p} \\ &+ \alpha^{-1/2}\exp(-\frac{\nu\lambda_1}{4(1+\lambda_1\alpha)}t)\left[\|u_0\|_{L^2} + \sqrt{\alpha}\|\nabla u_0\|_{L^2}\right] \\ &+ \frac{\alpha}{\nu}\|\mathrm{rot}\, f\|_{L^\infty(\mathbb{R}^+, L^p)} + \frac{\sqrt{2}(1+\lambda_1\alpha)^{1/2}}{\lambda_1\nu\sqrt{\alpha}}\|f\|_{L^\infty(\mathbb{R}^+, L^2)} ,\end{aligned} \tag{63}$$

This inequality implies the global existence of u in the case $1 < p \leq 2$.

In the case where $p = 2$, we obtain a better estimate than (63). Indeed, replacing ω by $\mathrm{rot}\,(u - \alpha\Delta u)$ in the equality (35) and taking the inner product of this equation with $\mathrm{rot}\,(u - \alpha\Delta u)$, we readily obtain, for $t \geq 0$,

$$\|\mathrm{rot}\,(u-\alpha\Delta u)(t)\|_{L^2}^2 \leq \exp(-\frac{\nu\lambda_1}{2(1+2\alpha\lambda_1)}t)\|\mathrm{rot}\,(u_0-\alpha\Delta u_0)\|_{L^2}^2 + \frac{2(1+2\alpha\lambda_1)^2}{\lambda_1^2\nu^2}\|\mathrm{rot}\,f\|_{L^\infty(\mathbb{R}^+,L^2)}^2 . \tag{64}$$

For more details, we refer the reader to [55, Section 2.2]. In the case where $2 < p < +\infty$, we remark that the continuous Sobolev embedding $H^1(\mathbb{T}^2) \subset L^p(\mathbb{T}^2)$ holds for any $1 < p < +\infty$. Thus, we directly deduce from the inequalities (60) and (64) that, for $2 < p < +\infty$, for $0 \leq t < T^*(u_0)$,

$$\begin{aligned}
&\|\mathrm{rot}\,(u-\alpha\Delta u)(t)\|_{L^p} \\
&\qquad \leq \exp(-\frac{\nu}{\alpha}t)\,\|\mathrm{rot}\,(u_0-\alpha\Delta u_0)\|_{L^p} + \frac{\alpha}{\nu}\|\mathrm{rot}\,f\|_{L^\infty(\mathbb{R}^+,L^p)} \\
&\qquad\quad + C_S(p)\min(\alpha^{-1},\alpha^{-\frac{1}{2}})\Big[\exp(-\frac{\nu\lambda_1}{4(1+2\alpha\lambda_1)}t)\|\mathrm{rot}\,(u_0-\alpha\Delta u_0)\|_{L^2} \\
&\qquad\quad + \frac{\sqrt{2}(1+2\alpha\lambda_1)}{\lambda_1\nu}\|f\|_{L^\infty(\mathbb{R}^+,L^2)}\Big] ,
\end{aligned} \tag{65}$$

where $C_S(p)$ is a positive constant depending on the above-mentioned Sobolev embedding. This inequality implies the global existence of u in the case where $2 < p < +\infty$.
Notice that the existence of solutions on the time interval $(-\infty,0]$ also holds if the forcing term belongs to $L^\infty((-\infty,0),W^{k+1,p})$, $k \geq 0$, $1 < p < +\infty$. But the solution $u(t)$ may blow-up at $-\infty$.

Assume now that the forcing term $f \in W^{1,p}_{per}$ does not depend on the time. Then, we introduce the map $S_\alpha(t) : u_0 \in V^{3,p} \mapsto u(t) \in V^{3,p}$, where $u(t)$ is the solution of the system (2). The properties that we obtained in Sections 2.3 and 2.4 imply that $S_\alpha(t)$ is a dynamical system (and also a non-linear continuous group). Moreover, due to the estimates (63) to (65), $S_\alpha(t)$ admits a bounded absorbing set $\mathscr{B}_\alpha$. We can choose for $\mathscr{B}_\alpha$ the ball $B_{V^{3,p}}(0,C\alpha^{-1}R_\alpha(p))$ of center 0 and radius $C\alpha^{-1}R_\alpha(p)$ in $V^{3,p}$, where C is a positive constant and where

$$\begin{aligned}
R_\alpha(p) &= \frac{\alpha}{\nu}\|\mathrm{rot}\,f\|_{L^p} + \frac{\sqrt{2}(1+\lambda_1\alpha)^{1/2}}{\lambda_1\nu\sqrt{\alpha}}\|f\|_{L^2} \text{ if } 1 < p < 2 \\
R_\alpha(p) &= \frac{\sqrt{2}(1+2\alpha\lambda_1)}{\lambda_1\nu}\|\mathrm{rot}\,f\|_{L^2} \text{ if } p = 2 \\
R_\alpha(p) &= \frac{\alpha}{\nu}\|\mathrm{rot}\,f\|_{L^p} + C_S(p)\min(\alpha^{-1},\alpha^{-\frac{1}{2}})\frac{\sqrt{2}(1+2\alpha\lambda_1)}{\lambda_1\nu}\|f\|_{L^2} \text{ if } 2 < p < +\infty .
\end{aligned} \tag{66}$$

Remark 4. The norm $\|\mathrm{rot}\,(u-\alpha\Delta u)\|_{L^p}$ is appropriate for estimating the $V^{3,p}$-norm of the solution u of the second grade fluid equations (2). The estimates (63), (64) and (65) are good if $\alpha>0$ is fixed or bounded away from zero. However, when α goes to 0, these estimates can be improved as we did it in [55, Section 2, Estimates (2.27) and (2.28)]. In order to obtain better estimates in the case where α is small, one proceeds in the following way. Instead of introducing the variable $\omega=\mathrm{rot}\,(u-\alpha\Delta u)$, one introduces the variable $\omega^*=-\mathrm{rot}\,\Delta u$ and one performs a priori estimates for ω^* by considering the transport equation:

$$\partial_t\omega^*+u\cdot\nabla\omega^*+\frac{\nu}{\alpha}\omega^*+\frac{1}{\alpha}\partial_t\mathrm{rot}\,u=\frac{1}{\alpha}u\cdot\nabla\mathrm{rot}\,u+\frac{1}{\alpha}\mathrm{rot}\,f\;, \tag{67}$$

and using the above estimates.

In [55, Section 4], using these better upper bounds, we have proved the convergence of the solutions of (2) to those of the Navier-Stokes equations on finite time intervals when α goes to 0. We have also obtained convergence results for the global attrractors. For the comparaison of periodic orbits or other invariant sets of (2) with those of the Navier-Stokes equations, when α is small, we refer to [35] and [49]. For another convergence result of solutions of (2) to those of the Navier-Stokes equations, we refer to [39].

Remark 5. The above global existence of solutions of (2) is still true, when the periodic boundary conditions are replaced by homogeneous Dirichlet ones, provided the domain Ω is smooth enough (of class C^2) and simply connected.

3 Dynamics of the second grade fluids in the 2D torus

In the whole section, we assume that the forcing term $f\in W^{1,p}_{per}$ does not depend on the time variable t. By (32), the solution $u(t)$ of (2) writes, for any $t\in\mathbb{R}$,

$$\begin{aligned}(u-\alpha\Delta u)(t,x)=\omega(t,x)=&e^{-\frac{\nu}{\alpha}t}\omega_0(\varphi(0;t,x))\\&+\int_0^t e^{-\frac{\nu}{\alpha}(t-s)}\Big(\mathrm{rot}\,f(\varphi(s;t,x))+\frac{\nu}{\alpha}\mathrm{rot}\,u(s,\varphi(s;t,x))\Big)ds\;.\end{aligned} \tag{68}$$

3.1 Existence of a compact global attractor

In the previous section, we have seen that $S_\alpha(t)$ admits a bounded absorbing set $\mathscr{B}_\alpha$ and that the trajectories of bounded sets are bounded. Thus, by [32, Theorem 3.4.6] or [58, Theorem 2.26], in order to establish the existence of a compact global attractor in $V^{3,p}$, it suffices to show that $S_\alpha(t)$ is asymptotically compact (or asymptotically smooth). Due to [32, Lemma 3.2.6] or [58, Theorem 2.31]), it is enough to prove that $S_\alpha(t)$ can be written as a sum

$$S_\alpha(t) = \Sigma_\alpha(t) + K_\alpha(t)\ , \tag{69}$$

where $\Sigma_\alpha(t)$ is an asymptotically uniformly contracting map on the bounded sets of $V^{3,p}$ and $K_\alpha(t)$ is a compact map from $V^{3,p}$ into itself. Actually, due to the equality (68), it suffices to show that, for any $u_0 \in V^{3,p}$,

$$S_\alpha(t)u_0 - \alpha\Delta S_\alpha(t)u_0 \equiv \omega(t;u_0) = \Sigma^*_\alpha(t)u_0 + K^*_\alpha(t)u_0\ , \tag{70}$$

where $\Sigma^*_\alpha(t)$ is an asymptotically uniformly contracting map on the bounded sets of $V^{3,p}$ into $V^{0,p}$ and $K^*_\alpha(t)$ is a compact map form $V^{3,p}$ into $V^{0,p}$.

The proof of the property (70) is simple. According to the equality (68), we set, for any $u_0 \in V^{3,p}$,

$$\begin{aligned}
\Sigma^*_\alpha(t)u_0 \equiv & e^{-\frac{\nu}{\alpha}t}\omega_0(\varphi_{u_0}(0;t,x);u_0)\\
K^*_\alpha(t)u_0 \equiv & K_{\alpha,1}(t)u_0 + K_{\alpha,2}(t)u_0\\
\equiv & \int_0^t e^{-\frac{\nu}{\alpha}(t-s)}\operatorname{rot} f(\varphi_{u_0}(s;t,x))ds + \int_0^t e^{-\frac{\nu}{\alpha}(t-s)}\frac{\nu}{\alpha}\operatorname{rot} u(s,\varphi_{u_0}(s;t,x);u_0)ds\ .
\end{aligned} \tag{71}$$

Since $\|\omega_0(\varphi_{u_0}(0;t,x);u_0)\|_{L^p} = \|\omega_0(x;u_0)\|_{L^p}$, it follows that, for any bounded set $B_0 \in V^{3,p}$, for any $u_0 \in B_0$, for any $t \geq 0$,

$$\|\Sigma^*_\alpha(t)u_0\|_{L^p} \leq C_1(\|B_0\|_{V^{3,p}})e^{-\frac{\nu}{\alpha}t}\ , \tag{72}$$

where $C_1(\|B_0\|_{V^{3,p}})$ depends only on the norm of B_0 in $V^{3,p}$.

We next show that $K_{\alpha,1}(t)$ is a compact map from $V^{3,p}$ into L^p_{per}. Let B_0 be a bounded subset of $V^{3,p}$. The set $[0,1]\times\overline{B}_0$ is a compact subset of $\mathbb{R}^+ \times V^{2,p}$. Since the map $(s,u_0) \in [0,t]\times\overline{B}_0 \mapsto \operatorname{rot} f(\varphi_{u_0}(s;t,\cdot)) \in L^p$ is a continuous mapping, the image is a compact subset of L^p_{per}. By Mazur's theorem, it follows that $\int_0^t e^{-\frac{\nu}{\alpha}(t-s)}\operatorname{rot} f(\varphi_{u_0}(s;t,x))ds$ belongs to a compact set of L^p_{per}. Thus $K_{\alpha,1}(t)$ is a compact map.

Finally, we prove that $K_{\alpha,2}(t)$ is a compact mapping from $V^{3,p}$ into L^p_{per} by showing that $K_{\alpha,2}(t)$ maps every bounded set $B_0 \subset V^{3,p}$ into a compact set of $W^{1,p}_{per}$ and thus into a relatively compact set of L^p_{per}. Since, by Lemma 1, the following estimate holds

$$\|\operatorname{rot} u(s,\varphi_{u_0}(s;t,\cdot);u_0)\|_{W^{1,p}} \leq \|\operatorname{rot} u(s,\cdot;u_0,\|_{W^{1,p}}\exp(\int_0^t \|\nabla u(\sigma,\cdot;u_0)\|_{L^\infty}d\sigma\ , \tag{73}$$

and that

$$\|\operatorname{rot} u(\cdot,\cdot;u_0)\|_{L^\infty((0,t),W^{1,p})}\exp(\int_0^t \|\nabla u(\sigma,\cdot;u_0)\|_{L^\infty}d\sigma \leq C_2(t,\|B_0\|_{V^{3,p}})\ , \tag{74}$$

(where $C_2(t,\|B_0\|_{V^{3,p}})$ depends only on t and on the norm of B_0 in $V^{3,p}$), it follows that $\int_0^t e^{-\frac{\nu}{\alpha}(t-s)}\frac{\nu}{\alpha}\operatorname{rot}u(s,\varphi_{u_0}(s;t,x);u_0)ds$ belongs to a bounded set of $W^{1,p}_{per}$ and thus to a compact set of L^p_{per}. And then $K_{\alpha,2}(t)$ is a compact mapping from $V^{3,p}$ into L^p_{per}.

We have thus proved that $S_\alpha(t)$ is asymptotically compact in $V^{3,p}$.

Remark 6. We can prove in the same way that $S_\alpha(t)$ admits a compact global attractor when the periodic boundary conditions are replaced by homogeneous Dirichlet ones, provided the domain Ω is smooth enough (of class C^2) and simply connected.

3.2 Regularity of the compact global attractor

We now consider a complete bounded orbit $u(t)$ (with $u(0)=u_0$) contained in the global attractor $\mathscr{A}_\alpha$. In particular, we know that $\omega(t)\equiv rot(u-\alpha\Delta u)(t)$ satisfies the following inequality, for any $1<p<+\infty$,

$$\|rot(u-\alpha\Delta u)(t)\|_{V^{1,p}}\le R_\alpha(p)\ ,\quad \forall t\in\mathbb{R}\ . \tag{75}$$

Let $\tau<0$. Since $\omega(t)\in V^{1,p}$ exists for any $t\in\mathbb{R}$ and, by (75), is uniformly bounded on $\mathbb{R}$, using the formula (68), we can write, for $t\ge\tau$,

$$\begin{aligned}\omega(t,x)=&e^{-\frac{\nu}{\alpha}(t-\tau)}\omega(\tau,\varphi(\tau;t,x))\\&+\int_\tau^t e^{-\frac{\nu}{\alpha}(t-s)}\Big(\operatorname{rot}f(\varphi(s;t,x))+\frac{\nu}{\alpha}\operatorname{rot}u(s,\varphi(s;t,x))\Big)ds\ ,\end{aligned} \tag{76}$$

where $\varphi(s;t,x)\equiv\varphi_{u_0}(s;t,x)$. Letting τ go to $-\infty$, one deduces from (75) and (76) that, for any $t\in\mathbb{R}$,

$$\omega(t,x)=\int_{-\infty}^t e^{-\frac{\nu}{\alpha}(t-s)}\Big(\operatorname{rot}f(\varphi(s;t,x))+\frac{\nu}{\alpha}\operatorname{rot}u(s,\varphi(s;t,x))\Big)ds\ . \tag{77}$$

If $f\in W^{2,p}_{per}$, the right-hand side member of (77) belongs to a smoother space than L^p_{per}. We now want to prove that indeed $\omega(t)$ is bounded in a smoother space $W^{\theta,p}$, where $0<\theta\le 1$. To this end, we introduce the integral

$$I(t,s,x)\equiv\int_s^t e^{-\frac{\nu}{\alpha}(t-\sigma)}g(\sigma,\varphi(\sigma;t,x))d\sigma\ ,\quad g(\sigma,x)=\operatorname{rot}f(x)+\frac{\nu}{\alpha}\operatorname{rot}u(\sigma,x)\ .$$

Since

$$\frac{\partial}{\partial x_k}I(t,s,x)=\int_s^t e^{-\frac{\nu}{\alpha}(t-\sigma)}\Big(\sum_{i=1}^2\partial_{x_i}g(\sigma,\varphi(\sigma;t,x))\partial_{x_k}\varphi_i(\sigma;t,x)\Big)d\sigma\ , \tag{78}$$

thus, by Lemma 1, we have,

$$
\begin{aligned}
\|\frac{\partial}{\partial x_k} I(t,s,x)\|_{L^p} \leq & \int_s^t e^{-\frac{\nu}{\alpha}(t-\sigma)} \|\nabla g\|_{L^p} \|\nabla \varphi(\sigma;t,\cdot)\|_{L^\infty} d\sigma \\
& \leq \|\nabla g\|_{L^\infty(L^p)} \int_s^t e^{-\frac{\nu}{\alpha}(t-\sigma)} \exp\Big(\int_\sigma^t \|\nabla u(\tau)\|_{L^\infty} d\tau \Big) d\sigma \ .
\end{aligned}
\tag{79}
$$

Assume now that

$$
\sup_{t\in\mathbb{R}} \|\nabla u(t)\|_{L^\infty} < \frac{\nu}{\alpha} \ , \tag{80}
$$

and set $a_0 \equiv \frac{\nu}{\alpha} - \sup_{\tau\in\mathbb{R}} \|\nabla u(\tau)\|_{L^\infty} > 0$. From the estimate (79) and the hypothesis (80), we deduce that, for any $s \leq t$,

$$
\|\frac{\partial}{\partial x_k} I(t,s,x)\|_{L^p} \leq a_0^{-1} \Big(\|\mathrm{rot}\, f\|_{W^{1,p}} + \frac{\nu}{\alpha} \sup_{\tau\in\mathbb{R}} \|\mathrm{rot}\, u(\tau)\|_{W^{1,p}} \Big) \ . \tag{81}
$$

Since this inequality holds for any t, s, we conclude that, for any $t \in \mathbb{R}$,

$$
\|\nabla(\mathrm{rot}\, u - \alpha\Delta \mathrm{rot}\, u)(t)\|_{L^p} \leq a_0^{-1} \Big(\|\mathrm{rot}\, f\|_{W^{1,p}} + \frac{\nu}{\alpha} \sup_{\tau\in\mathbb{R}} \|\mathrm{rot}\, u(\tau)\|_{W^{1,p}} \Big) \ . \tag{82}
$$

Assume now that $\sup_{v\in\mathscr{A}_\alpha} \|\nabla v\|_{L^\infty} < \frac{\nu}{\alpha}$ and set $a_1 \equiv \frac{\nu}{\alpha} - \sup_{v\in\mathscr{A}_\alpha} \|\nabla v\|_{L^\infty} > 0$. From the estimates (82) and (64), we deduce the following upper bound for any element u_0 in the global attractor $\mathscr{A}_\alpha$,

$$
\|\nabla(\mathrm{rot}\, u_0 - \alpha\Delta \mathrm{rot}\, u_0)\|_{L^p} \leq a_1^{-1} (\|\mathrm{rot}\, f\|_{W^{1,p}} + \frac{\nu}{\alpha} M_\alpha(p)) \ , \tag{83}
$$

where

$$
\begin{aligned}
M_\alpha(p) &= C_S \frac{\sqrt{2}(1+2\alpha\lambda_1)}{\max(\sqrt{\alpha},\alpha)\lambda_1 \nu} \|\mathrm{rot}\, f\|_{L^2} \text{ if } 2 < p < +\infty \\
M_\alpha(p) &= C_S \frac{\sqrt{2}(1+2\alpha\lambda_1)}{\alpha\lambda_1 \nu} \|\mathrm{rot}\, f\|_{L^2} \text{ if } p \geq 2 \ ,
\end{aligned}
\tag{84}
$$

and where $C_S > 0$ is a Sobolev embedding constant.

Assume now that, for the complete bounded orbit $u(t)$, the condition (80) is not true. Then we are not able to conclude that $\|\nabla(\mathrm{rot}\, u - \alpha\Delta \mathrm{rot}\, u)(t)\|_{L^p}$ is uniformly bounded for $t \in \mathbb{R}$. However, we can still show that there exists a positive number $0 < \theta < 1$, such that $\|(\mathrm{rot}\, u - \alpha\Delta \mathrm{rot}\, u)(t)\|_{W^{\theta,p}}$ is bounded by using an interpolation argument. Indeed, let $0 < \theta < 1$ such that

$$
a_\theta \equiv \frac{\nu}{\alpha} - \theta \sup_{\tau\in\mathbb{R}} \|\nabla u(\tau)\|_{L^\infty} > 0 \ . \tag{85}
$$

We next define the continuous linear map $\mathscr{T} : h \in L^\infty(\mathbb{R}, L^p_{per}) \mapsto w \in L^\infty(\mathbb{R}, L^p_{per})$, which is the solution of the integral equation

$$
\mathscr{T}(h)(t,x) \equiv w(t,x) = \int_{-\infty}^t e^{-\frac{\nu}{\alpha}(t-\sigma)} h(\sigma, \varphi(\sigma;t,x)) d\sigma \ .
$$

We remark that the vorticity ω, satisfying the equality (77), is given by $\omega = \mathscr{T}(g)$. The above computations show that $\mathscr{T}$ is also a continous map from $L^\infty(\mathbb{R}, W^{1,p}_{per})$ into itself, and thus, by interpolation, a continuous linear map from $L^\infty(\mathbb{R}, W^{\theta,p}_{per})$ into itself, for $0 \le \theta \le 1$. Moreover, for any $t \in \mathbb{R}$, we have,

$$\|\mathscr{T}(h)(t,\cdot)\|_{W^{\theta,p}} \le \int_{-\infty}^{t} e^{-\frac{\nu}{\alpha}(t-\sigma)} \|h(\sigma, \varphi(\sigma;t,\cdot))\|_{W^{\theta,p}} d\sigma \ .$$

As we will see, due to the condition (85), $g(\sigma, \varphi(\sigma;t,\cdot))$ belongs to $L^\infty(\mathbb{R}, W^{\theta,p}_{per})$ and thus ω satisfies the above inequality.
Remarking that

$$\begin{aligned} \|g(\sigma, \varphi(\sigma;t,\cdot))\|_{L^p} &\le \|g(\sigma,\cdot)\|_{L^p} \\ \|g(\sigma, \varphi(\sigma;t,\cdot))\|_{W^{1,p}} &\le \|g(\sigma,\cdot)\|_{W^{1,p}} \exp\big(\int_\sigma^t \|\nabla u(\tau)\|_{L^\infty} d\tau\big) \end{aligned} \tag{86}$$

we obtain by interpolation that,

$$\begin{aligned} \|\omega(t,\dot{})\|_{W^{\theta,p}} \le& \int_{-\infty}^{t} e^{-\frac{\nu}{\alpha}(t-\sigma)} \exp\theta\big(\int_\sigma^t \|\nabla u(\tau)\|_{L^\infty} d\tau\big) \|g(\sigma,\cdot)\|_{W^{\theta,p}} d\sigma \\ \le& a_\theta^{-1} \Big(\|\mathrm{rot} f\|_{W^{\theta,p}} + \frac{\nu}{\alpha} \sup_{\tau\in\mathbb{R}} \|\mathrm{rot}\, u(\tau)\|_{W^{\theta,p}} \Big) \ . \end{aligned} \tag{87}$$

And, we conclude that, for any $t \in \mathbb{R}$,

$$\|(\mathrm{rot}\, u - \alpha\Delta\mathrm{rot}\, u)(t)\|_{W^{\theta,p}} \le a_\theta^{-1} \Big(\|\mathrm{rot} f\|_{W^{\theta,p}} + \frac{\nu}{\alpha} \sup_{\tau\in\mathbb{R}} \|\mathrm{rot}\, u(\tau)\|_{W^{\theta,p}} \Big) \ . \tag{88}$$

If $\sup_{v\in\mathscr{A}_\alpha} \|\nabla v\|_{L^\infty} \ge \frac{\nu}{\alpha}$, we take $0 < \theta < 1$ so that

$$a_{1,\theta} \equiv \frac{\nu}{\alpha} - \theta \sup_{v\in\mathscr{A}_\alpha} \|\nabla v\|_{L^\infty} > 0 \ .$$

Thus, we obtain the following upper bound for any u_0 in the global attractor

$$\|\mathrm{rot}\, u_0 - \alpha\Delta\mathrm{rot}\, u_0\|_{W^{\theta,p}} \le a_{1,\theta}^{-1} (\|\mathrm{rot} f\|_{W^{\theta,p}} + \frac{\nu}{\alpha} M_\alpha(p)) \ , \tag{89}$$

Remark 7. One may wonder if the compact global attractors depend on p. Let $1 < p_1 < p_2 < +\infty$ and assume that the forcing term f belongs to $W^{1+\theta,p_2}$, $0 < \theta < 1$. We denote $\mathscr{A}_\alpha(p_1)$ and $\mathscr{A}_\alpha(p_2)$ the corresponding global attractors. It is clear that $\mathscr{A}_\alpha(p_2) \subset \mathscr{A}_\alpha(p_1)$. Taking into account the above regularity argument, we may show by using Sobolev embeddings and a bootstrap argument that $\mathscr{A}_\alpha(p_1) \subset \mathscr{A}_\alpha(p_2)$ and thus $\mathscr{A}_\alpha(p_1) = \mathscr{A}_\alpha(p_2)$.

Next we consider higher order derivatives of $\omega(t)$. Differentiating $\frac{\partial}{\partial x_k} I(t,s,x)$ with respect to x_l, we obtain

$$\frac{\partial^2}{\partial x_k \partial x_l} I(t,s,x) = \int_s^t e^{-\frac{\nu}{\alpha}(t-\sigma)} \Big[\sum_{i,j=1}^{2} \partial^2_{x_i x_j} g(\sigma, \varphi(\sigma;t,x)) \partial_{x_k} \varphi_i(\sigma;t,x) \partial_{x_l} \varphi_j(\sigma;t,x) + \sum_{i=1}^{2} \partial_{x_i} g(\sigma, \varphi(\sigma;t,x)) \partial^2_{x_k x_l} \varphi_i(\sigma;t,x) \Big] d\sigma \,, \tag{90}$$

from which we deduce that, for any $s \leq t$,

$$\|D_x^2 I(t,s,\cdot)\|_{L^p} \leq \int_s^t e^{-\frac{\nu}{\alpha}(t-\sigma)} \big[\|\nabla g\|_{L^p} \|D_x^2 \varphi(\sigma;t,\cdot)\|_{L^\infty} + \|D_x^2 g\|_{L^p} \|\nabla \varphi(\sigma;t,\cdot)\|^2_{L^\infty} \big] d\sigma \,. \tag{91}$$

Arguing as in Lemma 1 and using the inequality (14) of Lemma 1, we get the following estimate, for any $\sigma \leq t$,

$$\|D_x^2 \varphi(\sigma;t,\cdot)\|_{L^\infty} \leq \|D_x^2 u(\sigma)\|_{L^\infty} \exp\Big(3 \int_\sigma^t \|\nabla u(\tau)\|_{L^\infty} d\tau \Big) \,. \tag{92}$$

The estimates (91) and (92) imply, for any $s \leq t$,

$$\begin{aligned} \|D_x^2 I(t,s,\cdot)\|_{L^p} \leq & \|\nabla g\|_{L^\infty(L^p)} \|u\|_{L^\infty(W^{2,\infty})} \int_s^t e^{-\frac{\nu}{\alpha}(t-\sigma)} \exp\Big(3 \int_\sigma^t \|\nabla u(\tau)\|_{L^\infty} d\tau \Big) d\sigma \\ & + \|D_x^2 g\|_{L^\infty(L^p)} \int_s^t e^{-\frac{\nu}{\alpha}(t-\sigma)} \exp\Big(2 \int_\sigma^t \|\nabla u(\tau)\|_{L^\infty} d\tau \Big) d\sigma \,. \end{aligned} \tag{93}$$

Assume now that

$$\sup_{t \in \mathbb{R}} 3 \|\nabla u(t)\|_{L^\infty} < \frac{\nu}{\alpha} \,, \tag{94}$$

and set $a_2 \equiv \frac{\nu}{\alpha} - 3 \sup_{\tau \in \mathbb{R}} \|\nabla u(\tau)\|_{L^\infty} > 0$. Then, it follows from (93) and (94) that, for any $t \in \mathbb{R}$,

$$\begin{aligned} \|(\operatorname{rot} u - \alpha \Delta \operatorname{rot} u)(t)\|_{W^{2,p}} \leq & a_2^{-1} \Big[(\|\operatorname{rot} f\|_{W^{1,p}} + \frac{\nu}{\alpha} \|\operatorname{rot} u(\cdot)\|_{L^\infty(W^{1,p})}) \|u(\cdot)\|_{L^\infty(W^{2,\infty})} \\ & + (\|\operatorname{rot} f\|_{W^{2,p}} + \frac{\nu}{\alpha} \|\operatorname{rot} u(\cdot)\|_{L^\infty(W^{2,p})}) \Big] \\ \equiv & a_2^{-1} M_{2,\alpha}(p) \,. \end{aligned} \tag{95}$$

If $a_2 \equiv \frac{\nu}{\alpha} - 3 \sup_{v \in \mathscr{A}_\alpha} \|\nabla v\|_{L^\infty} > 0$, then the estimate (95) holds for any element u of $\mathscr{A}_\alpha$. By a recursion argument, we finally obtain the third assertion of Theorem 3.

Remark 8. The above regularity results of $\mathscr{A}_\alpha$ still hold, when the periodic boundary conditions are replaced by homogeneous Dirichlet ones, provided the domain Ω is smooth enough (of class C^2) and simply connected.

Remark 9. In the above regularity proofs, in order to get the V^{3+m} regularity, we need to assume that $\frac{\nu}{\alpha} - m\sup_{v\in\mathscr{A}_\alpha}\|\nabla v\|_{L^\infty} > 0$. This method does not allow to show that the attractor is bounded in C^∞ or in the set of analytic functions, even if f is analytic. So these regularity properties remain an open question. Note that if f is integrable in time and in a Gevrey class in the spatial variable and if the initial data are in a Gevrey class in the spatial variable, then the solutions of (2) also have Gevrey regularity (see [52] and [54]).

3.3 Finite-dimensional properties

We can also wonder if the dynamics of (2) has finite-dimensional properties. Using the methods of [61], we can certainly prove that the Hausdorff dimension of $\mathscr{A}_\alpha$ is finite. We leave it to the reader to check it.
We next want to recall a "finite-dimensional" property, which is well adapted to the Hilbert space setting, that is, to the case $p = 2$. Let P denote the classical orthogonal projection of $(L^2_{per}(\mathbb{T}^2))^2$ onto the subspace $H \equiv V^{0,2}$ of L^2-divergence-free vector fields. We also introduce the orthogonal projection P_n in H onto the space spanned by the eigenfunctions corresponding to the first n eigenvalues of the Stokes operator $A = -P\Delta$. Finally, we introduce the projection $Q_n = I - P_n$.

In [55], we have shown that there exists an integer N, such that, on the compact global attractor, the dynamics of (2) reduces to the dynamics of a system of N ordinary differential equations defined on $P_N V^{3,2}$ (see [55, Theorem 1.2]).
In [55], like in [34, Theorem 2.7], we deduced, from [55, Theorem 1.2], the so-called "finite number of determining modes property" for the system (2), when α is small enough. The property of "finite number of determining modes" was introduced and proved for the two-dimensional Navier-Stokes equations by Foias and Prodi in 1967 ([19]). This property means that the asymptotic behaviour in time of the second grade fluid system depends only on a finite number of parameters (called the determining modes).

Theorem 5. *Let f be given in $W^{1+d,2}_{per}$, $d > 0$.*
We assume that $\nu - 2\alpha(\sup_{z\in\mathscr{A}_\alpha}\|\nabla z\|_{L^\infty}) > 0$. Then System (2) has the property of finite number of determining modes, that is, there exists a positive integer N_0 such that, for any u_0, u_1 in $V^{3,2}$, the property

$$\|P_{N_0}S_\alpha(t)u_0 - P_{N_0}S_\alpha(t)u_1\|_{V^3} \longrightarrow_{t\to+\infty} 0$$

implies that

$$\|S_\alpha(t)u_0 - S_\alpha(t)u_1\|_{V^3} \longrightarrow_{t\to+\infty} 0\ .$$

One also could directly prove Theorem 5, by performing appropriate a priori estimates. But, showing Theorem 5 as a consequence of [55, Theorem 1.2] and of the proof of [34, Theorem 2.7] is more elegant.

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