# Die gedämpfte Klein-Gordon Gleichung und Invariante Mannigfaltigkeiten

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# The Klein-Gordon equation

**Klein** and **Gordon** (1926) independently derived a relativistic equation for a charged particle in an electromagnetic field, using ideas of quantum theory $\rightsquigarrow$ Klein-Gordon (or Klein-Gordon-Fock) equation:

$$\frac{1}{c^2}\psi_{tt} - \Delta\psi + (\frac{mc}{h})^2\psi = 0,$$

for the special case of a free particle in $\mathbb{R}^3$.
Mathematical generalisation:

$$\frac{1}{c^2}\psi_{tt} - \Delta\psi + \psi + V'(\psi) = 0,$$

where the potential $V$ is s.t. $V'$ is a nonlinear function. Invariance under the Lorentz transformations.

Examples: $V'(\psi) = \varepsilon|\psi|^{p-1}\psi$ where $2 < p \leq 5$, $\varepsilon = \pm 1$,
$V'(\psi) = -\psi + \sin\psi$ : sine-Gordon equation

The non-relativistic version leads to the Schrödinger equation

$$i\psi_t + \Delta\psi + V'(\psi) = 0$$

# The focusing Klein-Gordon equation: the subcritical case

Let $\mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

We consider the NLKG equation with or without damping $\alpha \geq 0$:

$$\begin{aligned} &u_{tt} + 2\alpha u_t - \Delta u + u - u^3 = 0, \ \ x \in \mathbb{R}^3, t \geq 0 \\ &(u(x,0), u_t(x,0)) = (u_0(x), u_1(x)) \in \mathcal{H}, \end{aligned} \qquad (1)$$

which writes as the first order system

$$U_t \equiv \frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & Id \\ \Delta - Id & -2\alpha Id \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ u^3 \end{pmatrix}, U(0) = U_0$$

or also

$$U_t = B_\alpha U + F(U)\ , U(0) = U_0.$$

The equation $U_t = B_\alpha U$ generates a linear $C^0$-group $e^{B_\alpha t}$ in $\mathcal{H}$.

The function $U(t) = (u(t), u_t(t)) \in C^0((-T,T), \mathcal{H})$ is a **mild (or integral) solution** of (1) if, for any $t \in (-T,T)$,

$$U(t) \equiv S_\alpha(t) U_0 = e^{B_\alpha t} U_0 + \int_0^t e^{B_\alpha (t-s)} F(U(s)) ds, \quad \text{Duhamel formula}$$

# Basic well-posedness I: $\alpha \geq 0$

Theorem (Local existence)

*1) For any $U_0 \in \mathcal{H}$, there exists a unique mild solution $S_\alpha(t)U_0 \in C^0([-T, T], \mathcal{H})$ for some $T \geq T_0(\|U_0\|_{\mathcal{H}}) > 0$.*

*2) Continuity w. r. to $U_0$; persistence of regularity.*

*3) The energy functional $E(u(t), u_t(t)) \in C^1([0, T))$ satisfies*

$$\frac{d}{dt}(E(u(t), u_t(t))) = -2\alpha\|u_t(t)\|_{L^2}^2 \leq 0,$$

*where*

$$E(\varphi, \psi) \equiv \int_{\mathbb{R}^3} (\frac{1}{2}(|\psi|^2 + |\nabla\varphi|^2 + |\varphi|^2) - \frac{1}{4}|\varphi|^4)dx.$$

Energy conservation if $\alpha = 0$. Strict Lyapunov functional if $\alpha > 0$

# Basic well-posedness II: $\alpha \geq 0$

If $U_0 \in \mathcal{H}$ and $\alpha \geq 0$ small, one writes ($\|.\|_{L^3_t L^6_x}$ is a Strichartz norm):

$$\|S_\alpha(t)U_0\|_{\mathcal{H}} \leq C_0(\|U_0\|_{\mathcal{H}} + \int_0^t \|u^3\|_{L^2} ds) \leq C(\|U_0\|_{\mathcal{H}} + \|u\|^3_{L^3_t L^6_x}).$$

Theorem (Global existence or blow-up)

*1) If $\|U_0\|_{\mathcal{H}} \ll 1$, then global existence and $\|u\|_{L^3(\mathbb{R}^+, L^6(\mathbb{R}^3))} < +\infty$*
*2) Let $T^* > 0$ be the maximal forward time of existence:*
*$T^* < +\infty \Longrightarrow \|u\|_{L^3((0,T^*),L^6(\mathbb{R}^3))} = \infty$.*
*3) Assume that $T^* = \infty$ and $\|u\|_{L^3((0,+\infty),L^6(\mathbb{R}^3))} < \infty$, then,*
*- if $\alpha = 0$, u scatters , i.e., there exists $(\tilde{u}_0, \tilde{u}_1) \in \mathcal{H}$ s.t.,*

$$S_\alpha(t)U_0 \equiv U(t) = e^{B_\alpha t}(\tilde{u}_0, \tilde{u}_1) + o_{\mathcal{H}}(1), \quad t \to \infty,$$

*- if $\alpha > 0$, $S_\alpha(t)U_0 \to 0, \quad t \to \infty$.*

**Small data:** global existence and scattering (or convergence to 0)
**Large data:** can have finite time blowup.
Simpler case of radial solutions: Restrict to the space $\mathcal{H}_{rad}$

# Forward scattering set and purpose of this talk

*Forward scattering set:*

$$\mathcal{S}_+ = \{(u_0, u_1) \in \mathcal{H}_{rad} \mid S_0(t)(u_0, u_1) \text{ exists globally and scatters}\}$$

$\mathcal{S}_+ \supset B(0, r)$ in $\mathcal{H}_{rad}$, $\mathcal{S}_+ \neq \mathcal{H}_{rad}$, $\mathcal{S}_+$ open in $\mathcal{H}_{rad}$.
**Questions of Nakanishi and Schlag when $\alpha = 0$:** is $\mathcal{S}_+$ bounded in $\mathcal{H}_{rad}$? Is $\partial\mathcal{S}_+$ a smooth manifold separating regions of finite time blowup and global existence? Same questions for $\alpha > 0$.

**First goal:** Describe transition between blowup/global existence and scattering or convergence. Results when the energy is at most slightly larger than the energy of the "ground state solution".

Case $\alpha = 0$: nice book "Invariant manifolds and dispersive Hamiltonian Evolution Equations" of K. Nakanishi, W. Schlag. See also J. Krieger, ...
Case $\alpha > 0$: work in progress of N. Burq, W. Schlag, G.R.

**Second goal:** Illustrate with this study how the simultaneous use of PDE technics and classical tools in the dynamical systems theory gives new results in PDE's

# Stationary solutions, ground state $\pm Q$

*Stationary solution* $u(t,x) = \varphi(x)$ of NLKG is a weak solution of

$$-\Delta\varphi + \varphi = \varphi^3 \tag{2}$$

Minimization problem: $\inf\left\{\|\varphi\|^2_{H^1} \,|\, \varphi \in H^1,\ \|\varphi\|_{L^4} = 1\right\}$
has a radial solution $\varphi_\infty > 0$, decaying exponentially, $Q = \lambda\varphi_\infty$ satisfies (2) for some $\lambda > 0$ (Z. Nehari, 1963).
Coffman (1972): **unique radial positive solution** $Q$

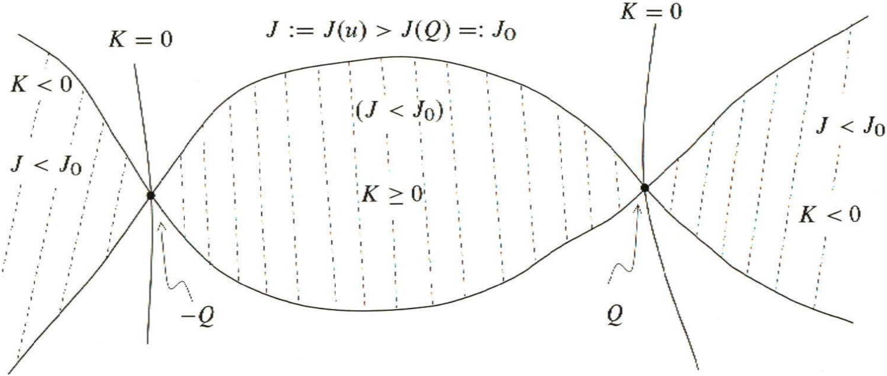
**Stationary energy:** $J(\varphi) := \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\varphi^2 - \frac{1}{4}\varphi^4\right) dx$

**Dilation functional:**
$K_0(\varphi) = \langle J'(\varphi)|\varphi\rangle = \int_{\mathbb{R}^3}(|\nabla\varphi|^2 + \varphi^2 - \varphi^4)(x)\, dx$

**Variational characterization**

$$J(Q) = \inf\{J(\varphi) \,|\, \varphi \in H^1 \setminus \{0\},\ K_0(\varphi) = 0\} \tag{3}$$

The infima are achieved uniquely by $\pm Q$, up to translations.
Existence of infinite number of smooth nodal solutions of (2).




## Splitting of $J(u) < J(Q)$ by the sign of $K = K_0$

Same picture for $E(u, u_t) < J(Q)$. The solutions are trapped by $K_0 \geq 0$ or $K_0 < 0$ in that case.

# Payne-Sattinger criterion: $\alpha \geq 0$

**Invariant decomposition** of $E < J(Q)$: (Payne-Sattinger 1975)

$$\mathcal{PS}_+ := \{(u_0, u_1) \in \mathcal{H} \mid E(u_0, u_1) < J(Q),\ K_0(u_0) \geq 0\}$$
$$\mathcal{PS}_- := \{(u_0, u_1) \in \mathcal{H} \mid E(u_0, u_1) < J(Q),\ K_0(u_0) < 0\}$$

In $\mathcal{PS}_+$, **global existence** for $t \in \mathbb{R}^+$: $K_0(u(t)) \geq 0$ implies:

$$\|u(t)\|_{H^1}^2 + \|u_t(t)\|_2^2 = 4E(U(t)) - (K_0(u(t)) + \|u_t(t)\|_2^2) \leq 4E(U(t)).$$

In $\mathcal{PS}_-$, **finite time blowup** for $t \in \mathbb{R}$ $(-K_0(u(t)) \geq \delta > 0)$
Convexity argument for $\alpha \geq 0$ with the auxiliary function
$y^\alpha(t) \equiv \frac{1}{2}\|u(t)\|_{L^2}^2 + \alpha \int_0^t \|u(s)\|_{L^2}^2 ds$:

$$y_{tt}^0 = \|u_t\|_2^2 - K_0(u) = 3\|u_t\|_2^2 + \|u\|_{H^1}^2 - 4E(U) > \delta.$$

Thus, $y_t^0(t)$ and $y^0(t)$ go to $+\infty$ as $t \to +\infty$. One proves that, for $t \geq t_0$, $\partial_t(y^{-1/2})(t) \leq \partial_t(y^{-1/2})(t_0) < 0$ and that there exists $t_1 > 0$ s.t. $(y^{-1/2})(t_1) = 0$.

# Linearized equation around $(Q, 0)$

If we plug $u = Q + w$ into the NLKG equation (1), we get

$$w_{tt} + 2\alpha w_t + L_+ w - 3Qw^2 - w^3 = 0, \qquad (4)$$

with $L_+ = -\Delta + Id - 3Q^2$ the linearized elliptic operator. One has

- $\langle L_+ Q | Q \rangle = -2\|Q\|_4^4 < 0$
- $L_+ \rho = -k^2 \rho$ unique (simple) negative eigenvalue, no kernel over radial functions
- Gap property: $L_+$ has no eigenvalues in $(0, 1]$, no threshold resonance (Demanet-Schlag, Costin-Huang-Schlag)
- $\sigma_{cont}(L_+) = [1, +\infty)$

Rewriting (4) as a system with $W = (w, w_t)^t$, we have

$$W_t = \begin{pmatrix} 0 & Id \\ -L_+ & -2\alpha Id \end{pmatrix} \begin{pmatrix} w \\ w_t \end{pmatrix} + \begin{pmatrix} 0 \\ 3Qw^2 + w^3 \end{pmatrix} = A_\alpha W + N(W)$$

# Non hyperbolic/Hyperbolic dynamics

**Spectrum of** $A_\alpha$: $\sigma(A_\alpha) = \{\mu_\alpha^-, \mu_\alpha^+\} \cup \sigma_{cont}(A_\alpha)$,
$\mu_\alpha^\pm = -\alpha \pm \sqrt{\alpha^2 + k^2}$ : simple eigenvalues; $\mu_\alpha^- < 0$ and $\mu_\alpha^+ > 0$
(the eigenprojectors are denoted $P_\alpha^\pm$).

**Case** $\alpha = 0$: $\sigma(A_0) = \{-k, +k\} \cup i(-\infty, 1] \cup i[1, +\infty)$
$\mathcal{H}_0^s = P_0^- \mathcal{H}$, $\mathcal{H}_0^u = P_0^+ \mathcal{H}$, $\mathcal{H}_0^c = (Id - P_0^- - P_0^+)\mathcal{H}$
Existence of a center space $\mathcal{H}_0^c$ or center stable space $\mathcal{H}_0^{sc} = \mathcal{H}_0^s \oplus \mathcal{H}_0^c$.
**Non hyperbolic dynamics**

**Case** $\alpha > 0$: **No central part. Hyperbolic dynamics**
Case $0 < \alpha \leq 1$:
$\sigma_{cont}(A_\alpha) = \{-\alpha + i(-\infty, \sqrt{1 - \alpha^2}]\} \cup \{-\alpha + i[\sqrt{1 - \alpha^2}, +\infty)\}$
Case $\alpha > 1$:
$\sigma_{cont}(A_\alpha) = [-\alpha - \sqrt{\alpha^2 - 1}, -\alpha + \sqrt{\alpha^2 - 1}] \cup \{-\alpha + i(-\infty, +\infty)\}$
Unstable and stable spaces $\mathcal{H}_\alpha^u = P_\alpha^+ \mathcal{H}$, $\mathcal{H}_\alpha^s = (Id - P_\alpha^+)\mathcal{H}$.

# Classical invariant manifolds theory in finite dimensions

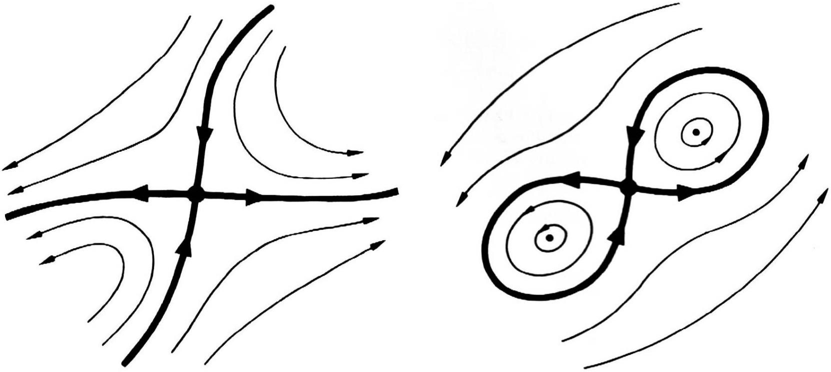
$y_t = Ay + f(y)$, $f(0) = 0, Df(0) = 0$, $\mathbb{R}^n = X^s \oplus X^c \oplus X^u$, where $X^s, X^c, X^u$ are $A$-invariant,
$\sigma(A_s) = \{\,\mathrm{Re}\,\lambda < 0\}$, $\sigma(A_u) = \{\,\mathrm{Re}\,\lambda > 0\}$, $\sigma(A_c) = \{\lambda \in i\mathbb{R}\}$.

Hyperbolic case $X^c = \{0\}$: Non hyperbolic case $X^c \neq \{0\}$:
Existence of **locally invariant** center, center stable manifolds $W^c$, $W^{cs}$ at 0, tangent to $X^c$ and $X^{cs}$ (non unique in general !)
In both cases: $\exists!$ local stable and unstable manifolds $W^s$, $W^u$, tangent at 0 to $X^s$, $X^u$. Invariance properties.

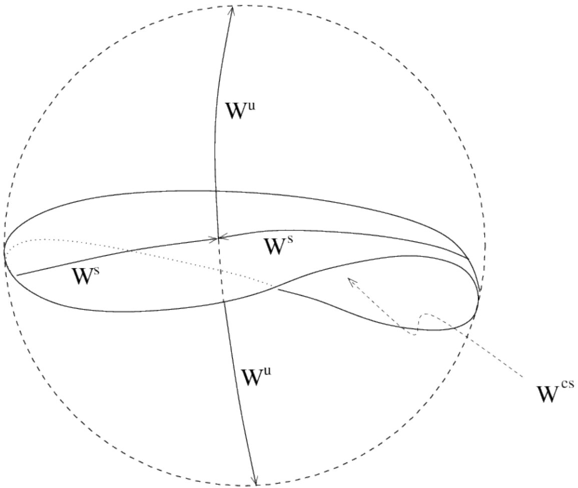
$$W^s = \{|y_0| < r \,|\, y(t) \to 0 \text{ exponential fast as } t \to \infty\}$$
$$W^u = \{|y_0| < r \,|\, y(t) \to 0 \text{ exponential fast as } t \to -\infty\}$$

$$y_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -a & 1 \\ 0 & 0 & -1 & -a \end{pmatrix} y + O(|y|^2),\ \sigma(A) = \{1, -1, -a-i, -a+i\}$$

# Stable and unstable manifolds: Hyperbolic case



# The invariant manifolds in the cases $\alpha = 0$ and $\alpha > 0$




$\alpha = 0$: Stable, unstable and center-stable manifolds $W^s$, $W^u$ and $W^{cs}$

$\alpha > 0$: $W^{cs} \rightsquigarrow W^s_\alpha$, $W^u \rightsquigarrow W^u_\alpha$ and $W^s \rightsquigarrow W^{ss}_\alpha$

# Theorem of Nakanishi and Schlag when $\alpha = 0$

Theorem (Nakanishi, Schlag)

*There exists $\varepsilon_0 > 0$ s.t. if*

$$E(u_0, u_1) < E(Q, 0) + \varepsilon_0^2, \quad \textit{Energy assumption}$$

*then, for $(u(t), u_t(t)) \equiv S_0(t)(u_0, u_1)$, one has, either*

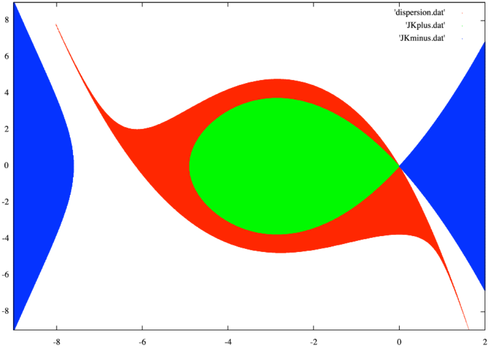
1. *finite time blowup*
2. *global existence and scattering to* $0$
3. *global existence and scattering to* $(\pm Q, 0)$*:* $(u(t), u_t(t)) = (\pm Q, 0) + (v(t), v_t(t)) + o_{\mathcal{H}}(1)$ *as* $t \to \infty$*, where* $(v(t), v_t(t)) = \Sigma_0(t)(v_0, v_1) \in \mathcal{H}$

*All 9 combinations of this trichotomy allowed as* $t \to \pm\infty$*.*

$\partial\mathcal{S}_+$ is the **unique center stable (smooth) manifold** of $(\pm Q, 0)$ (codimension 1), giving (3) and separating the open regions (1) and (2). Existence of 1-dimensional strongly stable, unstable manifolds of $(\pm Q, 0)$.

Stable manifold: Duyckaerts-Merle, Duyckaerts-Holmer-Roudenko

# Numerical 2-dim section through $\partial\mathcal{S}_+$ (Donninger, Schlag)




$$(Q + Ae^{-r^2}, Be^{-r^2})$$

- soliton at $(A, B) = (0, 0)$, $(A, B)$ vary in $[-9, 2] \times [-9, 9]$
- RED: global existence, WHITE: finite time blowup, GREEN: $\mathcal{PS}_+$, BLUE: $\mathcal{PS}_-$
- Results in a neighbourhood of $(Q, 0)$.

# The case $\alpha > 0$

Theorem (I: Burq, Schlag, G.R.)

*(i) There exists a continuous function* $\varepsilon : \alpha \in [0, +\infty) \mapsto \varepsilon(\alpha) > 0$ *s.t.,* $\inf_{0 \leq \alpha < +\infty} \varepsilon(\alpha) \geq \varepsilon_0 > 0$ *and s.t., if*

$$E(u_0, u_1) < E(Q, 0) + \varepsilon^2(\alpha), \quad \textit{Energy assumption}$$

*then,* $S_\alpha(t)(u_0, u_1)$ *satisfies, either*

1. *finite time blowup*
2. *global existence and convergence to* 0
3. *global existence and convergence to* $(\pm Q, 0)$

*(ii) The* **(smooth) stable manifold** *of* $(\pm Q, 0)$ *is of codimension* 1*, gives (3) and separates the open regions (1) and (2). Existence of a* 1*-dimensional unstable manifold of* $(\pm Q, 0)$.

At most, one sign change of $K_0(u(t))$ near $(Q, 0)$

Earlier convergence result to 0 of Keller (1983)

# The case $\alpha > 0$

Theorem (II: Burq, Schlag, G.R.)

*Let $(u_0, u_1) \in \mathcal{H}_{rad}$, then $S_\alpha(t)(u_0, u_1)$ satisfies, either*

1. *finite time blowup*
2. *or global existence and convergence to an equilibrium point $(\tilde{Q}, 0)$ of (1).*

Proof: functional and dynamical systems arguments

**The particular case $\alpha > 0$ large:** we set $u(t, x) = u_\varepsilon(\tau, x)$, where $\tau = \varepsilon^{1/2} t$, $\varepsilon = (2\alpha)^{-2}$:

$$\varepsilon u_{\varepsilon,\tau\tau} + u_{\varepsilon,\tau} - \Delta u_\varepsilon + u_\varepsilon - u_\varepsilon^3 = 0, \ (u_\varepsilon(0), u_{\varepsilon,\tau}(0)) = (u_0, \varepsilon^{-1/2} u_1).$$

Compare $(u_\varepsilon, u_{\varepsilon,t})$ with $(v_0(\tau), v_{0,\tau}(\tau))$ where $v_0(\tau)$ is the solution of the parabolic equation

$$v_{0,\tau} - \Delta v_0 + v_0 - v_0^3 = 0, \ v_0(0) = u_0.$$

## Dynamics near $(Q, 0)$, when $0 < \alpha_0 \leq \alpha \leq \alpha_1 < +\infty$

Proposition (A - Local manifolds, asymptotic phase)

*There exist $R_1 > 0$ and $\beta_1 > 0$ s. t., in $\tilde{B}((Q, 0), R_1)$,*

1. *$\exists!$ local stable manifold $W^s_\alpha(Q, 0)$ of codim. 1, tangent to $\mathcal{H}^s_\alpha$ at $(Q, 0)$.*
2. *$\exists!$ local unstable manifold $W^u_\alpha(Q, 0)$ of dimension 1, tangent to $\mathcal{H}^u_\alpha$ at $(Q, 0)$.*
3. *Let $U_0 \in \tilde{B}((Q, 0), R_1) \setminus W^s_\alpha(Q, 0)$. As long as $S_\alpha(t)U_0 \in \tilde{B}((Q, 0), R_1)$,*

$$\mathrm{dist}_{\mathcal{H}}(S_\alpha(t)U_0, W^u_\alpha(Q, 0)) \leq Ce^{-\beta_1 t}\mathrm{dist}_{\mathcal{H}}(U_0, W^u_\alpha(Q, 0))$$

4. *There exist $0 < r_1 < R_1$, $\eta_1 > 0$ and for $U_0 \in \tilde{B}((Q, 0), r_1) \setminus W^s_\alpha(Q, 0)$, a time $t_1 > 0$, s. t. $S_\alpha(t_1)U_0$ is in the Payne-Sattinger region $\mathcal{PS}_+$ or $\mathcal{PS}_-$ ($E(S_\alpha(t_1)U_0) \leq J(Q) - \eta_1$ and $S_\alpha(t_1)U_0 \in \tilde{B}((Q, 0), R_1)$)*

Property 4 follows from 3 and the strict decay of $E(U(t))$. Foliations

# Dynamics near $(Q,0)$ when $\alpha = 0$ (or $\alpha \geq 0$ small)

Proposition (B - Unique local manifolds - Nakanishi, Schlag)

*There exist $R_2 > 0$ s.t.*

1. $\exists!$ *local center stable manifold* $W_0^{cs}(Q,0)$ *of codimension* $1$ *in* $\tilde{B}((Q,0),R_2)$, *tangent to* $\mathcal{H}_0^{cs}$ *at* $(Q,0)$.
2. *If* $U_0 \in W_0^{cs}(Q,0)$, *then* $S_0(t)U_0 \equiv (Q+w,w_t)$ *satisfies*

$$\|(w,w_t)\|_{L_t^\infty(0,+\infty),\mathcal{H})} + \|w\|_{L^3((0,+\infty),L^6)} \lesssim R_1.$$

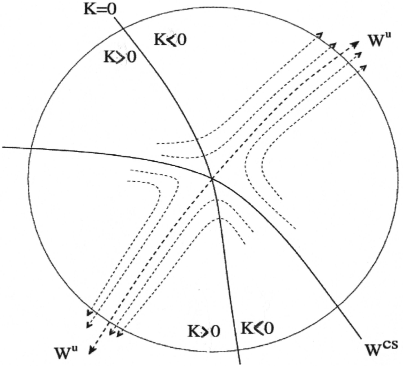
   $U(t)$ *scatters to* $(Q,0)$ , *i.e.,*
   $U(t) = (Q,0) + \Sigma_0(t)(v_0,v_1) + o_{\mathcal{H}}(1)$ *as* $t \to \infty$.
3. *If* $U(t) \in \tilde{B}((Q,0),R_2)$, $\forall t \geq 0$, *then* $U(t) \in W_0^{cs}$, $\forall t \geq 0$.
4. $\exists!$ *smooth local manifolds* $W_0^s(Q,0)$ *and* $W_0^u(Q,0)$ *of dimension* $1$ *in* $\tilde{B}((Q,0),R_2)$, *tangent to* $\mathcal{H}_0^s$ *and* $\mathcal{H}_0^u$ *at* $(Q,0)$.

The same proposition is true for $\alpha > 0$ small.

Use of Strichartz estimates for $\partial_{tt} + L_+$.

# Unstable dynamics off $W_0^{cs}(Q,0)$ when $\alpha \geq 0$ small

Ejection of trajectories, which are off $W_\alpha^{cs}$: proof of Nakanishi-Schlag or proof with foliations over $W_0^u(Q,0)$.
Stabilization of $\mathrm{sign}(K_0(u(t)))$ and $\mathrm{sign}(K_2(u(t)))$, where $K_2(u) = \int_{\mathbb{R}^3}(|\nabla u|^2 - 3/4u^4)dx$: virial




Sign of $K = K_0$ upon exit

# Important variational estimates above $J(Q)$

Proposition (C - Variational property)

*For any $r > 0$, there exist positive numbers $\varepsilon_0(r), \kappa_0, \kappa_1(r)$ s.t., for any $U \in \mathcal{H}$ satisfying*

$$E(U) < J(Q) + \varepsilon_0(r)^2, \; d_Q(U) \geq r,$$

*one has either*

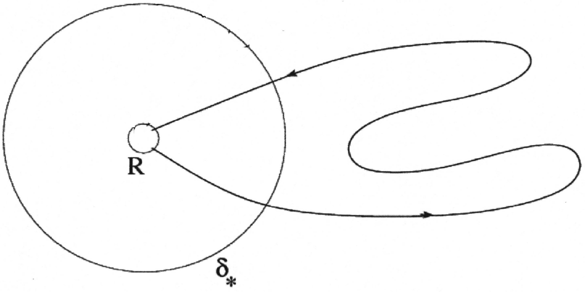
$$K_0(u) \leq -\kappa_1(r) \text{ and } K_2(u) \leq -\kappa_1(r), \text{ or}$$

$$K_0(u) \geq \min(\kappa_1(r), \kappa_0 \|u\|_{H^1}^2) \text{ and } K_2(u) \geq \min(\kappa_1(r), \kappa_0 \|\nabla u\|_{L^2}^2),$$

Propositions A and C allow to prove the main theorem in the case $\alpha_0 \leq \alpha \leq \alpha_1$.

# One-pass theorem when $\alpha = 0$ or $\alpha \geq 0$ small

**Crucial non-return property:** the trajectories do not return into small balls around $(\pm Q, 0)$. Generalisation of the argument of Nakanishi and Schlag by contradiction. In the $K_0(u(t)) < 0$ region, one integrates the quantity $\langle u(t), u_t(t)\rangle + \alpha \|u(t)\|_{L^2}^2$ between $T_1$ and $T_2$, which are exit and first re-entry times into a small $R$-ball. If $K_0(u(t)) > 0$, the proof is more involved (use of the virial $K_2(u(t))$).




One possible returning trajectory

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