Global Attractors in Partial Differential Equations

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1. Introduction

This survey is devoted to an introduction to the theory of global attractors for semigroups defined on infinite dimensional spaces, which has mainly been developed in the last three decades. A first purpose here is to describe the main ingredients leading to the existence of a global attractor. Once a global attractor is obtained, the question arises if it has special regularity properties, a particular shape etc... or if it has a finite-dimensional character. The second objective is thus naturally to give some of the most important properties of global attractors. Finally, we want to show the relevance of the abstract theory in applications to evolutionary equations.

Clearly, we can here neither describe all the related questions and results, nor give the detailed proofs of the main statements, although they are often very instructive. To keep the text elementary and self-contained, we have recalled all the needed basic concepts in the theory of dynamical systems and have included some proofs. In order to illustrate the general abstract results, we have chosen to discuss few equations, but in details, rather than to give a catalogue of applications to partial differential equations and functional differential equations.

The dynamical systems that arise in physics, chemistry or biology, are often generated by a partial differential equation or a functional differential equation and thus the underlying state space is infinite-dimensional. Usually these systems are either conservative or exhibit some dissipation. In the last case, one can hope to reduce the study of the flow to a bounded (or even compact) attracting set or *global attractor*, that contains much of the relevant information about the flow and often has some finite-dimensional character.

It is difficult to trace the origin of the concepts of dissipation and attractor. The word *attractor*, applied to a single invariant point, is ancient and probably appeared at the beginning of the century. One can find it, for example, in the book of Coddington and Levinson in 1955 [CoLe55] or in a paper of Mendelson in 1960 [Me60]. For a flow on a locally compact metric space, attractors consisting of more than one point have been studied by Auslander, Bhatia and Seibert in 1964 [ABS] (see the paper of [Mil] for various definitions in the finite-dimensional case). Several different notions of attractors are already found in the lecture notes of Bhatia and Hajek ([BhHa]) in 1969, for a semi-flow on an infinite-dimensional space. In 1968, Gerstein and Krasnoselskii [GeKr] studied the existence and properties of a maximal compact invariant set for the discrete system generated by a compact map S on a Banach space. In 1971, Billotti and LaSalle [BiLaS] described the maximal compact invariant set and proved stability results for maps whose iterates were eventually compact. The specific notion of compact global attractor, as used in this review, appeared in the papers of Oliva in the early 1970's (see [HMO]). The work of Ladyzenskaya ([La72], [La73]) in 1972 implied the existence of the compact global attractor for the semi-flow generated by the two-dimensional Navier-Stokes equations. In the same year 1972, Hale, LaSalle and Slemrod [HaLaSSl] gave general existence results of maximal compact invariant sets and introduced the concept of *asymptotically smooth systems*.

Let us now describe more precisely the concepts of *dissipation* and *global attractor*. In his study of the forced van der Pol equation, Levinson [Le44] introduced the concept *point dissipative* for maps S on the space $\mathbf{R}^n$. A map S is *point dissipative* if there exists a bounded set $B_0 \subset \mathbf{R}^n$ such that, for each $x \in \mathbf{R}^n$, there exists an integer $n_0(x, B_0)$ so that $S^n(x) \in B_0$, for $n \geq n_0$. Due to the local compactness of $\mathbf{R}^n$, any point dissipative map S is also *bounded dissipative* (or equivalently *uniformly ultimately bounded*); that is, there exists a bounded set $B_0 \subset \mathbf{R}^n$ such that, for each bounded set $B \subset \mathbf{R}^n$, there exists an integer $n_0(B, B_0)$ so that $S^n(B) \subset B_0$, for $n \geq n_0$. If S is bounded dissipative, the local compactness of $\mathbf{R}^n$ also implies that the ω-limit set $\omega(B) \equiv \cap_{m\geq 0}\mathrm{Cl}(\cup_{j\geq m} S^j(B))$ of any bounded set B is compact, invariant (i.e. $S(\omega(B)) = \omega(B)$) and attracts B, that is, $\delta_{\mathbf{R}^n}(S^n(B), \omega(B)) \to 0$ as $n \to +\infty$, where $\delta_{\mathbf{R}^n}(S^n(B), \omega(B)) = \sup_{x\in S^n(B)} \inf_{y\in\omega(B)} \|x-y\|_{\mathbf{R}^n}$. Therefore, if S is point dissipative, $\mathcal{A} = \omega(B_0)$ is the global attractor, that is, $\mathcal{A}$ is bounded, invariant and attracts every bounded set B of $\mathbf{R}^n$. Here, in addition, $\mathcal{A}$ is compact. Thus, in finite dimensions, point dissipativness implies the existence of a compact global attractor. Note that this definition of the global attractor implies that $\mathcal{A}$ is maximal with respect to inclusion and hence is unique.

Unfortunately, when the underlying space X is not locally compact, there are examples where point dissipativness does not imply that the orbits of bounded sets are bounded and where bounded dissipativness does not imply the existence of a global attractor. So the following question arises: are there interesting classes of dynamical sytems on non locally compact spaces that have properties similar to the ones mentioned above for dynamical systems on $\mathbf{R}^n$? To have a theory comparable to the one for maps on $\mathbf{R}^n$, one must impose a type of smoothing property on the operator $S : X \to X$. This is done by assuming, for example, that $S : X \to X$ or an iterate of S is compact (see [BiLaS]). More generally, it is sufficient to suppose that S is asymptotically smooth in the terminology of Hale, Lasalle and Slemrod [HaLaSSl] or equivalently, asymptotically compact in the terminology of Ladyzhenskaya [La87a].

In Section 2, we recall all the needed precise definitions, introduce the above concepts of dissipativness and asymptotic smooth or compact systems. We discuss some implications between these notions. The fundamental theorem of existence of a compact global attractor is stated and proved. Some basic properties like invariance, stability and connectedness of compact global attractors are also discussed. Finally, a large part of the section contains examples of asymptotically smooth systems.

Section 3 is devoted to a presentation of the most important properties of compact global attractors. Compact global attractors are robust objects with respect to perturbations. We give several continuity properties of the global attractors with respect to perturbation parameters and recall the stability of the flow on the global attractor under perturbations for Morse-Smale systems. The mentioned properties play an important role in the study of systems depending on several physical parameters and also in numerical approximations of these systems. Finally, we discuss the possibility of the flow on the compact global attractor $\mathcal{A}$ being finite-dimensional by first showing that, in most of the cases, $\mathcal{A}$ has finite Hausdorff or fractal dimension. The next question of interest is the reduction of the study of the flow on $\mathcal{A}$ to the discussion of the flow of some system on a finite-dimensional space. One effort in this direction is to assert the existence of an inertial manifold, that is a finite-dimensional Lipschitzian positively invariant manifold, that contains the global attractor. Unfortunately, the existence of inertial manifolds is rare in the general class of systems arising in applications. Another approach is to show the existence of a finite number of "modes", on which the corresponding dynamics approximates the dynamics of the original system on $\mathcal{A}$ (for example, Galerkin approximations). For evolutionary equations, this approach gives regularity with respect to time of the flow on $\mathcal{A}$ and regularity in "the spatial variables" when PDE's are involved.

So far, one has not yet given a description of the flow on the global attractor. In the general case, a qualitative description of the global attractor seems difficult. Section 4 is devoted to the class of gradient systems, that is systems which admit a strict Lyapunov functional. In this case, due to the *invariance principle of LaSalle*, the global attractor $\mathcal{A}$, if it exists, is the unstable set of the set of equilibrium points. If the equilibrium points are all hyperbolic, then $\mathcal{A}$ is the union of the unstable manifolds of each equilibrium point. Applications of the general abstract theory, in the frame of gradient systems, are then given to FDE's and to two representative classes of scalar partial differential equations, the reaction-diffusion equations and the (weakly) damped wave equations. Special emphasis is made on the scalar reaction-diffusion equation defined on a bounded interval of $\mathbf{R}$ and provided with separated boundary conditions. In this case, a result of Henry ([He85b], [An86]) says that the stable and unstable manifolds are always tranversal, which means that the global dynamical behaviour can only change by bifurcations of the equilibria. This important property was the starting point for the precise qualitative description of the flow on the global attractor. In this one-dimensional case, special properties like the strong maximum principle, the Sturm-Liouville theory and the Jordan curve theorem play a primordial role.

Finally, in Section 5, we illustrate the abstract theory of global attractors given in Section 2, by studying weakly damped dispersive equations, the prototype of which is the weakly damped Schrödinger equation.

Many topics have been left on the side, including the non autonomous evolutionary equations leading to the notions of processes and skew-product semi-flows (see [Da75], [Sell71], [MiSe], [Har91], [Vi92], [ChVi1], etc ...), the generalization of the concept of attractor to multivalued mappings (see [Ba2] for instance), the notion of random attractors for dissipative stochastic dynamical systems (see [CFl], [Deb] for example). Only few applications to the class of retarded functional differential equations have been given below (see [HVL] and [Nu00]). Finally, for further readings on global attractors and more examples, the reader should consult the books [BV89b], [Hal88], [Te], [La91], [ChVi2], [SeYou], for example.

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2. Fundamental Concepts

In this section, (X, d) (or simply X) denotes a metric space, with distance d. We use the semi-distance $\delta_X(\cdot,\cdot)$ defined on the subsets of X by

$$\begin{aligned} \delta_X(x, A) &= \inf_{a\in A} d(x,a) \ , \quad \forall x \in X \ , \quad \forall A \subset X \ , \\ \delta_X(A, B) &= \sup_{a\in A} \inf_{b\in B} d(a,b) = \sup_{a\in A} \delta_X(a, B) \ , \quad \forall A \, , B \subset X \ . \end{aligned}$$

For any subset A of X and any positive number ε, we introduce the open neighbourhood $N_X(A,\varepsilon) = \{z \in X \,|\, \delta_X(z,A) < \varepsilon\}$ (resp. the closed neighbourhood $\overline{N}_X(A,\varepsilon) = \{z \in X \,|\, \delta_X(z,A) \leq \varepsilon\}$). Finally, we define the *Hausdorff distance* $\mathrm{Hdist}_X(A,B)$, for any subsets A, B of X by

$$\mathrm{Hdist}_X(A,B) = \max(\delta_X(A,B), \delta_X(B,A)) \ .$$

In what follows, we mainly concentrate on continuous dynamical systems, or continuous semigroups, $S(t)$, $t \geq 0$ on X, whose definition we now recall.

Definition 2.1. A *continuous dynamical system* or *continuous semigroup* on X is a one-parameter family of mappings $S(t)$, $t \geq 0$ from X into X such that
1) $S(0) = \mathrm{I}$;
2) $S(t+s) = S(t)S(s)$ for any $t, s \geq 0$;
3) for any $t \geq 0$, $S(t) \in \mathcal{C}^0(X,X)$;
4) for any $u \in X$, $t \mapsto S(t)u \in \mathcal{C}^0((0,+\infty),X)$.
If the mappings $S(t)$ from X into X are defined for $t \in \mathbf{R}$, if the properties 2), 3) hold for any $t, s \in \mathbf{R}$ and if, in 4), $(0,+\infty)$ can be replaced by $\mathbf{R}$, then $S(t)$, $t \in \mathbf{R}$ is a continuous group. A one-parameter family of mappings $S(t)$, $t \geq 0$, satisfying only the properties 1), 2) and 3) will be simply called "a semigroup".

We recall that, if $S \in \mathcal{C}^0(X, X)$, the family S^n, $n \in \mathbf{N}$, is called a *discrete dynamical system* or *discrete semigroup*. If S is a C^0-diffeomorphism from X to X, then the family S^m, $m \in \mathbf{Z}$, forms a discrete group. Most of the properties described below are also valid for discrete dynamical systems. In the sequel, if we do not want to distinguish between discrete dynamical systems and (non discrete) semigroups, we simply refer to a semigroup S or $S(t)$, $t \in G^+$, where G^+ is either $[0, +\infty)$ or the set of nonnegative integers $\mathbf{N}$. Hereafter, G denotes either $\mathbf{R}$ or $\mathbf{Z}$.

The first example of continuous semigroups is given by ordinary differential equations $\dot{x} = f(x)$, $x \in \mathbf{R}^n$, where $f : x \in \mathbf{R}^n \mapsto f(x) \in \mathbf{R}^n$ is a globally Lipschitzian mapping. Another basic example is the class of retarded functional differential equations (see [HVL]). Evolutionary partial differential equations also give rise to continuous semigroups as shown in the following model example.

Example 2.2. Let X, Y be two Banach spaces, such that $Y \subset X$, with continuous injection. Let $\Sigma_0(t)$ be a linear $\mathcal{C}^0$- semigroup in X with infinitesimal generator A and $f : Y \to X$ be a Lipschitzian mapping on the bounded sets of Y. We assume that, either,
1) $Y = X$,
or
2) $\Sigma_0(t)$ is an analytic semigroup on X and $Y = X^\alpha = \mathcal{D}((\lambda Id - A)^\alpha)$, where $\alpha \in [0, 1)$ and λ is an appropriate real number.
We consider the semilinear differential equation in Y,

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(u(t)) \ , \quad t > 0 \ , \quad u(0) = u_0 \in Y \ . \tag{2.1}$$

It is well-known that this equation has a unique mild solution $u \in \mathcal{C}^0([0, T^*(u_0)), Y)$, where $T^*(u_0) \in (0, +\infty]$. If $T^*(u_0) = +\infty$, for any $u_0 \in Y$, then the family of mappings $S(t)$ defined by $S(t)u_0 = u(t)$ is a continuous semigroup on Y. In particular, the mapping $(u_0, t) \mapsto S(t)u_0$ is continuous from $[0, +\infty) \times Y$ into Y. We recall that $u(t)$ is a mild solution of (2.1) if, for $t \geq 0$,

$$S(t)u_0 = \Sigma_0(t)u_0 + \int_0^t \Sigma_0(t-s) f(S(s)u_0) \, \mathrm{d}s \equiv \Sigma_0(t)u_0 + U(t)u_0 \ . \tag{2.2}$$

Under the same hypotheses as above, assume now that there exists a subset Z_0 of Y such that $T^*(u_0) = +\infty$, for any $u_0 \in Z_0$, and that there is a positive constant C_0 such that

$$\|S(t)u_0\|_Y \leq C_0 \ , \quad \forall\, u_0 \in Z_0 \ , \quad \forall\, t \geq 0 \ .$$

We thus introduce the set $Z = \overline{\bigcup_{u_0 \in Z_0} \bigcup_{t \geq 0} \{S(t)u_0\}}^{Y}$, equipped with the distance d induced by the norm of Y. Then, (Z, d) is a complete metric space, $S(t)\big|_Z$ defines a continuous semigroup on Z and $\|S(t)u_0\|_Y \leq C_0$ for any $u_0 \in Z$, $t \geq 0$.

Remark. In the definition of continuous dynamical systems, several authors require also that, for any $u \in X$, the mapping $t \in [0, +\infty) \mapsto S(t)u \in X$ is continuous at $t = 0$. Actually, this hypothesis is unnecessary most of the time. In Section 4 below (see Equation (4.69)), we study an example where $S(t)$ is not continuous at $t = 0$.
A result of [CM] implies that, if $S(t)$ is a continuous dynamical system in the sense of Definition 2.1, then the mapping $(t, u) \in (0, +\infty) \times X \mapsto S(t)u \in X$ is continuous. If, moreover, the space X is locally compact and if, for any $u \in X$, $t \in [0, +\infty) \mapsto S(t)u \in X$ is continuous at $t = 0$, then, by a theorem of [Do], the mapping $(t, u) \in [0, +\infty) \times X \mapsto S(t)u \in X$ is continuous. If the space X is not locally compact, the joint continuity of $S(t)u$ at $t = 0$ may not be true (see the examples of [Ch] and [Ba2]).

2.1. Some definitions

In this subsection, we assume that $S(t)$, $t \in G^+$, is a semigroup on X. Here we define carefully the notions of invariance and attraction, which play a crucial role in the theory of global attractors.

Definition 2.3. A set A is *positively invariant* if $S(t)A \subset A$, for any $t \in G^+$. The set A is *invariant* if $S(t)A = A$, for any $t \in G^+$.

The following concept dealing with invariance and connectedness has been introduced in [LaS] and will be used later.

Definition. Let S be a semigroup of continuous maps from X into X. A closed invariant subset A of X is said to be *invariantly connected* if it cannot be represented as the union of two nonempty, disjoint, closed, positively invariant sets.

The *positive orbit* of $x \in X$ is the set $\gamma^+(x) = \{S(t)x \,|\, t \in G^+\}$. If $E \subset X$, the *positive orbit* of E is the set

$$\gamma^+(E) = \bigcup_{t \in G^+} S(t)E = \bigcup_{z \in E} \gamma^+(z) .$$

More generally, for $\tau \in G^+$, we define the orbit after the time τ of E by

$$\gamma^+_\tau(E) = \gamma^+(S(\tau)E) .$$

Let now I be an interval of $\mathbf{R}$ and $S(t)$, $t \geq 0$, a semigroup. We recall that a mapping u from I into X is a trajectory (or orbit) of $S(t)$ on I if $u(t+s) = S(t)u(s)$, for any $s \in I$ and $t \geq 0$ such that $t+s \in I$. In particular, if $I = (-\infty, 0]$ and $u(0) = z \in X$, u is called a *negative orbit* through z and is often denoted by $\gamma^-(z)$ or u_z. If $I = \mathbf{R}$ and $u(0) = z$, then u is called a *complete orbit* through z and is often denoted by $\gamma(z)$. We let $\Gamma^-(z)$ be the set of all negative orbits through z. If $\Gamma^-(z)$ is not empty, it may contain more than one negative orbit, because we have not assumed the property of *backward uniqueness*. We also let $\Gamma(z) = \Gamma^-(z) \cup \gamma^+(z)$ be the set of all complete orbits through z. In the same way, we define the sets $\gamma^-(E)$, $\Gamma^-(E)$ and $\Gamma(E)$, for any subset E of X. For later use, for any $z \in X$, we introduce the following set:

$$\begin{aligned} H(t,z) \;=\; \{y \in X \,|\, & \text{ there exists a negative orbit } u_z \text{ through } z \\ & \text{such that } u_z(0) = z \text{ and } u_z(-t) = y\} \;. \end{aligned}$$

We remark that $\Gamma^-(z) = \bigcup_{t \geq 0} H(t,z)$. Likewise, if $E \subset X$, we define the set $H(t,E) = \bigcup_{z \in E} H(t,z)$ and remark that $\Gamma^-(E) = \bigcup_{t \geq 0} H(t,E)$.

In a similar way, replacing $(-\infty, 0]$ (resp. $\mathbf{R}$) by $(-\infty, 0] \cap \mathbf{Z}$ (resp. $\mathbf{Z}$), we define the negative and complete orbits of maps S. In the framework of maps, it is very easy to give examples of non backward uniqueness. Consider the non injective logistic map $S : [0,1] \to [0,1]$, $Sx = \lambda x(1-x)$, with $2 < \lambda \leq 4$. The point $x_0 = (\lambda - 1)/\lambda$ is a fixed point of S and the point $y = \lambda^{-1}$ satisfies $Sy = x_0$. The iterates $S^{-n}y \in (0, x_0)$ are well defined and $\gamma(y) = \{S^{-n}y \,|, n = 0, 1, 2, \ldots\} \cup \{x_0\}$ is a complete orbit trough x_0.

The proof of the following lemma is elementary.

Lemma 2.4. *The set $A \subset X$ is invariant for the semigroup $S(t)$, $t \in G^+$ if and only if, for any $a \in A$, there exists a complete orbit u_a through a, with $u_a(G^+) \subset A$. If the semigroup $S(t)$, $t \geq 0$, is continuous, the complete orbits belong all to $\mathcal{C}^0(\mathbf{R}, A)$.*

In general, there may exist an invariant set A, which does not contain all complete orbits of S through each point in A. In the above example of the logistic map, the invariant set $A = \{x_0\}$ does not contain the complete orbit $\gamma(y)$.

Proposition 2.5. *Let $S(t)$ be a continuous semigroup on X and A be a compact invariant set. If the operators $S(t)$ are injective on A, for $t \geq 0$, then $S(t)\big|_A$ is a continuous group of continuous operators on A.*

Proof. By Lemma 2.4, for any $a \in A$, there exists a complete orbit $u_a \in \mathcal{C}^0(\mathbf{R}, A)$ such that $S(t)u_a(-t) = a$, for any $t \geq 0$. Since $S(t)\big|_A$ is one-to-one, we can set, for $a \in A$:

$$S(-t)a \;=\; S(t)^{-1}a \;=\; u_a(-t)\ ,$$

Clearly, $S(t)S(s) = S(t+s)$, for any $t, s \in \mathbf{R}$. Moreover, for any $t \geq 0$, $S(t) : A \to A$ is a continuous bijection on the compact set A, and therefore is an isomorphism from A to A. □

Of primary importance in the theory of dynamical systems is the set

$$\mathcal{J} = \{\text{ bounded complete orbits of } S\}\ .$$

If this set $\mathcal{J}$ is bounded, then, by Lemma 2.4, it is the *maximal bounded invariant set* that is; it is invariant, bounded and contains each bounded invariant set. If S has a global attractor $\mathcal{A}$, then $\mathcal{A}$ coincides with $\mathcal{J}$. However, in the general case, $\mathcal{J}$ needs not to be a global attractor, even if $\mathcal{J}$ is compact and attracts compact sets (see Example 2.24 below; examples involving continuous dynamical systems are also found in [Hal99]).

2.2. ω and α-limit sets

As indicated in the introduction, we are going to construct global attractors as ω-limit sets of bounded sets. For this reason, we now recall the definition and main properties of ω and α-limit sets.

Definition 2.6. Let E be a nonempty subset of X.
(i) We define the *ω-limit set* $\omega(E)$ of E as

$$\omega(E) \;=\; \bigcap_{s\in G^+} \overline{\gamma^+(S(s)E)}^X \;=\; \bigcap_{s\in G^+} \overline{(\bigcup_{t\geq s,\, t\in G^+} S(t)E)}^X\ . \tag{2.3}$$

(ii) We define the *α-limit set* $\alpha(E)$ of E as

$$\alpha(E) \;=\; \bigcap_{s\in G^+} \overline{(\bigcup_{t\geq s,\, t\in G^+} H(t,E))}^X\ . \tag{2.4}$$

Remark. Let $z \in X$ be such that there exists a negative orbit u_z through z. We define the *α_{u_z}-limit set* $\alpha_{u_z}(z)$ of the orbit u_z as

$$\alpha_{u_z}(z) \;=\; \bigcap_{s\in G^+} \overline{\{u_z(-t)\,|\, t\geq s,\; t\in G^+\}}^X\ . \tag{2.5}$$

An equivalent description of the ω and α-limits sets is given in terms of limits of sequences as follows:

Lemma 2.7. *(Characterization Lemma) Let E be a nonempty subset of X. Then,*

$$\begin{aligned}\omega(E) \;=\; \{y \in X \,|\, &\textit{there exist sequences } t_n \in G^+ \textit{ and } z_n \in E \textit{ such that}\\ &t_n \underset{n\to+\infty}{\longrightarrow} +\infty \textit{ and } S(t_n)z_n \underset{n\to+\infty}{\longrightarrow} y\}\ ,\\ \alpha(E) \;=\; \{y \in X \,|\, &\textit{there exist sequences } t_n \in G^+, x_n \in X \textit{ and } z_n \in E \textit{ such that}\\ &t_n \underset{n\to+\infty}{\longrightarrow} +\infty\ ,\ x_n \underset{n\to+\infty}{\longrightarrow} y \textit{ where } x_n = u_{z_n}(-t_n)\\ &\textit{and } u_{z_n} \textit{is a negative orbit through } z_n\}\ .\end{aligned}$$

Likewise, if z is a point in X such that there exists a negative orbit u_z through z, then

$$\begin{aligned}\alpha_{u_z}(z) = \{y \in X \,|\, &\textit{there exists a sequence } t_n \in G^+ \textit{ such that } t_n \underset{n\to+\infty}{\longrightarrow} +\infty\\ &\textit{and } u_z(-t_n) \underset{n\to+\infty}{\longrightarrow} y\}\ .\end{aligned} \tag{2.6}$$

We remark that if E is a nonempty subset of X, we have the following inclusions, for $t \in G^+$,

$$\begin{aligned}\omega(E) \;=\; \omega(S(t)E)\ ,\quad \alpha(E) \subset \alpha(S(t)E)\ ,\\ S(t)\omega(E) \subset \omega(E)\ ,\quad S(t)\alpha(E) \subset \alpha(E)\end{aligned} \tag{2.7}$$

Remark. If E is a nonempty subset of X, then, generally, $\omega(E) \neq \bigcup_{z\in E} \omega(z)$. Indeed, let us consider the flow $S(t)$ generated by the following ordinary differential equation

$$\dot{y} \;=\; y(1-y)(2+y)\ .$$

For any $y_0 \in \mathbf{R}$, $\lim_{t\to+\infty} S(t)y_0$ exists and $\lim_{t\to+\infty} S(t)y_0 = 1$ if $y_0 > 0$, $S(t)0 = 0$ and $\lim_{t\to+\infty} S(t)y_0 = -2$ if $y_0 < 0$. Thus, $\omega(y_0) = 1$ if $y_0 > 0$, $\omega(0) = 0$ and $\omega(y_0) = -2$ if $y_0 < 0$. However, for any $t \geq 0$, $S(t)[-2,1] = [-2,1]$ and therefore $\omega(E) = [-2,1]$.

Example 2.8. The ω-limit set can be empty as the following example, which appeared in the thesis of Cooperman [Coo] (see also [ChHa]) shows. Let H_0 be the Banach space of all real sequences $x = \{x_i, i \geq 1 \,|\, x_i \to 0 \text{ as } i \to +\infty\}$, equipped with the norm $\|x\|_{H_0} = \sup_{i\geq 1} |x_i|$. We introduce the map $T : x = (x_1, x_2, \ldots) \in H_0 \mapsto (1, x_1, x_2, \ldots) \in H_0$ and define the map $U : H_0 \to H_0$ by $U(x) = x/\|x\|_{H_0}$ if $\|x\|_{H_0} > 1$ and $U(x) = x$

if $\|x\|_{H_0} \leq 1$. Finally, we let $S = T \circ U$. We remark that, since $S^n = T^n \circ U$, for any $x \in H_0$, the first n terms in the sequence $S^n(x)$ are equal to 1. Clearly, for any $x_0 \in H_0$, the ω-limit set of x_0 is empty. Indeed, by the characterisation lemma, if $\omega(x_0) \neq \emptyset$, there exists $y \in H_0$ and a sequence $n_j \in \mathbf{N}$, $n_j \to +\infty$, such that $S^{n_j} x_0 \to y$. Since $y \in H_0$, there exists $i_0 \in \mathbf{N}$ such that, for $i \geq i_0$, $|y_i| \leq 1/2$. But, for $n_j \geq i_0$, $\|S^{n_j}(x_0) - y\|_{H_0} \geq 1/2$, which contradicts the convergence of $S^{n_j} x_0$ towards y. Likewise, $\omega(E) = \emptyset$ for any subset E of H_0.
Another obvious example is given by the flow generated on $\mathbf{R}$ by the ordinary differential equation $\dot{y} = 1$.

Thus, we can wonder when the ω-limit sets are nonempty and which are their properties.

Definition. Let A, E be two (nonempty) subsets of X. The set A is said to attract E if

$$\delta_X(S(t)E, A) \underset{t \to +\infty}{\to} 0 \ ,$$

that is, for any $\varepsilon > 0$, there exists a time $\tau = \tau(\varepsilon, A, E) \geq 0$ such that

$$S(t)E \subset \overline{N}_X(A, \varepsilon) \ , \quad t \geq \tau \ .$$

The following properties are elementary, yet fundamental.

Lemma 2.9. *Let E be a nonempty subset of X and S a semigroup on X. Assume that $\omega(E)$ is nonempty, compact and attracts E, then the following properties hold:*
1) $\omega(E)$ is invariant.
2) If moreover E is connected, $\omega(E)$ is invariantly connected. If, in addition, either $\omega(E) \subset E$ or $S(t)$ is a continuous semigroup, then $\omega(E)$ is connected.

Proof. Statement 1) as well as the connectedness of $\omega(E)$ in the case of a continuous semigroup are well-known and their proofs can be found, for instance, in [Hal88, Chapters 2 and 3]. The connectedness of $\omega(E)$ in the case where $\omega(E) \subset E$ is shown in [GoSa, Lemmas 4.1 and 4.2], for example. Here, adapting arguments given in [GoSa], we prove Statement 2).
(i) We begin by showing by contradiction, that, if $\omega(E)$ is a nonempty compact set, which attracts E, then $\omega(E)$ is invariantly connected, if E is connected. To simplify the notation, we assume, without loss of generality, that S is a discrete dynamical system. If $\omega(E)$ is not invariantly connected, then $\omega(E) = F_1 \cup F_2$, where F_1,

F_2 are disjoint, nonempty, compact, positively invariant sets. We fix $\varepsilon > 0$ so that $N_X(F_1, \varepsilon) \cap N_X(F_2, \varepsilon) = \emptyset$. Since F_1 and F_2 are invariantly connected, the continuity of the mapping S implies that there exists δ, $0 < \delta < \varepsilon$, such that $S(\overline{N}_X(F_i, \delta)) \subset N_X(F_i, \varepsilon)$, for $i = 1, 2$. As $\omega(E)$ attracts E, there exists $n_0 \in \mathbf{N}$, such that,

$$S^n E \subset \overline{N}_X(\omega(E), \delta) \ , \quad \forall n \geq n_0 \ . \tag{2.8}$$

In particular,

$$S^{n_0} E \subset (\overline{N}_X(F_1, \delta) \cap S^{n_0} E) \cup (\overline{N}_X(F_2, \delta) \cap S^{n_0} E) \ . \tag{2.9}$$

We note that, if $x \in E$ and $S^{n_0} x \in \overline{N}_X(F_i, \delta)$, then $S^{n_0+1} x \in N_X(F_i, \varepsilon) \cap N_X(\omega(E), \delta)$, which implies that $S^{n_0+1} x \in N_X(F_i, \delta)$. Thus, by recursion, $S^n x \in N_X(F_i, \delta)$, for $n \geq n_0$. It follows that, if there exists j, $j = 1$ or 2 such that $\overline{N}_X(F_j, \delta) \cap S^{n_0} E = \emptyset$, then $\overline{N}_X(F_j, \delta) \cap S^n E = \emptyset$, for $n \geq n_0$, which means that $F_j = \emptyset$. Hence $\mathcal{F}_i^0 \equiv \overline{N}_X(F_i, \delta) \cap S^{n_0} E$ is nonempty, for $i = 1, 2$. Thus, we have just proved that the connected set $S^{n_0} E$ is the union of the two nonempty, closed, disjoint subsets $\mathcal{F}_1^0$ and $\mathcal{F}_2^0$, which is a contradiction. Therefore, $\omega(E)$ is invariantly connected.

(ii) To prove that $\omega(E)$ is connected when E is connected and $\omega(E) \subset E$, we again argue by contradiction. If $\omega(E)$ is not connected, then $\omega(E) = F_1 \cup F_2$, where F_1, F_2 are disjoint, nonempty, compact sets. We fix $0 < \delta \leq \varepsilon$ so that $N_X(F_1, \varepsilon) \cap N_X(F_2, \varepsilon) = \emptyset$. As in (i), there exists $n_0 \in \mathbf{N}$, such that the inclusions (2.8) and (2.9) hold. The property $\omega(E) \subset E$ together with the invariance of $\omega(E)$ yields that $\omega(E) \subset S^{n_0} E$. Thus, we deduce from (2.9) that $F_i \subset \overline{N}_X(F_i, \delta) \cap S^{n_0} E$, for $i = 1, 2$. Hence, the connected set $S^{n_0} E$ is the union of the two nonempty, closed, disjoint subsets $\mathcal{F}_1^0$ and $\mathcal{F}_2^0$, which is a contradiction. Therefore, $\omega(E)$ is connected.

(iii) Finally, we prove by contradiction that $\omega(E)$ is connected, when $S(t)$ is a continuous semigroup. Arguing as in (ii), we show that there exists $n_0 \in \mathbf{N}$, such that the inclusions (2.8), for any $t \geq n_0$, and (2.9) hold. To obtain a contradiction, we need to prove, as in (ii), that $\mathcal{F}_i^0 \equiv \overline{N}_X(F_i, \delta) \cap S^{n_0} E$ is nonempty for $i = 1, 2$. Since $S(t)$ is a continuous semigroup and that F_1, F_2 are compact sets, there exists a positive time τ such that , for $0 \leq t \leq \tau$, $S(t)(\overline{N}_X(F_i, \delta)) \subset N_X(F_i, \varepsilon)$, for $i = 1, 2$. Let $x \in E$ be such that $S(n_0) x \in \overline{N}_X(F_i, \delta)$. Then, for $0 \leq t \leq \tau$, $S(n_0 + t) x \in \overline{N}_X(\omega(E), \delta) \cap N_X(F_i, \varepsilon)$, which implies that, for $0 \leq t \leq \tau$, $S(n_0 + t) x \in \overline{N}_X(F_i, \delta)$. Thus, by recursion, $S(t) x \in N_X(F_i, \delta)$, for $t \geq n_0$. We now conclude like in (i) that $\mathcal{F}_i^0 \equiv \overline{N}_X(F_i, \delta) \cap S^{n_0} E$ is nonempty, for $i = 1, 2$. $\square$

Remarks 2.10.
(i) We deduce from the above lemma that, if E is connected and if $\omega(E)$ contains only fixed points of S, then $\omega(E)$ is connected, which had been proved in [HR92b, Lemma 2.7]. This result is useful when one wants to show, in the case of gradient systems, that the ω-limit set $\omega(x_0)$ of an element $x_0 \in X$ is a single equilibrium point (see [HR92b] and [BrP97b] as well as Section 4.1 below).
(ii) Adapting the proof of Lemma 2.9 one shows that, if the α-limit set $\alpha(E)$ of the nonempty subset E of X is nonempty, compact and $\delta_X(H(t,E),\alpha(E)) \to 0$ as $t \to +\infty$ in G^+, then $\alpha(E)$ is invariant. If moreover, $H(t,E)$ is connected for any $t \in G^+$, $\alpha(E)$ is invariantly connected. If, in addition, either $\alpha(E) \subset E$ or $S(t)$ is a continuous semigroup, then $\alpha(E)$ is connected.
Of course, similar properties hold for the α_{u_z}-limit set of negative orbits u_z through $z \in X$.

The following property had already been proved for instance by Hale in 1969 ([Hal69]).

Proposition 2.11. *Let S be a semigroup on X. If E is a nonempty subset of X and there exists $\tau \in G^+$ such that $\gamma_\tau^+(E)$ is relatively compact, then $\omega(E)$ is nonempty, compact and attracts E.*

In the case $X = \mathbf{R}^n$, the hypotheses of Proposition 2.11 hold if we only assume that $\gamma_\tau^+(E)$ is bounded. If there exists $t_0 \in G^+$ such that $S(t)$ is compact for $t > t_0$ in G^+, then the hypotheses of Proposition 2.11 still hold, when $\gamma_\tau^+(E)$ is bounded, even if X is infinite-dimensional. Semigroups that are compact for $t > 0$ occur in the study of parabolic equations or retarded differential equations etc...However, there are examples, like the damped wave equation, where the associated semigroup is not compact and yet the properties given in Proposition 2.11 hold. For this reason, we consider the more general class of *asymptotically smooth* semigroups, which has been introduced in 1972 by Hale, LaSalle and Slemrod [HaLaSSl].

Definition 2.12. The semigroup S is *asymptotically smooth* if, for any nonempty, closed, bounded set $B \subset X$, there exists a nonempty compact set $J = J(B)$ such that J attracts $\{x \in B \,|\, S(t)x \in B, \forall\, t \in G^+\}$.

One remarks that S is asymptotically smooth if and only if, for any nonempty, closed, bounded set $B \subset X$ for which $S(t)B \subset B$, for any $t \in G^+$, there exists a compact set $J \subset B$ such that J attracts B. (see [Hal88]).

Obviously, if the semigroup $S(t)$ is compact for $t > t_0 \geq 0$, then $S(t)$ is asymptotically smooth. Asymptotically smooth semigroups have the following important property (see [Hal88, Chapters 2 and 3])

Proposition 2.13. *If S is an asymptotically smooth semigroup on X and E is a nonempty subset of X such that $\gamma_\tau^+(E)$ is bounded for some $\tau \in G^+$, then $\omega(E)$ is nonempty, compact, invariant and attracts E.*

In 1987, Ladyshenskaya [La87a] introduced the notion of *asymptotically compact semigroups*:

Definition 2.14. The semigroup S is asymptotically compact if, for any bounded subset B of X such that $\gamma_\tau^+(B)$ is bounded for some $\tau \in G^+$, every set of the form $\{S(t_n)z_n\}$, with $z_n \in B$ and $t_n \in G^+$, $t_n \rightarrow_{n \to +\infty} +\infty$, $t_n \geq \tau$, is relatively compact.

We remark that Proposition 2.13 at once implies that every asymptotically smooth semigroup S is asymptotically compact. On the other hand, Ladyshenskaya [La87a] had proved that, if $S(t)$ is an asymptotically compact semigroup on X and E is a nonempty subset of X such that $\gamma_\tau^+(E)$ is bounded for some $\tau \in G^+$, then $\omega(E)$ is nonempty, compact and attracts E. From this result, one immediately deduces that any asymptotically compact semigroup is asymptotically smooth, obtaining thus the following result.

Proposition 2.15. *Let S be a semigroup on X. Then, S is asymptotically smooth if and only if it is asymptotically compact.*

Since the concepts of *asymptotically compact* and *asymptotically smooth* are equivalent, I will not distinguish them in the sequel. I prefer to use the term *asymptotically smooth*, because it appeared first. Moreover, the term *asymptotically compact* is now misleading, because some authors, like Ball [Ba2], Sell and You [SeYou], call a semigroup *asymptotically compact* if for any bounded subset B of X, any set of the form $\{S(t_n)z_n\}$, with $z_n \in B$ and $t_n \in \mathbf{R}$, $t_n \rightarrow_{n \to +\infty} +\infty$, $t_n \geq \tau$, is relatively compact. The property of eventual boundedness of orbits of bounded sets is included in this definition, whereas, this is not the case for asymptotically smooth semigroups.

In 1982, in his study of the homotopy index for semiflows in non locally compact spaces, Rybakowski introduced the related concept of admissibility (see [Ry82] and also the appendix of [HMO]). Let $S(t)$ be a (local) continuous semigroup on X and N be a closed subset of X. The subset N is called *S-admissible* if for every sequence $\{z_n\} \subset N$ and every sequence $t_n \in \mathbf{R}$, $t_n \to_{n\to+\infty} +\infty$ such that $\{S(t)z_n \,|\, t \in [0, t_n]\} \subset N$, for any $n \in \mathbf{N}$, the set $\{S(t_n)z_n \,|\, n \in \mathbf{N}\}$ is relatively compact. As pointed out by Rybakowski, the notions of admissibility and asymptotically smooth semigroups are not equivalent [HMO].

Remark 2.16.
(i) One can also show that, if $S(t)$ is an asymptotically smooth semigroup on X and E is a nonempty subset of X such that $\Gamma^-(E)$ is nonempty and $\Gamma(E)$ is bounded, then $\alpha(E)$ is nonempty, compact, invariant, and $\delta_X(H(t,E), \alpha(E)) \to_{t\to+\infty} 0$ (for a proof, see [Hal88, Chapters 2 and 3] or [GR00]).
(ii) Likewise, one shows that, if $S(t)$ is asymptotically smooth and there exists a bounded complete orbit $u_z \in \mathcal{C}^0(\mathbf{R}, X)$ through some $z \in X$, then the α_{u_z}-limit set $\alpha_{u_z}(z)$ is nonempty, compact, invariant and $\delta_X(u_z(-t), \alpha_{u_z}(z)) \to_{t\to+\infty} 0$. Statement (ii) of Remarks 2.10 implies that $\alpha_{u_z}(z)$ is invariantly connected. If, moreover $S(t)$ is a continuous semigroup, then $\alpha_{u_z}(z)$ is connected.

Proposition 2.13 indicates that, if a global attractor $\mathcal{A}$ exists, then $\mathcal{A}$ contains the ω-limit set of any bounded set.

2.3. Global attractors

We are now ready to recall the definition of a global attractor and state its basic properties. In this paragraph, we also give the fundamental theorem of existence of compact global attractors.

Definition 2.17. A nonempty subset $\mathcal{A}$ of X is called a *global attractor* of the semigroup S if
1) $\mathcal{A}$ is a closed, bounded subset of X,
2) $\mathcal{A}$ is invariant under the semigroup S,
3) $\mathcal{A}$ attracts every bounded subset B of X under the semigroup S.

In the same way, one defines local attractors. A nonempty subset J of X is a *local attractor* if J is closed, bounded, invariant and attracts a neighbourhood of itself. In the past, a special class of global attractors has mainly been studied: it is the class of *compact global attractors*.

If $S(t)$ is a continuous semigroup and the global attractor $\mathcal{A}$ is compact, it is straightforward to show that, given a trajectory $\gamma^+(x_0)$, there exist sequences of positive numbers ε_n, t_n and a sequence of points $y_n \in \mathcal{A}$, such that $\varepsilon_n \to_{n\to+\infty} 0$, $t_{n+1} > t_n$, $t_{n+1} - t_n \to_{n\to+\infty} +\infty$, and $\delta_X(S(t)x_0, S(t-t_n)y_n) \leq \varepsilon_n$, for any $t_n \leq t \leq t_{n+1}$. Moreover, $\delta_X(y_{n+1}, S(t_{n+1}-t_n)y_n)$ decays to 0.
We also remark that if $\mathcal{A}$ is the global attractor of a semigroup $S(t)$, $t \in \mathbf{R}$, then, for any $t_0 > 0$, $\mathcal{A}$ is the global attractor of the discrete semigroup generated by $S(t_0)$. Conversely, if $S(t_0)$ has a compact global attractor $\mathcal{A}_0$ and that $S(t)$ is a continuous semigroup, then $\mathcal{A}_0$ is also the global attractor of $S(t)$, $t \in \mathbf{R}$.

Before giving the main theorem of existence of global attractors, we describe some fundamental properties of the global (and local) attractors.

Properties of global attractors.

The following properties of a global attractor are a direct consequence of its definition and of Lemma 2.4.

Lemma 2.18. *If the semigroup S admits a global attractor $\mathcal{A}$, the following properties hold:*
a) If B is an invariant bounded subset of X, then $B \subset \mathcal{A}$ (maximality property).
b) If B is a closed subset of X, which attracts every bounded subset B of X, then $\mathcal{A} \subset B$ (minimality property).
c) $\mathcal{A}$ is unique.
d) $\mathcal{A} = \{$bounded complete orbits of $S(t)\}$.

In the case of a Banach space X, as an immediate consequence of Lemma 2.9 2), we have the following connectedness result, due to Massat [Ma83a]:

Proposition 2.19. *Let S be a semigroup on a Banach space X and A be a compact, invariant set attracting any compact set of X, then A is connected. In particular, if S has a compact global attractor $\mathcal{A}$, then $\mathcal{A}$ is connected.*

Even when X is not a Banach space, we obtain some connectedness properties of the global attractor. The next proposition generalizes Theorem 4.2 of [Ba2] as well as

Theorem 3.1 of [GoSa].

Proposition 2.20. *If S is a semigroup on a connected metric space X and if A is a compact, invariant set attracting any compact set of X, A is invariantly connected. If, in addition, $S(t)$ is a continuous semigroup on X, then A is connected.*

Proof. We begin by showing by contradiction that if A is a compact, invariant set attracting any compact set of X, A is invariantly connected. To simplify the notation, we assume, without loss of generality, that S is a discrete semigroup. If A is not invariantly connected, then $A = A_1 \cup A_2$, where A_1, A_2 are nonempty, disjoint, compact, positively invariant subsets of X. We fix $\varepsilon > 0$ so that $N_X(A_1, \varepsilon) \cap N_X(A_2, \varepsilon) = \emptyset$. Since A_1 and A_2 are invariantly connected, the continuity of the mapping S implies that there exists δ, $0 < \delta < \varepsilon/2$, such that $S(\overline{N}_X(A_i, \delta)) \subset N_X(A_i, \varepsilon)$, for $i = 1, 2$. We set, for $i = 1, 2$,

$$X_i = \{x \in X \,|\, \text{ there exists } n_0 = n_0(x) \text{ such that } S(n)x \in N_X(A_i, \delta),\ n \geq n_0\}$$

Clearly, $X_1 \cap X_2 = \emptyset$. Moreover, the following properties hold:
(i) $X = X_1 \cup X_2$. Indeed, let $x \in X$. As x is attracted by A, there exists $n_0(x) \equiv n_0$ such that $\{S(n)x \,|\, n \geq n_0\} \subset N_X(A, \delta)$. Assume that $S(n_0)x \in N_X(A_1, \delta)$. Then, $S(n_0 + 1)x \in N_X(A, \delta) \cap N_X(A_1, \varepsilon)$, which implies that $S(n_0 + 1)x \in N_X(A_1, \delta)$. By recursion, it follows that $S(n)x \in N_X(A_1, \delta)$, for $n \geq n_0$ and thus $x \in X_1$.
(ii) $X_i \neq \emptyset$, for $i = 1, 2$. Indeed, due to the positive invariance of A_i, we have the inclusion $A_i \subset X_i$.
(iii) X_1 and X_2 are closed sets. If $x_1 \in \overline{X_1}$, there exist $y_m \in X_1$, $m \in \mathbf{N}$, such that y_m converges to x_1 as m goes to ∞. Since A attracts every compact set of X, there exists $n_0 > 0$ such that $\{S(n)K \,|\, n \geq n_0\} \subset N_X(A, \delta)$, where $K = \{x_1\} \cup \{y_m \,|\, m \in \mathbf{N}\}$. The arguments of (i) imply that $S(n)y_m \in N_X(A_1, \delta)$, for $n \geq n_0$ and $m \in \mathbf{N}$. Due to the continuity of the map $S(n_0)$, there exists $m_0 > 0$ such that $d(S(n_0)y_m, S(n_0)x_1) \leq \delta$, for $m \geq m_0$. It follows that $x_1 \in N_X(A_1, 2\delta) \subset N_X(A_1, \varepsilon)$. Hence, $\{S(n)x_1 \,|\, n \geq n_0\} \subset N_X(A_1, \delta)$ and $x_1 \in X_1$.
We have thus proved that the connected set X is the union of the two closed, disjoint compact subsets X_1 and X_2, which is a contradiction. Therefore A is invariantly connected.

To prove that A is connected when, in addition, $S(t)$ is a continuous semigroup, we argue by contradiction and follow the lines of the above proof. If A is not connected, then $A = A_1 \cup A_2$, where A_1, A_2 are disjoint, nonempty, compact sets. Like before, one introduces two positive constants ε and δ and, for $i = 1, 2$, the set

$$X_i = \{x \in X \mid \text{ there exists } \tau_0 = \tau_0(x) \text{ such that } S(t)x \in N_X(A_i, \delta),\ t \geq \tau_0\}$$

Clearly $X_1 \cap X_2 = \emptyset$ and the following properties hold:
(a) $X = X_1 \cup X_2$. Let $x \in X$. Since x is attracted by A, there exists $\tau > 0$ such that $\{S(t)x \mid t \geq \tau\} \subset N_X(A, \delta)$. As $S(t)$ is a continuous semigroup, $\{S(t)x \mid t \geq \tau\}$ is connected and therefore is either completely contained in $N_X(A_1, \varepsilon)$ or $N_X(A_2, \varepsilon)$.
(b) $A_i \subset X_i$, for $i = 1, 2$. Indeed, let $a \in A_i$. Due to the invariance of A, there exists $b \in A$ and $T > 0$ such that $S(T)b = a$ and $S(t+T)b = S(t)a \in A$, for any $t \geq 0$. Since $S(t)$ is a continuous semigroup, $\{S(t)a = S(T+t)b \mid t \geq 0\}$ is connected and therefore completely contained in A_i.
(c) One shows like in (iii) that X_1 and X_2 are closed subsets of X.
Thus, X is the union of the two closed, disjoint compact subsets X_1 and X_2, which is a contradiction. Hence A is connected. $\square$

Remarks.
1. As an immediate consequence of Proposition 2.20, we notice that if the set A satisfies the hypotheses of Proposition 2.20, if, for any $x \in X$, the map $t \in [0, +\infty) \mapsto S(t)x \in X$ is continuous and if X is not connected, then every connected component of X contains *exactly* one connected component of A.
2. If $S(t)$ is not a continuous semigroup on X, then, in general, under the hypotheses of Proposition 2.20, the set A is only invariantly connected. Gobbino and Sardella [GoSa] give an example of a discrete semigroup defined on a connected metric space X which has a non connected compact global attractor. Modifying the proof of [GoSa, Proposition 4.3] in the above way, one shows that, if $S(t)$ is any semigroup on X and if A is a compact, invariant set attracting every compact set of X, then either A is connected or A has infinitely many connected components. This fact directly implies that, when X is connected and locally connected, the compact global attractor $\mathcal{A}$ is connected.

Before giving the fundamental theorem of existence of compact global attractors, we introduce the important notions of stability and dissipativness.

Let S be a semigroup on the metric space X. A set $J \subset X$ attracts points locally if there exists a neighbourhood U of J such that J attracts each point of U under S.

We recall that $E \subset X$ is (*Lyapunov*) *stable* if, for any neighbourhood V of E, there exists a neighbourhood $W \subset V$ of E such that

$$S(t)W \subset V \ , \quad \forall t \in G^+ \ . \tag{2.10}$$

We say that E is stable for $t \geq \tau$ in G^+ if the above inclusion (2.10) holds only for any $t \in G^+$, $t \geq \tau$. The set E is asymptotically stable if E is stable and attracts points locally. The set E is uniformly asymptotically stable if it is asymptotically stable and attracts a neighbourhood of itself.

The next theorem describes a stability result for compact global attractors.

Theorem 2.21. *Let S be a semigroup on X. If A is a compact, positively invariant set, which attracts a neighbourhood of itself, the following properties hold:*
(i) if the mapping $(t, z) \in G^+ \times X \mapsto S(t)z \in X$ is continuous, then A is stable and thus uniformly asymptotically stable. In particular, A is uniformly asymptotically stable if S is a discrete semigroup.
(ii) if $S(t)$ is a continuous semigroup, then, for any $\tau > 0$, A is stable for $t \geq \tau$.

Proof. (i) Let $\varepsilon > 0$ be a fixed positive number. By assumption, there exists a neighbourhood W_1 of A and $t_1 \in G^+$ such that, for any $t \in G^+$, $t \geq t_1$,

$$S(t)W_1 \subset N_X(A, \varepsilon) \ .$$

Moreover, the joint continuity of the mapping $(t, z) \in G^+ \times X \mapsto S(t)z \in X$ implies that, for any $a_{0i} \in A$, there exists $\eta(a_{0i}) > 0$ such that, for any $z \in X$ with $\delta_X(z, a_{0i}) < \eta(a_{0i})$ and any $t \in G^+ \cap [0, t_1]$,

$$\delta_X(S(t)z, S(t)a_{0i}) < \varepsilon \ ,$$

and hence, due to the positive invariance of A

$$S(t)z \in N_X(S(t)A, \varepsilon) \subset N_X(A, \varepsilon) \ . \tag{2.11}$$

Since the set A is compact, it can be covered by a finite number k of neighbourhoods $N_X(a_{0i}, \eta(a_{0i}))$. Hence, there exists a positive number η such that $A \subset N_X(A, \eta) \subset \cup_{i=1}^{i=k} N_X(a_{0i}, \eta(a_{0i}))$. We deduce from (2.11) that

$$S(t)N_X(A, \eta) \subset N_X(A, \varepsilon) \ , \quad \forall t \in G^+ \cap [0, t_1] \ .$$

If we set $W = W_1 \cap N_X(A,\eta)$, then

$$S(t)W \subset N_X(A,\varepsilon) \ , \quad \forall t \in G^+ \ .$$

We finally remark that, for any discrete dynamical system, the mapping $(t,z) \in \mathbf{N} \times X \mapsto S(t)z \in X$ is continuous.
(ii) We recall that, due to a result of [CM], if $S(t)$ is a continuous semigroup, the mapping $(t,z) \in [\tau,+\infty) \times X \mapsto S(t)z \in X$ is continuous, for any $\tau > 0$. The arguments used in (i) then show that A is stable for $t \geq \tau$. $\square$

Other results involving stability properties are given in [Hal88].
We finally introduce the concept of dissipativness.

Definition 2.22. The semigroup S is *point* (*compact*) (*locally compact*) (*bounded*) *dissipative* on X if there exists a bounded set $B_0 \subset X$, which attracts each point (compact set) (a neighbourhood of each compact set) (bounded set) of X.

If the semigroup is bounded dissipative, there exists a bounded set $B_1 \subset X$ with the property that, for any bounded set $B \subset X$, there exists $\tau = \tau(B) \in G^+$ such that $\gamma_\tau^+(B) \subset B_1$. Such a set B_1 is called an *absorbing set* for S.

If S is a semigroup on X, which admits a global attractor, then S is bounded dissipative. Of course, if S is bounded dissipative, it has not necessarily a global attractor, as it is easily seen in Example 2.8, where the unit ball in H_0 is an absorbing set and nevertheless there does not exist a global attractor.

An interesting implication of dissipativness for asymptotically smooth semigroups is as follows ([Hal00], [ChHa]):

Theorem 2.23. *Let $\mathcal{F}$ be a family of subsets of X. If S is an asymptotically smooth semigroup on X and there is a bounded set B in X that attracts each element of $\mathcal{F}$, then there exists a compact invariant set which attracts every element of $\mathcal{F}$.*

Example 2.24. The notions of point dissipative, compact dissipative and bounded dissipative are not equivalent in general, as is shown in this example, which is a modification of an unpublished example of the thesis of Cooperman ([Coo]) and is described in [ChHa].
Let $H = l^2$ be the Hilbert space of square summable series $\{x = (x_1, x_2, \ldots) \,|\, x_i \in \mathbf{R}\,, i =$

$1, 2, \ldots,\ \|x\|^2 \equiv \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ with the orthonormal basis $e_j = (0, \ldots, 0, 1, 0, \ldots)$, $j = 1, 2, \ldots$, where the number 1 appears at the j^{th} position. Also, we denote by θ the zero element of l^2 and introduce the following points

$$x_{1,j} = \frac{1}{2} e_j \ , \quad j \geq 2 \ , \quad \text{and} \quad x_{n,j} = \frac{1}{2^n} e_j + 2^{n-2} e_1 \ , \quad j \geq n \geq 2 \ ,$$

We define an auxiliary map $T : H \to H$ by first setting

$$T(x_{n,j}) = \frac{1}{2^n} e_j + 2^n e_1 \ , \quad n \geq 1 \ , \quad j \geq \max(n, 2) \ . \tag{2.12}$$

Then, we extend T to a map from H to H as follows.
If x belongs to the ball $B_H(x_{n,j}, c_{n,j})$ of center $x_{n,j}$ and radius $c_{n,j}$, $n \geq 1$, $j \geq \max(n, 2)$, where $c_{n,j} = \frac{1}{4}\frac{1}{2^j}$, we set, using (2.12):

$$T(x) = \frac{\|x - x_{n,j}\| x_{n,j} + c_{n,j}(x - x_{n,j}) + (c_{n,j} - \|x - x_{n,j}\|) T(x_{n,j})}{c_{n,j}} \ ,$$

otherwise, we simply set $T(x) = x$.
We finally introduce the mapping $S : H \to H$ given by

$$S(x) = T(\frac{1}{2} x) \ , \quad x \in H \ .$$

Cholewa and Hale [ChHa] have shown that $S : H \to H$ is continuous and asymptotically smooth. Moreover, each point of H has a neighbourhood which is attracted by the zero element θ in l^2, which implies that S is compact dissipative. Yet, there is no bounded set in H attracting each bounded set of H. Indeed, one at once proves by recursion that

$$S^j(e_j) = \frac{1}{2^j} e_j + 2^j e_1 \ , \quad j \geq 2 \ ,$$

which implies that, for each $k \geq 1$, the set $\gamma_k^+(\{x \in H \mid \|x\| = 1\})$ is unbounded in H.

However, the different notions of dissipativness can be equivalent in some cases. For example, it has been proved by Massat ([Ma83a], see also [HVL]) that point dissipativness and compact dissipativness are equivalent for certain classes of neutral functional differential equations which arise in connection with the telegraph equation. More generally, if the semigroup S is asymptotically smooth, one can relate the different types of dissipativeness. Assume that S is asymptotically smooth and point dissipative. If, for any compact subset K of X, there exists $\tau \geq 0$ such that $\gamma_\tau^+(K)$ is bounded, then

S is locally compact dissipative. If, for any bounded subset B of X, there exists $\tau \geq 0$ such that $\gamma_\tau^+(B)$ is bounded, then S is bounded dissipative. (see [Ma83a] and also [Hal88]). The existence of compact global attractors is actually a consequence of the latter property and of Theorem 2.23, although we shall give a different proof below.

Existence of global attractors.

Let us recall that a set A is said to be a *maximal compact invariant set* if it is a compact invariant set under S and is maximal with respect to these properties. One of the basic results concerning the existence of a maximal compact invariant set (see [HaLaSSl]) is the following:

Theorem 2.25. *If S is a semigroup on X such that there is a nonempty compact set K that attracts each compact set of X, and $A = \cap_{t \in G^+} S(t)K$, then*
(i) *A is independent of K,*
(ii) *A is the maximal compact invariant set in X,*
(iii) *A attracts compact sets.*
And the connectedness properties given in Proposition 2.19 and Proposition 2.20 hold.

Remark. The above set A is invariant and attracts compact sets. It is expected to be the compact global attractor. Unfortunately, without additional hypotheses, this is not the case, as we have already seen in Example 2.24.
A simpler example where A satisfies the properties of Theorem 2.25 and is not a compact global attractor is described in [Hal99]. Let H be a separable Hilbert space with orthonormal basis e_j, $j \geq 1$ and let λ_j, $j \geq 1$ be a sequence of real numbers, $0 < \lambda_j < 1$, for $j \geq 1$. We introduce the linear mapping defined on the basis vectors by $Se_j = \lambda_j e_j$. Since $\|Sx\|_H \leq \|x\|_H$, for all $x \in H$, $\gamma^+(B)$ is bounded if B is bounded. Clearly, $S^n x \to 0$ as $n \to +\infty$, for any $x \in H$; and $\{0\}$ attracts a neighbourhood of every point and thus every compact set. If $\lambda_j \leq \lambda < 1$, then $\{0\}$ is the global attractor of S. But, if, for instance, $\lambda_j \to 1$, when $j \to +\infty$, then $\{0\}$ cannot be a global attractor. One remarks that, in the first case, the radius $r(\sigma(S))$ of the spectrum of S is strictly less than 1, while in the second case $r(\sigma(S)) = r(\sigma_{\text{ess}}(S))$ is equal to 1, where $r(\sigma_{\text{ess}}(S))$ is the radius of the essential spectrum of S. This example is actually a simple illustration of the general theorem (see [Hal99] and also [Hal88, Section 2.3]), which states that if S is a bounded linear point dissipative map on a Banah space X, then $\{0\}$ attracts compact sets. It is the compact global attractor if and only if $r(\sigma(S)) < 1$ or also if and

only if $r(\sigma_{\text{ess}}(S)) < 1$.

We now state and prove the fundamental theorem of existence of a compact global attractor. A first version of it is due to [HaLaSSl]. Several different proofs of the statements below can be found in [Hal85, Theorem 2.2], [Hal88, Theorem 2.4.6], [La87a, Theorem3.1] (see also [Te] and, for a generalization to multivalued maps, [Ba2]). Here we give a simple proof, using mainly Proposition 2.13.

Theorem 2.26. *(*Existence of a compact global attractor*) The semigroup $S(t)$, $t \in G^+$, on X admits a* compact *global attractor $\mathcal{A}$ in X if and only if*
(i) $S(t)$ is asymptotically smooth,
(ii) $S(t)$ is point dissipative,
(iii) For any bounded set $B \subset X$, there exists $\tau \in G^+$ such that $\gamma_\tau^+(B)$ is bounded.
Moreover,

$$\mathcal{A} = \bigcup \{\omega(B) \,|\, B \text{ bounded subset of } X\} \ . \tag{2.13}$$

And the connectedness properties given in Proposition 2.19 and Proposition 2.20 hold.

In the proof of Theorem 2.26, we use the following auxiliary result.

Lemma 2.27. *Assume that the semigroup $S(t)$, $t \in G^+$, on X is point dissipative and that, for any bounded set $B \subset X$, there exists $\tau \in G^+$ such that $\gamma_\tau^+(B)$ is bounded. Then, there is a bounded set $B_1 \subset X$ such that, for any compact subset K of X, there exist $\varepsilon = \varepsilon(K) > 0$ and $t_1 = t_1(K) \in G^+$ such that*

$$S(t)(N_X(K,\varepsilon)) \subset B_1 \ , \quad \forall t \geq t_1(K) \ , \quad t \in G^+ \ . \tag{2.14}$$

Proof. Without loss of generality, we may assume that $G^+ = [0, +\infty)$. Since $S(t)$ is point dissipative, there exists a bounded set B_0, which may be assumed to be open, such that, for any $x_0 \in X$, there is a time $t^*(x_0) \geq 0$ with

$$S(t)x_0 \subset B_0 \ , \quad \forall t \geq t^*(x_0) \ .$$

As $S(t^*(x_0))$ is continuous from X into X, we can find $\varepsilon(x_0) > 0$ such that

$$S(t^*(x_0))(N_X(x_0, \varepsilon(x_0))) \subset B_0 \ ,$$

and thus, for $s \geq \tau_0$, where τ_0 is chosen so that $\gamma_{\tau_0}^+(B_0)$ is bounded,

$$S(s + t^*(x_0))(N_X(x_0, \varepsilon(x_0))) \subset \gamma_{\tau_0}^+(B_0) \equiv B_1 \ .$$

If K is a compact set in X, we can cover K by a finite number of neighbourhoods $N_X(x_{0i}, \varepsilon(x_{0i}))$, $1 \leq i \leq k$, where $x_{0i} \in K$. Moreover, there exists $\varepsilon(K) > 0$, such that $K \subset N_X(K, \varepsilon(K)) \subset \bigcup_{i=1}^{i=k} N_X(x_{0i}, \varepsilon(x_{0i}))$. Finally, if we set $t_1(K) = \max_{1 \leq i \leq k}(\tau_0 + t^*(x_{0i}))$, we have

$$S(t)(N_X(K, \varepsilon(K))) \subset B_1 \ , \quad \forall t \geq t_1 \ . \tag{2.15}$$

□

Proof of Theorem 2.26 Clearly, if $\mathcal{A}$ is a compact global attractor, then the properties (i), (ii), (iii) hold.

Conversely, we assume now that the properties (i), (ii), (iii) are satisfied; without loss of generality, we may also suppose that $G^+ = [0, +\infty)$. Let B_1 be the bounded set which has been constructed in Lemma 2.27. We set $\mathcal{A} = \omega(B_1)$. Due to Proposition 2.13 and Lemma 2.9, $\mathcal{A}$ is nonempty, compact, invariant and attracts B_1. Actually, $\mathcal{A}$ attracts any bounded set B of X. Indeed, again according to Proposition 2.13, the ω-limit set $K = \omega(B)$ is nonempty, compact and attracts B. Let ε be a real number such that $0 < \varepsilon < \varepsilon(K)$, where $\varepsilon(K)$ has been introduced in Lemma 2.27. Since K attracts B, there exists $t_0 > 0$ such that

$$S(t)B \subset N_X(K, \varepsilon) \ , \quad t \geq t_0 \ ,$$

and therefore, for $t \geq 0$,

$$S(t)S(t_1(K) + t_0)B \subset S(t)S(t_1(K))N_X(K, \varepsilon) \subset S(t)B_1 \ ,$$

where $t_1(K)$ has been defined in Lemma 2.27. Since $\mathcal{A}$ attracts B_1, it follows that $\mathcal{A}$ also attracts B.

Due to Lemma 2.18, $\mathcal{A}$ contains any bounded invariant set and, in particular, the ω-limit set $\omega(B)$ of any bounded set B. Since $\mathcal{A} = \omega(\mathcal{A})$, the equality (2.13) holds. □

Example 2.28. The existence of a global attractor does not necessarily imply that, for any bounded set $B \subset X$, the orbit $\gamma^+(B)$ is bounded. For instance, if $S(t)$ is a continuous semigroup, which is not continuous at $t = 0$, then, due to the lack of continuity at $t = 0$, the size of $\gamma^+_\tau(B)$ can grow to ∞, when $\tau \to 0$; such a semigroup generated by an evolutionary equation will be introduced in Section 4.5. Here we construct a continuous map $S : l^2 \to l^2$, which has a compact global attractor and yet the image through S of the unit ball is unbounded. Like before, we consider the Hilbert space $H = l^2$ of square summable series, with the orthonormal basis e_j, $j = 1, 2, \ldots$.

Let $\varphi : s \in [0, +\infty] \mapsto \varphi(s) \in [0, 1]$ be a continuous function such that $\varphi(s) = 1$, if $0 \leq s \leq \frac{1}{8}$ and $\varphi(s) = 0$, if $s \geq \frac{1}{4}$. We define the map S in the following way:

$$\begin{aligned} S(x) &= j\varphi(\|\frac{1}{2}e_j - x\|)e_1 \ , \quad \forall x \in B_H(\frac{1}{2}e_j, \frac{1}{4}) \ , \quad j \geq 2 \ , \\ S(x) &= \theta \ , \quad \text{otherwise} \ , \end{aligned}$$

where θ is the zero element of l^2. Clearly, $S^k(x) = \theta$, for any $x \in H$ and $k \geq 2$; thus $\{\theta\}$ is the compact global attractor. On the other hand, $S(B_H(\theta, 1))$ is unbounded in H.

We recall that an *equilibrium point* of the semigroup $S(t)$ is a point $x \in X$ such that $S(t)x = x$, for any $t \in G^+$.

The following result had already been proved by Billotti and LaSalle in 1971 (see [BiLaS]). Here, we deduce it from Theorem 2.26.

Theorem 2.29. *Assume that $S(t)$ is either a continuous semigroup or a discrete semigroup and that S is point dissipative. If there exists $t_1 \in G^+$ such that the semigroup $S(t)$ is compact for $t \in G^+$, $t > t_1$, then there exists a compact global attractor $\mathcal{A}$ in X. Moreover, if X is a Banach space and $t_1 = 0$, there exists (at least) an equilibrium point of $S(t)$.*

Proof. We give the proof in the case where $G^+ = [0, +\infty)$. Theorem 2.29 is a direct consequence of Theorem 2.26, if we show that, for any bounded set $B \subset X$, there exists $\tau \geq 0$ such that $\gamma_\tau^+(B)$ is bounded. Since $S(t)B$ is relatively compact, for $t > t_1$, it is sufficient to show this property for any compact set. As $S(t)$ is point dissipative, there exists a bounded set B^*, which may be assumed to be open, which attracts every point of X. Let $t_2 = t_1 + 2$ and $B_0 = N_X(B^*, \varepsilon_0)$, where $\varepsilon_0 > 0$. Arguing as in the proof of Lemma 2.27, one shows that, for any compact subset H of X, there exists $t_3(H) > t_2$ such that $S(t)(H) \subset \gamma^+(S(t_2)(B_0))$, for any $t \geq t_3(H)$. Thus, it remains to show that $\gamma^+(S(t_2)(B_0))$ is bounded. Let $K_0 = \overline{S(t_2)B_0}$; as in the proof of Lemma 2.27, for any $x_0 \in K_0$, there are $\varepsilon(x_0) > 0$ and $t(x_0) > 0$ such that $S(t(x_0))N_X(x_0, \varepsilon(x_0)) \subset B_0$. Since K_0 is compact, we can cover it by a finite number k of neighbourhoods $N_X(x_{0i}, \varepsilon(x_{0i}))$, $1 \leq i \leq k$. We set $t_3(K_0) = \max_{1 \leq i \leq k}(t_2 + t(x_{0i}))$ and $K_3 = \bigcup_{s=1}^{s=t_3(K_0)+1} S(s)\overline{S(t_1+1)B_0}$. If $S(t)$ is a continuous semigroup on X, then K_3 is a compact set. Morover, one at once checks that $S(s)S(t_2)B_0 \subset K_3$, for any

$s \geq 0$. The last statement of the theorem can be found in [GeKr] and in [BiLaS] (see also [HaLo]).
The proof of the above theorem is similar when S is a discrete semigroup. $\square$

Remark 2.30. Often, one can define a global attractor in a weak sense, when the semigroup is bounded dissipative. For example, let X be a separable, reflexive Banach space; we denote by X_w the space X endowed with the corresponding weak topology. Assume that $S(t)$ is a bounded dissipative continuous semigroup, such that, for each fixed t, $S(t) : X_w \to X_w$ is continuous. Then there is a positive number r such that the ball $B_0 = B_X(0, r)$ of center 0 and radius r is an absorbing set for $S(t)$. In particular, there exists $\tau \geq 0$ such that $S(t)B_0 \subset B_0$, for $t \geq \tau$. Since X is reflexive and separable, the weak topology on $\overline{B_0}^{\text{weak}}$ is metrizable. We denote by d_{w} the corresponding metric. Since $\overline{B_0}^{\text{weak}}$ is compact for the weak topology, the semigroup $S(t)\big|_Y$ restricted to $Y = \overline{\gamma_\tau^+ B_0}^{\text{weak}}$ is obviously compact on Y for the weak topology and point dissipative. Therefore, by Theorem 2.29, there exists a global attractor $\mathcal{A}$, bounded in X, compact in X_w, invariant, such that, for any bounded set $B \subset X$,

$$\lim_{t \to +\infty} \delta_w(S(t)B, \mathcal{A}) \, = \, 0 \; .$$

2.4. Examples of asymptotically smooth semigroups

We now give some examples of asymptotically smooth semigroups. The motivation of the hypotheses in the first example probably comes from the Duhamel formula for evolutionary equations.
If B is a bounded subset of a Banach space X, we set $\|B\|_X = \sup_{b \in B} \|b\|_X$.

Theorem 2.31. *Let X be a Banach space and $S(t)$, $t \in G^+$, be a semigroup defined on a closed positively invariant subset M of X. Assume that one can write, for any $t \in G^+$ and any $u \in M$,*

$$S(t)u \, = \, U(t)u + V(t)u \; , \tag{2.16}$$

where $U(t)$ and $V(t)$ are mappings of M into X, with the property that, for any bounded set $B \subset M$, there exists $\tau_0(B) \in G^+$ such that,
(i) *$U(t)B$ is relatively compact for any t in G^+, $t > \tau_0(B)$,*
and
(ii) *for any t in G^+, $t > \tau_0(B)$,*

$$\|V(t)u\|_X \, \leq \, k(t, \|B\|_X) \; , \quad \forall u \in B, \tag{2.17}$$

where $k : (s, r) \in [0, +\infty) \times [0, +\infty) \mapsto [0, +\infty)$ *is a function such that* $k(s, r) \to 0$ *as* $s \to +\infty$.

Then $S(t)$ *is asymptotically smooth.*

Conversely, if a semigroup $S(t)$, $t \in G^+$, *admits a compact global attractor* $\mathcal{A}$ *on a Banach space* X, *then* $S(t)$ *must have the representation (2.16) with* $U(t)$, $V(t)$ *satisfying* (i) *and* (ii).

Proof. To simplify the notation, we assume that $G^+ = [0, +\infty)$. Let B be a bounded subset of M and $\tau = \tau(B) \geq 0$ such that $\gamma_\tau^+(B)$ is bounded. We consider a set of the form $\{S(t_n)z_n\}$, with $z_n \in B$ and $t_n \in [\tau, +\infty)$, $t_n \to_{n \to +\infty} +\infty$. It suffices to show that, for any $\varepsilon > 0$, we can cover the set $\{S(t_n)z_n\}$ by a finite number of balls of radius $r \leq \varepsilon$. Due to the condition (2.17), there exists $t_1 = t_1(\varepsilon, B) > 0$ such that

$$\|V(t)u\|_X \leq k(t, \|\gamma_\tau^+(B)\|_X) \leq \varepsilon/2 , \quad \forall u \in \gamma_\tau^+(B) , \quad \forall t \geq t_1 . \tag{2.18}$$

We set $t_2 = \tau_0(\gamma_\tau^+(B))$ and choose $t_3 > \sup(t_1, t_2)$. Let $n_1 \in \mathbf{N}$ be such that, for $n \geq n_1$, $t_n > \tau + t_3$; clearly the set $\{S(t_n)z_n \,|\, n \leq n_1\}$ can be covered by a finite number of balls of radius ε. It remains to show that the set $B_1 = \{S(t_n)y \,|\, y \in B,\ t_n \geq \tau + t_3\}$ can be covered by a finite number of balls of radius $r \leq \varepsilon$. Each element of B_1 can be written as

$$S(t_3)S(t_n - t_3)y = U(t_3)S(t_n - t_3)y + V(t_3)S(t_n - t_3)y .$$

But $U(t_3)S(t_n - t_3)y \subset U(t_3)\gamma_\tau^+(B)$, which is compact and thus can be covered by a finite number of balls of radius $\varepsilon/2$. Furthermore, due to (2.18), $\|V(t_3)S(t_n - t_3)y\|_X \leq \varepsilon/2$, for any $n \geq n_1$. Thus B_1 can be covered by a finite number of balls of radius ε.

Conversely, assume that the semigroup $S(t)$ admits a compact global attractor $\mathcal{A}$ on the Banach space X. Then, due to the compactness of $\mathcal{A}$, for any $z \in X$, there exists at least one element $a \in \mathcal{A}$ such that $d_X(z, a) = \delta_X(z, \mathcal{A})$; we thus choose such an element a and denote it by $a = Pz$. The mappings $U(t)u = PS(t)u$ and $V(t)u = (Id - P)S(t)u$ clearly satisfy the conditions (i) and (ii) of Theorem 2.31. $\square$

Remark 2.32. In the case of a Hilbert space (see [Te, Remark I.1.5]) and more generally, in the case of a uniformly convex Banach space, we can choose $U(t)u = P_0S(t)u$ and $V(t)u = (Id - P_0)S(t)u$, where P_0 is the projection onto the closed convex hull $\overline{\mathrm{co}}(\mathcal{A})$ of the global attractor $\mathcal{A}$. In this case, $U(t)$ and $V(t)$ are continuous functions of u.

Remark 2.33. The following typical example of semigroup satisfying the hypotheses of Theorem 2.31 is often encountered in the study of semilinear equations. It has been first studied by Webb ([Web79a]). Let $S(t)$, $t \geq 0$, be a continuous semigroup on a Banach space X such that, for $t \geq 0$,

$$S(t)u \;=\; \Sigma(t)u + \int_0^t \Sigma(t-s)K(S(s)u)\,ds \;,$$

for any $u \in M$, where M is a closed positively invariant subset of X, $\Sigma(t)$ is a $\mathcal{C}^0$-semigroup of linear mappings from X into X and K is a compact map from X into X. We assume that $\Sigma(t) = \Sigma_1(t) + \Sigma_2(t)$, $t \geq 0$, where $\Sigma_1(t)$ is a compact linear map for $t > t_0 \geq 0$ and the linear map $\Sigma_2(t)$ satisfies

$$\|\Sigma_2(t)\|_{L(X,X)} \;\leq\; k_2(t) \;, \tag{2.19}$$

with $k_2 : t \in [0,+\infty) \mapsto [0,+\infty)$ is a function such that $k_2(t) \to 0$ as $t \to +\infty$.
If the positive orbit of any bounded subset B of M is bounded, one easily shows ([Web79a], [Web79b], [Hal88]) that then $S(t)$ is written as a sum $S(t)u = U(t)u + V(t)u$, with U, V satisfying the hypotheses of Theorem 2.31.

In the applications, it is often difficult to determine the decomposition $U(t)+V(t)$ given in Theorem 2.31. For this reason, we shall give other criteria of asymptotical smoothness. The following result is due to Ceron and Lopes (see [CeLo]). We recall that a pseudometric $\rho(.,.)$ is precompact (with respect to the norm of X), if any bounded sequence in X has a subsequence which is a Cauchy sequence with respect to ρ.

Proposition 2.34. *Let X be a Banach space and $S(t)$, $t \in G^+$, be a semigroup defined on a closed positively invariant subset M of X. Assume that, for any bounded set $B \subset M$, there exists $\tau_0(B)$ in G^+, such that, for any u_1, u_2 in B and for any t in G^+, $t \geq \tau_0(B)$,*

$$\|S(t)u_1 - S(t)u_2\|_X \;\leq\; k(t, \|B\|_X)\|u_1 - u_2\|_X + \rho_{t,\|B\|_X}(u_1,u_2) \;, \tag{2.20}$$

where $k : (s,r) \in [0,+\infty) \times [0,+\infty) \mapsto [0,+\infty)$ is a function such that $k(s,r) \to 0$ as $s \to +\infty$ and $\rho_{t,\|B\|_X}$ is a precompact pseudometric, for t in G^+, $t > \tau_0(B)$.
Then $S(t)$ is asymptotically smooth.

The proof of Proposition 2.34 is very similar to the one of Theorem 2.31.

We now give a third criterium of asymptotical smoothness, which deals with functionals and has first been introduced by J. Ball ([Ba1]) and then applied to dispersive equations by Abounouh ([Ab1]), Ghidaglia ([Gh94]) etc....

Proposition 2.35. *Let Y be a topological space and X be a uniformly convex Banach space such that X is continuously embedded into Y. We consider a semigroup $S(t)$, $t \geq 0$, on X satisfying the following properties:*
(i) *for any $t \geq 0$, the mapping $S(t)$ is continuous on the bounded subsets of X, for the topology of Y;*
(ii) *for any bounded set B of X such that $\gamma_\tau^+(B)$ is bounded in X for some $\tau \geq 0$, every sequence $S(t_j)b_j$, where $b_j \in B$ and $t_j \to_{j\to\infty} +\infty$, is relatively compact in Y;*
(iii) *for any $x_0 \in X$ and $t \geq 0$, we have*

$$\mathcal{F}(S(t)x_0) = \exp(-\gamma t)\mathcal{F}(x_0) + \int_0^t \exp(-\gamma(t-s))\mathcal{F}_1(S(s)x_0)\,ds\ , \tag{2.21}$$

where $\gamma > 0$, $\mathcal{F}(x) = \|x\|_X^p + \mathcal{F}_0(x)$, $p > 0$ and $\mathcal{F}_0$, $\mathcal{F}_1$ are continuous functionals on the bounded sets of X for the topology of Y and are bounded on the bounded sets of X. Then the semigroup $S(t)$ is asymptotically smooth in X.

Proof. We recall that any uniformly convex Banach space is reflexive. Let B be a bounded set in X such that $\gamma_\tau^+(B)$ is bounded in X for some $\tau \geq 0$ and let $S(t_j)b_j$ be a sequence such that $b_j \in B$ and $t_j \to_{j\to\infty} +\infty$. Since X is reflexive, there exists a subsequence, still labelled by j, such that $S(t_j)b_j \rightharpoonup z$ weakly in X, where $z \in B_0 \equiv \overline{\mathrm{co}}(\gamma_\tau^+(B))$, the closed convex hull of $\gamma_\tau^+(B)$. Due to (ii), we can also suppose that $S(t_j)b_j \to z$ in Y as $j \to +\infty$. We want to show that $S(t'_j)b_{j'} \to z$ in X, where j' is a subsequence of j, $j' \to +\infty$. Since X is uniformly convex, it suffices to show that $\lim_{j'\to+\infty} \|S(t_{j'})b_{j'}\|_X = \|z\|_X$, for some subsequence j' of j. As $S(t_j)b_j$ converges weakly to z, we already know that $\|z\|_X \leq \liminf_{j\to+\infty} \|S(t_j)b_j\|_X$. Thus, it remains to show that $\limsup_{j'\to+\infty} \|S(t_{j'})b_{j'}\|_X \leq \|z\|_X$, for some subsequence j' of j.
For each $n \in \mathbf{N}$, there exists a subsequence j^n such that $S(t_{j^n} - n)b_{j^n} \rightharpoonup z_n$ weakly in X and $S(t_{j^n} - n)b_{j^n} \to z_n$ in Y. By a diagonalization argument, we can construct a subsequence j' such that

$$S(t_{j'}-n)b_{j'} \rightharpoonup z_n\ , \text{ weakly in } X\ , \quad \text{and} \quad S(t_{j'}-n)b_{j'} \to z_n \in Y\ ,\ \forall n \in \mathbf{N}\ . \tag{2.22}$$

We infer from (i) that, for any $t \geq 0$, $S(t_{j'} - n + t)b_{j'} \to S(t)z_n$ in Y. In particular, $S(n)z_n = z$. We consider now the equality (2.21) for $t = n$ and $x_0 = S(t_{j'} - n)b_{j'}$, when

$t_{j'} - n \geq \tau$. From (2.21), (2.22) and the dominated convergence theorem of Lebesgue, we deduce that

$$\limsup_{j' \to +\infty} (\|S(t_{j'})b_{j'}\|_X^p) + \mathcal{F}_0(z) \leq \exp(-\gamma n) \sup_{x \in B_0} (|\mathcal{F}(x)|) + \int_0^n \exp(-\gamma(n-s))\mathcal{F}_1(S(s)z_n)\, ds \ ,$$

or also, since $\mathcal{F}(z) = \exp(-\gamma n)\mathcal{F}(z_n) + \int_0^n \exp(-\gamma(n-s))\mathcal{F}_1(S(s)z_n)\, ds$,

$$\limsup_{j' \to +\infty} (\|S(t_{j'})b_{j'}\|_X^p) \leq 2\exp(-\gamma n) \sup_{x \in B_0} (|\mathcal{F}(x)|) + \|z\|_X^p \ .$$

Letting n go to $+\infty$, we obtain that

$$\limsup_{j' \to +\infty} \|S(t_{j'})b_{j'}\|_X \leq \|z\|_X \ .$$

The proposition is thus proved. □

Remarks 2.36.

(i) In the applications, the space Y is often the space X endowed with the weak topology of X. Then the condition (ii) is always satisfied. In this case, we can also assume that X is only a reflexive Banach space and replace the conditions on the functional $\mathcal{F}$ by the hypothesis:

(H.1) $\mathcal{F} : X \to \mathbf{R}^+$ is continuous, bounded on the bounded sets of X and the properties $S(t_j)b_j \rightharpoonup z$ weakly in X, where b_j is bounded in X, $t_j \to_{j \to +\infty} +\infty$, and $\limsup_{j \to +\infty} \mathcal{F}(S(t_j b_j)) \leq \mathcal{F}(z)$ imply that $S(t_j)b_j$ converges strongly to z in Z, where Z is a Banach space such that $X \subset Z$, with continuous injection.

Then, one shows like in the proof of Proposition 2.35 that, if B is a bounded set in X such that $\gamma_\tau^+(B)$ is bounded in X for some $\tau \geq 0$ and if $S(t_j)b_j$ is a sequence such that $b_j \in B$ and $t_j \to_{j \to \infty} +\infty$, then there exists a subsequence j', such that $S(t_{j'})b_{j'}$ converges strongly in Z to some element $z \in B_0 \equiv \overline{\mathrm{co}}^X(\gamma_\tau^+(B))$.

(ii) Moise, Rosa and Wang [MRW] consider more general functionals on Y, in the case where Y is the space X endowed with the weak topology.

(iii) A similar result holds for discrete semigroups S provided that the equality (2.21) is replaced by

$$\mathcal{F}(S^n x_0) = \exp(-\gamma n)\mathcal{F}(x_0) + \sum_{m=0}^{n} \exp(-\gamma(n-m))\mathcal{F}_1(S^m x_0)\, ds \ , \quad \forall n \in \mathbf{N} \ . \quad (2.23)$$

Further example of asymptotically smooth semigroups: α-contracting and condensing semigroups.

Another class of examples of asymptotically smooth semigroups is given by the α-contracting and condensing semigroups. Let now X be a Banach space and $\mathcal{B}$ be the set of its bounded subsets. The mapping $\alpha : \mathcal{B} \to [0, +\infty)$, defined by

$$\alpha(B) \ = \ \inf\{l > 0 \,|\, B \, \text{admits a finite cover by sets of diameter } \ \leq l\} \ ,$$

is called the (Kuratowski)-measure of noncompactness or, shorter, the α-measure of noncompactness. It has the following properties (see [De], for example):
(a) $\alpha(B) = 0$ if and only if $\overline{B}$ is compact;
(b) $\alpha(\cdot)$ is a seminorm, i.e., $\alpha(\lambda B) = |\lambda|\alpha(B)$ and $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$;
(c) $B_1 \subset B_2$ implies $\alpha(B_1) \leq \alpha(B_2)$; $\alpha(B_1 \cup B_2) = \max(\alpha(B_1), \alpha(B_2))$;
(d) $\alpha(B) = \alpha(\mathrm{co}(B))$;
(e) α is continuous with respect to the Hausdorff distance Hdist_X.

A continuous map $S : X \to X$ is a *conditional α-contraction of order k*, $0 \leq k < 1$, with respect to the measure α if $\alpha(S(B)) \leq k\alpha(B)$, for all bounded sets $B \subset X$ for which $S(B)$ is bounded. The map S is an *α-contraction of order k* if it is a conditional α-contraction of order k and a bounded map. A continuous map $S : X \to X$ is a *conditional α-condensing map*, with respect to the measure α if $\alpha(S(B)) < \alpha(B)$, for all bounded sets $B \subset X$ for which $S(B)$ is bounded and $\alpha(B) > 0$. The map S is α-condensing if it is conditional α-condensing and bounded.

Every bounded linear operator S can be written in the form $S = U + V$, where the linear map U is compact and the spectral radius of V is the same as the radius of the essential spectrum of S. Also there exists an equivalent norm on X such that $\|S\|_{L(X,X)} = r(\sigma_{\mathrm{ess}}(S))$, with respect to this new norm. From a result of Nussbaum [Nu1] stating that $r(\sigma_{\mathrm{ess}}(S)) = \lim_{n\to+\infty}(\alpha(S^n))^{1/n}$, it follows that S is an α-contraction with respect to a norm equivalent to the one of X if and only if $r(\sigma_{\mathrm{ess}}(S)) < 1$.

The prototype of α-contraction is given by the nonlinear map $S = U + V$, where U is a nonlinear compact map and V is a globally Lipschitz mapping with Lipschitz constant k, $0 \leq k < 1$. In this case, S is an α-contraction of order k. If S is a mapping satisfying the condition (2.20), for any bounded set $B \subset X$, where $0 \leq k < 1$ is independent of B, S is an α-contraction of order k.

Conditional α-condensing maps are asymptotically smooth (for a proof, see [Ma80], [Hal88]). One should notice that asymptotically smooth maps are not necessarily conditional α-condensing, as it is shown by an example in [ChHa, Proposition 1.1]. This fact is somehow expected since the definition of α-condensing involves the metric whereas the definition of asymptotically smooth only involves the topology. So we can wonder if for any asymptotically smooth map S on a Banach space, there exists another equivalent norm for which S is conditional α-condensing.

The following interesting property also holds:

Theorem 2.37. *Let X be a Banach space. If $S : X \to X$ is an α-condensing and compact dissipative mapping, then S has a fixed point.*

This theorem was discovered independently by Nussbaum [Nu2] and by Hale and Lopes [HaLo].

Similar definitions and properties hold for continuous semigroups $S(t)$, $t \in [0, +\infty)$. A semigroup $S(t)$ on X is a *conditional α-contraction* if there exists a continuous function $k : [0, +\infty) \to [0, +\infty)$ such that $k(t) \to 0$ as $t \to +\infty$ and, for each $t > 0$ and each bounded set $B \subset X$ for which $S(t)B$ is bounded, one has $\alpha(S(t)B) \leq k(t)\alpha(B)$. The function $k(t)$ is called the *contracting function* of $S(t)$. The semigroup $S(t)$ is an *α-contraction* if it is a conditional α-contraction and, for each $t > 0$, the set $S(t)B$ is bounded if B is bounded. Likewise, the semigroup $S(t)$, $t \geq 0$, is *α-condensing* if, for any bounded set B in X and for any $t > 0$, the set $S(t)B$ is bounded and $\alpha(S(t)B) < \alpha(B)$ if $\alpha(B) > 0$.
It is shown in [CeLo] that if $S(t)$ is a semigroup satisfying the assumptions of Proposition 2.34 with $M = X$ and the function k is independent of $\|B\|_X$, then $S(t)$ is an α-contraction. In particular, if $S(t)$ is a semigroup satisfying the assumptions of Theorem 2.31, where $M = X$ and $V(t)$ is a globally Lipschitz function with Lipschitz constant $k(t) \to_{t \to +\infty} 0$, then $S(t)$ is an α-contraction.
Conditional α-contractions $S(t)$ are asymptotically smooth (see [Hal88]). Theorem 2.37 and Theorem 2.26 imply the following result (for a proof, see [Hal88, Section 3.4]):

Theorem 2.38. *Let X be a Banach space and $S(t)$, $t \geq 0$, be a continuous semigroup on X. If moreover $S(t)$, $t \geq 0$, is an α-contraction with contracting function $k(t) \in [0, 1)$, is point dissipative and if, for any bounded set $B \subset X$, there exists $\tau \geq 0$ such that $\gamma_\tau^+(B)$ is bounded, then $S(t)$ has (at least) an equilibrium point.*

2.5. Minimal global B-attractors

In some applications, it happens that there exists an unbounded invariant set that attracts all bounded sets. For this reason, we recall here the more general definition of a minimal global attractor.

Let X be a metric space and S be a semigroup on X. Following Ladyzenskaya [La87a], we say that a set $\mathcal{A} \subset X$ is a *minimal global B-attractor* if it is a nonempty, closed, set that attracts all bounded sets of X and is minimal with respect to these properties.

The following result was noted in [HR93a].

Proposition 2.39. *The semigroup S on X admits a minimal global B-attractor $\mathcal{A}_X$ on X if S is asymptotically smooth and if for any bounded set $B \subset X$, there exists $\tau \in G^+$ such that $\gamma_\tau^+(B)$ is bounded. Moreover, $\mathcal{A}_X$ is invariant and*

$$\mathcal{A}_X \;=\; \mathrm{Cl}_X\big(\bigcup\{\omega(B) \,|\, B \textit{ bounded subset of } X\}\big) \;.$$

Proof. It is a direct consequence of Proposition 2.13. □

If, under the assumptions of Proposition 2.39, the union of the ω-limit sets of all the bounded sets is bounded, then $\mathcal{A}_X$ is the compact global attractor. It was asserted in [HR93a] that, under the hypotheses of Proposition 2.39, the minimal global B-attractor is always locally compact. However, this is not the case as has been shown with an example by Valero (see [ChHa]).

Consider the flow $S(t)$ of the linear ODE $\dot{x} = Bx$ where B is a $n \times n$ matrix, which, for example, has one positive eigenvalue and $n-1$ negative eigenvalues, then the one-dimensional unstable manifold of the origin is an unbounded minimal B-attractor. Another simple example is the flow $S(t) : \mathbf{R}^2 \to \mathbf{R}^2$ of the ODE $\dot{x} = 0$, $\dot{y} = -y$. The minimal global B-attractor for $S(t)$ is the x-axis. This equation has a first integral $\Phi(x, y) = x$. On every level set $\Phi_c = \{(x, y) \,|\, \Phi(x, y) = c\}$, $S(t)$ has a compact global attractor $(c, 0)$. This example is a special case of an evolutionary equation on a space X, which has a continuous first integral Φ. It is often the case that on each level set Φ_c, the associated semigroup $S(t)$ admits a compact global attractor $\mathcal{A}_c$. Then the minimal global B-attractor $\mathcal{A}_X$ is given by $\mathcal{A}_X = \mathrm{Cl}_X(\bigcup_{c \in \mathbf{R}} \mathcal{A}_c)$. Other examples of such systems with first integrals are studied in [HR93b, Section 6].

Examples of minimal global B-attractors, that are not necessarily global attractors, arise in the study of damped wave equations with local damping (see [HR93a]). An example of an unbounded minimal global B-attractor is also given in [Qi].

Different notions of global attractors involving two different spaces are considered in [BV89b], [MiSc], [Fe96], [Hal99] and [Mi00], for instance. Often, the semigroups $S(t)$ generated by evolutionary partial differential equations on unbounded domains are no longer asymptotically smooth on function spaces, which are large enough to contain all the interesting dynamics. One way to overcome this difficulty is to introduce two different adequate topologies on the space X, so that, on the bounded sets for the first topology, $S(t)$ is asymptotically smooth for the second topology (see [Mi00], for details).

2.6. Periodic systems

In this paragraph, we very briefly indicate that the notion of global attractor can be extended to evolutionary equations, which are nonautonomous.

Let X be a Banach space. We consider, for instance, the nonautonomous evolutionary equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = f(t,u) \ , \quad u(s) = u_0 \in X \ , \tag{2.24}$$

where f is a continuous map from $\mathbf{R} \times X$ into X and is Lipschitz-continuous in u on the bounded sets of X. Then, through each point (s, u_0) of $\mathbf{R} \times X$, there exists a unique local solution $u(t, s, u_0, f)$ of (2.24). Under appropriate hypotheses on f, this solution is global and we set $u(t, s, u_0, f) = S(t, s)u_0$. The operator $S(t, s) : X \to X$ satisfies the relations

$$S(s,s) = Id \ , \quad \forall s \in \mathbf{R} \ , \quad S(t,s) = S(t,\tau)S(\tau,s) \ , \quad \text{for any } t \geq \tau \geq s \ , \tag{2.25}$$

and has also has the following properties

$$\begin{aligned} S(t,s) &\in \mathcal{C}^0(X;X) \ , \quad \forall s \in \mathbf{R} \ , \quad \forall t \geq s \ , \\ S(t,s)u_0 &\in \mathcal{C}^0([s,+\infty);X) \ , \quad \forall s \in \mathbf{R} \ , \quad \forall u_0 \in X \ . \end{aligned} \tag{2.26}$$

For later use, we introduce a subset $\mathcal{F} \subset \mathcal{C}^0(\mathbf{R} \times X, X)$, which consists of functions $g(t, u)$ satisfying the above conditions.
If there exists $\omega > 0$ such that $f(t + \omega, u) = f(t, u)$, for any $(t, u) \in \mathbf{R} \times X$, then

$$S(t+\omega, s+\omega) = S(t,s) \ , \quad \text{for any } t \geq s \ . \tag{2.27}$$

More generally, let (X, d) be a metric space and let us consider a family of operators $S(t, s) : X \to X$, $s \in \mathbf{R}$, $t \geq s$ satisfying the conditions (2.25), (2.26) and (2.27). Since

$S(t,s)$ is periodic, it is meaningful to introduce the associated period map $T_0 = S(\omega, 0)$ and study the existence of a global attractor for T_0. We begin with a lemma, which shows that for period maps, there is an equivalent form of point dissipativness which is easier to verify (see [Pl66] in the finite-dimesional case and [Hal00] in the general case).

Lemma 2.40. *Let $S(t,s) : X \to X$, $s \in \mathbf{R}$, $t \geq s$, satisfying the conditions (2.25), (2.26) and (2.27). Assume that, for any bounded set $B \subset X$, there exists $n_0 = n_0(B)$ such that $\bigcup_{n \geq n_0} T_0^n(\cup_{0 \leq s < \omega} S(\omega, s)B)$ is bounded. Then, the existence of a positive number R such that, for any $(s, u_0) \in [0, +\infty) \times X$,*

$$\limsup_{t \to +\infty} \|S(t,s)u_0\|_X \leq R \,, \tag{2.28}$$

is equivalent to the existence of a positive number r such that, for any $(s_0, u_0) \in [0, +\infty) \times X$, there exists a time $\tau > s_0$ such that

$$\|S(\tau, s_0)u_0\|_X < r \,, \tag{2.29}$$

As a consequence of Lemma 2.40 and Theorem 2.26, we obtain the following result:

Theorem 2.41. *Let $S(t,s) : X \to X$, $s \in \mathbf{R}$, $t \geq s$, satisfying the conditions (2.25), (2.26) and (2.27). Assume that the property (2.29) holds, that T_0 is asymptotically smooth and that, for any bounded set $B \subset X$, there exists $n_0 = n_0(B)$ such that $\bigcup_{n \geq n_0} T_0^n(\cup_{0 \leq s < \omega} S(\omega, s)B)$ is bounded. Then, T_0 admits a compact global attractor $\mathcal{A}_0$. Moreover, for any $0 \leq \sigma < \omega$, $T_\sigma = S(\sigma + \omega, \sigma)$ has a compact global attractor $\mathcal{A}_\sigma = S(\sigma, 0)\mathcal{A}_0$.*

Remark. If $S(t,s) : X \to X$, $s \in \mathbf{R}$, $t \geq s$, is a family of operators satisfying the conditions (2.25), (2.26) and (2.27), we obtain a particular case of a *periodic process* by setting $U(\tau, s)u_0 = S(\tau + s, s)u_0$. We recall that a family of operators $U(\tau, s) : X \to X$ for $\tau \geq 0$, $s \in \mathbf{R}$, is a process on X, if $U(0, s) = Id$, $U(t, \sigma + s)U(\sigma, s) = U(\sigma + t, s)$, for any $s \in \mathbf{R}$ and for any $\sigma > 0$, $t > 0$, $U(t,s) \in \mathcal{C}^0(X, X)$ and $U(t,s)u \in \mathcal{C}^0([0, +\infty), X)$, for any $u \in X$ and $s \in \mathbf{R}$. The process is periodic if there exists a positive number ω such that $U(t, s + \omega) = U(t, s)$, for any $t \geq 0$, $s \in \mathbf{R}$.
Let us remark that processes are natural extensions of the notion of continuous semigroups. Indeed, if one defines the operators $\Sigma(t) : [s, u] \in \mathbf{R} \times X \mapsto [s + t, U(t,s)u] \in \mathbf{R} \times X$, $\Sigma(t)$, $t \geq 0$, is a continuous semigroup under the aditionnal hypothesis that $U(t,s)u$ is jointly continuous in (s, u).

Unfortunately, since the time variable does not belong to a compact set, the semigroup $\Sigma(t)$ will never have a compact global attractor. Thus, when $U(\tau, s)$ is not periodic, one needs to find another way to generalize the notion of compact attractor to nonautonomous systems.

Let us come back to the evolutionary equation (2.24). For any $t \geq 0$, we introduce the *translation* $\sigma(t)(f)$ defined by $\sigma(t)(f)(s,x) = f(t+s,x)$. We suppose now that the set $\mathcal{F}$ is a metric space, with the property that $f \in \mathcal{F}$ implies that the translation $\sigma(t)(f)$ belongs to $\mathcal{F}$. For any $t \geq 0$, we define $\pi(t) : X \times \mathcal{F} \to X \times \mathcal{F}$ by

$$\pi(t)(u_0, f) \;=\; (u(t,0,u_0,f), \sigma(t)(f)) \; . \tag{2.30}$$

One easily shows that $\pi(0) = Id$ and that $\pi(t+s)(u_0,f) = \pi(t)\pi(s)(u_0,f)$. If the family $\mathcal{F}$ is chosen so that π satisfies the continuity properties required in Definition 2.1, we have thus defined a continuous semigroup on $X \times \mathcal{F}$, which is called the *skew-product flow of* $S(t,s)$ or the *skew-product flow of the associated process* $U(\tau,s)$. Under appropriate compactness hypotheses on $\mathcal{F}$, one can thus study the existence of global attractors for the continuous semigroup $\pi(t)$. Skew-product flows had been first exploited by Miller [Mill65] and Sell [Sell67] (see also [Sell71]) in the frame of ordinary differential equations. Skew-product flows have also been associated to more general processes $U(\tau, s)$ (see [Da75]). For further study of global attractors associated to nonautonomous systems, we refer to [ChVi1], [ChVi2], [MiSe], [Hal88], [Har91] and [SeYou].

3. General properties of global attractors

In the previous section, only a few properties of global attractors have been given. In general, the invariance and attractivity of the global attractors, combined with some additional hypotheses, imply interesting robustsness and regularity properties. Often also, the flow restricted to the global attractor shows finite dimensional behaviour. In this section, we are going to briefly describe such additional properties for compact global attractors.

3.1. Dependence on parameters

One of the basic problems in dynamical systems is to compare the flows defined by different semigroups. In the study of semigroups restricted to a finite dimensional compact manifold (with or without boundary), this comparison is made very often through the notion of *topological equivalence* (for the definition, see below). If the semigroups are defined on a finite or infinite dimensional vector space for example, then considerable care must be taken in order to discuss the behaviour of orbits at infinity. If each of the semigroups has a compact global attractor, one can hope to consider the topological equivalence of the flows restricted to the global attractors. This is the strongest type of comparison of flows that can be expected in the sense that it uses the very detailed properties of the flows. In particular, it requests the knowledge of transversality properties, that are very difficult to show in the infinite-dimensional case. For this reason, we begin with much weaker concepts of comparison, like estimates of the Hausdorff distance between the global attractors. We shall mainly give general comparison results and refer the reader to Section 4 and to [BV89b], [Hal88], [Hal98], [HLR], [HR89], [HR90], [HR92b], [HR93b], [Ko90], [Ra95], [ST] and [Vi92] for applications to (singularly) perturbed systems and discretised equations.

In this paragraph, (X, d) still denotes a metric space and we consider a family of semigroups $S_\lambda(t)$, $t \in G^+$, depending on a parameter $\lambda \in \Lambda$, where $\Lambda = (\Lambda, d_\lambda)$ is a metric space. For sake of clarity, we assume that all the semigroups S_λ are defined on the same space X, although in many applications, each S_λ may be defined on a different space X_λ. Then one has to determine first how to relate these spaces in order to have a concept of convergence of the semigroups S_λ, which replaces the hypotheses (H.1a), (H.1b) or (H.1c) given below. Such situations arise in the discretisation of partial differential equations, in problems on thin domains etc ... (for more details see [HLR], [HR89],[HR93b] and [HR95]).

We assume that each semigroup S_λ has a global attractor $\mathcal{A}_\lambda$ and, if λ_0 is a non-isolated point of Λ, we are interested in the behavior of $\mathcal{A}_\lambda$ when $\lambda \to \lambda_0$.
We say that the sets $\mathcal{A}_\lambda$ are *upper semicontinuous* (resp. *lower semicontinuous*) on Λ at $\lambda = \lambda_0$ if

$$\lim_{\lambda\in\Lambda\to\lambda_0} \delta_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) = 0 \ , \quad (\text{resp. } \lim_{\lambda\in\Lambda\to\lambda_0} \delta_X(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda) = 0) \ . \tag{3.1}$$

We say that the sets $\mathcal{A}_\lambda$ are *continuous* at λ_0 if they are both upper and lower semi-continuous at λ_0. Due to the strong attractivity property and the invariance of the attractors, the upper semicontinuity property holds if the dependence in the parameter is not too bad. To get upper semicontinuity, one often assumes that either

(H.1a) There exist $\eta > 0$, $\tau_0 \in G^+$ with $\tau_0 > 0$ and a compact set $K \subset X$ such that

$$\bigcup_{\lambda \in N_\Lambda(\lambda_0,\eta)} \mathcal{A}_\lambda \subset K \ , \tag{3.2}$$

and if $\lambda_k \to \lambda_0$, $x_k \in \mathcal{A}_{\lambda_k}$, for $k \neq 0$ and $x_k \to x_0$, then $S_{\lambda_k}(\tau_0)x_k \to S_{\lambda_0}(\tau_0)x_0$;

or

(H.1b) There exist $\eta > 0$, $t_0 \in G^+$ with $t_0 > 0$ and a bounded set $B_0 \subset X$ such that

$$\bigcup_{\lambda \in N_\Lambda(\lambda_0,\eta)} \mathcal{A}_\lambda \subset B_0 \ , \tag{3.3}$$

and, for any $\varepsilon > 0$, any $\tau \in G^+$, $\tau \geq t_0$, there exists $0 < \theta = \theta(\varepsilon, \tau) < \eta$ such that

$$\delta_X(S_\lambda(\tau)x_\lambda, S_{\lambda_0}(\tau)x_\lambda) \leq \varepsilon \ , \quad \forall x_\lambda \in \mathcal{A}_\lambda \ , \forall \lambda \in N_\Lambda(\lambda_0, \theta) \ . \tag{3.4}$$

In most of the cases, the hypotheses (H.1a) or (H.1b) are rather easy to check. Actually, in the case of evolutionary partial differential equations, the compactness condition (3.2) is often proved by showing that, due to an asymptotic smoothing effect, the attractors $\mathcal{A}_\lambda$ are uniformly bounded with respect to λ in a metric space X_2 which is compactly embedded in the space X. Often also, the stronger convergence property

(H.1c) there exists $t_0 \in G^+$ with $t_0 > 0$ such that $S_\lambda(t)x \to S_{\lambda_0}(t)x$ uniformly for (t, x) in bounded sets of $G^+ \times X_2$, as $\lambda \in \Lambda \to \lambda_0$,

holds. Hypotheses (H.1a) or (H.1b) imply upper semicontinuity of the attractors at $\lambda = \lambda_0$ (see [Hal88, Theorem 2.5.2]).

Proposition 3.1. *Let λ_0 be a nonisolated point of Λ. If the hypothesis (H.1a) or (H.1b) holds, the global attractors $\mathcal{A}_\lambda$ are upper semicontinuous on Λ at $\lambda = \lambda_0$; that is, $\lim_{\lambda \in \Lambda \to \lambda_0} \delta_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) = 0$.*

Proof. 1) We give only the proof in the case when $G^+ = [0, +\infty)$. Under the hypothesis (H.1a), the global attractors $\mathcal{A}_\lambda$ for $\lambda \in N_\Lambda(\lambda_0, \eta)$ are compact. We remark that $\mathcal{A}_\lambda$ is also the compact global attractor of the discrete semigroup $S_\lambda(\tau)$. To prove upper semicontinuity, it suffices to show that, for any sequences $\lambda_k \in N_\Lambda(\lambda_0,\eta)$, $k \geq 1$, $x_k \in \mathcal{A}_{\lambda_k}$, $k \geq 1$, such that $\lambda_k \to \lambda_0$, $x_k \to x_0$, the limit $x_0 \in \mathcal{A}_{\lambda_0}$. Since $\mathcal{A}_{\lambda_k}$ is invariant under $S_{\lambda_k}(\tau_0)$, there exists $x_k^1 \in \mathcal{A}_{\lambda_k}$ such that $x_k = S_{\lambda_k}(\tau_0)(x_k^1)$. Without loss of generality, due to the compactness of K, we can assume that the sequence x_k^1 converges to some element x_0^1. The hypothesis (H.1a) implies that $x_0^1 = S_{\lambda_k}(\tau_0)x_0$. Using a recursion argument, one thus obtains an infinite sequence $x_0^j \in K$, $j \to +\infty$, where $S_{\lambda_k}^j(\tau_0)x_0^j = x_0$. Clearly, the complete orbit $\gamma(x_0) = \{S_{\lambda_k}^n(\tau_0)x_0 \,|\, n \in \mathbf{Z}\}$ is bounded in X, which implies that $x_0 \in \mathcal{A}_{\lambda_0}$.

2) Let $\varepsilon > 0$ be fixed. Since $\mathcal{A}_{\lambda_0}$ is the global attractor of $S_{\lambda_0}(t)$, there exists a time $\tau_\varepsilon \geq t_0$ such that

$$S_{\lambda_0}(t)B_0 \subset N_X(\mathcal{A}_{\lambda_0}, \varepsilon/2)\ , \quad \forall t \geq \tau_\varepsilon\ . \tag{3.5}$$

By hypothesis (H.1b), there exists $\theta > 0$, such that, for $\lambda \in N_\Lambda(\lambda_0,\theta)$,

$$\delta_X(S_\lambda(\tau_\varepsilon)x_\lambda, S_{\lambda_0}(\tau_\varepsilon)x_\lambda) \leq \varepsilon/2\ , \quad \forall x_\lambda \in \mathcal{A}_\lambda\ ,$$

which, together with (3.5), implies that $S_\lambda(\tau_\varepsilon)\mathcal{A}_\lambda \subset N_X(\mathcal{A}_{\lambda_0},\varepsilon)$. Since $\mathcal{A}_\lambda$ is invariant, $\mathcal{A}_\lambda \subset N_X(\mathcal{A}_{\lambda_0},\varepsilon)$. □

In general, the lower semicontinuity property does not hold, as shown by the simple ODE

$$\dot{x} = (1-x)(x^2-\lambda)\ , \tag{3.6}$$

where $\lambda \in [-1,1]$. Here, $\mathcal{A}_0 = [0,1]$, $\mathcal{A}_\lambda = 1$, for $\lambda < 0$ and $\mathcal{A}_\lambda = [-\sqrt{\lambda},1]$, for $\lambda > 0$ and a bifurcation phenomenon occurs. In general, lower semicontinuity at a given point λ_0 is obtained only by imposing additional conditions on the flow. It is mainly known to hold in the case of gradient like systems, when all the equilibrium points are hyperbolic (see Section 4 below). However, as pointed out in [BaPi], the lower semicontinuity property is generic, under simple compactness assumptions. We recall that a subset Q of a topological space Λ is *residual* if Q contains a countable intersection of open dense sets in Λ. We say that a property (P) of elements of Λ is *generic* if the set $\{\lambda \in \Lambda \,|\, \lambda \text{ satisfies } (P)\}$ is residual. Let K be a compact metric space and K^c be the set of compact subsets of K. It is well-known (see [Ku]), that, if Λ is a topological space, and $f : \Lambda \to K^c$ is upper semicontinuous at any $\lambda \in \Lambda$, then there exists a residual subset $\Lambda_0 \subset \Lambda$ such that f is continuous at every $\lambda_0 \in \Lambda_0$. Here we apply this property to compact global attractors $\mathcal{A}_\lambda$, satisfying the hypothesis (H.1a), with $\lambda \in N_\Lambda(\lambda_0,\eta)$ replaced by $\lambda \in \Lambda$.

Corollary 3.2. *If the hypothesis (H.1a) holds, with $\lambda \in N_\Lambda(\lambda_0, \eta)$ replaced by $\lambda \in \Lambda$, there exists a residual subset Λ_0 of Λ such that the sets $\mathcal{A}_\lambda$ are continuous at any $\lambda_0 \in \Lambda_0$.*

The parameter λ can be the domain $\Omega \subset \mathbf{R}^n$ on which one defines a partial differential equation. In [BaPi], Babin and Pilyugin have applied Corollary 3.2 to prove generic continuity of global attractors of nonlinear heat equations with respect to the domain Ω, thus recovering some of the earlier results of Henry ([He85a], [He87]).

The hypotheses (H.1a) or (H.1b) do not allow to estimate the semidistance $\delta_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0})$. It becomes possible, if one imposes stronger attractivity properties on $\mathcal{A}_{\lambda_0}$ (see [HLR], [BV89b, Chapter 8]):

Proposition 3.3. *Assume that λ_0 is a nonisolated point of Λ and that Hypothesis (H.1b) holds. Suppose also that there exist positive constants α_0, β_0, γ_0, c_0 and c_1 such that*

$$\delta_X(S_{\lambda_0}(t)B_0, \mathcal{A}_{\lambda_0}) \leq c_0 \exp(-\alpha_0 t) \ , \forall t \geq t_0 \ , \quad t \in G^+ \ , \tag{3.7}$$

and, for any $\lambda \in N_\Lambda(\lambda_0, \eta)$, for any $x \in B_0$,

$$\delta_X(S_\lambda(t)x, S_{\lambda_0}(t)x) \leq c_1 \delta_\Lambda(\lambda, \lambda_0)^{\gamma_0} \exp(\beta_0 t) \ , \forall t \geq t_0 \ , \quad t \in G^+ \ , \tag{3.8}$$

then, there exist $c > 0$ and $\eta_1 \leq \eta$ such that, for any $\lambda \in N_\Lambda(\lambda_0, \eta_1)$, we have,

$$\delta_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \leq c\delta_\Lambda(\lambda, \lambda_0)^{\frac{\alpha_0\gamma_0}{\alpha_0+\beta_0}} \ . \tag{3.9}$$

Proof. We introduce the time $t_1 = -\ln(\frac{c_1}{c_0}\delta(\lambda, \lambda_0)^{\frac{\alpha_0\gamma_0}{\alpha_0+\beta_0}})/\alpha_0$. We remark that $t_1 \geq t_0$, if $\eta_1 > 0$ is small enough. From the estimates (3.7) and (3.8), we deduce that $\delta_X(S_\lambda(t_1)\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \leq c\delta_\Lambda(\lambda, \lambda_0)^{\frac{\alpha_0\gamma_0}{\alpha_0+\beta_0}}$, which implies (3.9) by invariance of $\mathcal{A}_\lambda$. $\square$

The property (3.7) is difficult to verify. However, we shall prove it below for gradient systems, whose equilibria are all hyperbolic.

Remark 3.4. If the conditions (3.7) and (3.8) hold for every $\lambda \in \Lambda$, with constants α_0, β_0, γ_0, c_0 and c_1 independent of λ, we obtain the estimate

$$\mathrm{Hdist}_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \leq c\delta_\Lambda(\lambda, \lambda_0)^{\frac{\alpha_0\gamma_0}{\alpha_0+\beta_0}} \ . \tag{3.10}$$

In particular, the attractors $\mathcal{A}_\lambda$ are continuous at every $\lambda \in \Lambda$.

Other concepts of comparison of the attractors $\mathcal{A}_\lambda$, which are weaker than continuity, have been introduced in [HR93b] and may be more appropriate when the limiting system S_{λ_0} is not neccessarily dissipative. For sake of clarity, we assume now that $G^+ = [0, +\infty)$; obviously the results below also hold for maps.

Definition 3.5. Let $\lambda_0 \in \bar{L}$, where L is a subset of Λ. The ω-limit set $\tilde{\omega}_{\mathrm{L}}(\mathcal{A}., \lambda_0)$ of the family of sets $\mathcal{A}_\lambda$, $\lambda \in N_\Lambda(\lambda_0, \eta) \cap L$, $\eta > 0$ is defined by

$$\tilde{\omega}_L(\mathcal{A}., \lambda_0) \;=\; \bigcap_{0<\delta<\eta} \mathrm{Cl}_X \bigcup_{\lambda \in N_\Lambda(\lambda_0,\delta)\cap L} \mathcal{A}_\lambda \;. \tag{3.11}$$

Several properties of the set $\tilde{\omega}_L(\mathcal{A}., \lambda_0)$ are given in [HR93b]. If $\lambda_0 \in L$ and, for each $\lambda \in N_\Lambda(\lambda_0, \eta) \cap L$, S_λ has a compact global attractor, then the upper semicontinuity (resp. the lower semicontinuity) of the attractors at $\lambda = \lambda_0$ implies that $\tilde{\omega}_L(\mathcal{A}., \lambda_0) \subset \mathcal{A}_{\lambda_0}$ (resp. $\mathcal{A}_{\lambda_0} \subset \tilde{\omega}_L(\mathcal{A}., \lambda_0)$). If $\mathrm{Cl}_X(\bigcup_{\lambda \in L \cap N_\Lambda(\lambda_0,\eta)} \mathcal{A}_\lambda)$ is compact, it follows from the inclusion $\tilde{\omega}_L(\mathcal{A}., \lambda_0) \subset \mathcal{A}_{\lambda_0}$ that the attractors are upper semicontinuous at λ_0. We remark that the inclusion $\mathcal{A}_{\lambda_0} \subset \tilde{\omega}_L(\mathcal{A}., \lambda_0)$ does not imply lower semicontinuity of the attractors $\mathcal{A}_\lambda$. Indeed, consider the ODE $\dot{x} = -x((-1)^n \lambda_n + (x-1)^2)$ with $\lambda_n = 1/n$, that is $L = \{1, 1/2, ..., 1/n, ...\}$. There is no continuity of the attractors at $\lambda = 0$; however, $\tilde{\omega}_L(\mathcal{A}., 0) = \mathcal{A}_0 = [0, 1]$.
We notice that $\tilde{\omega}_L(\mathcal{A}., \lambda_0)$ does not involve directly the semigroup S_{λ_0}. In particular, S_{λ_0} could be conservative. The following question then arises: how much information can we obtain about a conservative system by considering the limit of dissipative systems, when the dissipation goes to zero? We cannot hope to obtain too many specific properties of the dynamics of the limit system in this way, but one should be able to obtain some information about the manner in which the orbits of the dissipative systems wander over the level sets of the energy of the limit system.
Consider the ODE $\dot{u} = v$, $\dot{v} = f(u) - \beta v$, where $\beta \geq 0$ is a constant, $f \in \mathcal{C}^2(\mathbf{R}, \mathbf{R})$ has only simple zeros and $f(u)$ is dissipative (i.e. $\limsup_{|u| \to +\infty} \frac{f(u)}{u} \leq \alpha < 0$). The energy functional is $\Phi(u, v) = (1/2)v^2 - \int_0^u f(s)ds$. For $\beta > 0$, the ODE is a gradient system and has a global attractor $\mathcal{A}_\beta$. Let $\{s_j,\ j = 1, 2, ...M\}$ be the set of the saddle equilibrium points of the system. If $\Phi(s_j) \neq \Phi(s_k)$, for $j \neq k$, $j, k = 1, 2, ...M$, then, for any interval $L = (0, \beta_0]$,

$$\tilde{\omega}_L(\mathcal{A}., 0) \;=\; \{(u, v) \in \mathbf{R}^2 \,|\, \Phi(u, v) \;\leq\; c_M\} \;,$$

where $c_M = \max\{\Phi(s_j)\, j = 1, 2, \ldots M\}$ (for details, see [HR93b]).

The limit $\tilde{\omega}_L(\mathcal{A}., \lambda_0)$ only uses information about the attractors. As a consequence, the transient behaviour of the semigroups S_λ for initial data not on the attractors is completely ignored. To gain some information about this transient behaviour, one can consider the following concept of ω-limit set:

Definition 3.6. Let $\lambda_0 \in \bar{L}$, where L is a subset of Λ and let $S_\lambda(t)$ be a family of semigroups on the metric space X. For a given subset B of X, the *ω-limit set of B with respect to the family of semigroups $S_\lambda(t)$, $\lambda \in L \cap N_\Lambda(\lambda_0, \eta)$, $\eta > 0$*, is denoted by $\hat{\omega}_L(B)$ and is defined in the following way: a point $y \in \hat{\omega}_L(B)$ if and only if there are sequences $\lambda_n \in L \cap N_\Lambda(\lambda_0, \eta)$, $\lambda_n \to \lambda_0$, $t_n \to +\infty$ and $x_n \in B$ such that $S_{\lambda_n}(t_n)x_n \to y$.

One remarks that the definition of $\hat{\omega}_L(B)$ treats λ as if it were also a time parameter; it does not prescribe an order in the limits. The notion of continuity of global attractors or the definition of $\tilde{\omega}_L(\mathcal{A}., \lambda_0)$ prescribes the limit $t \to +\infty$ before the limit $\lambda \to \lambda_0$. However, in many practical situations, it is not clear that an order in the limits should be imposed (see, for instance, the discussion in [Mi99]).

Suppose that $\lambda_0 \in \Lambda$ and that Hypothesis (H.1c) holds. If B is a bounded set such that the ω-limit set $\omega_{\lambda_0}(B)$ of B with respect to S_{λ_0} exists, is nonempty, compact and attracts B, and if, either $\omega_{\lambda_0}(B) \subset B$ or $\omega_{\lambda_0}(B)$ attracts a neighbourhood of B, then, $\omega_{\lambda_0}(B) = \hat{\omega}_L(B)$. In particular, if the semigroup S_{λ_0} has a compact global attractor $\mathcal{A}_{\lambda_0} \subset B$, then the equality $\hat{\omega}_\Lambda(B) = \mathcal{A}_{\lambda_0}$ holds. Applications of this property to situations, where the limit system S_{λ_0} has a first integral, are given in [HR93b]. For example, consider the retarded differential difference equation

$$\dot{x} = -(1+\varepsilon) f(x(t)) + f(x(t-1)) \;, \tag{3.12}$$

where $\varepsilon \geq 0$ is a parameter, $f \in \mathcal{C}^1(\mathbf{R}, \mathbf{R})$ satisfies $f(0) = 0$ and $f'(x) \geq \delta > 0$, for all $x \in \mathbf{R}$. For any $\varepsilon \geq 0$, one defines a semigroup $S_\varepsilon(t)$ on the space $X = \mathcal{C}^0([-1, 0], \mathbf{R})$ by the relation $(S_\varepsilon(t)\varphi)(\theta) = x(t + \theta, \varphi)$, $\theta \in [-1, 0]$, where $x(t, \varphi)$ is the unique solution of (3.12) with initial data φ. It is shown ([HR93b]) that, for $\varepsilon > 0$, the global attractor $\mathcal{A}_\varepsilon$ reduces to $\{0\}$, which implies that $\tilde{\omega}_{(0,\varepsilon_0]}(\mathcal{A}., 0) = \{0\}$, for any positive number ε_0. On the other hand, for $\varepsilon = 0$, the function $\Phi(\varphi) = \varphi(0) + \int_{-1}^{0} f(\varphi(s))\, ds$ is a first integral. On each level set $\Phi^{-1}(c)$, there is a unique equilibrium point $e(c)$ of (3.12), for $\varepsilon = 0$, and the ω-limit set $\omega_0(\Phi^{-1}(c))$ with respect to the semigroup $S_0(t)$ reduces to

$\{e(c)\}$. It is then proved that, if B is an arbitrary closed, bounded set in X, we have $\hat{\omega}_{(0,\varepsilon_0]}(B) = \omega_0(B) = I(B)$, where $I(B) = \{e(c_\varphi) \,|\, \varphi \in B\}$ and $c_\varphi = \Phi(\varphi)$.

We next want to study how the flows -restricted to the compact global attractors-vary with the perturbation parameters. If the semigroups are defined on an infinite-dimensional vector space, we are lead to make severe restrictions on the flows, that we consider. For this reason, we restrict our discussion to Morse-Smale systems. Since general comparison results are mainly available in the frame of discrete Morse-Smale systems, we shall restrict our study to this class.

Morse-Smale maps.

As we have already explained, in the infinite dimensional case, the strongest expected comparison of the dynamics of two different semigroups is the *topological equivalence* of the flows restricted to the compact global attractors. In the case of discrete semigroups, the notion of topological equivalence is replaced by the *conjugacy* of the trajectories. In [Pal69] and [PaSm], it has been proved that any Morse-Smale $\mathcal{C}^r$-diffeomorphism S, defined on a compact manifold M is stable, that is, there exists a neighbourhood $N^r(S)$ of S in the set $\mathrm{Diff}^r(M)$ of all $\mathcal{C}^r$-diffeomorphisms, $r \geq 1$, such that, for each $T \in N^r(S)$, there exists a homeomorphism $h \equiv h(T) : M \to M$ and $h \circ T = T \circ h$ holds on M. This important stability property has been generalized to the Morse-Smale maps defined on a Banach manifold by Oliva (see [Ol82] and [HMO]). For sake of simplicity, we describe this result only in the case of a Banach space X. We begin with some definitions and notations.
Let $S \in \mathcal{C}^r(U, X)$, $r \geq 1$ and U be an open subset of X. For any fixed point x_0 of S, we introduce the stable and unstable sets of S at x_0 by

$$\begin{aligned} W^s(x_0, S) &= \{y \in X \,|\, S^n(y) \to x_0 \text{ as } n \to +\infty\} \\ W^u(x_0, S) &= \{y \in X \,|\, \text{there exists a negative orbit } u_y \text{ of } S \text{ such that } u_y(0) = y \\ &\qquad\qquad \text{and } u_y(-n) \to x_0 \text{ as } n \to +\infty\} \end{aligned} \tag{3.13}$$

A fixed point x_0 of S is *hyperbolic* if the spectrum $\sigma(DS(x_0))$ does not intersect the unit circle $\{z \in \mathbf{C} \,|\, |z| = 1\}$ in $\mathbf{C}$ with center 0.

Remark 3.7. If x_0 is a hyperbolic fixed point of $S \in \mathcal{C}^r(U, X)$, $r \geq 1$, then there exists a neighbourhood V of x_0 in U such that the sets

$$\begin{aligned} W^s_{\mathrm{loc}}(x_0,S) &= W^s(x_0,S,V) = \{y \in W^s(x_0,S) \mid S^n(y) \in V,\, n \geq 0\} \\ W^u_{\mathrm{loc}}(x_0,S) &= W^u(x_0,S,V) = \{y \in W^u(x_0,S) \mid S^{-n}(y) \text{ exists and } S^{-n}(y) \in V,\, n \geq 0\} \end{aligned} \tag{3.14}$$

are embedded $\mathcal{C}^r$-submanifolds of X. These sets are called the *local stable* and *local unstable manifolds* of x_0. The manifold $W^s_{\mathrm{loc}}(x_0,S)$ is positively invariant, whereas $W^u_{\mathrm{loc}}(x_0,S)$ is negatively invariant. Moreover $W^u_{\mathrm{loc}}(x_0,S)$ is locally positively invariant. If the part of the spectrum $\sigma(DS(x_0))$ lying outside the unit circle is composed of a finite set of m eigenvalues, then $W^u_{\mathrm{loc}}(x_0,S)$ (resp. $W^s_{\mathrm{loc}}(x_0,S)$) is an embedded $\mathcal{C}^r$-submanifold of dimension m of X (resp. of codimension m of X).
If S and the derivative $DS(y)$ are injective at each point y of $\cup_{n\geq 0}(S^n(W^u_{\mathrm{loc}}(x_0,S))$, then $W^u(x_0,S) = \cup_{n\geq 0}(S^n(W^u_{\mathrm{loc}}(x_0,S))$ is an injectively immersed $\mathcal{C}^r$-submanifold of X, of the same dimension as $W^u_{\mathrm{loc}}(x_0,S)$, and is invariant. If $W^s_{\mathrm{loc}}(x_0,S)$ is of finite codimension m, if S is injective and the derivative $DS(y)$ has dense range at each point y of $\cup_{n\geq 0}(S^{-n}(W^s_{\mathrm{loc}}(x_0,S))$, then $W^s(x_0,S) = \cup_{n\geq 0}(S^{-n}(W^s_{\mathrm{loc}}(x_0,S))$ is an injectively immersed $\mathcal{C}^r$-submanifold of codimension m of X (see [He81, Theorem 6.1.9]). Moreover, $W^s(x_0,S)$ is invariant under S. For further details, see also Section 4.1.

A point x_0 is a periodic point of period p if $S^p(x_0) = x_0$, $S^n(x_0) \neq x_0$, for $0 < n \leq p-1$. A periodic point x_0 of period p is hyperbolic if the (finite) orbit $\mathcal{O}(x_0) = \{x_0, S(x_0), ..., S^{p-1}(x_0)\}$ of x_0 is hyperbolic, that is, if every point $y \in \mathcal{O}(x_0)$ is a hyperbolic fixed point of S^p. As above, one introduces the sets $W^s_{\mathrm{loc}}(y,S)$ and $W^u_{\mathrm{loc}}(y,S)$ and $W^u(y,S) = \cup_{n\geq 0}(S^{np}(W^u_{\mathrm{loc}}(y,S))$, for every $y \in \mathcal{O}(x_0)$. These stable and unstable sets have the properties mentioned in Remark 3.7. Hereafter, we denote by $\mathrm{Per}(S)$ the set of periodic points of S.
Let $S \in \mathcal{C}^r(X,X)$, $r \geq 1$. The nonwandering set $\Omega(S)$ of S is the set of all points $x \in \mathcal{J}(S)$ (where $\mathcal{J}(S)$ is the maximal bounded invariant set of S) such that, given a neighbourhood V of x in $\mathcal{J}(S)$ and any integer n_0, there exists $n \geq n_0$ with $S^n(V) \cap V \neq \emptyset$. If $\Omega(S)$ is finite, then $\Omega(S) = \mathrm{Per}(S)$. One also notices that, if $\mathcal{J}(S)$ is compact and S is injective on $\mathcal{J}(S)$, then $\Omega(S)$ is compact and invariant.

Following [Ol82] and [HMO], we introduce a topological subspace $\mathcal{KC}^r(X,X)$ of $\mathcal{C}^r_b(X,X)$, $r \geq 1$, with the following properties:
(KC1) $S \in \mathcal{KC}^r(X,X)$ implies that $\mathcal{J}(S)$ is compact;
(KC2) the sets $\mathcal{J}(S)$ are uppersemicontinuous on $\mathcal{KC}^r(X,X)$, that is, for any $S \in \mathcal{KC}^r(X,X)$, given a neighbourhood U of $\mathcal{J}(S)$ in X, there exists a neighbourhood $V(S)$

of S in $\mathcal{KC}^r(X,X)$ such that $\mathcal{J}(T) \subset U$, for any $T \in V(S)$;
(KC3) for any $S \in \mathcal{KC}^r(X,X)$, S and DS are injective at each point of $\mathcal{J}(S)$.

Example. Let $S_\lambda \in \mathcal{C}_b^r(X,X)$, $r \geq 1$, be a family of maps depending on a parameter $\lambda \in \Lambda$, where Λ is a metric space. Assume that each map S_λ admits a compact global attractor $\mathcal{A}_\lambda$ and satisfies the hypothesis (H.1a) or (H.1b) at every point $\lambda_0 \in \Lambda$. Then, $\mathcal{J}(S_\lambda) = \mathcal{A}_\lambda$ and the sets $\mathcal{A}_\lambda$ are uppersemicontinuous in λ. If moreover the above condition (KC3) holds, the family $S_\lambda, \lambda \in \Lambda$ can be chosen as a $\mathcal{KC}^r(X,X)$-space.

Finally, we introduce the class of Morse-Smale maps:

Definition 3.8. A map $S \in \mathcal{C}_b^r(X,X)$, $r \geq 1$, is a Morse-Smale map if the above conditions (KC1) and (KC3) as well as the following conditions are satisfied:
(i) $\Omega(S)$ is finite (hence $\Omega(S) = \mathrm{Per}(S)$);
(ii) every periodic point x_0 of S is hyperbolic and $\dim W^u(x_0,S)$ is finite;
(iii) $W^u(x_0,S)$ is tranversal to $W^s_{\mathrm{loc}}(x_1,S)$, for any periodic points x_0 and x_1 of S.

If S is a Morse-Smale map, then $\mathcal{J}(S) = \cup_{x_0 \in \mathrm{Per}(S)} W^u(x_0,S)$. The Morse-Smale maps have a remarkable property, namely they are $\mathcal{J}$-stable.

Definition 3.9. A map $S \in \mathcal{KC}^r(X,X)$ is *$\mathcal{J}$-stable* or simply *stable* if there exists a neighbourhood $V(S)$ of S in $\mathcal{KC}^r(X,X)$, such that each $T \in V(S)$ is *conjugate* to S, that is, there exists a homeomorphism $h = h(T) : \mathcal{J}(S) \to \mathcal{J}(T)$ satisfying the conjugacy condition $h \circ S = T \circ h$ on $\mathcal{J}(S)$.

Adapting the arguments used in [Pal69] and in [PaSm], Oliva showed, mutatis mutandis, the following basic result (see [Ol82] and [HMO]):

Theorem 3.10. *Let a subspace $\mathcal{KC}^r(X,X)$ of $\mathcal{C}_b^r(X,X)$, $r \geq 1$, be given. The set of all r-differentiable Morse-Smale maps is open in $\mathcal{KC}^r(X,X)$. Moreover, every Morse-Smale map S in $\mathcal{KC}^r(X,X)$ is $\mathcal{J}$-stable.*

This result has important applications in the study of partial differential equations depending on various parameters, including time or space discretisations. In Section 4, we shall apply it to gradient systems. If $S_\lambda \in \mathcal{C}^r(X,X)$, $r \geq 1$, is a family of maps depending on a parameter $\lambda \in \Lambda$ and S_{λ_0} is a Morse-Smale map, Theorem 3.10 allows to conclude that, for λ close to λ_0, S_λ has the same type of connecting orbits. If S_{λ_0} is no longer a Morse-Smale map, this persistence of connecting orbits can still be proved in some cases, with the help of the Conley index (see [Co], [MiMr], [MR], for example).

If $S_1(t)$ and $S_2(t)$ are two continuous semigroups on a Banach space X, we say that $S_1(t)$ is *topologically equivalent* or simply *equivalent* to $S_2(t)$, if there exists a homeomorphism $h : \mathcal{J}(S_1) \to \mathcal{J}(S_2)$, which preserves the orbits and the sense of orientation in time t. A continuous semigroup $S_1(t)$ is *stable* if there exists a neighbourhood $N(S_1(\cdot)$ of $S_1(\cdot)$ within a given class of continuous semigroups such that every semigroup $T(\cdot) \in N(S_1(\cdot))$ is equivalent to $S_1(\cdot)$. Like above, one can define Morse-Smale continuous semigroups. Very recently, Oliva [Ol00] has given a proof of a stability result for Morse-Smale continuous semigroups in the infinite-dimensional case. Also, stability of certain continuous Morse-Smale semigroups $S_\lambda(t)$ generated by evolutionary equations has been proved by reducing $S_\lambda(t)$ to a Morse-Smale system $\Sigma_\lambda(t)$, defined by a finite-dimensional system of ODE's depending smoothly enough on the parameter λ (see Section 3.4 below on inertial manifolds).

3.2. Dimension of compact global attractors

The existence of a (compact) global attractor $\mathcal{A} \subset X$ leads to the question of whether there exists a finite-dimensional dynamical system whose dynamics on its global attractor reproduces the dynamics on $\mathcal{A}$ or at least whose attractor has the same topological properties as $\mathcal{A}$. Also, from the computational point of view, one is interested in knowing if the solutions on the attractor can be recovered by solving numerically a large enough system of ODE's and how big should be this system. A first step in this direction consists in showing that the "dimension" of the set $\mathcal{A}$ is finite and in giving a good estimate of it. Various notions of dimension have been studied in conjunction with global attractors. Among them, the Hausdorff and fractal dimensions have played a primordial role. We will briefly describe both notions and state some results. For an exhaustive study in the Hilbertian framework, we refer to the book of Temam ([Te]).

Let E be a topological space. We say that E has *finite topological dimension* if there exists an integer n such that, for every covering $\mathcal{U}$ of E, there exists another open covering $\mathcal{U}'$ refining $\mathcal{U}$ so that every point of E belongs to at most $n+1$ sets of $\mathcal{U}'$. In this case, the topological dimension $\dim(E)$ is defined as the minimal integer n satisfying this property. It is a classical result that, if E is a compact space with $\dim(E) \leq n$, where n is an integer, then it is homeomorphic to a subset of $\mathbf{R}^{2n+1}$. Moreover, the set of such homeomorphisms is residual in the set of all maps from E

into $\mathbf{R}^{2n+1}$. However, special properties of such embeddings are not known, and, in the case where E is contained in a Banach space, it could be more convenient to deal with linear projections (simply called projections in what follows). As generalizations of the topological dimension, there are several stronger fractional measures of dimension applicable to sets which have no regular structure. The most commonly used are the Hausdorff and fractal dimensions.

Let $E \subset X$, where X is a metric space. The Hausdorff dimension is based on approximating the d-dimensional volume of the set E by a covering of a finite number of balls with radius smaller than ε, that is,

$$\mu(E,\alpha,\varepsilon) \;=\; \inf\left\{\sum_i r_i^\alpha \,|\, r_i \le \varepsilon \text{ and } E \subset \bigcup_i B_X(x_i,r_i)\right\} \;,$$

where $B_X(x_i,r_i)$ is the ball of center x_i and radius r_i. The α-dimensional Hausdorff measure of E is then defined as

$$\mu^\alpha(E) \;=\; \lim_{\varepsilon\to 0} \mu(E,\alpha,\varepsilon) \;,$$

and the Hausdorff dimension $\dim_H(E)$ of E is essentially the value of α for which $\mu^\alpha(E)$ is a finite nonzero number,

$$\dim_H(E) \;=\; \inf\{\alpha > 0 \,|\, \mu^\alpha(E) = 0\} \;.$$

It is known [HW] that $\dim(E) \le \dim_H(E)$ and that $\dim(E) = \dim_H(E)$ if E is a submanifold of a Banach space. There are also examples of sets E in $\mathbf{R}^n$, for which $\dim(E) = 0$ and $\dim_H(E) = n$.

The fractal dimension also called *limit capacity* or *box dimension* is a stronger measure than the Hausdorff dimension; here all the balls in the covering are required to have the same radius. Given $\varepsilon > 0$, let $n(\varepsilon, E)$ be the minimal number of balls $B_X(x_i,\varepsilon)$ of radius ε needed to cover E. One defines the fractal dimension $\dim_F(E)$ as

$$\dim_F(E) \;=\; \limsup_{\varepsilon\to 0} \frac{\log n(\varepsilon,E)}{\log(1/\varepsilon)} \;.$$

It is easily proved that $\dim_H(E) \le \dim_F(E)$ (see [Man]). The quantities $\dim_H(E)$ and $\dim_F(E)$ can be different (see [Man] for an example of a compact subset K of l^2 with finite Hausdorff dimension and infinite fractal dimension).

One of the first estimates of the Hausdorff dimension of compact invariant sets has been given by Mallet-Paret in 1976. In [MP76], he showed that, if K is a compact subset of a separable Hilbert space H and is negatively invariant under a mapping $S \in \mathcal{C}^1(U, H)$, where U is a given open neighbourhood of K and where, for any $z \in K$, the derivative $DS(z)$, restricted to some linear subspace $Y \subset X$ of finite codimension, is a strict contraction, then $\dim_H(K)$ is finite. From this property, he deduced that, if the contraction hypothesis is replaced by the condition that the derivative $DS(z)$ is a compact operator, for any $z \in K$, then $\dim_H(K)$ is finite. The same conclusion holds if the property on DS is replaced by the property that $S(U)$ is relatively compact in H. This implies that the compact global attractor of a large class of delay equations and of parabolic equations, including the two-dimensional Navier-Stokes equations, has finite Hausdorff dimension (see [MP76], for an application to RFDE and the heat equation). The hypothesis " $DS(z)\big|_Y$ is a strict contraction, for any $z \in K$" is a so called *flattening condition* onto a finite dimensional subspace of X. Using the same ideas as Mallet-Paret and a squeezing property of the semigroup $S(t)$, Foias and Temam [FT79] showed that any compact invariant subset under the flow generated by the two-dimensional Navier-Stokes equations has finite Hausdorff dimension and gave estimates of the dimension. Assuming flattening conditions similar to [MP76], but on the mapping S itself, Ladyzhenskaya ([La82]) has improved the estimate of the Hausdorff dimension for the global attractor of the two-dimensional Navier-Stokes equations given in [FT79].
Notice that, in [HMO, Theorem 6.8], there is an interesting result of existence of retarded functional differential equations the attractors of which have infinite Hausdorff dimension.

In 1981, Mañé ([Man]) generalized the abstract result of Mallet-Paret to the case of a Banach space and weakened the "flattening condition" in the following way.

Let X be a Banach space and $L_\lambda(X, X)$ be the space of bounded linear mappings Σ that can be decomposed as $\Sigma = \Sigma_1 + \Sigma_2$ where $\Sigma_1 \in L(X, X)$ is compact and $\|\Sigma_2\|_{L(X,X)} < \lambda$. One remarks that, for $\Sigma \in L_{\lambda/2}(X, X)$, there exists a finite-dimensional subspace $Y \subset X$ such that, if $\Sigma_Y : Y \subset X \to X$ is the map induced by Σ on Y, then $\|\Sigma_Y\|_{L(X)} < \lambda$. This amounts to define the number $\nu_\lambda(\Sigma)$ as the minimal integer n such that there exists a subspace $Y \subset X$ of dimension n with $\|\Sigma_Y\|_{L(X)} < \lambda$, when $\Sigma \in L_{\lambda/2}(X, X)$.

Theorem 3.11. *Let X be a Banach space, K be a compact negatively invariant set* *under a mapping $S \in \mathcal{C}^1(U,X)$, where U in an open neighbourhood of K in X. If $DS(x)$ belongs to the space $L_1(X,X)$, for any $x \in K$, then*

$$\dim_F(K) < +\infty \ .$$

Theorem 3.11 is proved in [Man]. There, the following explicit bound for $\dim_F(K)$ is given in the case where $DS(x)$ belongs to the space $L_{1/4}(X,X)$, for any $x \in K$,

$$\dim_F(K) \leq \frac{\log \lambda_1}{\log(1/2(1+\varepsilon)\lambda)} \ ,$$

where $\lambda_1 = \nu 2^\nu (1+\frac{k+\lambda}{\lambda\varepsilon})^\nu$, $0 < \lambda < 1/2$, $0 < \varepsilon < (1/2\lambda)-1$, $k = \sup_{x\in K} \|DS(x)\|_{L(X,X)}$ and $\nu = \sup_{x\in K} \nu_\lambda(DS(x))$. One then remarks that, if $DS(x)$ belongs to the space $L_1(X,X)$ for any $x \in K$, there exists an integer n, sufficiently large, such that $DS^n(x) \in L_{1/4}(X,X)$, for any $x \in K_n \equiv \cap_0^n S^{-j}(K)$, which implies that $\dim_F(K) = \dim_F(K_n)$ is finite.

Remark. In practice, Theorem 3.11 is very useful to show that global attractors have finite fractal dimension, even if the bounds of the dimension given in the proof are not so accurate. Indeed, in most of the cases where the semigroup satisfies the hypotheses of Theorem 2.31, one can also show that the hypothesis, required on DS in Theorem 3.11, holds.

In the case of Hilbert spaces, the notion of m-dimensional volume element and its evolution are easily expressed, which gives much more accurate bounds of the various dimensions of the attractor (see [DO], [CFT85], where $DS(x)$ is a compact mapping and [GT87b] for the non compact case). In particular, the *Lyapunov exponents* of the flow on the attractor have become a standard tool in the description of the evolution of volumes under the semigroup S. A result, first proved by Constantin and Foias [CF85] in the case of the attractor of the two-dimensional Navier-Stokes equations, states that, if the sum of the first m global Lyapunov exponents on an invariant compact set K is negative, then the Hausdorff dimension of K is less than m and the fractal dimension of K is finite and bounded above, up to the product by a universal constant, by m (see [CFT85, Chapter 3] and, for refinements using the local Lyapunov exponents, [EFT]). We refer especially the reader to the book of Temam [Te], where these topics are well explained and estimates of the dimension of the global attractors of numerous partial differential equations, including the reaction-diffusion equations, the damped wave equations, the Navier-Stokes, Kuramoto-Sivashinsky and Cahn-Hilliard equations, are given in terms of physical parameters. Other various estimates of dimensions of attractors are contained in [BV89b], [La87b], [La90] (see also [BN] in this volume). Finally, we note that Thieullen [Th] has given estimates of the Hausdorff dimension of K, involving "Lyapunov exponents" in the frame of Banach spaces.

If K is a compact subset of a Banach space X, with $\dim_F(K) < m/2$, $m \in \mathbf{N}$, then, for every subspace $Y \subset X$ of dimension $\dim Y \geq m$, the set of projections $P : X \to Y$ such that $P\big|_Y$ is injective on Y is a residual subset of the space of all the (continuous) projections from X onto Y, endowed with the norm topology. This result has been given in [Man], where, by an unfortunate mistake, $\dim_F(K)$ has been replaced by $\dim_H(K)$. One notices that the statement is no longer true with the hypothesis $\dim_H(K) < m/2$ (see [SYC]). Recently, in the Hilbertian case, the above result has been improved by Foias and Olson [FO], who showed that the inverse $(P\big|_Y)^{-1}$ of most projections P are Hölder continuous mappings. In general, these inverse are not Lipschitz-continuous ([ML]).

Theorem 3.12. *(Mañé; Foias and Olson). Let X be a Hilbert space and K be a compact subset of X with fractal dimension $\dim_F(K) < m/2$, $m \in \mathbf{N}$. Then, for any projection (resp. orthogonal projection) P_0 onto a subspace Y of X of dimension m and for any $\varepsilon > 0$, there exist a projection (resp. orthogonal projection) P onto Y and a positive number $\theta \leq 1$, such that $P\big|_K$ is injective, $\|P - P_0\|_{L(X,Y)} \leq \varepsilon$ and $P\big|_K$ has Hölder inverse, i.e.,*

$$\|P^{-1}x - P^{-1}y\|_{L(X,X)} \leq C\|x - y\|_X^\theta \ , \quad x, y \in P(K) \ .$$

As an application of this theorem, we go back to Example 2.2, in the Hilbertian case, and write the equation (2.1) under the following short form, where we assume, for sake of simplicity, that $Y = X$,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{F}(u) \ , \quad u(0) = u_0 \in X \ . \tag{3.15}$$

We assume that the continuous semigroup $S(t)$ has a compact global attractor $\mathcal{A}$ of finite fractal dimension and that $\mathcal{F}\big|_{\mathcal{A}}$ is Hölder continuous from X into X. Let P be a projection given by the above theorem. On $P\mathcal{A}$, the following dynamical system is well defined,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = P\mathcal{F}(P^{-1}z) \equiv \tilde{F}(z) \ , \quad z \in P\mathcal{A} \ , \tag{3.16}$$

where, under the above hypothesis, $\tilde{F}$ is Hölder continuous from $P\mathcal{A}$ into $P\mathcal{A}$. The next step is to extend this system to a system of differential equations defined everywhere in $\mathbf{R}^m$, by using a standard extension theorem like the theorem of Stein. Since $\tilde{F}$ is only Hölder continuous, the solutions of this generalized system of ODE's may not be unique and differentiable. However, one can show that the solutions of this extended system exist globally and are attracted by $P\mathcal{A}$ (see [EFNT, Chapter 10]).
One can also use Theorem 3.12 to show that, under the above hypotheses, it is always possible to reproduce "approximately" the dynamics of (3.15) on the global attractor $\mathcal{A}$ by a system of ODE's in $\mathbf{R}^3$. This is done in [Ro], where it is proved that, if $\mathcal{F}$ is bounded and continuous on $\mathcal{A}$, then, for any $T > 0$ and any $\varepsilon > 0$, there exist two functions $g : \mathbf{R}^3 \to \mathbf{R}^3$ and $\Phi : \mathbf{R}^3 \to X$, which are Lipschitz and Hölder continuous, respectively, such that, for any solution $u(t) \in \mathcal{A}$, there exists a solution $z(t)$ of

$$\frac{\mathrm{d}z}{\mathrm{d}t} = g(z) \ ,$$

with $\|\Phi(z(t)) - u(t)\|_X \leq \varepsilon$, for any $t \in [0, T]$.

If one drops the requirement of obtaining a conjugacy between the flows on the attractors, one can construct a homeomorphism between the attractor $\mathcal{A}$ and the one of a finite sytem of ODE's. More precisely, if $\mathcal{A}$ is the compact global attractor of a continuous semigroup generated by an evolutionary equation on a Hilbert space X, with finite Hausdorff dimension, then there exists a finite system of ODE's with a global attractor K, homeomorphic to $\mathcal{A}$. Conversely, if K is the global attractor of a finite system of ODE's, there exists an infinite-dimensional continuous semigroup $S(t)$ generated by an evolutionary equation on X, with a compact global attractor $\mathcal{A}$ homeomorphic to K and having finite Hausdorff dimension (see [Ro], where also a review of other related results is given).

3.3. Regularity of the flow on the attractor and determining modes

If $S(t)$ is a continuous semigroup on X, which has a compact global attractor $\mathcal{A}$, the semigroup $S(t)$ restricted to $\mathcal{A}$ can have interesting "regularity" properties that are not shared by the semigroup $S(t)$ and are a consequence of the compactness and invariance of $\mathcal{A}$. For instance, if $S(t)$ is generated by a partial differential equation defined on a domain $\Omega \subset \mathbf{R}^n$, $\mathcal{A}$ can be composed of functions $u(y)$ which are more regular in the space variable $y \in \Omega$ than the typical elements of the space X. On the other hand, for semigroups defined by evolutionary equations in the infinite-dimensional case,

it is not expected, in general, that, for each $x \in X$, $S(t)x$ is differentiable in t, for $t \geq 0$. In many cases, however, the function $t \mapsto S(t)x$ will be differentiable in t if x belongs to a compact invariant set. Time regularity results in a general setting had been obtained in 1985 by Hale and Scheurle ([HS]) and have recently been generalized in [HR00], through the use of a Galerkin decomposition. We present here these results in the simplified situation of Example 2.2. In the same time, we want to emphasize the relation between regularity and finite-dimensionality character. We recall that X is a Banach space, A is the infinitesimal generator of a $\mathcal{C}^0$-semigroup $\Sigma_0(t)$ in X, and that $f : Y \to X$ is a Lipschitzian mapping on the bounded sets of Y, where either
1) $Y = X$, or
2) $\exp(At)$ is an analytic linear semigroup on X and $Y = X^\alpha = \mathcal{D}((\lambda Id - A)^\alpha)$, with $\alpha \in [0, 1)$, λ an appropriate real number.
In the case 1), we set $Y = X^\alpha$ for $\alpha = 0$. Assuming that all of the solutions of the following equation on Y,

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) \,=\, Au(t) + f(u(t)) \ , \quad t > 0 \ , \quad u(0) \,=\, u_0 \in Y \ , \tag{3.17}$$

are global, we obtain a continuous semigroup $S(t)$ on Y.

If $\Sigma_0(t)$ is an analytic semigroup, time regularity properties of the mapping $S(t)x$, for $x \in Y$, are well-known (see [He81, Chapter 3]). Actually, if $f : u \in Y \mapsto f(u) \in X$ is of class $\mathcal{C}^k$ or analytic, then, for $t > 0$, the mapping $t \to S(t)x$, $x \in Y$, is of class $\mathcal{C}^k$ or analytic. Time regularity properties have been especially addressed in the case of the Navier-Stokes equations and other parabolic systems (see [FT79], [FT89], [Pr91], for example). For this reason, in the next theorem, we consider only the case where $Y = X$. We suppose that there exists a positive number θ such that the radius of the essential spectrum $r(\sigma_{\text{ess}}(\exp At))$ satisfies

$$r(\sigma_{\text{ess}}(\exp At)) \,\leq\, \exp(-\theta t) \ , \quad \forall t \geq 0 \ . \tag{3.18}$$

The following regularity result is given in [Hal88] and follows from results proved in [HS].

Theorem 3.13. *Suppose that A satisfies the condition (3.18), $f \in \mathcal{C}^k(X, X)$, $1 \leq k < +\infty$ (resp. f is analytic from X to X) and that $S(t)$ has a compact global attractor $\mathcal{A}$. Then, there exists a positive number η depending on $\mathcal{A}$ such that, if*

$$\|Df(u)\|_{L(X,X)} \,\leq\, \eta \ , \ \textit{for any } u \textit{ in a neighbourhood of } \mathcal{A} \ , \tag{3.19}$$

the mapping $t : \mathbf{R} \mapsto S(t)u \in X$ *is of class* $\mathcal{C}^k$ *(resp. analytic), for any* $u \in \mathcal{A}$.

The proof of Theorem 3.13 is rather long. The most important steps of the proof will be described below, when we indicate how the above theorem can be generalized. The above time regularity property in Theorem 3.13 cannot be true if f is not at least a $\mathcal{C}^1$-function, as illustrated by the contre example given in [Hal88, page 57].

The smallness condition (3.19) is rather restrictive and limits the applications of this result. However, if one exploits further the ideas contained in [HS] and uses a Galerkin decomposition, one can generalize Theorem 3.13 in a significant way. For example, if X is a Hilbert space and the eigenfunctions of the operator A form a complete orthonormal system, we can decompose any solution u of (3.17) into the sum of two functions v_n and w_n, with v_n being the sum of the first n terms in the expansion of u and w_n being the remainder. Under additional natural conditions on A and f, one shows that there exists an integer N_1 such that each solution on the compact global attractor can be represented in the form

$$u(t) \;=\; v_{N_1}(t) + w_{N_1}(t) \;=\; v_{N_1}(t) + w^*(v_{N_1})(t) \;, \tag{3.20}$$

where the function $w^*(v_{N_1})(t)$ depends on $v_{N_1}(s)$, $s \leq t$ and is as smooth in v_{N_1} as the vector field f. The equation (3.17) is thus reduced to a finite-dimensional system of N_1 (nonautonomous) functional differential equations with infinite delay. It is very natural to refer to the coefficients of the projection $v_{N_1}(t)$ of $u(t)$ onto the first N_1 eigenfunctions as the *determining modes of the equation* (3.17) *on the attractor* $\mathcal{A}$. Furthermore, $w_{N_1}(t) = w^*(v_{N_1})(t)$ is a solution of an equation similar to (3.17), where, if N_1 is large enough, the nonlinearity satisfies the smallness condition (3.19). This property, together with the reduction of (3.17) to a finite-dimensional system of FDE's, will imply the regularity in time of the solutions on the attractor $\mathcal{A}$.

We now state these results more precisely. We keep the assumptions of Example 2.2, which have been recalled above and assume further that

H1 $f : Y \to X$ is a $\mathcal{C}^k$ function, $k \geq 1$;

H2 the continuous semigroup $S(t)$ has a compact invariant set $\mathcal{J}$;

H3 For any $n \geq 1$, $n \in \mathbf{N}$, there is a continuous linear map $P_n : Z \to Z$, where $Z = X$ or Y, such that $AP_n = P_nA$ on $\mathcal{D}(A)$, and the following additional properties hold:

(i) P_n converges strongly to the identity in Z as n goes to infinity;

(ii) there exists a positive constant $K_0 \geq 1$ such that, if $Q_n = Id - P_n$, then,

$$\|P_n\|_{L(Z,Z)} \;\leq\; K_0 \;, \quad \|Q_n\|_{L(Z,Z)} \;\leq\; K_0 \;, \quad \forall n \in \mathbf{N} \;; \tag{3.21}$$

(iii) there exist an integer n_1, two positive constants δ_1 and $K_1 \geq 1$ such that, for $n \geq n_1$, $t > 0$, we have,

$$\begin{aligned} \|e^{At}w\|_Z &\leq K_1 e^{-\delta_1 t}\|w\|_Z \ , \forall w \in W_n^Z \ , \\ \|e^{At}w\|_Y &\leq K_1 e^{-\delta_1 t} t^{-\alpha}\|w\|_X \ , \forall w \in W_n^X \ , \end{aligned} \tag{3.22}$$

where $W_n^Z = Q_n Z$, for $Z = X$ or Y. We also set $V_n^Z = P_n Z$.

In addition, our main result will depend upon either,

H4 there are a positive constant $K_2 \geq 1$, independent of $n \in \mathbf{N}$, and a sequence of positive numbers δ_n, $\delta_n \rightarrow_{n \rightarrow +\infty} +\infty$, such that, for $t > 0$,

$$\begin{aligned} \|e^{At}w\|_Z &\leq K_2 e^{-\delta_n t}\|w\|_Z \ , \forall w \in W_n^Z \ , \\ \|e^{At}w\|_Y &\leq K_2 e^{-\delta_n t}(t^{-\alpha} + \delta_n^{\alpha})\|w\|_X \ , \forall w \in W_n^X \ , \end{aligned} \tag{3.23}$$

or

H5 the set $\{Df(u_1)u_2 \,|\, u_1 \in \mathcal{J} \ , \ \|u_2\|_Y \leq 1\}$ is relatively compact in X.

Remark 3.14. In several concrete situations, there exists a neighbourhood U of $\mathcal{J}$ in Y such that $f : U \rightarrow X$ is completely continuous. If X is a reflexive Banach space, one shows, by arguing as in [MP76, page 339] that this property implies the hypothesis **H5**.

We recall that, for any $k > 0$, $B_Z(0, k)$ denotes the open ball in Z of center 0 and radius k. Let r be a positive constant such that $\|u\|_Y \leq r/K_0$, for any $u \in \mathcal{J}$. In what follows, we denote by $\mathcal{C}^k_{bu}(B_Y(0, 4r); X)$ the subset of $\mathcal{C}^k(B_Y(0, 4r); X)$ of the mappings g whose derivatives $D^j g(u)$, $j \leq k$, are bounded for $u \in B_Y(0, 4r)$ and the derivative $D^k g(u) : u \in B_Y(0, 4r) \mapsto D^k g(u) \in L^k(Y, X)$ is uniformly continuous. To prove that the restriction of the flow $S(t)$ to $\mathcal{J}$ is of class $\mathcal{C}^k$, $k \geq 1$, we shall assume that, $f \in \mathcal{C}^k(Y, X)$ belongs to the space $\mathcal{C}^k_{bu}(B_Y(0, 4r); X)$.

If $u(t)$ is a mild solution of (3.17) contained in $\mathcal{J}$ and

$$u(t) = P_n u(t) + Q_n u(t) \equiv v(t) + w(t) \ , \tag{3.24}$$

then $(v(t), w(t))$ is a mild solution of the following system

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= Av + P_n f(v + w) \ , \\ \frac{\mathrm{d}w}{\mathrm{d}t} &= Aw + Q_n f(v + w) \ . \end{aligned} \tag{3.25}$$

If $n \geq n_1$, the property (3.22) implies that

$$w(t) \;=\; \int_{-\infty}^{t} e^{A(t-s)} Q_n f(v(s)+w(s))\,\mathrm{d}s \;. \tag{3.26}$$

In what follows, we shall always choose $n \geq n_1$. For $d > 0$ and $n \geq 1$, we introduce the following sets,

$$\begin{aligned} U_{P_nY}(\mathcal{J},d) \;&=\; \{v \in V_n^Y \mid \|v\|_Y \;<\; 2r \;,\; \mathrm{dist}_Y(v,P_n\mathcal{J}) \;<\; d\} \;, \\ U_{Q_nY}(d) \;&=\; \{w \in W_n^Y \mid \|w\|_Y \;<\; \inf(d,2r)\} \;. \end{aligned}$$

as well as the subsets

$$\begin{aligned} \mathcal{C}^0_{P_nY}(\mathcal{J},d) \;&=\; \mathcal{C}^0(\mathbf{R};U_{P_nY}(\mathcal{J},d)) \;, \quad \mathcal{C}^{k,u}_{P_nY}(\mathcal{J},d) \;=\; \mathcal{C}^k_{bu}(\mathbf{R};U_{P_nY}(\mathcal{J},d)) \;, \\ \mathcal{C}^0_{Q_nY}(d) \;&=\; \mathcal{C}^0(\mathbf{R};U_{Q_nY}(d)) \;, \quad \mathcal{C}^{k,u}_{Q_nY}(d) \;=\; \mathcal{C}^k_{bu}(\mathbf{R};U_{Q_nY}(d)) \;, \end{aligned}$$

where $k \geq 1$.

The equality (3.26) suggests that, given $v \in \mathcal{C}^0_{P_nY}(\mathcal{J},d)$, the function $w(t)$ can be obtained as a fixed point of the operator $T_v(w) : \mathcal{C}^0_{Q_nY}(d) \to \mathcal{C}^0_{Q_nY}(d)$ defined by

$$T_v(w) \;=\; \int_{-\infty}^{t} e^{A(t-s)} Q_n f(v(s)+w(s))\,\mathrm{d}s \;. \tag{3.27}$$

The problem consists now in finding $d > 0$ and $N_0 \geq n_1$, such that, under the above hypotheses, the mapping T_v is indeed a uniform contraction from $\mathcal{C}^0_{Q_nY}(d)$ into itself, for $n \geq N_0$. Actually, motivated by the next remark, we shall also prove that T_v is a uniform contraction from $\mathcal{C}^{0,u}_{Q_nY}(d)$ into itself, for $n \geq N_0$.

Remark. If $\mathcal{J}$ is a compact invariant set for $S(t)$, the set of the complete orbits $u(t)$ of (3.17) contained in $\mathcal{J}$ is uniformly equicontinuous; that is, for any positive number η_0, there exists a positive number η_1 such that, for any $t \in \mathbf{R}$, for any complete orbit $u(\mathbf{R}) \subset \mathcal{J}$, $\|u(t+\tau)-u(t)\|_Y \leq \eta_0$ if $|\tau| \leq \eta_1$.

If the hypotheses **H4** or **H5** hold, one can prove that, for n large and d small enough, the map T_v is a uniform contraction from $\mathcal{C}^0_{Q_nY}(d)$ into itself [HR00].

Theorem 3.15. *Assume that the hypotheses* **H1**, **H2**, **H3** *and either* **H4** *or* **H5** *hold. Then, there is a positive constant d_1 such that, for $0 < d \leq d_1$, there exist an integer $N_0(d)$, and, for $n \geq N_0(d)$, a (unique) Lipschitz-continuous function*

$$w^* : v \in \mathcal{C}^0_{P_nY}(\mathcal{J},d) \mapsto w^*(v) \in \mathcal{C}^0_{Q_nY}(d) \;,$$

which is a mild solution of

$$\frac{\mathrm{d}w^*(v)}{\mathrm{d}t}(t) \;=\; Aw^*(v)(t) + Q_n f(v(t) + w^*(v)(t)) \; . \tag{3.28}$$

The mapping $w^*(v)(t)$ depends only upon $v(s)$, $s \le t$, and the map w^* is also Lipschitz-continuous from $\mathcal{C}^{0,u}_{P_nY}(\mathcal{J}, d)$ into $\mathcal{C}^{0,u}_{Q_nY}(d)$.
Moreover, $w^*(v)$ is as smooth in v as the vector field f; that is, if $f \in \mathcal{C}^k_{bu}(B_Y(0,4r); X)$, $k \ge 1$, then the mapping w^* is in $\mathcal{C}^k_{bu}(\mathcal{C}^{0,u}_{P_nY}(\mathcal{J}, d); \mathcal{C}^{0,u}_{Q_nY}(d))$.
Furthermore, if $f \in \mathcal{C}^k_{bu}(B_Y(0,4r); X)$, $k \ge 1$, then w^* is a uniformly continuous map from $\mathcal{C}^{k,u}_{P_nY}(\mathcal{J}, d)$ into $\mathcal{C}^{k,u}_{Q_nY}(d)$.

Remark 3.16. Under the hypotheses of Theorem 3.15, we can choose $N_1 = N_1(d_1) \ge N_0(d_1)$, such that, if $u(t) \in \mathcal{J}$, for $t \in \mathbf{R}$, is a mild solution of (3.17), then, for $n \ge N_1$, $w(t) = Q_n u(t)$ belongs to $\mathcal{C}^0_{Q_nY}(d_1)$ and thus, by uniqueness of the mild solutions, $u(t)$ must be represented as

$$u(t) \;=\; v(t) + w^*(v)(t) \; , \tag{3.29}$$

where $v(t) = P_n u(t)$ is the mild solution of the system of functional differential equations

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) \;=\; Av(t) + P_n f(v(t) + w^*(v)(t)) \; . \tag{3.30}$$

Moreover, $v(t)$ and $w(t)$ are in $\mathcal{C}^{0,u}_{P_nY}(\mathcal{J}, d_1)$ and $\mathcal{C}^{0,u}_{Q_nY}(d_1)$ respectively.
If, for any n, the range of P_n is of dimension n, the above property leads to say that *the flow on the compact invariant set $\mathcal{J}$ is determined by a finite number N_1 of modes.*

More classically, one says that the equation (3.17) has *a finite number N_1 of determining modes* if the range of P_{N_1} is finite-dimensional of dimension N_1 and if the condition

$$\|P_{N_1}u_1(t) - P_{N_1}u_2(t)\|_Y \to_{t\to+\infty} 0 \text{ implies that } \|u_1(t) - u_2(t)\|_Y \to_{t\to+\infty} 0 \; .$$

The property of finite number of determining modes has been extensively studied for parabolic equations and, especially for equations arising in fluid dynamics, like the Navier-Stokes equations. We refer to Foias and Prodi [FP67] and to Ladyzhenskaya [La72] for the earliest results on the two-dimensional Navier-Stokes equations. Later, various estimates for the minimal number of determining modes in terms of the Reynolds number or the Grashof number have been established (see the estimates in [FMTeTr], which have been improved in [JTi93]). A generalisation of the concept of determining modes to the one of determining functionals together with applications to dissipative parabolic equations has been recently given by Cockburn, Jones and Titi [CJTi97]. Based on the ideas of [CJTi97], Chueshov [CHu98] extended their results to other dissipative equations, including the damped wave equation.

Here the property of finite number of determining modes is a direct corollary of Theorem 3.15, if the compact invariant set $\mathcal{J}$ is moreover the global attractor ([HR00]). In the next theorem, we do not need the assumption that P_{N_1} has finite-dimensional range. Of course, in view of the applications, the case when P_{N_1} has finite-dimensional range is more interesting.

Theorem 3.17. *Assume that the compact invariant set $\mathcal{J}$ is the global attractor of the equation (3.17) and that the hypotheses* **H1**, **H2**, **H3** *and either* **H4** *or* **H5** *hold. If $u_1(t)$ and $u_2(t)$ are any two solutions of (3.17) satisfying*

$$\|P_{N_1}u_1(t) - P_{N_1}u_2(t)\|_Y \to_{t\to+\infty} 0 \ , \tag{3.31}$$

where the integer N_1 has been defined in Remark 3.16, then

$$\|u_1(t) - u_2(t)\|_Y \to_{t\to+\infty} 0 \ . \tag{3.32}$$

We now go back to the regularity in time of the complete orbits contained in $\mathcal{J}$. Arguing as in [HS, page 154] by considering the auxiliary differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}s} = P_{N_1}Av + P_{N_1}f(v + w^*(v)) \ , \quad v(0) = v_0 \ ,$$

with v_0 given in the Banach space $\mathcal{C}^0_{bu}(\mathbf{R}; P_{N_1}Y)$ and using Theorem 3.15 together with Remark 3.16, we obtain [HR00]:

Theorem 3.18. *Assume that the hypotheses* **H1**, **H2**, **H3** *and either* **H4** *or* **H5** *hold and that, for any $n \geq 1$, AP_n is a linear bounded mapping from Y into Y. If f belongs to $\mathcal{C}^k_{bu}(B_Y(0,4r);X)$, $k \geq 1$, then, for any $u_0 \in \mathcal{J}$ of (3.17), the mapping $t \in \mathbf{R} \mapsto S(t)u_0 = u_0(t) \in Y$ belongs to $\mathcal{C}^k_{bu}(\mathbf{R};Y)$ and $u_0(t)$ is a classical solution of (3.17). Moreover, there exists a positive constant $K_{k,\mathcal{J}}$ such that, for any $u_0 \in \mathcal{J}$,*

$$\sup_{t\in\mathbf{R}} \|\frac{\mathrm{d}^j u_0}{\mathrm{d}t^j}(t)\|_Y \leq K_{k,\mathcal{J}} \ , \quad \forall j \ , \ 1 \leq j \leq k \ . \tag{3.33}$$

To show that the restriction of the flow $S(t)$ to $\mathcal{J}$ is analytic when f is analytic, we complexify the spaces X and Y, the operators A, P_n, Q_n etc ... and assume, as in [HS], that

H6 there exists a real number $\rho > 0$ such that f has an holomorphic extension from $D_Y(4r,\rho) = \{u_1 + iu_2 \mid u_1 \in B_Y(0,4r)\ ,\ u_2 \in B_Y(0,\rho)\}$ into the complexified space X and f is bounded on the bounded sets of $\overline{D_Y(4r,\rho)}$.

We also complexy the time variable. Given a small positive number θ, we introduce the complex strip $D_\theta = \{t \in \mathbf{C} \,|\, |\mathrm{Im}\, t| < \theta\}$ and the Banach space $C_\theta(Z)$, defined by

$$C_\theta(Z) = \{u : \overline{D_\theta} \to Z \,|\, \text{u is continuous, bounded in } \overline{D_\theta}, \text{ holomorphic in } D_\theta, \\ \text{and } \ u(t) \in \mathbf{R}\ ,\ \forall t \in \mathbf{R}\}\ ,$$

and equipped with the norm $|||u|||_Z = \sup_{t\in\overline{D_\theta}} \|u(t)\|_Z$.

Showing first an analytic analog of Theorem 3.15 and arguing as in the proof of Theorem 3.18 yield:

Theorem 3.19. *Assume that the hypotheses* **H1**, **H2**, **H3**, **H6** *and either* **H4** *or* **H5** *hold and that, for any $n \geq 1$, AP_n is a linear bounded mapping from Y into Y. Then, there exists $\theta > 0$, such that any solution $u_0(t) \subset \mathcal{J}$ of (3.17) belongs to $C_\theta(Y)$. In particular, $t \in \mathbf{R} \mapsto u_0(t) \in Y$ is an analytic function.*

Theorem 3.18 and Theorem 3.19 can be applied to several evolutionary PDE's, including the heat and damped wave equations, and even to PDE's with delay. In these cases, regularity in time (up to analyticity) on the compat attractor is obtained, even if the solutions are not regular in the spatial variable, that is, even if the domain Ω, on which the equation is given, is not regular (see Section 4.5 below, for the time regularity of the flow on the global attractor of the linearly damped wave equation).
In [HR00], Theorem 3.15, Theorem 3.18 and Theorem 3.19 are proved under more general conditions than **H4** or **H5**, which allows applications to weakly dissipative equations like the weakly damped Schrödinger equation. Also there the commutation condition $AP_n = P_nA$ is relaxed and application to the wave equation with local damping is given.

When the evolutionary equation (3.17) arises from a partial differential equation defined on a domain $\Omega \in \mathbf{R}^n$, one deduces regularity in the spatial variable from Theorem 3.15 and Theorem 3.18. Indeed, under the hypotheses of Theorem 3.18, one shows that, for $1 \leq j \leq k-1$, for any solution $u_0(t) \in \mathcal{J}$ of (3.17), the derivative $\frac{\mathrm{d}^j u_0}{\mathrm{d}t^j}$ belongs to $\mathcal{C}_b^0(\mathbf{R};\mathcal{D}(A))$ and is a classical solution of the equation

$$\frac{\mathrm{d}^{j+1}u}{\mathrm{d}t^{j+1}}(t) \,=\, A\frac{\mathrm{d}^j u}{\mathrm{d}t^j}(t) + \frac{\mathrm{d}^j f(u)}{\mathrm{d}t^j}(t) \;, \quad t>0 \;, \quad \frac{\mathrm{d}^j u}{\mathrm{d}t^j}(0) \text{ given in } Y \;, \tag{3.34}$$

In the general case, due to the boundary conditions, $u \in \mathcal{J}$ is not expected to belong to $\mathcal{D}(A^j)$, for $j \geq 2$. However, we can obtain higher order regularity results ([HR00]), as it is illustrated in the next example where A is the generator of a $\mathcal{C}^0$-semigroup only, $Y = X$ and $\alpha = 0$.
We introduce a family of spaces Z_l, $l \in \mathbf{N}$, with $Z_{l+1} \subset Z_l$, $Z_0 = X$, $\mathcal{D}(A) \subset Z_1$, such that,

$$Au \,=\, g \;, \quad u \in \mathcal{D}(A) \;, \quad g \in Z_l \;, \quad \text{implies} \quad u \in Z_{l+1} \;. \tag{3.35}$$

A simple recursion argument using Theorem 3.18 shows the next regularity result.

Theorem 3.20. *Assume that $Y = X$, $\alpha = 0$, that (3.35) as well as the hypotheses* **H1**, **H2**, **H3** *and either* **H4** *or* **H5** *hold and that, for any $n \geq 1$, AP_n is a linear bounded mapping from X into X. Suppose that f belongs to $\mathcal{C}_{bu}^k(B_X(0,4r);X)$, $k \geq 1$. If, for $k \geq 2$, f is in $\mathcal{C}_b^{k-j-1}(Z_j;Z_j)$, for $1 \leq j \leq k-1$, then, for any orbit $u_0(t) \subset \mathcal{J}$, the mapping $t \in \mathbf{R} \mapsto u_0(t)$ belongs to $\mathcal{C}_b^j(\mathbf{R};Z_{k-j})$, $0 \leq j \leq k$. And there exists a positive constant $\tilde{K}_{\mathcal{J}}^k$ such that, for any $u_0(t) \subset \mathcal{J}$,*

$$\|u_0\|_{\mathcal{C}_b^j(\mathbf{R};Z_{k-j})} \,\leq\, \tilde{K}_{\mathcal{J}}^k \;, \quad \forall j \;, \quad 0 \leq j \leq k \;. \tag{3.36}$$

We remark that regularity in Gevrey classes can also be deduced from Theorem 3.15, Theorem 3.18 and Theorem 3.20 (for earlier regularity results in Gevrey classes in the case of dissipative equations, we refer to [FT89], [Pr91], [FeTi98], [OTi], for example).
Finally, it should be noticed that the proofs of time regularity in [FT79], [FT89], [Pr91] use a classical Galerkin procedure. Also, in his proof of the space regularity of the global attractor of the weakly damped Schrödinger equation, Goubet has introduced a Galerkin decomposition [Go96], [Go98], in a spirit different from the above one.

3.4. Inertial manifolds

Theorem 3.12 and Theorem 3.15 of the previous sections allow to embed the compact global attractor of some classes of systems into the one of a finite-dimensional system of differential equations. In the second case, we obtained functional differential equations with infinite delay, while in the first case, we got ordinary differential equations, the solutions of which may not be unique. It is therefore natural to try to exhibit classes of dissipative systems, for which these ODE's define a flow. This is actually the purpose of the theory of inertial manifolds, introduced by Foias, Sell and Temam [FST].

Suppose that we are given a continuous semigroup $S(t)$ on a Banach space X, which is generated by an evolutionary equation on X and has a compact global attractor $\mathcal{A}$. One can define an inertial manifold $\mathcal{M}$ of $S(t)$ as a finite-dimensional Lipschitzian submanifold of X, which contains $\mathcal{A}$ and is positively invariant under $S(t)$ (i.e. $S(t)\mathcal{M} \subset \mathcal{M}$, for any $t \geq 0$). If the semigroup $S(t)$ is one-to-one on $\mathcal{M}$, then the flow restricted to $\mathcal{M}$ is determined by a finite-dimensional system of ODE's with locally Lipschitzian vector field. This finite-dimensional system is called *inertial form*. As indicated below, in the process of constructing inertial manifolds, one shows that $\mathcal{M}$ attracts bounded sets exponentially. For this reason probably, Foias, Sell and Temam have included the exponential attraction property of bounded sets, in their definition.

Up to the present time, one of the basic ways to construct inertial manifolds has been to obtain the inertial manifold as a Lipschitzian graph over a finite-dimensional space and to apply the classical methods of center manifold theory. With such methods, the construction of inertial manifolds encounters the same technical problems and the same obstructions as the one of global center manifolds. In order to prove the existence of such global center manifolds, one needs some normal hyperbolicity property of the manifold $\mathcal{M}$, that is, the flow in X towards $\mathcal{M}$ must be stronger than the dispersion of the flow on $\mathcal{M}$. Since $\mathcal{M}$ does not only contain the neighbourhood of equilibrium points (as it is the case for local center manifolds of equillibria) and that $\mathcal{A}$ may be large, the dispersion of the flow on $\mathcal{M}$ may be large. In the frame of semigroups generated by partial differential equations, this strong normal hyperbolicity requirement leads to the so-called *cone condition* ([MPS]) and *gap condition* (see [FST], [CL88], for example), that we explain below. Unfortunately, these conditions are shown to be satisfied only by some partial differential equations in one space variable, including the reaction-diffusion equations (see [FST], [Te]), the Ginzburg-Landau equation [Te], the Cahn-Hilliard equation, the Kuramoto–Sivashinsky equation, etc $\cdots$ and by few reaction-diffusion equations on special domains in dimensions 2 and 3 ([MPS], [HR92c]).

To be more specific, we go back to the evolutionary equation of Example 2.2 with $Y = X^\alpha$, $\alpha \in [0,1)$. We assume that the spectrum of A consists of a sequence of real eigenvalues λ_n, $n \in \mathbf{N}$, in increasing order. For any integer n, we introduce the spectral projection $P_n \in L(X,X)$ onto the space generated by the eigenfunctions associated with the first n eigenvalues and assume that the hypotheses **H1**, **H2**, **H3** and **H4** of Section 3.3 hold with $\delta_n = \lambda_{n+1}$ and $\mathcal{J} = \mathcal{A}$. In particular, these assumptions hold if A is a positive self-adjoint operator with compact inverse.
In order to obtain a globally Lipschitzian nonlinear function on the right hand side of (2.1), we truncate the function f. Let $m : \mathbf{R} \to [0,1]$ be a $\mathcal{C}^\infty$-function such that $m(y) = 1$ if $y \in [0,1]$, $m(y) = 0$ if $y \geq 2$. Let $r > 0$ be chosen so that $\mathcal{A} \subset \overline{B}_Y(0,r)$. We then set

$$f_m(u) \;=\; m\left(\frac{\|u\|_Y^2}{4r^2}\right) f(u) \;,$$

and consider the modified equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} \;=\; Au(t) + f_m(u(t)) \;, \quad t > 0 \;, \quad u(0) \;=\; u_0 \in Y \;. \tag{3.37}$$

Clearly, the function f_m is globally Lipschitzian and bounded from Y into X and (3.37) also defines a continuous semigroup on Y, denoted by $S_m(t)$. One may construct an inertial manifold for $S_m(t)$ by using a Galerkin method. Like in Section 3.3, we choose an integer n and write any solution $u(t)$ of (3.37) as $u(t) = P_n u(t) + Q_n u(t) = v(t) + w(t)$, where (v,w) is a solution of the system

$$\frac{\mathrm{d}v}{\mathrm{d}t} \;=\; Av + P_n f_m(v+w) \;, \quad \frac{\mathrm{d}w}{\mathrm{d}t} \;=\; Aw + Q_n f_m(v+w) \;. \tag{3.38}$$

One way for obtaining an inertial manifold $\mathcal{M}$ of $S_m(t)$ as a graph over V_n^Y is to solve, for every $v_0 \in V_n^Y$ the system (3.38) on $(-\infty, 0]$, under the condition

$$v(0) \;=\; v_0 \;, \quad w \in \mathcal{C}_b^0((-\infty,0), W_n^Y) \;. \tag{3.39}$$

Due to (3.26), given $v_0 \in V_n^Y$, $(v(t), w(t))$ is a solution of (3.38) and (3.39) if and only if $w(t)$ is a fixed point of the map $T_{v_0} : \mathcal{C}_b^0((-\infty,0), W_n^Y) \to \mathcal{C}_b^0((-\infty,0), W_n^Y)$ given by

$$T_{v_0}(w) \;=\; \int_{-\infty}^{t} e^{A(t-s)} Q_n f_m(v(s) + w(s))\,\mathrm{d}s \;,$$

where $v(t)$ is the solution of

$$\frac{\mathrm{d}v}{\mathrm{d}t} = Av + P_n f_m(v+w) \ , \quad v(0) = v_0 \ . \tag{3.40}$$

If we prove that $T_{v_0} : \mathcal{C}_b^0((-\infty,0), W_n^Y) \to \mathcal{C}_b^0((-\infty,0), W_n^Y)$ is a strict contraction, when $\mathcal{C}_b^0((-\infty,0), W_n^Y)$ is equipped with the norm $\|w\|_\mu = \sup_{t\le 0}(e^{\mu t}\|w(t)\|_Y)$, where $\mu > 0$ is well chosen, then T_{v_0} has a unique fixed point $w_{v_0}(t)$. One then checks that the graph of the Lipschitzian mapping $\Psi : v_0 \in V_n^Y \mapsto \Psi(v_0) = v_0 + w_{v_0}(0) \in Y$ defines an inertial manifold $\mathcal{M}$ of $S_m(t)$. For example, T_{v_0} is a strict contraction, if

$$\lambda_{n+1} - \lambda_n \ \geq \ C\lambda_{n+1}^\alpha \ , \tag{3.41}$$

where C is a positive constant depending on the Lipschitz constant of f_m and on α. The condition (3.41) is a *gap condition* on the eigenvalues of A and is rather restrictive.

Due to the positive invariance of the inertial manifold $\mathcal{M}$, the equation (3.37) on $\mathcal{M}$ reduces to the finite system of ODE's:

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) = Av(t) + f_m(\Psi(v(t))) \equiv g(v(t)) \ , \quad t > 0 \ , \quad v(0) \text{ given in } V_n^Y \ . \tag{3.42}$$

This system defines a flow on $V_n^Y = P_nY$ and has a compact global attractor $\mathcal{A}_n = P_n\mathcal{A}$. The solutions of (3.42) on $P_n\mathcal{A}$ are written as $v(t) = P_nS_m(t)\Psi(v(0))$, that is, the flow of (3.42) on $P_n\mathcal{A}$ is conjugate to $S_m(t)$.

Suppose now that $S(t) = S_\lambda(t)$ and also $S_m(t)$ depend on a parameter λ in a Banach space Λ and that, for each λ, one can construct an inertial manifold $\mathcal{M}_\lambda$ over V_n^Y and an inertial form

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g_\lambda(v(t)) \ , \quad t > 0 \ , \quad v(0) \text{ given in } V_n^Y \ ,$$

where g_λ and Dg_λ are continuous functions of v, λ. If the flow defined by the vector field g_{λ_0} is structurally stable, then we know that each flow $S_\lambda(t)\big|_{\mathcal{A}_\lambda}$ is equivalent to the flow of $S_{\lambda_0}(t)\big|_{\mathcal{A}_{\lambda_0}}$, for λ close to λ_0 (see [HR92c] and [MSM]).

4. Gradient systems

Until now, we have not described the behaviour of the flow on the global attractor. Even in the finite-dimensional case, this behaviour is often not known. In the case of the gradient systems, a partial description of the flow restricted to the attractor can be given. We first recall some general properties of gradient systems and then present a few examples of such systems.

4.1. General properties of gradient systems

We recall that the set G^+ denotes either $[0, +\infty)$ or $\mathbf{N}$

Definition 4.1. Let $S(t)$, $t \in G^+$ be a semigroup on X.
1) A function $\Phi \in \mathcal{C}^0(X, \mathbf{R})$ is *a Lyapunov functional* if

$$\Phi(S(t)u) \leq \Phi(u) \ , \quad \forall t \in G^+ \ , \quad \forall u \in X \ . \tag{4.1}$$

2) A Lyapunov functional Φ is a strict Lyapunov functional if, moreover,

$$\Phi(S(t)u) = \Phi(u) \ , \ \forall t \in G^+, \text{ implies that } u \text{ is an equilibrium point .} \tag{4.2}$$

3) A semigroup $S(t)$ is *a gradient system* if it has a strict Lyapunov functional and if, either $G^+ = \mathbf{N}$ or $G^+ = [0, +\infty)$ and $S(t)$ is a continuous semigroup. In the later case, $S(t)$ is called *a continuous gradient system.*

The simplest example of discrete gradient system is a monotone map S on $\mathbf{R}$ (for example, $Sx \leq Sy$ if $x \leq y$).

Notation. We denote $\mathcal{E} = \{z \in X \mid S(t)z = z \,, \forall t \geq 0\}$ the set of equilibrium points of $S(t)$. Clearly, $\mathcal{E}$ is an invariant and closed set. If S is a discrete semigroup, $\mathcal{E}$ is simply the set of fixed points $\mathrm{Fix}(S)$ of S.

The following result, known as the Invariance Principle of LaSalle, plays a basic role in the theory of gradient systems. We set $G^- = \{-g \mid g \in G^+\}$.

Proposition 4.2. *Invariance Principle of LaSalle*
Let $S(t)$ be a gradient system on X with a Lyapunov functional Φ.
1) If z is an element of X such that $\gamma_\tau^+(z)$ is relatively compact in X, then,
i) $l = \lim_{t \to +\infty} \Phi(S(t)z)$ exists and $\Phi(v) = l$, for any $v \in \omega(z)$,

ii) $\omega(z) \subset \mathcal{E}$*; in particular,* $\mathcal{E} \neq \emptyset$*, and* $\delta_X(S(t)z, \mathcal{E}) \to_{t\to+\infty} 0$.
2) Let $u_z \in \mathcal{C}^0(G^-, X)$ *be a negative orbit through some* $z \in X$*. If the negative orbit* $u_z(G^-)$ *is relatively compact, then the set* $\alpha_{u_z}(z)$ *is non-empty, compact, invariant, consists only of equilibrium points and* $\delta_X(u_z(-t), \alpha_{u_z}(z)) \to_{t\to+\infty} 0$*. Furthermore,* $\alpha_{u_z}(z)$ *is connected.*

Proof. The function $t \mapsto \Phi(S(t)z)$ is non increasing and bounded from below, since $\Phi(.)$ is a continuous function on X and that $\gamma_\tau^+(z)$ is relatively compact in X, hence $l \equiv \lim_{t\to+\infty} \Phi(S(t)z)$ exists. If $v \in \omega(z)$, there exists a sequence $t_n \to +\infty$ such that $S(t_n)z \to v$. As $\Phi(.)$ is continuous on X, $\Phi(S(t_n)z) \to \Phi(v)$ and $\Phi(v) = l$.
One remarks that the Property i) also holds if the Lyapunov functional Φ is not a strict Lyapunov functional.
Since $\gamma_\tau^+(z)$ is relatively compact, the semigroup $S(t)$ restricted to $\overline{\gamma_\tau^+(z)}$ is asymptotically smooth. From Proposition 2.13, we then deduce that $\omega(z) \neq \emptyset$ and that $\delta_X(S(t)z, \omega(z)) \to_{t\to+\infty} 0$. The inclusion $S(t)\omega(z) \subset \omega(z)$ implies that $\Phi(S(t)v) = l = \Phi(v)$, for any $t \geq 0$ and $v \in \omega(z)$. Thus, $v \in \mathcal{E}$.
Since $\alpha_{u_z}(z)$ is a nonincreasing intersection of nonempty compact sets, it is a nonempty compact set. The attractivity property is proved at once by a contradiction argument using the compactness of the closure of $u_z(G^-)$. The invariance property follows from Remarks 2.10 (ii). The fact that $\alpha_{u_z}(z) \subset \mathcal{E}$ is proved like above. Finally, by Remark 2.16 (ii), $\alpha_{u_z}(z)$ is invariantly connected. Since $\alpha_{u_z}(z) \in \mathcal{E}$, $\alpha_{u_z}(z)$ is connected.
□

Remarks 4.3.
1) Assume that the hypotheses of Proposition 4.2 hold. The above assertions i) and ii) simply mean that $\omega(z) \subset \mathcal{E}_l$, where $\mathcal{E}_l = \{v \in \mathcal{E} \mid \Phi(v) = l\}$. If $\mathcal{E}_l$ is *discrete*, it follows from Lemma 2.9 that there exists only one element $v_0 \in \mathcal{E}_l$ such that $\omega(z) = v_0$, that is, that $S(t)z \to v_0$ as $t \to +\infty$; and the orbit through z is said to be *convergent*. In general, if the sets $\mathcal{E}_l$ are not discrete, the positive orbits are not convergent (for examples, see [PaMe] in the finite-dimensional case or [PoRy] in the infinite-dimensional case). However, in the frame of Example 2.2, the positive orbit through z is shown to be convergent if there is $v \in \omega(z)$ such that the spectrum of the linear map $\Sigma_v(1)$ intersects the unit circle in $\mathbf{C}$ at most at the point 1 and 1 is a simple eigenvalue ([HR92b], [BrP97b]), where $\Sigma_v(t)$ is the $\mathcal{C}^0$-semigroup generated by the linear operator $A + Df(v)$. Below, we shall give some examples of reaction-diffusion equations and damped wave

equations, for which the convergence property holds even if the sets $\mathcal{E}_l$ are not discrete (see [Po00] for a review of convergence results).
2) Assume that $S(t)$ is an asymptotically smooth gradient system, which has the property that, for any bounded set $B \subset X$, there exists $\tau \geq 0$ such that $\gamma_\tau^+(B)$ is bounded. Then, $S(t)$ is point dissipative if and only if $\mathcal{E}$ is *bounded*.
3) More generally, let $S(t)$ be a continuous semigroup and Φ be a Lyapunov functional associated with the semigroup $S(t)$. Let $H = \{x \in X \mid \Phi(S(t)x) = \Phi(x) \ , \forall t \geq 0\}$ and M be the maximal invariant subset of H. One shows like in the proof of Proposition 4.2 that, if $z \in X$ is such that $\gamma^+(z)$ is relatively compact in X, then the ω- limit set $\omega(z)$ is contained in M.
4) The simplest example of a gradient system, from which the term "gradient" actually comes, is the O.D.E. $\dot{y} = \nabla F(y)$, where $F \in \mathcal{C}^{1,1}(\mathbf{R}^n, \mathbf{R})$. The associated strict Lyapunov functional is $\Phi(y) = -F(y)$. If F is an analytic function, the bounded orbits are convergent ([Loj]).

For a continuous semigroup, the definitions of stable and unstable sets are analogous to those given for a discrete system in (3.13). We recall that

$$\begin{aligned} W^u(\mathcal{E}) \equiv W^u(\mathcal{E}, S(t)) \ &= \ \{v \in X \mid \text{there exists a negative orbit } u_v \text{ of } S(t) \\ &\qquad \text{such that } u_v(0) = v \text{ and } \delta_X(u_v(-t), \mathcal{E}) \to_{t\to+\infty} 0\} \ , \end{aligned}$$

$$\begin{aligned} W^u(e) \equiv W^u(e, S(t)) \ &= \ \{w \in X \mid \text{there exists a negative orbit } u_w \text{ of } S(t) \\ &\qquad \text{such that } u_w(0) = w \text{ and } u_w(-t) \to_{t\to+\infty} e\} \ , \\ W^s(e) \equiv W^s(e, S(t)) \ &= \ \{v \in X \mid S(t)v \to_{t\to+\infty} e\} \ , \end{aligned}$$

where e is any element of $\mathcal{E}$.

Two important remarks should be made about the sets $W^u(e, S(t))$ and $W^s(e, S(t))$.

Remark 4.4. If $S(t)$ is a continuous semigroup, then, for any $t_0 > 0$, e is a fixed point of $S(t_0)$ and we have the equalities

$$W^u(e, S(t)) \ = \ W^u(e, S(t_0)) \ , \quad W^s(e, S(t)) \ = \ W^s(e, S(t_0)) \ .$$

Remark 4.5. Assume that X is a Banach space. We recall that a continuous semigroup $S(t)$ is said to be of class $\mathcal{C}^r$, $r \geq 1$, if, for any $t \in G^+$, $S(t) \in \mathcal{C}^r(X, X)$. An equilibrium point e of a continuous semigroup $S(t)$, $t \geq 0$, is *hyperbolic* if the linear map $D_e S(t)\big|_{t=1} = DS_1(e)$, where $S_1 = S(1)$ satisfies:

$$\sigma(DS_1(e)) \cap \{z \in \mathbf{C} \,|\, |z| = 1\} = \emptyset \,. \tag{4.3}$$

The condition (4.3) means that e is a hyperbolic fixed point of S_1. We set $L = DS_1(e)$. Let $\sigma_+ = \{z \in \sigma(L) \,|\, |z| > 1\}$ and $\sigma_- = \{z \in \sigma(L) \,|\, |z| < 1\}$. If σ_+ is a finite set of n_e elements, then n_e is called the index $\mathrm{ind}(e)$ of e. Let P_+ and P_- be the spectral projections corresponding to the sets σ_+ and σ_-, let $X_\pm = P_\pm X$ and $L_\pm = LP_\pm$. There exist two small positive numbers $\delta_\pm$ such that $\sup\{|z| \,|\, z \in \sigma_-\} < 1 - 2\delta_-$ and $\sup\{|z| \,|\, z \in \sigma(L_+^{-1})\} < 1 - 2\delta_+$. For any $u_+ \in X_+$, $u_- \in X_-$, we set

$$\|u_+\|_1 = \sup_{n \geq 0} \frac{\|L_+^{-n} u_+\|_X}{(1-\delta_+)^n}, \quad \|u_-\|_1 = \sup_{n \geq 0} \frac{\|L_-^{n} u_-\|_X}{(1-\delta_-)^n} \,,$$

and, for $u \in X$, we introduce the norm $\|\cdot\|_1$ on X, defined by

$$\|u\|_1 = \sup(\|P_+ u\|_1, \|P_- u\|_1) \,, \tag{4.4}$$

which is equivalent to the original norm $\|\cdot\|_X$.
For any $R > 0$, let $O_R(e) = \{y \in X \,|\, \|y - e\|_1 \leq R\}$, $O_R^\pm = O_R(0) \cap X_\pm$. One shows that there exist a positive number R, two functions $g_+ \in \mathcal{C}^1(O_R^+, O_R^-)$, $g_- \in \mathcal{C}^1(O_R^-, O_R^+)$, such that the local stable and unstable manifolds of S_1 at e are given by

$$\begin{aligned} W^s(e, S_1, O_R(e)) &= \{u \in X \,|\, u = e + P_-(u-e) + g_-(P_-(u-e)) \,, P_-(u-e) \in O_R^-\} \\ W^u(e, S_1, O_R(e)) &= \{u \in X \,|\, u = e + P_+(u-e) + g_+(P_+(u-e)) \,, P_+(u-e) \in O_R^+\}. \end{aligned} \tag{4.5}$$

The functions $g_\pm$ satisfy $g_\pm(0) = 0$ and $Dg_\pm(0) = 0$. If the index $\mathrm{ind}(e)$ is finite, then $W^u(e, S_1, O_R(e))$ is a manifold of dimension $\mathrm{ind}(e)$.
Furthermore, there exist positive constants R_e, K_e and β_e such that, if $S_1^n y \in O_{R_e}(e)$, for $n = 1, \ldots, m$, then,

$$\delta_X(S_1^n y, W^u(e, S_1, O_{R_e}(e))) \leq K_e \exp(-\beta_e n) \delta_X(y, W^u(e, S_1, O_{R_e}(e))) \,, \quad 1 \leq n \leq m \,, \tag{4.6}$$

where β_e depends on $\delta_\pm$ (see [Wel], [BV83], [ChHaTa], for example).
Assume now that $S(t)$ is a semigroup of class $\mathcal{C}^1$ on the Banach space X and a gradient system with a Lyapunov functional Φ satisfying, for any $t > 0$,

$$\Phi(S(t)x) < \Phi(x) \,, \quad \forall x \in X \,, \quad x \notin \mathcal{E} \,. \tag{4.7}$$

Suppose also that e is a hyperbolic equilibrium point of $S(t)$ with finite index $\mathrm{ind}(e)$. Then one can show that, for any $\rho > 0$, there exists a positive number r such that

$$W^u(e, S(t)) \cap B_X(e, r) \subset W^u(e, S_1, O_\rho(e)) , \tag{4.8}$$

where $B_X(e, r) = \{x \in X \mid \|x - e\|_X < r\}$.

Often, it is easier to construct and study invariant manifolds of time-one maps rather than those of the flow $S(t)$. Remark 4.4 and Remark 4.5 show that, in the case of continuous gradient systems with a Lyapunov functional satisfying the condition (4.7), the local and global unstable manifolds of $S(t)$ at e can indeed be replaced by those of the map S_1.

From Theorem 2.26 and Proposition 4.2, one easily deduces the following result:

Theorem 4.6. *Let $S(t)$, $t \in G^+$, be an asymptotically smooth gradient system, which has the property that, for any bounded set $B \subset X$, there exists $\tau \geq 0$ such that $\gamma_\tau^+(B)$ is bounded. If the set of equilibrium points $\mathcal{E}$ is bounded, then $S(t)$ has a compact global attractor $\mathcal{A}$ and $\mathcal{A} = W^u(\mathcal{E})$.*
Furthermore, if $\mathcal{E}$ is a discrete set, $\mathcal{E}$ is a finite set $\{e_1, e_2, \ldots e_{n_0}\}$ and $\mathcal{A} = \bigcup_{e_j \in \mathcal{E}} W^u(e_j)$.

If $\mathcal{E}$ is a discrete set and $u_z \in \mathcal{C}^0(G^+, \mathcal{A})$ is a complete orbit in $\mathcal{A}$ through z, there exist equilibrium points e_j and e_k such that $\alpha_{u_z}(z) = e_j$ and $\omega(z) = e_k$. If z is not an equilibrium point, $\Phi(e_k) < \Phi(a) \leq \Phi(e_j)$. The orbit which joins the points e_j and e_k is called a *heteroclinic orbit*.

Under the hypotheses of Theorem 4.6, we introduce the m_0 distinct values $v_1 > v_2 > \ldots > v_{m_0}$ of the set $\{\Phi(e_1), \Phi(e_2), \ldots, \Phi(e_{n_0})\}$ and let

$$\mathcal{E}^j = \{e_{ji} \in \mathcal{E} \mid \Phi(e_{ji}) = v_j\} , j = 1, \ldots, m_0 .$$

The sets $\mathcal{E}^1$, $\mathcal{E}^2$, ..., $\mathcal{E}^{m_0}$ define a *Morse decomposition* of the attractor $\mathcal{A}$, i.e.,
(i) the subsets $\mathcal{E}^j$ are compact, invariant and disjoint;
(ii) for any $a \in \mathcal{A} \setminus \bigcup_j \mathcal{E}^j$ and every complete orbit u_a through a, there exist k and l, depending on u_a, so that $k < l$, $\alpha_{u_a}(a) \in \mathcal{E}^k$ and $\omega(a) \in \mathcal{E}^l$.

For $1 \leq k \leq m_0$, one defines

$$\mathcal{A}^k = \bigcup \{W^u(e) \mid e \in \bigcup_{j=k}^{m_0} \mathcal{E}^j\} ,$$

and, for $0 < d < d_0 \equiv \inf_{2 \leq k \leq m_0}(v_{k-1} - v_k)$,

$$F_d^k = \{u \in X \,|\, \Phi(u) \leq v_{k-1} - d\} ,$$

where v_0 is choosen so that $v_0 > v_1 + d_0$. Assume now that the hypotheses of Theorem 4.6 hold and that $\mathcal{E}$ is a discrete set. Then, the same arguments as those used to prove Theorem 4.6, show that, for any $0 < d < d_0$ and any k, $1 \leq k \leq m_0$, $\mathcal{A}^k$ is the (compact) global attractor of $S(t)/_{F_d^k}$.

If X is a Banach space and all the equilibrium points of $S(t)$ are hyperbolic, then, using the above Morse decomposition and Remark 4.5, one shows that the global attractor exponentially attracts a neighbourhood of it. This property plays an important role in the lower semicontinuity of global attractors. Its proof is implicitely contained in [HR89] and can be found in [BV89a] (See also [Hal88], [BV89b], [Ko90] and [GR00]).

Theorem 4.7. *Let X be a Banach space. Assume that the hypotheses of Theorem 4.6 hold and that the Lyapunov functional satisfies (4.7), for any $t \in G^+$, $t \neq 0$. Suppose moreover that, either $S(t)$ is a continuous semigroup of class $\mathcal{C}^1$ or S is a $\mathcal{C}^1$-mapping from X to X and that, in both cases, all the equilibrium points of $S(\cdot)$ are hyperbolic. Then, there exist a bounded neighbourhood B_1 of the global attractor $\mathcal{A}$ and positive constants C_1, $d_1 < d_0$, γ, such that,*

$$\delta_X(S(t)(\tilde{F}_{d_1}^1), \mathcal{A}) \leq C_1 \exp(-\gamma t) , \quad \forall t \in G^+ , \tag{4.9}$$

where $\tilde{F}_{d_1}^1 = F_{d_1}^1 \cap B_1$.

The number $\gamma > 0$ in (4.9) depends on the minimum, over $e \in \mathcal{E}$, of the distance of the spectrum of $DS_1(e)$ to the unit circle in $\mathbf{C}$. The $\mathcal{C}^1$-regularity hypotheses in Theorem 4.7 and in Remark 4.5 can be weakened and replaced by the following assumption $S(1)(y + e) = e + Ly + Q(e, y)$, where $L \in L(X, X)$ satisfies the spectral hypothesis (4.3), $Q(e, 0) = 0$ and $Q(e, \cdot) : X \to X$ is Lipschitz-continuous on the bounded sets of X. We also assume that the Lipschitz constant of $Q(e, \cdot)$ on the balls $B_X(0, r)$ is a continuous, nondecreasing function of r, vanishing at $r = 0$. In this case, the mappings $g^{\pm}$ of Remark 4.5 are only Lipschitzian mappings.
The proof of Theorem 4.7 actually shows that, for any $u_0 \in \tilde{F}_{d_1}^1$, there exists a finite number of trajectories $S(t)u_{0j}$, $u_{0j} \in \mathcal{A}$, $t \in [t_j, t_{j+1})$, $j = 0, .., k(u_0)$, with $t_0 = 0$ and $t_{k(u_0)} = +\infty$ such that $\|S(t)u_0 - S(t)u_{0j}\|_X \leq C_1 \exp(-\gamma t)$, for any $t \in [t_j, t_{j+1})$.

The "trajectory" $\tilde{u}(t) = \cup_{j=0}^{k(u_0)-1}(\cup_{t_j}^{t_{j+1}} S(t)u_{0j})$ is called a *finite-dimensional combined trajectory* by Babin and Vishik in [BV89a]. Actually, under additional conditions, it is shown there, that, for any $\eta > 0$, one can construct a finite-dimensional combined trajectory $\tilde{u}_\eta(t) \in \mathcal{A}$ such that $\|S(t)u_0 - \tilde{u}_\eta(t)\|_X \leq C(\eta)\exp(-\eta t)$, where $C(\eta) > 0$ depends on η. This trajectory $\tilde{u}_\eta(t)$ belongs piecewise to invariant manifolds, whose dimension increases when η decays to zero.

The assumption (4.7) implies that, for any hyperbolic equilibrium point $e \in \mathcal{E}$, there is a neighbourhood U_e of e such that, if $x_0 \in U_e \backslash W^s(e, S(t))$, then there exists $t_0 = t_0(x_0) > 0$ so that $S(t)x_0 \notin U_e$, for $t \geq t_0$, that is, $S(t)x_0$ eventually leaves U_e never to return. This property plays an important role in the proof of (4.9) and also in the proof of the following theorem, stating that the unstable and stable manifolds of a hyperbolic equilibrium point are embedded submanifolds of X.

Theorem 4.8. *Assume that the hypotheses of Theorem 4.7 hold and that $S_1 = S(1)$ as well as the linear map $DS_1(y)$ are injective at each point y of the global attractor $\mathcal{A}$, then, for each $e \in \mathcal{E}$, the unstable set $W^u(e, S(t))$ is an embedded $\mathcal{C}^1$-submanifold of X of finite dimension equal to* ind(e), *which implies that the Hausdorff dimension* $\dim_H(\mathcal{A})$ *is finite and equal to* $\max_{e\in\mathcal{E}} \mathrm{ind}(e)$. *If furthermore, for each $e \in \mathcal{E}$, S_1 is injective and $DS_1(y)$ has dense range at each point y of $W^s(e, S(t))$, then the stable set is an embedded $\mathcal{C}^1$-submanifold of X of codimension* ind(e).

The above theorem is a consequence of Remark 3.7, Remark 4.4 and [He81, Theorem 6.1.9, Theorem 6.1.10] (see also [BV83]). Under the hypotheses of Theorem 4.8, one shows that, for any hyperbolic equilibrium point e, there is a neighbourhood Q_e of $W^s(e, S)$ such that, if $x_0 \in Q_e \backslash W^s(e, S(t))$, then there exists $t_0 = t_0(x_0) > 0$ so that $S(t)x_0 \notin Q_e$, for $t \geq t_0$.

Remarks.

1. Assume that the gradient system $S(t)$ is generated by the evolutionary equation (2.1). Under additional hypotheses, Theorem 4.8 implies that the Hausdorff dimension of the global attractor of (2.1) is estimated by the maximum of the number of eigenvalues with positive real part of $A + Df(e)$, $e \in \mathcal{E}$. If in (2.1), f is replaced by λf, one thus obtains asymptotic estimates of $\dim_H(\mathcal{A})$, when $\lambda \to +\infty$, by using asymptotic estimates of the number of positive eigenvalues of the operators $A + \lambda Df(e)$ (see [BV89b, Chapter 10, Section 4]).

2. If $S(t)$ is generated by an evolutionary equation, the injectivity of S_1 (resp. DS_1) comes usually from a backward uniqueness result of the solutions of the evolutionary equation (resp. the corresponding linearized equation). And one shows that the range of DS_1 is dense by proving that the adjoint map is injective. Backward uniqueness results are known to hold for a large class of parabolic equations (see [BT], [He81, Chapter 7, Section 6], [Gh86]). The hyperbolicity of the equilibrium points is usually a generic property with respect to the various parameters involved in the definition of the semigroup $S(t)$.

Gradient systems depending on parameters.

As in Section 3.1, we consider a family of semigroups $S_\lambda(t) : X \to X$, $t \in G^+$, depending on a parameter $\lambda \in \Lambda$, where $\Lambda = (\Lambda, d_\lambda)$ is a metric space.
Let λ_0 be a nonisolated point of Λ. We assume that $S_{\lambda_0}(t)$ is a gradient system satisfying the hypotheses of Theorem 4.7, the conditions (H.1b) and (3.8) of Section 3.1, then Proposition 3.1, Proposition 3.3 and Theorem 4.7 imply that, for λ close enough to λ_0, we have,

$$\delta_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \le c\delta_\Lambda(\lambda, \lambda_0)^{\frac{\gamma\gamma_0}{\gamma+\beta_0}} , \tag{4.10}$$

for some positive constant c.

We now turn to lower semicontinuity results and estimates of the semi-distance $\delta_X(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda)$. Lower semicontinuity properties have been first proved in a very general setting in [BV86] and [HR89] (see also [Hal88, Chapter 4, Section 10]). To show some of the ingredients, which are necessary, we begin with a very simple result. Hereafter, we denote by $\mathcal{E}_\lambda$ the set of equilibria of S_λ. Besides the condition (H.1b) of Section 3.1, we introduce the following hypotheses:

(H.2) The set $\mathcal{E}_{\lambda_0}$ is a finite set, say $\mathcal{E}_{\lambda_0} = \{e_1^{\lambda_0}, \dots, e_{n_{\lambda_0}}^{\lambda_0}\}$;

(H.3) the global attractor $\mathcal{A}_{\lambda_0}$ is compact and,

$$\mathcal{A}_{\lambda_0} = \bigcup_{j=1}^{n_{\lambda_0}} W^u(e_j^{\lambda_0}, S_{\lambda_0}(t)) = \bigcup_{j=1}^{n_{\lambda_0}} W^u(e_j^{\lambda_0}, S_{\lambda_0}(1)) ;$$

and

(H.4) for $j = 1, \dots, n_{\lambda_0}$, there exists a neighbourhood O_j of $e_j^{\lambda_0}$ in X such that

$$\lim_{\lambda \to \lambda_0} \delta_X(W^u(e_j^{\lambda_0}, S_{\lambda_0}(1), O_j), \mathcal{A}_\lambda) = 0 .$$

Proposition 4.9. *If the hypotheses (H.1b), (H.2), (H.3) and (H.4) hold, the global attractors $\mathcal{A}_\lambda$ are lower semicontinuous at $\lambda = \lambda_0$, i.e., $\delta_X(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda) \to 0$ as $\lambda \to \lambda_0$, which, together with Proposition 3.1, implies that the attractors $\mathcal{A}_\lambda$ are continuous at $\lambda = \lambda_0$.*

Proof. We give the proof only when $G^+ = [0, +\infty)$. Without loss of generality, we suppose that $t_0 = 1$ in the hypothesis (H.1b).
Assume that the sets $\mathcal{A}_\lambda$ are not lower semicontinuous at $\lambda = \lambda_0$. Then, there exist $\varepsilon > 0$ and, for any $m \in \mathbf{N}$, an element $\lambda_m \in N_\Lambda(\lambda_0, 1/m)$ and an element $\varphi_m \in \mathcal{A}_{\lambda_0}$ such that $\delta_X(\varphi_m, \mathcal{A}_{\lambda_m}) > \varepsilon$. Since $\mathcal{A}_{\lambda_0}$ is compact, the sequence φ_m converges to an element $\varphi_0 \in \mathcal{A}_{\lambda_0}$ and

$$\delta_X(\varphi_0, \mathcal{A}_{\lambda_m}) > \varepsilon/2 \ , \quad \forall m \in \mathbf{N} \ , \quad m \geq m_0 \ , \tag{4.11}$$

where $m_0 \in \mathbf{N}$, $m_0 > 0$.
If $\varphi_0 \in W^u(e_j^{\lambda_0}, S_{\lambda_0}(1), O_j)$, for some $j = 1, \ldots, n_{\lambda_0}$, (4.11) contradicts the hypothesis (H.4). Thus, assume that $\varphi_0 \notin W^u(e_j^{\lambda_0}, S_{\lambda_0}(1), O_j)$, for any $j = 1, \ldots, n_{\lambda_0}$, then there are n_0 and $\psi_0 \in W^u(e_i^{\lambda_0}, S_{\lambda_0}(1), O_i)$, for some i, so that $\varphi = S_{\lambda_0}(n_0)\psi_0$. Since $\mathcal{A}_{\lambda_0}$ is compact, for any $\varepsilon > 0$, there exists $\delta > 0$, depending only on $\mathcal{A}_{\lambda_0}$, such that, if $\delta_X(v, \psi_0) \leq \delta$, then,

$$\delta_X(S_{\lambda_0}(n_\varphi)v, S_{\lambda_0}(n_\varphi)\psi_0) \leq \varepsilon/4 \ . \tag{4.12}$$

The hypotheses (H.4) and (H.1b) imply that we can choose $\theta = \theta(\varepsilon, n_\varphi) > 0$, such that, for $\lambda \in N_\Lambda(\lambda_0, \theta)$, there exists $v_\lambda \in \mathcal{A}_\lambda$ so that

$$\begin{aligned} \delta_X(\psi_0, v_\lambda) &\leq \delta \ , \\ \delta_X(S_\lambda(n_\varphi)v_\lambda, S_{\lambda_0}(n_\varphi)v_\lambda) &\leq \varepsilon/4 \ . \end{aligned} \tag{4.13}$$

We deduce from (4.12) and (4.13) that, for $\lambda \in N_\Lambda(\lambda_0, \theta)$,

$$\delta_X(S_\lambda(n_\varphi)v_\lambda, \varphi) \leq \varepsilon/2 \ ,$$

which contradicts the property (4.11). □

Remark 4.10. Usually, when X is a Banach space, the hypothesis (H.4) is shown by proving that, for λ close enough to λ_0, $S_\lambda(t)$ has, at least, n_{λ_0} equilibrium points e_j^λ, $j = 1, \ldots, n_{\lambda_0}$ and that, for each $j = 1, \ldots, n_{\lambda_0}$, there exists a neighbourhood O_j^λ of e_j^λ such that

$$\lim_{\lambda \to \lambda_0} \delta_X(W^u(e_j^{\lambda_0}, S_{\lambda_0}(1), O_j), W^u(e_j^\lambda, S_\lambda(1), O_j^\lambda)) = 0 \ . \tag{4.14}$$

If, for instance, the equilibrium points $e_j^{\lambda_0}$ are all hyperbolic and that the mapping $S_\lambda(1)$ converges to $S_{\lambda_0}(1)$ in $\mathcal{C}^1(B, X)$, as $\lambda \to \lambda_0$, where B is a bounded neighbourhood of $\mathcal{A}_{\lambda_0}$, then the property (4.14) holds (see [Wel] and [BV86], for example). In the general case, if $e_j^{\lambda_0}$ is not hyperbolic, the hypothesis (H.4) is much more difficult to prove. However, in a particular regular perturbation of the Chafee-Infante equation, Kostin [Ko95] proved this condition.

If we want to estimate the semi-distance $\delta_X(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda)$, we need stronger hypotheses. Actually, general results are only known in the case where S_{λ_0} is a gradient system. The next theorem summarizes the above discussion and gives an estimate of the distance $\mathrm{Hdist}_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0})$. To simplify, we do not state the optimal hypotheses (for a more general statement and various applications to perturbed problems, see [HR89], [Hal88], [Ko90], [Ra95]).

Theorem 4.11. *Let be given a family of semigroups S_λ, $\lambda \in \Lambda$, on a Banach space X, whith compact global attractors $\mathcal{A}_\lambda$. Assume that the hypothesis (H.1b) holds, that S_{λ_0} is a gradient system with a Lyapunov functional satisfying (4.7). Suppose moreover that, either $S_{\lambda_0}(t)$ is a continuous semigroup of class $\mathcal{C}^1$ or S_{λ_0} is a $\mathcal{C}^1$-mapping from X to X, and that all the equilibrium points of $S_{\lambda_0}(\cdot)$ are hyperbolic.*
1. If $S_\lambda(1)$ converges to $S_{\lambda_0}(1)$ in $\mathcal{C}^1(B_1, X)$, where B_1 is a bounded neighbourhood of $\mathcal{A}_{\lambda_0}$, then the global attractors $\mathcal{A}_\lambda$ are continuous at $\lambda = \lambda_0$.
2. If, moreover, there are positive constants C and p such that

$$\|S_{\lambda_0}(1) - S_\lambda(1)\|_{\mathcal{C}^1(B_1,X)} \leq C\delta_X(\lambda, \lambda_0)^p \ , \tag{4.15}$$

for any λ in some neighbourhood of λ_0, then there exist a neighbourhood $N_\Lambda(\lambda_0, \varepsilon)$, two positive constants $\hat{C}$ and $\hat{p}$, $0 < \hat{p} \leq p$, such that, for $\lambda \in N_\Lambda(\lambda_0, \varepsilon)$,

$$\mathrm{Hdist}_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \leq \hat{C}\delta_X(\lambda, \lambda_0)^{\hat{p}} \ . \tag{4.16}$$

Theorem 4.11 gives good information about the size of the compact global attractor $\mathcal{A}_\lambda$ for λ near λ_0. Under the condition of hyperbolicity of the equilibria, the size of the global attractor does not change. However, the flows may not stay the same for each λ. As in Section 3.1, one gets more precise information, when $S_{\lambda_0}(t)$ is a Morse-Smale semigroup. In the context of gradient systems, Theorem 3.10 implies the following statement, which is very useful in the applications.

Theorem 4.12. *Let be given a family of semigroups S_λ, $\lambda \in \Lambda$ on a Banch space X, whith compact global attractors $\mathcal{A}_\lambda$. Assume that the hypothesis (H.1b) holds, that, for any λ, S_λ is a gradient system with Lyapunov functional satisfying (4.7). Suppose moreover that, either, for any λ, $S_\lambda(t)$ is a continuous semigroup of class $\mathcal{C}^1$ or S_{λ_0} is a $\mathcal{C}^1$-mapping from X to X and that $S_\lambda(1)$ converges to $S_{\lambda_0}(1)$ in $\mathcal{C}^1(B_1, X)$, where B_1 is a bounded neighbourhood of $\mathcal{A}_{\lambda_0}$. In addition, suppose that:*
1) $S_\lambda(1)$ and $DS_\lambda(1)(y)$ are injective at each point y of $\mathcal{A}_\lambda$, $\lambda \in \Lambda$,
2) every equilibrium point $e_{\lambda_0} \in \mathcal{E}_{\lambda_0}$ is hyperbolic and $W^u(e_{\lambda_0,1}, S_{\lambda_0}(1))$ is transversal to $W^s_{\text{loc}}(e_{\lambda_0,2}, S_{\lambda_0}(1))$, for any equilibria $e_{\lambda_0,1}$, $e_{\lambda_0,2} \in \mathcal{E}_{\lambda_0}$,
then, there exist $n_0 \in \mathbf{N}$ and a neighbourhood $N_\Lambda(\lambda_0, \varepsilon)$ of λ_0, such that, for any $\lambda \in N_\Lambda(\lambda_0, \varepsilon)$, $S_\lambda(t)$ is a Morse-Smale system (i.e. $S_\lambda(n_0)$ is a Morse-Smale map) and $S_\lambda(n_0)$ is conjugate to $S_{\lambda_0}(n_0)$.

In the remaining parts of this section, we are going to describe classical examples of gradient systems generated by evolution equations.

4.2. Retarded functional differential equations

In many problems in physics and biology, the future state of a system depends not only on the present state, but also on past states of the system. The theory of functional differential equations probably started with the work of Volterra [Vo28] [Vo31], who, in his study of models in viscoelasticity and population dynamics, introduced some rather general differential equations incorporating the past states of the system. Since then, retarded functional differential equations (RFDE's) play an important role in biology (predator-prey models, spread of infections, circummutation of plants, etc ...) and in mechanics. Here, we mainly describe a model RFDE arising in viscoelasticity. For further studies and generalizations to neutral functional differential equations, we refer the reader to the book of Hale and Lunel [HVL] as well as to [Hal88] and [Nu00].

For a given $\delta > 0$ and $n \in \mathbf{N}\backslash\{0\}$, let $\mathcal{C} = \mathcal{C}^0([-\delta, 0); \mathbf{R}^n)$ be the space of continuous functions from $[-\delta, 0]$ into $\mathbf{R}^n$ equipped with the norm $\| \cdot \| = \| \cdot \|_{\mathcal{C}}$. For any $\alpha \geq 0$, for any function $x : [-\delta, \alpha) \to \mathbf{R}^n$ and any $t \in [0, \alpha)$, we let x_t denote the function from $[-\delta, 0]$ to $\mathbf{R}^n$ defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-\delta, 0]$.
Suppose that $f \in \mathcal{C}^k(\mathcal{C}, \mathbf{R}^n)$, $k \geq 1$, and that f is a bounded map in the sense that f takes bounded sets into bounded sets. An autonomous retarded functional differential equation with finite delay is a relation

$$\dot{x}(t) = f(x_t) , \tag{4.17}$$

where $\dot{x}(t)$ is the right hand derivative of $x(t)$ at t.
For a given $\varphi \in \mathcal{C}$, one says that $x(t, \varphi)$ is a *solution of (4.17) on the interval* $[0, \alpha_\varphi)$, $\alpha_\varphi > 0$, *with initial value* φ *at* $t = 0$, if $x(t, \varphi)$ is defined on $[-\delta, \alpha_\varphi)$, satisfies (4.17) on $[0, \alpha_\varphi)$, $x_t(\cdot, \varphi) \in \mathcal{C}$ for $t \in [0, \alpha_\varphi)$ and $x_0(\cdot, \varphi) = \varphi$. Using the contraction fixed point theorem, one shows that, for any $\varphi \in \mathcal{C}$, there exists a unique mild solution $x(t, \varphi)$ defined on a maximal interval $[-\delta, \alpha_\varphi)$. Moreover, $x(t, \varphi)$ is continuous in (t, φ), of class $\mathcal{C}^k$ in φ and, for $t \in (k\delta, \alpha_\varphi)$, of class $\mathcal{C}^k$ in t. If $\alpha_\varphi < +\infty$, then $\|x_t(\cdot, \varphi)\| \to +\infty$ as $t \to \alpha_\varphi^-$.

We assume now that all the solutions of (4.17) are defined for $t \in [0, +\infty)$. Then, the one-parameter family of maps $S(t)$, $t \geq 0$ on $\mathcal{C}$ defined by $S(t)\varphi = x_t(\cdot, \varphi)$ is a continuous semigroup on $\mathcal{C}$. We also introduce the linear semigroup $V(t) : \mathcal{C} \to \mathcal{C}$, $t \geq 0$, given by

$$\begin{aligned} (V(t)\varphi)(\theta) &= \varphi(t + \theta) - \varphi(0) , \quad t + \theta < 0 , \\ &= 0 , \quad t + \theta \geq 0 . \end{aligned} \tag{4.18}$$

The following theorem states the basic qualitative properties of the semigroup $S(t)$ and can be found in [HVL], for example.

Theorem 4.13. *If the positive orbits of bounded sets are bounded, then $S(t)$ is a compact map for $t \geq \delta$. Moreover, for $t \geq 0$, one has*

$$S(t)\varphi = U(t)\varphi + V(t)\varphi \ , \tag{4.19}$$

where $U(t)$ is a compact map from X into X for $t \geq 0$ and $V(t)$ has been defined in (4.18). Furthermore, for any $\beta > 0$, there is an equivalent norm $|\cdot|$ on $\mathcal{C}$ so that $|S(t)\varphi| \leq \exp(-\beta t)|\varphi|$, $t \geq 0$, and $S(t)$ is an α-contraction in this norm for $t \geq 0$.

The fact that $S(t)$ can be written as (4.19) had been remarked by Hale and Lopes [HaLo]. The next result is mainly a consequence of Theorem 2.26, Theorem 2.38 and Theorem 4.13. The analyticity property in the third statement is due to [Nu73].

Theorem 4.14. *If the positive orbits of bounded sets are bounded and if $S(t)$ is point dissipative, then*
(i) $S(t)$ has a connected compact global attractor $\mathcal{A} \subset \mathcal{C}$;
(ii) there is at least an equilibrium point (a constant solution) of (4.17);
(iii) if $f \in \mathcal{C}^k(\mathcal{C}, \mathbf{R}^n)$, $k \geq 0$, (resp. analytic), then any element u of the attractor $\mathcal{A}$ is a $\mathcal{C}^{k+1}$ function (resp. analytic);
(iv) if f is analytic, then $S(t)$ is one-to-one on $\mathcal{A}$.

We now present an example of a gradient system generated by a RFDE, which arises in viscoelasticity [LN]. We now let $\delta = 1$ and suppose that b is a function in $\mathcal{C}^2([-1,0], \mathbf{R})$, such that $b(-1) = 0$, $b(s) > 0$, $b'(s) \geq 0$, $b''(s) \geq 0$ and that

$$b''(\theta_0) > 0 \quad \text{for some } \theta_0 \in (-1, 0) \ . \tag{4.20}$$

Let $g \in \mathcal{C}^1(\mathbf{R}, \mathbf{R})$ be such that

$$G(x) \equiv \int_0^x g(s)\,\mathrm{d}s \to +\infty \ \text{ as } |x| \to +\infty \ . \tag{4.21}$$

We consider the equation

$$\dot{x}(t) = -\int_{-1}^{0} b(\theta) g(x(t+\theta))\,\mathrm{d}\theta \ . \tag{4.22}$$

Equation (4.22) is a special case of (4.17) with $f(\varphi) = -\int_{-1}^{0} b(\theta)g(\varphi(\theta))\,\mathrm{d}\theta$. Let $S(t)$ be the local semigroup on $\mathcal{C}$ defined by $S(t)\varphi = x_t(\cdot,\varphi)$, where $x(t,\varphi)$ is the local solution of (4.22) through φ at $t = 0$. To show that the solutions of (4.22) exist globally, one introduces the functional on $\mathcal{C}$

$$\Phi(\varphi) \;=\; G(\varphi(0)) + \frac{1}{2}\int_{-1}^{0} b'(\theta)\left(\int_{\theta}^{0} g(\varphi(s))\,\mathrm{d}s\right)^{2} \mathrm{d}\theta \;.$$

We set $L_\theta(\psi) = \int_\theta^0 g(\psi(s))\,\mathrm{d}s$. A short computation shows that, for $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi(S(t)\varphi)) \;=\; -\frac{1}{2}\big(b'(-1)[L_{-1}(S(t)\varphi)]^2 + \int_{-1}^{0} b''(\theta)[L_\theta(S(t)\varphi)]^2\,\mathrm{d}\theta\big)\;. \qquad (4.23)$$

The hypotheses on b and g imply that $\Phi(\varphi) \to +\infty$ as $\|\varphi\| \to +\infty$ and, due to (4.23), that $\Phi(S(t)\varphi) \leq \Phi(\varphi)$, for $t \geq 0$. Therefore, the solutions of (4.22) exist globally and the orbits of bounded sets are bounded. The next theorem summarizes the properties of the semigroup $S(t)$ ([Hal88], [HRy], [Hal00]).

Theorem 4.15. *Assume that the conditions (4.20) and (4.21) hold. Then, the ω-limit set of any orbit is a single equilibrium point.*
If moreover, the set $\mathcal{E}$ of the zeros of g is bounded, the semigroup $S(t)$ generated by (4.22) is a continuous gradient system and admits a compact connected global attractor $\mathcal{A}_{bg}$ in $\mathcal{C}$. If, in addition, each element of $\mathcal{E}$ is hyperbolic, then $\dim W^u(x_0) = 1$, *for any $x_0 \in \mathcal{E}$ and $\mathcal{A}_{bg} = \bigcup_{x_0\in\mathcal{E}} W^u(x_0)$.*

Sketch of the proof. We first observe that any solution $x(t)$ of (4.22) satisfies

$$\ddot{x}(t) + b(0)(g(x(t)) \;=\; b'(-1)L_{-1}(x_t) + \int_{-1}^{0} b''(\theta)L_\theta(x_t)\,\mathrm{d}\theta\;. \qquad (4.24)$$

Suppose that $\Phi(S(t)\varphi) = \Phi(\varphi)$, for $t \geq 0$, then (4.23) and (4.24) imply that the solution $x(t)$ through φ satisfies

$$\ddot{x}(t) + b(0)(g(x(t)) \;=\; 0\;,$$

together with $L_s(x_t) = 0$ for s in some interval I_0 containing θ_0. It follows that $\dot{x}$ is a constant. The boundedness of $x(t)$ implies that $x(t)$ is a constant and thus φ is a zero of g. Hence Φ is a strict Lyapunov functional. The existence of the compact global attractor $\mathcal{A}_{bg}$ is a direct consequence of Theorem 4.6 and Theorem 4.14.

If $g(c) = 0$, the linear variational equation about c is $\dot{y}(t) = -\int_{-1}^{0} b(s)g'(c)y(t+s)\,\mathrm{d}s$ and the eigenvalues λ of the linear variational equation are given by

$$\lambda = -\int_{-1}^{0} b(s)g'(c)\exp(\lambda s)\,\mathrm{d}s \ .$$

It is possible to show that the equilibrium point c is hyperbolic if and only if $g'(c) \neq 0$. If $g'(c) = 0$, $\lambda = 0$ is a simple eigenvalue. Property 1) of Remarks 4.3 then implies that the ω-limit set of any positive orbit is a singleton. If $g'(c) > 0$, then c is stable. Finally, one easily shows that, if $g'(c) < 0$, then c is unstable with $\dim W^u(c) = 1$. □

Suppose now that b is a fixed function and consider the global attractor $\mathcal{A}_{bg}$ as a function of the parameter g. Semicontinuity and continuity results of $\mathcal{A}_{bg}$ with respect to g are proved in [HR89] and are actually an application of Proposition 4.9 and Theorem 4.11. We have seen that, for each zero c with $g'(c) < 0$, c is an unstable equilibrium point with $\dim W^u(c) = 1$; this means that there are two distinct complete orbits $\varphi_e(t)$ and $\psi_e(t)$ which approach e as $t \to -\infty$. Since the ω-limit set of $\varphi_e(t)$ (resp. $\psi_e(t)$) is a single equilibrium point e_φ (resp. e_ψ), the next problem is to determine if e_φ (resp. e_ψ) is smaller or larger than e. If $\mathcal{E} = \{e_1, e_2, e_3\}$ with $e_1 < e_2 < e_3$, the flow on $\mathcal{A}_{bg}$ preserves the natural order since $\mathcal{A}_{bg}$ is connected. The case when $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$ with $e_1 < e_2 < e_3 < e_4 < e_5$, has been studied by Hale and Rybakowski [HRy]. To state their result, it is convenient to use the notation $j[k,l]$ to mean that the unstable point e_j is connected to e_k and e_l by a trajectory. If g has five simple zeros, then e_2, e_4 are unstable, while e_1, e_3, e_5 are stable.

Theorem 4.16. *Let b be fixed. One can realize each of the following flows on $\mathcal{A}_{bg}$ by an appropriate choice of g with five simple zeros:*
(i) $2[1,3]$, $4[3,5]$*;*
(ii) $2[1,4]$, $4[3,5]$*;*
(iii) $2[1,5]$, $4[3,5]$*;*
(iv) $2[1,3]$, $4[2,5]$*;*
(v) $2[1,3]$, $4[1,5]$*;*

4.3. Scalar parabolic equations

The simplest and most studied gradient partial differential equation is the semilinear heat or reaction-diffusion equation, which models several physical phenomena like heat conduction, population dynamics, etc...The heat equation belongs to the class of parabolic equations, where smoothing effects take place in finite positive time. Here, we study this equation under very simplified hypotheses on the nonlinearity.

Let Ω be a bounded domain in $\mathbf{R}^n$, with Lipschitzian boundary. We consider the following heat equation

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \Delta u(x,t) + f(u(x,t)) + g(x) \ , \quad x \in \Omega \ , \quad t > 0 \ , \\ u(x,t) &= 0 \ , \quad x \in \partial\Omega \ , \quad t > 0 \ , \\ u(x,0) &= u_0(x) \ , \quad x \in \Omega \ , \end{aligned} \tag{4.25}$$

where g is in $L^2(\Omega)$ and $f : \mathbf{R} \to \mathbf{R}$ is a locally Lipschitz continuous function. We introduce the operator $A = -\Delta_D$ with domain $\mathcal{D}(A) = \{v \in H_0^1(\Omega) \,|\, -\Delta v \in L^2(\Omega)\}$ and set $V = H_0^1(\Omega) \equiv \mathcal{D}(A)^{1/2}$.
In the case $n \geq 2$, we assume that the locally Lipschitz continuous function f also satisfies the following growth condition:

(A.1) there exist positive constants C_0 and α, with $(n-2)\alpha \leq 2$ such that

$$|f(y_1) - f(y_2)| \leq C_0(1 + |y_1|^\alpha + |y_2|^\alpha)|y_1 - y_2| \ , \quad \forall y_1, y_2 \in \mathbf{R} \ . \tag{4.26}$$

The restriction $(n-2)\alpha \leq 2$ has been made only for sake of simplicity. Most of the results of Section 4.3 also hold if $(n-2)\alpha \leq 4$ (see Remark 4.18 1. below). The hypothesis (A.1) together with the Sobolev embeddings properties, allow to define the mapping $u \in V \mapsto f(u) \in L^2(\Omega)$, by $(f(u))(x) = f(u(x))$, for almost every $x \in \Omega$.
This mapping is Lipschitzian on the bounded sets of V. With the above definitions of A and V, we can rewrite (4.25) as an abstract evolutionary equation in V:

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = -Au(t) + f(u(t)) + g \ , \quad t > 0 \ , \quad u(0) = u_0 \ , \tag{4.27}$$

Since A is a sectorial operator and $f : V \to L^2(\Omega)$ is Lipschitzian on the bounded sets of V, for any $r \geq 0$, there exists $T \equiv T(r) > 0$ such that, for any $u_0 \in V$, with $\|u_0\|_V \leq r$, the equation (4.27) has a unique classical solution $u \in \mathcal{C}^0([0,T],V) \cap \mathcal{C}^1((0,T],L^2(\Omega)) \cap$

$\mathcal{C}^0((0,T],\mathcal{D}(A))$.
Later, when we need more regularity on f, we suppose in addition that

(A.2) $f \in \mathcal{C}^1_u(\mathbf{R},\mathbf{R})$, f' is locally Hölder continuous and, if $n \geq 2$, there exist nonnegative constants C_1, α_1, β_1, such that

$$|f'(y_1) - f'(y_2)| \,\leq\, C_1(1+|y_1|^{\beta_1}+|y_2|^{\beta_1})|y_1-y_2|^{\alpha_1} \;, \quad \forall y_1, y_2 \in \mathbf{R} \;, \tag{4.28}$$

where $\alpha_1 > 0$, $(\alpha_1+\beta_1)(n-2) \leq 2$ if $n \geq 2$. In this case, f is a $\mathcal{C}^{1,\alpha_1}$-mapping from V into $L^2(\Omega)$.

Remark.
For sake of simplicity, we have provided the above heat equation with homogeneous Dirichlet boundary conditions. All the assertions of this subsection remain true if we replace them in (4.25) by homogeneous Neumann conditions, in which case $V = H^1(\Omega)$. Even, much more general boundary conditions may be chosen. Furthermore, we can replace the Laplacian operator by any second order operator $\sum_{i,j=1}^{i,j=n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) + a_0(x)$, where a_{ij}, a_0 are smooth enough functions of x and the matrix $[a_{ij}(x)]_{i,j}$ is symmetric, positive definite, for any $x \in \Omega$.

To obtain global existence of the solutions of (4.27), we need to impose, for example, a dissipation condition. Here we assume that

(A.3) there exist constants $C_2 \geq 0$ et $\mu \in \mathbf{R}$ such that

$$yf(y) \,\leq\, C_2 + \mu y^2 \;, \quad F(y) \,\leq\, C_2 + \frac{1}{2}\mu y^2 \;, \forall y \in \mathbf{R} \;, \tag{4.29}$$

with

$$\mu \,<\, \lambda_1 \;, \tag{4.30}$$

where λ_1 is the first eigenvalue of the operator A and where F is the primitive $F(y) = \int_0^y f(s)\,\mathrm{d}s$ of f. Global existence of solutions of (4.25) already holds under the hypothesis (4.29). Condition (4.30) will ensure that all the solutions are uniformly bounded. Indeed, introducing the functional $\Phi_0 \in \mathcal{C}^0(V,\mathbf{R})$ given by

$$\Phi_0(u) \,=\, \int_\Omega \left(\frac{1}{2}|\nabla u(x)|^2 - F(u(x)) - g(x)u(x)\right)\,\mathrm{d}x \;, \tag{4.31}$$

one shows that, if $u(t)$ is a classical solution of (4.27) for $t \leq T$, then $\Phi_0(u(t)) \in \mathcal{C}^0([0,T]) \cap \mathcal{C}^1((0,T])$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0(u(t)) \;=\; -\|u_t(t)\|^2_{L^2} \;, \quad \forall t \in (0,T] \;, \tag{4.32}$$

which implies that Φ_0 is a strict Lyapunov function of (4.27). Moreover, using the assumptions (A.1), (A.3) and the property (4.32), we obtain, for any $0 < \varepsilon < \lambda_1 - \mu$,

$$\begin{aligned}\frac{1}{2}(1 - \frac{\mu+\varepsilon}{\lambda_1})\|\nabla u(t)\|^2_{L^2} - C|\Omega| - \frac{1}{2\varepsilon}\|g\|^2_{L^2} \;\leq\; \Phi_0(u(t)) \;\leq\; \Phi_0(u(0)) \\ \leq C^*(1 + \|g\|_{L^2} + \|u(0)\|^{1+\alpha}_{H^1})\|u(0)\|_{H^1} \;,\end{aligned} \tag{4.33}$$

where C, C^* are positive constants. This implies that all the solutions of (4.27) are global and the orbits of bounded sets are bounded. We notice that the set $\mathcal{E}_P$ of the equilibrium points of (4.27) is given by

$$\mathcal{E}_P \;=\; \{u \in V \,|\, \Delta_D u + f(u) + g = 0\} \;.$$

Due to the dissipative condition (A.3), this set is bounded in V.

If we let $S(t)u_0$ denote the solution $u(t)$ of (4.27), with initial data $u_0 \in V$, we have defined a continuous gradient system $S(t)$ on V. We remark that the map $(t, u_0) \mapsto S(t)u_0$ belongs to $\mathcal{C}^0([0,+\infty) \times V, V)$. Due to a backward uniqueness result of Bardos and Tartar [BT], the mapping $S(t)$ is injective on V, for any $t \geq 0$. Furthermore, since for any bounded set $B \subset V$, the orbit $\gamma^+(B)$ is bounded in V, one shows, by using the smoothing properties of $S(t)$, that the orbit $\gamma^+_\tau(B)$ is bounded in $\mathcal{D}(A)$, for any $\tau > 0$ ([He81, Theorem 3.5.2]), which implies in particular that $S(t)$ is compact for $t > 0$. Applying Proposition 2.5 and Theorem 4.6, one obtains the existence of a compact global attractor.

Theorem 4.17. *Assume that the assumptions (A.1) and (A.3) hold. Then, the semigroup $S(t)$ generated by (4.27) has a compact connected global attractor $\mathcal{A} = W^u(\mathcal{E}_P)$ in V. The semigroup $S(t)\big|_{\mathcal{A}}$ is a continuous group of continuous operators. Moreover, the global attractor $\mathcal{A}$ is bounded in $\mathcal{D}(A)$ and thus in $H^2(\Omega)$, if the domain Ω is either convex or of class $\mathcal{C}^{1,1}$.*

Properties of the compact attractor $\mathcal{A}$

In what follows, we assume that the assumptions (A.1), (A.2) and (A.3) hold.

First, we recall that $\mathcal{A}$ is often bounded in a higher order Sobolev space. If, for instance, the nonlinearity f belongs to $\mathcal{C}^k(V, L^2(\Omega))$ (resp. is analytic), then the map $(t, u_0) \in [\tau, +\infty) \times V \mapsto S(t)u_0 \in V$ is of class $\mathcal{C}^k$, for $\tau > 0$ (resp. is analytic) (see [He81, Corollary 3.4.6]). Due to the invariance of $\mathcal{A}$, it follows that $S(t)\big|_{\mathcal{A}} : t \in \mathbf{R} \mapsto S(t)u_0 \in V$ is of class $\mathcal{C}^k$ (resp. analytic). Arguing by recursion on k, one deduces from the regularity in time property that, if $f \in \mathcal{C}_b^{k-j}(H^{j+1}(\Omega), H^j(\Omega))$, $1 \leq j \leq k$, (resp. $\mathcal{C}^\infty(H^{j+1}(\Omega), H^j(\Omega))$, for $j \geq 1$) and if moreover Ω is of class $\mathcal{C}^{k+1,1}$ (resp. $\mathcal{C}^\infty$) and $g \in H^k(\Omega)$ (resp. $\mathcal{C}^\infty(\overline{\Omega})$), then $\mathcal{A}$ is bounded in $H^{k+2}(\Omega)$ (resp. $\mathcal{C}^\infty(\overline{\Omega})$). If $\Omega \subset \mathbf{R}^n$, $n = 1, 2, 3$, is either convex or of class $\mathcal{C}^{1,1}$, applying for example the results of Section 3.3, one shows that, if $f \in \mathcal{C}_{bu}^k(\mathbf{R}, \mathbf{R})$ and $f^{(k)}$ is locally Hölder continuous (respectively f is analytic), then the map $t \mapsto S(t)u_0$ is in $\mathcal{C}^k((0, +\infty), V)$ (respectively analytic), for any $u_0 \in V$ and thus, that $\mathcal{A}$ is bounded in $H^{k+2}(\Omega)$ (resp. $\mathcal{C}^\infty(\overline{\Omega})$) if moreover Ω is of class $\mathcal{C}^{k+1,1}$ (resp. $\mathcal{C}^\infty$).
For regularity in Gevrey spaces, when the Dirichlet boundary conditions are replaced by periodic ones, we refer to [FT89], [Pr91] and [FeTi98] and the references therein.

The fractal dimension of $\mathcal{A}$ is finite. An explicit bound is given in [Te, Chapter 6], for example. By Theorem 3.17, the property of finite number of determining modes holds.

We remark that e is a hyperbolic equilibrium point of (4.27) if and only if the spectrum $\sigma(-A + Df(e))$ of $(-A + Df(e))$ does not intersect the imaginary axis in $\mathbf{C}$ or, equivalently here, if 0 is not an eigenvalue of $(-A + Df(e))$. Moreover, the index $\mathrm{ind}(e)$ is equal to the number of positive eigenvalues $l_j(e)$ of $-A + Df(e)$. We recall that the hyperbolicity of the equilibria of (4.25) is a generic property in $g \in L^2(\Omega)$ ([BV83], [BV89b, Chapter 6, Theorem 3.4]). Generic hyperbolicity of the equilibrium points also holds with respect to the domain Ω ([He87]). In the one-dimensional case, generic hyperbolicity with respect to f has also been proved in [BrCh] and [SW], for example.

If all of the equilibrium points $(e, 0)$ of (4.27) are hyperbolic, we deduce from Theorem 4.6 and Theorem 4.8 that $\mathcal{A} = \bigcup_{e_j \in \mathcal{E}_P} W^u(e_j)$ and that the Hausdorff dimension $\dim_H(\mathcal{A})$ is equal to $\max_{e \in \mathcal{E}_P} \mathrm{ind}(e)$. The unstable and stable manifolds $W^u(e, S(t))$ and $W^s(e, S(t))$ are embedded $\mathcal{C}^1$-submanifolds of V of dimension $\mathrm{ind}(e)$ and codimension $\mathrm{ind}(e)$, respectively. We shall see later that these stable and unstable manifolds always intersect transversally if Ω is an interval of $\mathbf{R}$. This property is not known in higher dimension space. If we allow the function f to depend upon x and assume that this function $f(x,.)$ satisfies the assumptions (A.1), (A.2) and (A.3) uniformly in x, then the resulting semigroup $S(t)$ still admits a compact global attractor. Brunovsky and Poláčik have proved that the semigroup defined by (4.27) is a Morse-Smale system, generically in such non linearities $f(x,.)$ (see [BrP97a]). Furthermore, for the unit ball in $\mathbf{R}^2$, Poláčik has shown that there exists a function $f(x,u)$ for which the tranversality of the stable and unstable manifolds does not hold (see [Po94]).

In the one-dimensional case, the eigenvalues λ_i of the operator A fulfill the gap condition needed in the construction of inertial manifolds. In this case, (4.27) has an inertial manifold (see [FST]). Mallet Paret and Sell [MPS] have proved that this gap condition can be replaced by a cone condition, which is less restrictive. As a consequence, they showed that (4.27) has an inertial manifold of class $\mathcal{C}^1$, if Ω is either a rectangular domain $(0,2\pi/a_1)\times(0,2\pi/a_2)$, where a_1, a_2 are arbitrary positive numbers or Ω is the cube $(0,2\pi)^3$ and if $f:(x,y)\in\overline{\Omega}\times\mathbf{R}\mapsto f(x,y)\in\mathbf{R}$ is of class $\mathcal{C}^3$. These results are valid for the equation (4.25) supplemented with homogeneous Dirichlet or Neumann boundary conditions or periodic boundary conditions. The existence of an inertial manifold can also be proved in the case of domains in $\mathbf{R}^{n+1}$, which are thin in n directions ([HR92c], [Ra95]).

If $\Omega\subset\mathbf{R}$, the positive orbit of every point $u_0\in V$ is convergent ([Ze]). If Ω is a domain in $\mathbf{R}^{n+1}$, which is thin in n directions, the positive orbits are still convergent ([HR92b]). In the case $n\geq 1$, all the orbits of (4.27) are still convergent if $f:\mathbf{R}\to\mathbf{R}$ and g are analytic functions ([Si]). For further details on convergence properties, see [HR92b], [BrP97b] and [Po00].

Remarks 4.18.
1. For sake of simplicity in the exposition, we have assumed that the exponent α in (A.1) satisfies the condition $(n-2)\alpha\leq 2$, when $n\geq 3$. We can still associate with (4.27) a continuous semigroup $S(t)$ on V, provided $(n-2)\alpha\leq 4$ and that the domain Ω is either convex or of class $\mathcal{C}^{1,1}$. In this case, the proof of the existence of the associated continuous semigroup $S(t)$ is less straightforward and uses a fixed point argument introduced by Fujita and Kato (see [FK] and also [GR00]). This semigroup admits a compact global attractor $\mathcal{A}$ in V, with the same properties as above. The restriction of the above semigroup $S(t)$ to $X_2=H^2(\Omega)\cap H^1_0(\Omega)$ is also a continuous semigroup on X_2 and $\mathcal{A}$ is the global attractor of $S(t)|_{X_2}$.

2. If we do not want to introduce a limitation on the growth rate of f, we can also

consider the equation (4.25) in the space $X = \mathcal{C}_0^0 \equiv \{v \in \mathcal{C}^0(\overline{\Omega}) \,|\, v = 0 \text{ in } \partial\Omega\}$ and introduce the operator $A_c = -\Delta$ with domain $\mathcal{D}(A_c) = \{v \in X \cap H_0^1(\Omega) \,|\, \Delta v \in X\}$. If $g = 0$ and $f(0) = 0$, local existence and uniqueness of the solutions of (4.25) are well known. If, moreover, there exists a positive constant κ such that,

$$yf(y) \leq 0 \,, \quad \forall |y| \geq \kappa \,, \tag{4.34}$$

one shows, by using the maximum principle and truncature arguments, that the solutions of (4.25) are all global, which allows to associate to (4.25) a continuous semigroup $S_c(t)$ on X. One proves under the assumption (4.34), that, for any bounded set $B \subset X$, any $\tau > 0$ and any $0 < \beta < 1$, the orbit $\gamma_\tau^+(B)$ is bounded in $\mathcal{C}^{1,\beta}(\overline{\Omega})$ and hence that $S_c(t)$ has a compact global attractor in X (see [HK], [Har91] for further details, and also [Po00] for a setting in the spaces $W^{s,p}(\Omega)$, $p \geq 2$, $1 \leq s < 2$).

4.4. One-dimensional scalar parabolic equations

In the one-dimensional case, detailed properties of the flow on the compact attractor can be obtained by using tools like the Sturm-Liouville theory, the Jordan curve theorem as well as the strong maximum principle. Here, we are going to distinguish two types of boundary conditions. Because of lack of space, we describe only a few results and refer to the fairly complete review of Hale [Hal97] for further results.

The case of separated boundary conditions.

For sake of simplicity, we consider the following reaction diffusion equation on $\Omega = (0,1)$, provided with homogeneous Neumann boundary conditions:

$$u_t \,=\, u_{xx} + f(x, u, u_x) \;\text{ in } \Omega \,=\, (0,1) \,, \quad u_x(0) \,=\, u_x(1) \,=\, 0 \,. \tag{4.35}$$

We could consider more general separated boundary conditions like

$$b_0 u_x(0,t) + \beta_0 u(0,t) \,=\, b_1 u_x(1,t) + \beta_1 u(1,t) \,=\, 0 \,, \tag{4.36}$$

where b_0, β_0, b_1, β_1 are normalized so that $b_0^2 + \beta_0^2 = b_1^2 + \beta_1^2 = 1$, and also replace u_{xx} with $a(x)u_{xx}$, where $a \in \mathcal{C}^2(\overline{\Omega})$ is a positive function on $\overline{\Omega}$.

We assume now that $f : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is a $\mathcal{C}^2$-function satisfying both conditions:

(C1) there exist γ, $0 \leq \gamma < 2$, and a continuous function $\kappa : [0, +\infty) \to [0, +\infty)$ such that

$$|f(x, y, \xi)| \,\leq\, \kappa(r)(1 + |\xi|^\gamma) \,, \quad \forall (x, y, \xi) \in [0,1] \times [-r, r] \times \mathbf{R} \,; \tag{4.37}$$

(C2) there exists a positive constant K such that

$$y f(x,y,0) \,\leq\, 0 \;, \quad \forall (x,y) \in [0,1] \times \mathbf{R} \;, \quad |y| > K \;. \tag{4.38}$$

Under these hypotheses, the equation (4.35) generates a continuous semigroup $S(t)$ on the space $X^s = \mathcal{D}((-\Delta_N + I)^s)$, $1/2 \leq s \leq 1$, where Δ_N is the Laplacian operator with homogeneous Neumann boundary conditions. We recall that $\mathcal{D}(-\Delta_N + I) = \{u \in H^2(\Omega) \,|\, u_x(0) = u_x(1) = 0\}$. The dissipation condition (C2) implies that the orbits of bounded sets are bounded.

In the one-dimensional case, the presence of gradient terms in the nonlinearity does not prevent the gradient structure as it has been proved by Zelenyak [Ze].

Proposition 4.19. *The continuous semigroup $S(t)$ is a gradient system on X^s, $s \in [1/2, 1]$. Moreover, every positive orbit is convergent.*

Sketch of the proof. The first step of the proof consists in finding a Lyapunov functional $\Phi_0(u)$ for (4.35). For $u \in X^s$, one considers functionals

$$\Phi_0(u) \,=\, \int_0^1 G(x,u,u_x)\,\mathrm{d}x,$$

where $G : (x,y,\xi) \in [0,1] \times \mathbf{R}^2 \mapsto G(x,y,\xi) \in \mathbf{R}^2$ and one observes that, for any solution $u(x,t)$ of (4.35),

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0(u(t)) \,=\, -\int_0^1 G_{\xi\xi}(x,u,u_x) u_t^2 \,\mathrm{d}x \;, \tag{4.39}$$

provided the mapping G satisfies

$$\begin{aligned} \xi G_{\xi y} - f G_{\xi\xi} + G_{\xi x} \,&=\, G_y \;, \quad \forall (x,y,\xi) \in [0,1] \times \mathbf{R}^2 \;, \\ u_t(0,t) G_\xi(0,y,0) \,&=\, u_t(1,t) G_\xi(1,y,0) \,=\, 0 \;, \quad \forall y \in \mathbf{R} \;. \end{aligned} \tag{4.40}$$

One then shows that there exists a solution G of class $\mathcal{C}^2$ of (4.40) such that $G_{\xi\xi}(x,y,\xi) > 0$. Thus, Φ_0 is a strict Lyapunov functional and $S(t)$ is a gradient system.
To prove that the ω-limit set $\omega(\varphi)$ is a singleton, for every $\varphi \in X^s$, we apply the general result of [HR92b] mentioned in Remarks 4.3. Indeed, for any equilibrium point $e \in \mathcal{E}_P$ of (4.35), the eigenvalue problem for the linearization of (4.35) at e (called Sturm-Liouville problem)

$$\lambda v \,=\, v_{xx} + f_y(x,e,e_x) v + f_\xi(x,e,e_x) v_x \;, \quad \forall x \in (0,1) \;, \quad v_x(0) \,=\, v_x(1) \;, \tag{4.41}$$

has the following well-known properties. All the eigenvalues $\lambda_j(e)$ are real and algebraically simple. The (normalized) eigenfunction $\varphi_j(e)$ corresponding to the j-th eigenvalue $\lambda_j(e)$ has exactly $j-1$ zeros. In particular, if 0 is an eigenvalue of (4.41), it must be simple and the general convergence result of [HR92b] applies. $\square$

Probably, the most important property of the scalar equation (4.35), which is the starting point of the qualitative description of the global attractor $\mathcal{A}$ is the transversality property of the stable and unstable manifolds of the equilibria. This result is due to Henry [He85b] (see also [An86] for another proof).

Theorem 4.20. *If e and e^* are hyperbolic equilibria of (4.35), then $W^u(e)$ is transversal to $W^s(e^*)$. Thus, $S(t)$ is a Morse-Smale system and is structurally stable, when the equilibria are all hyperbolic.*

Here we can only give an idea of the proof of this result (for more details, see [He85b], [An86] and [Hal88]). It involves the following two basic results. For a continuous function $v : [0,1] \to \mathbf{R}$, let $z(v)$ denote the number (possibly infinite) of zeros of v in $(0,1)$. We say that a differentiable function v has a multiple zero at $x_0 \in [0,1]$ if $v(x_0) = v_x(x_0) = 0$. An application of the maximum principle in two-dimensional domains together with the Jordan curve theorem yields the following result (see [Ni], [Mat82], [Che]).

Lemma 4.21. *Let $v(x,t) \in \mathcal{C}^0([0,+\infty), X^s)$ be a solution of the linear nonautonomous equation*

$$v_t = v_{xx} + a(x,t)v + b(x,t)v_x \ , \quad x \in (0,1) \ , \quad v_x(0) = v_x(1) = 0 \ , \tag{4.42}$$

where a, b are functions in $L^\infty((0,1)\times\mathbf{R})$. Then, if v is not identically zero, the following properties hold:
(i) $z(v(\cdot,t))$ is finite for any $t > 0$;
(ii) $z(v(\cdot,t))$ is nonincreasing in t;
(iii) if $v(x_0,t_0) = v_x(x_0,t_0) = 0$ for some $t_0 > 0$, $x_0 \in (0,1)$, then $z(v(\cdot,t))$ drops strictly at $t = t_0$.

Notice that the above lemma holds as well for other separated boundary conditions and for periodic boundary conditions. The nonincrease of the zero number together with the above Sturm-Liouville properties imply the following restriction on the connecting orbits.

Lemma 4.22. *If $e \in X^s$, $e^* \in X^s$ are hyperbolic equilibrium points of (4.35) and there is an element $u_0 \in X^s$ such that $\alpha(u_0) = e$ and $\omega(u_0) = e^*$, then* $\dim W^u(e) > \dim W^u(e^*)$.

The main ingredients of the proof of Lemma 4.22 are as follows. If $u(t)$ is a solution of (4.35) through u_0, then $u_t(t)$ satisfies a linear equation of the form (4.42), whose coefficients converge exponentially to those of the linearized equations around e and e^*, when $t \to -\infty$ and $t \to +\infty$ respectively. Since no solution of (4.42) approaches zero faster than any exponential, one can show that $u_t(t) \to 0$ as $t \to -\infty$ along the direction of one of the eigenvectors of the operator $\partial^2/\partial x^2 + f_y(x, e, e_x)I + f_\xi(x, e, e_x)\partial/\partial x$. It follows from the above Sturm-Liouville properties that $z(u_t(t)) \leq \mathrm{ind}(e) - 1$ for t close to $-\infty$. Likewise, one shows that $z(u_t(t)) \geq \mathrm{ind}(e^*)$ for t close to $+\infty$. Then Lemma 4.21 implies that $\dim W^u(e) > \dim W^u(e^*)$. $\square$

To complete the proof of Theorem 4.20, one assumes that the manifolds are not transversal, uses the characterization of the tangent space $TW^s(e^*)$ of $W^s(e^*)$ in terms of the adjoint of the linearized equation around $u(t)$ and argues as in the proof of Lemma 4.22 for this adjoint equation to show that $\dim W^u(e) < \dim W^u(e^*)$, which contradicts Lemma 4.22.

Lemma 4.22 naturally leads to the problem of connecting orbits, when all of the equilibrium points are hyperbolic. We say that $C(e, e^*)$ is an orbit connecting e to e^* if, for any point $u_0 \in C(e, e^*)$, we have $\alpha(u_0) = e$ and $\omega(u_0) = e^*$. This problem has been discussed for a long time in the special case of the Chafee-Infante equation:

$$u_t = u_{xx} + \mu^2(u - u^3) \text{ in } (0,1)\ , u_x(0) = u_x(1) = 0\ . \tag{4.43}$$

It has been shown by Chafee and Infante [CI] that the only stable equilibrium points of (4.43) are the constant functions ± 1. Furthermore, for each $j = 1, 2, \cdots$ two equilibrium points $e_j^\pm$ of index j bifurcate supercritically from 0 at $\mu_j = j\pi$. In the interval $(0, \pi)$, there are three equilibrium points 0, ± 1; in the interval (μ_j, μ_{j+1}), 0 has index $j+1$ and there are exactly $2j+3$ equilibria 0, ± 1, $e_k^\pm$, $k = 1, \cdots, j$. The complete description of the attractor $\mathcal{A}_\mu$ has been given by Henry in [He85b]. For $\mu \in (\mu_j, \mu_{j+1})$, the attractor $\mathcal{A}_\mu$ is the closure of $W^u(0)$, and, for each $1 \leq k \leq j$, there exists an orbit connecting $e_k^\pm$ to $e_l^\pm$, for $1 \leq l < k$, and to ± 1.

Before presenting general results on the existence of connecting orbits, we describe another important consequence of the properties of the zero number; that is the existence of an inertial manifold of (4.35) of minimal dimension, when the equilibria are all hyperbolic. More precisely, let $N = \max_{e_j \in \mathcal{E}_P} \operatorname{ind}(e_j)$. Using the zero number, Rocha [Ro91] has shown that, for any equilibria e_j and e_k, with $j \neq k$, one has

$$z(e_j - e_k) \, < \, N \, .$$

As a consequence, he proves the existence of a Lipschitz continuous inertial manifold.

Theorem 4.23. *If the hypotheses (C1) and (C2) hold and the equilibria are all hyperbolic, there exists an (Lipschitzian) inertial manifold of (4.35) of minimal dimension N and it is a graph over the linearized unstable manifold of maximal dimension. If $f(x,y,\xi) = f(x,y)$, the inertial manifold is of class $\mathcal{C}^1$.*

This result had been proved before by Jolly [Jo] in the case $f(x,y,\xi) = y - y^3$ and by Brunovsky [Br90] in the general case $f(x,y,\xi) = f(y)$.

We now go back to the problem of connecting orbits in the global attractor $\mathcal{A}$ of (4.35). We consider here the semigroup $S(t)$ acting on X^1 and assume that all of the equilibria are hyperbolic. The case of a nonlinearity f, depending only on u was solved by Brunovsky and Fiedler ([BrFi89]). Recently, the general case of a nonlinearity $f(x,y,\xi)$ has been mainly considered by Fusco and Rocha, Fiedler and Rocha, Wolfrum ([Wo]). To determine the set $\mathcal{E}_P$ of the equilibrium points of (4.35), one solves the ODE

$$u_x \, = \, v \; , \quad v_x \, = \, -f(x,u,v) \; , \quad u(0) \, = \, u_0 \; , \quad v(0) \, = \, 0 \; . \tag{4.44}$$

Since $\mathcal{A}$ is compact, the set $\mathcal{E}_P$ is a finite set of k elements $\{e_1, \ldots, e_k\}$ that we have ordered so that

$$e_1(0) < e_2(0) < \ldots < e_k(0) \; . \tag{4.45}$$

By uniqueness of the solutions of (4.44), these values are distinct. At the other boundary point $x = 1$, this order may have changed. We thus obtain a permutation $\pi_f \equiv \pi$ of the set $\{1, \ldots, k\}$ given by

$$e_{\pi_f(1)}(1) < e_{\pi_f(2)}(1) < \ldots < e_{\pi_f(k)}(1) \; . \tag{4.46}$$

This *shooting* permutation was introduced by Fusco and Rocha [FuRo]. It characterizes the existence of connecting orbits as proved by Fiedler and Rocha ([FiRo96]).

Theorem 4.24. *Let $f(x,y,\xi)$ be a function in $\mathcal{C}^2([0,1] \times \mathbf{R} \times \mathbf{R}; \mathbf{R})$ satisfying the conditions (C1) and (C2). Assume that the corresponding equation (4.35) has only* *hyperbolic equilibria. Let π_f be the permutation defined by (4.45) and (4.46). Then π_f determines, in an explicit constructive process, which equilibria are connected and which are not. In other words, this permutation determines which of the sets $C(e_i, e_j)$ are nonempty.*

The proof of Theorem 4.24 uses, in a crucial way, the zero number of differences of solutions of equations of type (4.35), the transversality of stable and unstable manifolds, the shooting surface, the above mentioned Sturm-Liouville properties and the Conley index.

The proof of Theorem 4.24 relies on constructive lemmas that we state, without comments. The first lemma shows that the existence of a connecting orbit from e to e^* implies a particular type of *cascading*.

Lemma 4.25. *(Cascading) Under the assumptions of Theorem 4.24, assume that e, e^* are two equilibria with $n = \operatorname{ind}(e) - \operatorname{ind}(e^*) > 0$. Then $C(e, e^*) \neq \emptyset$ if and only if, there exists a sequence (cascade)*

$$e = v_0 , \quad v_1, \ \dots , \quad v_n = e^*$$

of equilibria such that, for every j, $0 \le j < n$, we have:
(i) $\operatorname{ind}(v_{j+1}) = \operatorname{ind}(v_j) - 1$,
(ii) $C(v_j, v_{j+1}) \neq \emptyset$.

Due to the cascading lemma, it suffices to check all of the possible connections from e to e^*, when $\operatorname{ind}(e) - \operatorname{ind}(e^*) = 1$.

Definition 4.26. If e, e^* are two equilibria of (4.35) with $\operatorname{ind}(e) - \operatorname{ind}(e^*) = 1$, we say that the connections between e and e^* are *blocked* if one of the following conditions holds:
(i) $z(e - e^*) \neq \operatorname{ind}(e^*)$,
(ii) there exists a third equilibrium w with $w(0)$ between $e(0)$ and $e^*(0)$ such that $z(e - w) = z(e^* - w) = z(e - e^*)$.

Brunovsky and Fiedler [BrFi89] had already shown that blocking prevents connections. In [FiRo96], the reverse property is proved.

Lemma 4.27. *(Liberalism) Let e, e^* be two equilibria of (4.35) with* $\operatorname{ind}(e) - \operatorname{ind}(e^*) = 1$. *Then, $C(e, e^*) \neq \emptyset$ if and only if the connections from e to e^* are not blocked.*

Theorem 4.24 raises the question, whether the shooting permutation determines the global attractor. More precisely, let us denote by $\mathcal{A}_f$ and π_f the global attractor and the shooting permutation of the semigroup $S_f(t)$ generated by (4.35), corresponding to the nonlinearity f. Fiedler and Rocha [FiRo97] have shown that the shooting permutation π_f completely determines the global attractor, up to an orbit preserving homeomorphism.

Theorem 4.28. *Let $f_1(x,y,\xi)$ and $f_2(x,y,\xi)$ be two functions in $\mathcal{C}^2([0,1]\times\mathbf{R}\times\mathbf{R};\mathbf{R})$ satisfying the conditions (C1) and (C2). Assume that the corresponding equations (4.35) have only hyperbolic equilibria. Then the equality $\pi_{f_1}=\pi_{f_2}$ implies that the global attractors $\mathcal{A}_{f_1}$ and $\mathcal{A}_{f_2}$ are topologically equivalent.*

Theorem 4.24 and Theorem 4.28 have been given in the frame of homogeneous Neumann boundary conditions. Although the global attractor for a given nonlinearity depends on the choice of the boundary conditions, the set of their topological equivalence classes is independent of the boundary conditions in the following sense. Let us consider again the reaction diffusion equation

$$u_t \;=\; u_{xx}+f(x,u,u_x) \;\text{ in } (0,1)\ . \tag{4.47}$$

For $\tau=(\tau_0,\tau_1)$ given in $[0,1]^2$, we provide (4.47) with the boundary conditions:

$$-\tau_0 u_x(0,t)+(1-\tau_0)u(0,t) \;=\; \tau_1 u_x(1,t)+(1-\tau_1)u(1,t) \;=\; 0\ , \tag{4.48}$$

If $f(x,y,\xi)$ is a function in $\mathcal{C}^2([0,1]\times\mathbf{R}\times\mathbf{R};\mathbf{R})$ satisfying the conditions (C1) and (C2), then the equations (4.47) and (4.48) generate a continuous semigroup $S_f^\tau(t)$ on X^1, which admits a compact global attractor $\mathcal{A}_f^\tau$. Using a homotopy argument together with the Morse-Smale property of the global attractors, Fiedler [Fi96] has obtained the equality of the sets of topological equivalence.

Theorem 4.29. *Let $\tau=(\tau_0,\tau_1)\in[0,1]^2$ and $\sigma=(\sigma_0,\sigma_1)\in[0,1]^2$ be given. If $f(x,y,\xi)$ is a function in $\mathcal{C}^2([0,1]\times\mathbf{R}\times\mathbf{R};\mathbf{R})$ satisfying the conditions (C1) and (C2) and if $S_f^\tau(t)$ has only hyperbolic equilibria, then there exists a function $g(x,y,\xi)\in\mathcal{C}^2([0,1]\times\mathbf{R}\times\mathbf{R};\mathbf{R})$ satisfying the conditions (C1) and (C2), such that $S_g^\sigma(t)$ has only hyperbolic equilibria and that the global attractors $\mathcal{A}_f^\tau$ and $\mathcal{A}_g^\sigma$ are topologically equivalent.*

The case of periodic boundary conditions.

If we allow periodic boundary conditions for the reaction-diffusion equation in (4.35), then the structure of the flow can be different. Let us consider the equation

$$u_t = u_{xx} + f(x, u, u_x) \ , \forall x \in S^1 \ , \tag{4.49}$$

where $f : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is a $\mathcal{C}^2$-function satisfying the conditions (C1). The equation (4.49) defines a local continuous semigroup $S(t)$ on $X^1 = H^2(S^1)$ and, if moreover the condition (C2) holds, $S(t)$ admits a compact global attractor $\mathcal{A}$. In the case of separated boundary conditions, we have seen in Proposition 4.19 that the ω-limit set of any $\varphi \in X^1$ is a singleton. Here we may have closed orbits, as it can be shown in some explicit examples (see [AnFi88]). Furthermore, Fiedler and Mallet-Paret [FiMP89] have proved the following somehow surprising generalization of the classical Poincaré-Bendixson theorem.

Theorem 4.30. *If the conditions (C1) and (C2) hold, then the ω-limit set of any $\varphi \in X^1$ satisfies exactly one of the following alternatives:*
(i) either $\omega(\varphi)$ consists in precisely one periodic orbit of minimal period $p > 0$, or
(ii) $\alpha(\psi) \subset \mathcal{E}_P$ and $\omega(\psi) \subset \mathcal{E}_P$, for any $\psi \in \omega(\varphi)$.

The alternative (ii) means that $\omega(\varphi)$ consists of equilibria and connecting (homoclinic or heteroclinic) orbits. Again, the main tool in the proof of Theorem 4.30 is the zero number (for more details, see [FiMP89] and also [Po00]).

If f is independent of x, all periodic orbits are actually *rotating waves*, i.e. solutions of the form $u = u(x - ct)$. Independently, Massat [Ma86] had proved that, in this particular case, either $\omega(\varphi)$ is a single rotating wave or a set of equilibria which differ only by shifting x. Matano [Mat88] had shown that, if $f(y, \xi) = f(y, -\xi)$, then $\omega(\varphi)$ is a single equilibrium point. In the case where f is analytic in his arguments, Angenent and Fiedler [AnFi88] had proved before that, if $\psi \in \omega(\varphi)$, then $\omega(\psi)$ and $\alpha(\psi)$ contain a periodic orbit or an equilibrium point and that every periodic orbit is a rotating wave. Furthermore, heteroclinic orbits between rotating waves are constructed in [AnFi88].

4.5. A damped hyperbolic equation

We now illustrate some of the additional difficulties encountered when one considers partial differential equations which do not smooth in finite time but are still dissipative and have a global attractor. As a model we choose the linearly damped wave equation, which arises as mathematical model in biology and in physics ([Za]). The equation with non linearity $f(u) = \sin u$ is called Sine-Gordon equation and is used to model the dynamics of a Josephson junction driven by a current source. The equation with non linearity $f(u) = |u|^\alpha u$ arises in relativistic quantum mechanics.

The equation with constant positive damping.

We begin the analysis with the following equation with constant positive damping:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) + \gamma \frac{\partial u}{\partial t}(x,t) \; &= \; \Delta u(x,t) + f(u(x,t)) + g(x) \; , \quad x \in \Omega \; , \quad t > 0 \; , \\ u(x,t) \; &= \; 0 \; , \quad x \in \partial\Omega \; , \quad t > 0 \; , \\ u(x,0) \; &= \; u_0(x) \; , \quad \frac{\partial u}{\partial t}(x,0) \; = \; v_0(x) \; , \quad x \in \Omega \; , \end{aligned} \tag{4.50}$$

where γ is a positive constant, Ω is a bounded domain in $\mathbf{R}^n$, with Lipschitzian boundary. We assume that g belongs to $L^2(\Omega)$ and that $f : \mathbf{R} \to \mathbf{R}$ is a locally Lipschitz continuous function satisfying the assumption (A.1). As in Section 4.3, we introduce the operator $A = -\Delta_D$ with domain $\mathcal{D}(A) = \{v \in H_0^1(\Omega) \mid -\Delta v \in L^2(\Omega)\}$ and the mapping $f : v \in V \mapsto f(v)(x) \in L^2(\Omega)$. We write (4.50) as a system of first order

$$\frac{\mathrm{d}U}{\mathrm{d}t}(t) \; = \; BU(t) + f^*(U(t)) + G \; , \quad t > 0 \; , \quad U(0) \; = \; U_0 \; , \tag{4.51}$$

where

$$B \; = \; \begin{pmatrix} 0 & I \\ -A & -\gamma \end{pmatrix} \; , \quad f^*(U) \; = \; \begin{pmatrix} 0 \\ f(u) \end{pmatrix} \; , \quad G \; = \; \begin{pmatrix} 0 \\ g \end{pmatrix} \; , \quad U \; = \; \begin{pmatrix} u \\ v \end{pmatrix} \; ,$$

and introduce the Hilbert space $X = V \times L^2(\Omega) = H_0^1(\Omega) \times L^2(\Omega)$, equipped with the norm $\|U\|_X = (\|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2)^{1/2}$. Since the operator $B_0 : (u,v) \in \mathcal{D}(B_0) \mapsto (v, -Au) \in X$ is a skew-adjoint operator on X, where $\mathcal{D}(B_0) = \mathcal{D}(B) = \mathcal{D}(A) \times H_0^1(\Omega)$ and that $f^* : X \to X$ is Lipschitz continuous on the bounded sets of X, for any $r > 0$, there exists $T \equiv T(r) > 0$ such that, for any $U_0 \in X$, with $\|U_0\|_X \le r$, the equation (4.51) has a unique mild (or integral) solution $U \in \mathcal{C}^0([-T,T], X)$. If moreover

$U_0 \in \mathcal{D}(B)$, then $U \in \mathcal{C}^0([-T,T],\mathcal{D}(B)) \cap \mathcal{C}^1([-T,T],X)$ is a classical solution of (4.51). We now introduce the functional $\Phi \in \mathcal{C}^0(X,\mathbf{R})$ defined by

$$\Phi(U) \;=\; \Phi((u,v)) \;=\; \int_\Omega \left(\frac{1}{2}v^2(x) + \frac{1}{2}|\nabla u(x)|^2 - F(u(x)) - g(x)u(x)\right) \mathrm{d}x \;. \qquad (4.52)$$

One easily shows that, if $U \in \mathcal{C}^0([0,T],X)$ is a solution of (4.51), $\Phi(U(t)) \in \mathcal{C}^1([0,T])$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(U(t)) \;=\; -\gamma\|v(t)\|^2_{L^2} \;, \quad \forall t \in [0,T] \;, \qquad (4.53)$$

which implies that Φ is a strict Lyapunov function for (4.51). One remarks that the set of equilibria $\mathcal{E}_H$ of (4.51) is given by

$$\mathcal{E}_H \;= \mathcal{E}_P \times \{0\} \;=\; \{(u,0) \in X \,|\, \Delta_D u + f(u) + g \;=\; 0\} \;.$$

If f and F satisfy the assumptions (A.1) and (A.3), we deduce from the property (4.53), by arguing as in (4.33), that all the solutions of (4.51) are global and the orbits of bounded sets are bounded. If, for any $U_0 \in X$, we let $S(t)U_0$ denote the solution $U(t)$ of (4.51), we have defined a continuous gradient system $S(t)$ on X. In addition, the mapping $(t,U_0) \mapsto S(t)U_0$ belongs to $\mathcal{C}^0([0,+\infty) \times X, X)$.

In 1979, Webb has proved that each positive orbit $\gamma^+(U_0)$ is relatively compact in X, by using the variation of constants formula and the arguments leading to the proof of Remark 2.33 (see [Web79a] and [Web79b]). Actually, under the assumptions (A.1) and (A.3), the semigroup $S(t)$ has a compact global attractor $\mathcal{A}$. In the non critical case, that is, under the additional assumption $(n-2)\alpha < 2$ when $n \geq 3$, the existence of a compact global attractor has been proved, in 1985, independently by Hale [Hal85] and Haraux [Har85] (see also [GT87b]). In his proof, Hale showed that the assumptions of Remark 2.33 are satisfied, whereas Haraux proved that the complete orbits belong to a more regular space than X, when the domain Ω is more regular. One notices that the proof of Hale does not require regularity of the domain and also works for more general operators than the Laplacian, with less regular coefficients. In the critical case $(n-2)\alpha = 2$ when $n \geq 3$, the existence of a compact global attractor has been first given by Babin and Vishik [BV89b] under an additional assumption on f and later by Arrieta, Carvalho and Hale [ACH] in the general case. Another proof using functionals has been outlined by Ball [Ba1]. Here, we give a sketch of these proofs and explain their comparative advantages. We begin with two preliminaries remarks.

From the assumption (A.1), we at once deduce that

$$\|f(u)\|_{L^2} \leq c_0(\|u\|_{L^2} + \|u\|^{\alpha+1}_{L^{2(\alpha+1)}} + \|f(0)\|_{L^2}) \; . \tag{4.54}$$

If Ω is a bounded domain in $\mathbf{R}^n$ with Lipschitzian boundary, the embedding from $H^1(\Omega)$ into $L^{2(\alpha+1)}(\Omega)$ is compact, if $n = 1, 2$ or if $n \geq 3$ and $(n-2)\alpha < 2$, which implies that $f^* : X \to X$ is a compact map.

We also recall that the operator B is the infinitesimal generator of a linear $\mathcal{C}^0$-group e^{tB}. Using adequate functionals as below or spectral arguments, one shows that there exist positive constants c_1 and c_2 such that

$$\|e^{tB}\|_{L(X,X)} \leq c_1 \exp(-c_2\gamma t) \; , \quad t \geq 0 \; . \tag{4.55}$$

The next theorem of existence of a compact global attractor is fundamental.

Theorem 4.31. *Assume that the assumptions (A.1) and (A.3) hold. Then, the semigroup $S(t)$ generated by (4.51) has a compact connected global attractor $\mathcal{A} \subset X$, given by $\mathcal{A} = W^u(\mathcal{E}_P \times \{0\})$.*
Furthermore, if the domain Ω is either convex or of class $\mathcal{C}^{1,1}$, and if the additional assumption (A.2) holds, then $\mathcal{A}$ is compact in $X_2 = (H^2(\Omega) \cap V) \times V$ and is the global attractor of $S(t)$ restricted to X_2.

Proof. Since $S(t)$ is a gradient system, whose set $\mathcal{E}_H$ of equilibrium points is bounded in X, and since the orbits of bounded sets are bounded, we may deduce the existence of a compact global attractor from Theorem 4.6, as soon as we have shown that $S(t)$ is asymptotically smooth. We shall present three different proofs of this property.
1. Since $S(t)U_0 = U(t)$ is a mild solution of (4.51), one can write

$$S(t)U_0 = e^{tB}U_0 + \int_0^t e^{(t-s)B}(f^*(S(s)U_0) + G)\,\mathrm{d}s \; . \tag{4.56}$$

If $n = 1, 2$ or if $n \geq 3$ and $(n-2)\alpha < 2$, we deduce from (4.54), (4.55) and (4.56) that $S(t)$ satisfies the hypotheses of Remark 2.33. Hence $S(t)$ is asymptotically smooth and has a compact global attractor $\mathcal{A}$ in X. As f is actually a bounded map from V into $H^s(\Omega)$, for some positive s, one can either use a "bootstrap" argument (see [Har85]) or apply Theorem 3.18 or Theorem 3.20 to show that, under the additional smoothness assumptions on f and Ω, the global attractor $\mathcal{A}$ is bounded in X_2.

2. In the case $n \geq 3$ and $(n-2)\alpha = 2$, the mapping $f^* : X \to X$ is no longer compact and a more complicated argument is needed. We first present the functional argument of [Ba1]. To this end, we introduce the space $Y = L^2(\Omega) \times V'$. One easily checks that $S(t)$ is continuous on the bounded subsets of X for the topology of Y and that, for any bouded set $\mathcal{B} \subset X$, the orbit $\gamma^+(\mathcal{B})$ is relatively compact in Y. We set, for any $U \in X$,

$$\begin{aligned} E_0(U) &= \int_\Omega \Big(\frac{\gamma}{2}u^2 + uv\Big)\,\mathrm{d}x\ , \\ \mathcal{F}(U) &= \gamma E_0(U) + 2\Phi(U) \equiv \|v\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \mathcal{F}_0(U)\ . \end{aligned} \tag{4.57}$$

A simple computation and a density argument imply

$$\frac{\mathrm{d}\mathcal{F}(U(t))}{\mathrm{d}t} + \gamma\mathcal{F}(U(t)) = \mathcal{F}_1(U(t))\ , \tag{4.58}$$

where $U(t) = S(t)U_0$ and

$$\mathcal{F}_1(U) = \int_\Omega (\frac{\gamma^3}{2}u^2 + \gamma^2 uv + \gamma f(u)u - 2\gamma F(u) - \gamma g u)\,\mathrm{d}x\ . \tag{4.59}$$

Integrating (4.58), we obtain the equality (2.21) of Proposition 2.35. Clearly, the functionals $\mathcal{F}_0$ and $\mathcal{F}_1$ are bounded on the bounded sets of X and continuous on the bounded sets of X for the topology of Y, not only in the case $(n-2)\alpha \leq 2$, but also in the case $(n-2)\alpha < 4$. Indeed, the continuity of these functionals is proved by showing, that, if u_n converges to u in $L^2(\Omega)$ and is bounded in V, then terms like $\int_\Omega |u_n|^{\alpha+1}|u_n - u|\,\mathrm{d}x$ converge to 0. Using the classical Sobolev inequalities, one gets

$$\int_\Omega |u_n|^{\alpha+1}|u_n - u|\,\mathrm{d}x \leq \|u_n\|_{L^{2n/(n-2)}}^{\alpha+1}\|u - u_n\|_{L^q} \leq \|u_n\|_{H^1}^{\alpha+1}\|u - u_n\|_{H^s}\ , \tag{4.60}$$

where $2 \leq q < 2n/(n-2)$ and $0 < s < 1$, which implies that $\int_\Omega |u_n|^{\alpha+1}|u_n - u|\,\mathrm{d}x$ converges to 0. By Proposition 2.35, $S(t)$ is asymptotically smooth and hence has a global attractor $\mathcal{A}$ in X. The boundedness of $\mathcal{A}$ in X_2 is a consequence of [HR00].

3. Unfortunately the previous proof can hardly be generalized to the cases where the damping term γu_t is replaced by $\gamma(x)h(u_t)$ with $\gamma(x) \geq 0$ and $h(\cdot)$ a nonlinear adequate function. In the critical case, the splitting (4.56) of $S(t)$, which is just the linear variation of constants formula does not directly imply that $S(t)$ is asymptotically smooth. Thus, as in [BV89b] and in [ACH], we introduce another type of splitting, which relies on a non linear variation of constants formula (see [ACH] for further details). In [ACH], it was first remarked that, if f satisfies the conditions (A.1), (A.2) with $\alpha_1 = 1$ and (A.3),

then f can be written as a sum $f = f_1 + f_2$ where f_1 et f_2 are two functions in $\mathcal{C}^1(\mathbf{R}, \mathbf{R})$ with locally Lipschitzian derivatives and satisfy

$$\begin{aligned} |f_1'(y_1) - f_1'(y_2)| &\leq c_1(1 + |y_1|^{\beta_1} + |y_2|^{\beta_1})|y_1 - y_2|^{\alpha_1} \ , \ \forall y_1 \, , \, y_2 \in \mathbf{R} \ , \\ f_1'(0) &= 0 \ , \\ |f_2(y)| &\leq c_2 \ , \quad |f_2'(y)| \leq c_2 \ , \quad \forall y \in \mathbf{R} \ , \\ y f_1(y) &\leq \mu_1 y^2 \ , \quad \forall y \in \mathbf{R} \ , \quad \mu_1 < \lambda_1 \ , \end{aligned} \tag{4.61}$$

where c_1, c_2 are positive constants. We now introduce the continuous semigroup $S_1(t) : U_0 \in X \mapsto U_1(t) \in X$ defined by the equation

$$\frac{\mathrm{d}U_1}{\mathrm{d}t}(t) = BU_1(t) + f_1^*(U_1(t)) \ , \quad t > 0 \ , \quad U_1(0) = U_0 \ , \tag{4.62}$$

where $f_1^*(U_1) = (0, f_1(u_1))$. The mapping $S_1(t)$ is asymptotically contracting, that is, for any $r > 0$, there exist positive numbers $k_1(r)$ and $k_2(r)$ such that, if $\|U_0\|_X \leq r$, we have, for any $t \geq 0$,

$$\|S_1(t)U_0\|_X \leq k_1(r) \exp(-k_2(r)t) \ . \tag{4.63}$$

The property (4.63) is easily proved by using the functional $E_1(U) = \gamma E_0(U) + 2\Phi_1(U)$, where Φ_1 is nothing else as the functional Φ, in which $F(u(x)) + g(x)u(x)$ has been replaced by $F_1(u(x)) = \int_0^{u(x)} f_1(s)\mathrm{d}s$. Indeed, using the properties (4.61) of f_1, one shows that

$$\begin{aligned} \nu_1 \|U_1(t)\|_X^2 &\leq E_1(U_1(t)) \leq (1 + C(r))\|U_1(t)\|_X^2 \ , \\ \frac{\mathrm{d}}{\mathrm{d}t} E_1(t) &\leq -\nu_1 \gamma \|U_1(t)\|_X^2 \ , \end{aligned}$$

where $\nu_1 = \min(1/2, 1 - \mu_1/\lambda_1)$. From these inequalities, we easily deduce (4.63).
We next consider the solution $U_2(t) = (u_2(t), u_{2t}(t)) = S_2(t)U_0$ of the following equation

$$\begin{aligned} \frac{\mathrm{d}^2 u_2}{\mathrm{d}t^2}(t) + \gamma \frac{\mathrm{d}u_2}{\mathrm{d}t}(t) &= \Delta_D u_2(t) + f_2(u(t)) + g + f_1(u(t)) - f_1(u_1(t)) \ , \quad t > 0 \ , \\ u_2(0) &= 0 \ , \quad \frac{\mathrm{d}u_2}{\mathrm{d}t}(0) = 0 \ , \end{aligned} \tag{4.64}$$

Since $S(t)U_0 = S_1(t)U_0 + S_2(t)U_0$ and that the semigroup $S_1(t)$ is asymptotically contracting, $S(t)$ will be asymptotically smooth, if we show that, for any bounded set B in X, the set $\{S_2(t)U_0 \,|\, U_0 \in B\}$ is relatively compact in X, for $t > 0$. Classical energy estimates arguments show that there exists θ, with $1/2 < \theta < 1$, such that, for any $U_0 \in B_X(0, r)$ and any $t \geq 0$,

$$\|A^{1/2-\theta/2} \frac{\mathrm{d}u_2}{\mathrm{d}t}(t)\|_{L^2} + \|A^{1-\theta/2} u_2(t)\|_{L^2} \leq k_3(r) \ , \tag{4.65}$$

where $k_3(r)$ is a positive constant depending only on r (for more details, see [BV89b, Chapter 2, Section 6], [ACH] or [GR00]). This estimate implies in particular that $S_2(t)$ is a compact mapping for $t > 0$. It then follows from Theorem 2.31 that $S(t)$ is asymptotically smooth and that $S(t)$ admits a compact global attractor in X. The estimate (4.65), which is independent of $t > 0$, as well that the invariance of $\mathcal{A}$ imply that $\mathcal{A}$ is actually bounded in $H^{2-\theta}(\Omega) \times H^{1-\theta}(\Omega)$. Finally a "bootstrap" argument shows that $\mathcal{A}$ is bounded in X_2.

4. Under the additional smoothness hypotheses, the semigroup $S(t)\big|_{X_2}$ is also bounded dissipative in X_2 and asymptotically smooth in X_2 (for a detailed proof see [HR88] or [La86]). The asymptotic smoothness of $S(t)$ in X_2 is proved like in 1. Indeed, for $(n-2)\alpha = 2$, $f^* : X_2 \to X_2$ is a compact map. Thus, $S(t)$ has a compact global attractor $\mathcal{A}_2$ in X_2. Obviously, $\mathcal{A}_2 \subset \mathcal{A}$. On the other hand, $\mathcal{A}_2$ attracts the bounded invariant set $\mathcal{A}$ and thus, $\mathcal{A} \subset \mathcal{A}_2$. The theorem is proved. $\square$

Remark 4.32. In the part 2 of the proof of Theorem 4.31, we have seen that the critical exponent for the energy estimates is actually given by $(n-2)\alpha = 4$. Unfortunately, local existence of solutions of the equation (4.51), when Ω is a bounded domain, is not known if $2 < (n-2)\alpha < 4$. However, due to the Strichartz inequalities, local existence of solutions of the wave equation in the whole space $\mathbf{R}^n$ is known, if $2 < (n-2)\alpha < 4$. To obtain local existence of solutions of (4.50), one can use these techniques, if one is able to extend the equation (4.50) to an equation on the whole space in an appropriate way. This can be done in the case of Neumann boundary conditions for special domains and in the case of periodic boundary conditions if $\Omega = (0, L)^n$, $L > 0$, for example.
Here let us consider only the case of periodic boundary conditions, when $\Omega = (0, L)^n$, $n \geq 3$ and assume that the inequality (4.26), with $2 < (n-2)\alpha < 4$, and the property (4.29), with $\mu < 0$, hold. Extending the solutions of (4.50) to the whole space $\mathbf{R}^n$ and using the Strichartz inequalities allow to show global existence and uniqueness of the solutions of (4.50) as well as the continuity of the mapping $(t, U_0) \in [0, +\infty) \times X \mapsto S(t)U_0 \in X$. One also shows that $S(t)$ is continuous on the bounded sets of X for the topology of Y. Like in the case $(n-2)\alpha \leq 2$, $S(t)$ is a gradient system with Lyapunov functional Φ. Actually, the functional introduced in (4.57) allows to prove that $S(t)$ is bounded dissipative in X. The same functional argument as in the part 2 of the proof of Theorem 4.31 implies that $S(t)$ has a compact global attractor $\mathcal{A}$ in X (for further details, see [Ka95], [Fe95], [GR00] and [Ra00]; see also [Lo] for earlier results). In the case when f satisfies the additional condition (A.2), Kapitanski [Ka95] had proved the

existence of the compact global attractor $\mathcal{A}$ in X by using a splitting method similar to the one used above and by showing that $\mathcal{A}$ is bounded in a more regular space than X. A "bootstrap" argument finally implies that $\mathcal{A}$ is bounded in X_2. In the case of $\Omega = \mathbf{R}^n$, the existence of a compact global attractor in the case $2 < (n-2)\alpha < 4$ had been proved by Feireisl [Fe95].

Properties of the compact attractor $\mathcal{A}$.

We have seen in Section 3.3 that, under additional conditions, the restriction of the flow to the compact global attractor $\mathcal{A}$ is a more regular function of the time variable. In the case of the damped wave equation, Theorem 3.18 and Theorem 3.19 apply. Indeed, **H1**, **H2** and **H3** are easily proved and the boundedness of $\mathcal{A}$ in $H^{s+1}(\Omega) \times H^s(\Omega)$ for some $s > 0$ implies the compactness condition **H5**. From Theorem 3.20, we deduce that the elements in $\mathcal{A}$ are more regular functions of the spatial variable x (see [HR00] for further details). We recall that, in the case of a smooth domain, such regularity results had been proved by Ghidaglia and Temam [GT87a], when $(n-2)\alpha < 2$. Gevrey regularity results for the orbits contained in $\mathcal{A}$ in the case of periodic boundary conditions are also given in [HR00]. For sake of simplicity, we assume in the next theorem that $n = 1, 2, 3$ (for details, see [HR00]).

Theorem 4.33. *Assume that the conditions (A.1), (A.2) and (A.3) hold and that Ω is a bounded domain of class $\mathcal{C}^{1,1}$ in $\mathbf{R}^n$, $n = 1, 2, 3$.*
1. If $f \in \mathcal{C}^k_{bu}(\mathbf{R}, \mathbf{R})$, $k \geq 1$ and $f^{(k)}$ is locally Hölder continuous (resp. $f : \mathbf{R} \to \mathbf{R}$ is a real analytic function), then, for any $U_0 \in \mathcal{A}$, $t \in \mathbf{R} \mapsto S(t)U_0 \in X$ is of class $\mathcal{C}^k$ (resp. analytic).
2. If moreover Ω is of class $\mathcal{C}^{k-1,1}$ and $g \in H^{k-1}(\Omega)$, then $\mathcal{A}$ is bounded in $(H^{k+1}(\Omega) \cap V) \times H^k(\Omega)$.

Since the assumptions **H1**, **H2**, **H3** and **H4** hold, Theorem 3.17 implies that the equation (4.51) has the property of finite number of determining modes.

Generalizing the results of [CFT85] to the non compact case, Ghidaglia and Temam have shown that, under the hypotheses (A.1), with $(n-2)\alpha < 2$, (A.2) and (A.3), the global attractor $\mathcal{A}$ of (4.51) has finite fractal dimension. If $(n-2)\alpha = 2$ or if $2 < (n-2)\alpha < 4$ in the periodic case, the same type of proof shows that $\mathcal{A}$ has finite fractal dimension.

In what follows, we assume that the three assumptions (A.1), (A.2) and (A.3) hold.

We remark that $(e, 0)$ is a hyperbolic equilibrium point of (4.51) if and only if e is a hyperbolic equilibrium point of (4.27) and that $\mathrm{ind}((e, 0)) = \mathrm{ind}(e)$. Indeed, if $l_j(e)$, $j \geq 1$, denote the eigenvalues of the operator $-A + Df(e)$, then the eigenvalues of the operator $B + (Df(e))^* \in L(X, X)$ are given by

$$\begin{aligned} \mu_j^{\pm} &= \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 + 4l_j(e)}\right) \ , \quad \text{if } \gamma^2 + 4l_j(e) \geq 0 \ , \\ \mu_j^{\pm} &= \frac{1}{2}\left(-\gamma \pm i\sqrt{|\gamma^2 + 4l_j(e)|}\right) \ , \quad \text{if } \gamma^2 + 4l_j(e) < 0 \ . \end{aligned} \tag{4.66}$$

Thus, if all the equilibrium points $(e, 0)$ are hyperbolic, Theorem 4.6 and Theorem 4.8 imply that

$$\mathcal{A} = \bigcup_{e_j \in \mathcal{E}_P} W^u(e_j) \ ,$$

and that the Hausdorff dimension $\dim_H(\mathcal{A})$ is equal to $\max_{e \in \mathcal{E}_P} \mathrm{ind}(e)$. Moreover, the unstable and stable manifolds $W^u((e, 0), S(t))$ and $W^s((e, 0), S(t))$ are embedded $\mathcal{C}^1$-submanifolds of X of dimension $\mathrm{ind}(e)$ and codimension $\mathrm{ind}(e)$, respectively. In general, one does not know if the stable and unstable manifolds intersect transversally, even when Ω is an interval of $\mathbf{R}$. As in the parabolic case, we can replace the function $f(\cdot)$ by a function $f(x, \cdot)$ depending on the spatial variable $x \in \Omega$. If one assumes that the conditions (A.1), (A.2) and (A.3) hold uniformly in x, then the semigroup $S(t)$ still admits a compact global attractor $\mathcal{A}$. Generalizing the proof of [BrP97a] to the damped wave equation, one shows that (4.51) is a Morse-Smale system, generically in the pair of parameters $(\gamma, f(x, \cdot))$ (see [BrR]).

Unfortunately, even in the one-dimensional case, the orbit structure on $\mathcal{A}$ is not really known. Indeed, unlike the parabolic case, arguments using the zero number are not applicable. At this time, no good tools seem to be available. Till now, we do not know, for instance, if $\mathcal{A}$ can be written as a graph, nor if it is contained in a Lipschitzian manifold. Moreover, one does not expect that the stable and unstable manifolds intersect transversally for all the values of γ. However, if $\Omega \subset \mathbf{R}$, we deduce from the Sturm Liouville theorem and from (4.66) that all the eigenvalues of $B + (Df(e))^*$, $e \in \mathcal{E}_P$, are simple, which, together with Remarks 4.3, implies that all the orbits of (4.51) are convergent ([HR92b]). Furthermore, if $n = 1$ and $f(u) = \mu^2(au - bu^3)$ for example, the bifurcation diagram for the global attractor $\mathcal{A}_\mu$ with respect to the parameter μ is essentially the same as the one given by Chafee-Infante for the corresponding parabolic equation ([Web79b]). In the case $n \geq 1$, all the orbits of (4.51) are still convergent if $f : \mathbf{R} \to \mathbf{R}$ is an analytic function ([HJ]).

However, one expects that the flow on the global attractor $\mathcal{A}$ for $\gamma > 0$ very large is equivalent to the flow on the global attractor of the corresponding parabolic equation, when this system is Morse-Smale. This is the case, indeed. To prove it, it is easier to consider the rescaled wave equation

$$\begin{aligned} \varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2}(x,t) + \frac{\partial u^\varepsilon}{\partial t}(x,t) \,&=\, \Delta u^\varepsilon(x,t) + f(u^\varepsilon(x,t)) + g(x) \;, \quad x \in \Omega \;, \quad t > 0 \;, \\ u^\varepsilon(x,t) \,&=\, 0 \;, \quad x \in \partial\Omega \;, \quad t > 0 \;, \\ u^\varepsilon(x,0) \,&=\, u_0(x) \;, \quad \frac{\partial u^\varepsilon}{\partial t}(x,0) \,=\, v_0(x) \;, \quad x \in \Omega \;, \end{aligned} \tag{4.67}$$

where $\varepsilon = \gamma^{-2} > 0$. The formal limit of (4.67) is the parabolic equation (4.25). Hereafter, we denote by $S_\varepsilon(t)$ the continuous semigroup generated on X by (4.67) and by $\mathcal{A}_\varepsilon$ the global attractor of $S_\varepsilon(t)$. We let $S_P(t)$ be the semigroup on V, defined by the equation (4.25) and denote by $\mathcal{A}_P$ the global attractor of $S_P(t)$ on V. To compare the attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_P$, we introduce the set $\mathcal{A}_0 = \{(u,v) \in X \mid u \in \mathcal{A}_P \,,\, v = \Delta u + f(u) + g\}$, which is bounded in X. If Ω is either convex or of class $\mathcal{C}^{1,1}$, $\mathcal{A}_0$ is also bounded in X_2.

Beginnning with the papers of Zlamal ([Zl59], [Zl60]) on the telegrapher's equation $\varepsilon u^\varepsilon_{tt} + u^\varepsilon_t - u^\varepsilon_{xx} = 0$, the dependence in ε of (4.67) has been extensively analysed (see [BV87], [HR88], [HR90], [MSM], [Ko90],[Wit] and [Ra99]).
At first glance, (4.67) appears as a singular perturbation of the equation (4.25). Actually, it is not the case, if we compare adequate time-τ maps instead of comparing the continuous semigroups. For any $(u_0, v_0) \in X$, we write the solution $u^\varepsilon(t)$ of (4.67) as $u^\varepsilon = u^\varepsilon_1 + u^\varepsilon_2$, where u^ε_1 and u^ε_2 are the solutions of

$$\begin{aligned} \varepsilon u^\varepsilon_{1tt} + u^\varepsilon_{1t} + A u^\varepsilon_1 \,&=\, 0 \;, \quad (u^\varepsilon_1(0), u^\varepsilon_{1t})(0) \,=\, (0, v_0) \;, \\ \varepsilon u^\varepsilon_{2tt} + u^\varepsilon_{2t} + A u^\varepsilon_2 \,&=\, f(u^\varepsilon) + g \;, \quad (u^\varepsilon_2(0), u^\varepsilon_{2t})(0) \,=\, (u_0, 0) \;. \end{aligned} \tag{4.68}$$

Using a priori estimates on $(u^\varepsilon_1, u^\varepsilon_{1t})(t)$ and comparing $(u^\varepsilon_2, u^\varepsilon_{2t})(t)$ with $(u(t), u_t(t))$, where $u(t) = S_P(t)u_0$, we obtain the following result [MR], [Ra99]:

Lemma 4.34. *There exist a positive constant ε_0 and, for any $r > 0$, a positive number $C(r)$, such that, for $0 \leq \varepsilon \leq \varepsilon_0$ and, for any $(u_0, v_0) \in X$, satisfying $\|u_0\|^2_{H^1(\Omega)} + \varepsilon \|v_0\|^2_{L^2(\Omega)} \leq r^2$, we have, for $t \geq 0$,*

$$\varepsilon \| \frac{\mathrm{d}}{\mathrm{d}t}(tu^\varepsilon(t) - tu(t))\|^2_{L^2} + \|t(u^\varepsilon - u)(t))\|^2_{H^1} \,\leq\, C(r)\varepsilon^2(1 + \|(u_0,v_0)\|^2_X) \exp C(r)t \;,$$

where u^ε and u are the solutions of (4.67) and (4.25) respectively.

Similar estimates hold for the linearized semigroups $DS_\varepsilon(t)$ and $DS_0(t)$. Lemma 4.34 leads to define the semigroup $S_0(t)$ on X by

$$S_0(t)(u_0,v_0) = (S_P(t)u_0, \frac{\mathrm{d}}{\mathrm{dt}}(S_P(t)u_0)) \ , \quad t>0 \ , \quad S_0(0)(u_0,v_0) = (u_0,v_0) \ . \quad (4.69)$$

Due to the smoothing properties of the parabolic equation (4.25), $S_0 \in \mathcal{C}^0((0,+\infty)\times X, X)$ and, for $t \geq 0$, $S_0(t) \in \mathcal{C}^1(X,X)$. Clearly, $S_0(t)$ is a gradient system with Lyapunov functional $\Phi_0(u,v) = \int_\Omega (1/2|A^{1/2}u|^2 - F(u) - g(x)u)\mathrm{d}x$ and $\mathcal{A}_0$ is the global attractor of $S_0(t)$. For $\tau > 0$ a fixed number, we introduce the C^1-mapping $S_\varepsilon = S_\varepsilon(\tau)$ from X into X, for $\varepsilon \geq 0$. Lemma 4.34 and its analogue for the linearized semigroups $DS_\varepsilon(t)$ and $DS_0(t)$ imply that S_ε converges to S_0 in a C^1-sense, when ε goes to 0. In particular, there exists a positive constant $C_0(r,\tau)$ depending only on r and τ so that, if $\|(u_0,v_0)\|_X \leq r$,

$$\|S_\varepsilon(u_0,v_0) - S_0(u_0,v_0)\|_X + \|DS_\varepsilon(u_0,v_0) - DS_0(u_0,v_0)\|_{L(X,X)} \ \leq \ C_0(r,\tau)\varepsilon^{1/2} \ . \quad (4.70)$$

As a direct consequence of (4.70), Theorem 4.11 and Theorem 4.12, we obtain the next result.

Theorem 4.35. (i) *The global attractors $\mathcal{A}_\varepsilon$ are upper semicontinuous at $\varepsilon = 0$.*
(ii) *If all the equilibrium points e of (4.25) are hyperbolic, the global attractors $\mathcal{A}_\varepsilon$ are lower semicontinuous at $\varepsilon = 0$ and there exist positive constants ε_1, C and $\kappa \leq 1/2$, such that, for $0 \leq \varepsilon \leq \varepsilon_1$,*

$$\delta_X(\mathcal{A}_0, \mathcal{A}_\varepsilon) + \delta_X(\mathcal{A}_\varepsilon, \mathcal{A}_0) \ \leq \ C\varepsilon^\kappa \ .$$

(iii) *Assume that the continuous semigroup $S_P(t)$ is a Morse-Smale system. Then, there exist positive numbers τ and ε_2, such that, for $0 \leq \varepsilon \leq \varepsilon_2$, $S_\varepsilon(t)$ is a Morse-Smale system and there is a homeomorphism $h_\varepsilon : \mathcal{A}_0 \to \mathcal{A}_\varepsilon$ satisfying the conjugacy condition $h_\varepsilon \circ S_0(\tau) = S_\varepsilon(\tau) \circ h_\varepsilon$, for any $\varepsilon > 0$.*

This example illustrates well the relevance of replacing the comparison of continuous semigroups by the one of maps.

For more details, we refer to [Ra99], [Ra00] and [MR]. Assertions (i) and (ii) had been proved earlier in [HR88] and [HR90] respectively (see also [Ko90]). Using the assertion (iii) in the case $n = 1$, we obtain, for ε small enough, the same orbit structure on $\mathcal{A}_\varepsilon$ as in the parabolic case. In the case $n = 1$, Mora and Solà Morales [MSM] had proved that, for ε small enough, the semigroup $S_\varepsilon(t)$ admits an inertial manifold and that this inertial manifold converges, in a C^1-sense, to the one of $S_0(t)$, when ε goes to 0, which reduces the comparison of (4.67) and (4.25) to a finite-dimensional perturbation problem. This result can also be deduced directly from the general theorems of $\mathcal{C}^1$-dependence of inertial manifolds with respect to parameters.

Remarks 4.36.

1. All the above assertions (except Remark 4.32) remain true if we replace the homogeneous Dirichlet boundary conditions in (4.50) by homogeneous Neumann boundary conditions. In this case, $V = H^1(\Omega)$. With some small changes, these assertions also hold even if we consider more general boundary conditions.
2. If we replace the equation (4.50) by a system of m damped wave equations, that is not necessarily gradient, one can still show the existence of a compact global attractor in $(V \times L^2(\Omega))^m$, under adequate dissipative hypotheses on the non linearity f. In this case, one shows directly that the associated semigroup is bounded dissipative (see [Hal88] and [Te]).
3. If, in (4.50), one replaces $\gamma \frac{\mathrm{d}u}{\mathrm{d}t}$ by $\gamma(-\Delta + Id)\frac{\mathrm{d}u}{\mathrm{d}t}$, one obtains the so called strongly damped wave equation. The linear operator $B : (u, v) \in \mathcal{D}(A) \times \mathcal{D}(A) \mapsto (v, -Au - Av) \in X$ generates an analytic semigroup on X. Under the conditions (A.1) and (A.3), one shows that the corresponding nonlinear semigroup $S(t)$ can be written in the form (2.16) and has a compact global attractor in X (see [Web80], [Fit81], [Ma83b], [Hal88]). In the one-dimensional case, all the orbits are convergent [HR92b].

The equation with a variable non negative damping.

One can now wonder what happens if the damping term γu_t in (4.50) is replaced by a function $h(u_t)$ or more generally by $\gamma(x)h(u_t)$, where $\gamma(x)$ is a non negative function on the spatial variable x. The existence of a compact global attractor $\mathcal{A}$ in X has been proved by Ceron and Lopes [CeLo], under the assumptions (A.1), with $(n-2)\alpha < 2$, and (A.3), in the case where $\gamma > 0$ is a constant and $h \in \mathcal{C}^1(\mathbf{R}, \mathbf{R})$ satisfies

$$h(0) = 0 \,, \quad 0 < a \le h'(y) \le b \,, \quad \forall y \in \mathbf{R} \,, \tag{4.71}$$

where a, b are positive constants. In their proof, they introduced the criterium (2.20) of asymptotic smoothness and applied Proposition 2.34 (for a generalization to the case

$\gamma(x)h(u_t)$ and $(n-2)\alpha \leq 2$, see [FZ]).

We now try to present the difficulties encountered when the positive constant γ is replaced by a nonnegative function $\gamma(x) \in \mathcal{C}^1(\overline{\Omega}, [0, +\infty))$, which is not identically zero on the closure of Ω. Assume that the conditions (A.1) and (A.3) are satisfied. In this case, global existence and uniqueness of mild solutions of (4.51), with γ replaced by $\gamma(x)$, still hold. As before, we denote by $S(t)$ the associated semigroup on X. If Φ is the functional introduced in (4.52) and $U \in \mathcal{C}^0([0,T], X)$ is a solution of (4.51), the equality (4.53) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(U(t)) \;=\; -\int_\Omega \gamma(x)|u_t(t,x)|^2 \,\mathrm{d}x \ , \quad \forall t \in [0,T] \ , \tag{4.72}$$

which implies, as in the case of a constant positive damping, that the orbits of bounded sets are bounded. Unfortunately, without additional conditions on $\gamma(x)$, we cannot deduce from (4.72) that Φ is a strict Lyapunov functional.
As a direct consequence of Proposition 2.39, Theorem 2.26 and the part 1 of the proof of Theorem 4.31, we obtain the following result.

Proposition 4.37. *Assume that the hypotheses (A.1), (A.3) and $(n-2)\alpha < 2$ hold. If we suppose that there exist positive constants K and θ such that*

$$\|e^{Bt}\|_{L(X,X)} \;\leq\; Ke^{-\theta t} \ , \quad \forall t \geq 0 \ , \tag{4.73}$$

then the semigroup $S(t)$ is asymptotically smooth and has a minimal global B-attractor $\mathcal{A}_X$. If, in addition, $S(t)$ is point dissipative, then $\mathcal{A}_X$ is the compact global attractor of $S(t)$.

To apply Proposition 4.37, one must first obtain conditions that will imply that the linear semigroup e^{Bt} satisfies (4.73). If $\gamma(x_0) > 0$ at some point $x_0 \in \Omega$, then each solution $e^{Bt}U_0$ approaches zero as $t \to +\infty$ (see [Iw], [Da78]). However, as remarked by Dafermos [Da78], one can construct examples, for $n \geq 2$, where the approach to zero is not uniform with respect to initial data in a ball and so (4.73) is not satisfied. Using geometric optics arguments, Bardos, Lebeau and Rauch [BLR] have shown that, if Ω and γ are of class $\mathcal{C}^\infty$, the property (4.73) holds if the following condition is satisfied:

(BLR) *There exists $\tau > 0$ such that every ray of geometric optics intersects the set* $\mathrm{s}(\gamma) \times (0,\tau)$, *where* $\mathrm{s}(\gamma)$ *is the support of* γ.

The condition (BLR) is true in particular if $\mathrm{s}(\gamma)$ is a neighbourhood of $\partial\Omega$. Condition (BLR) gives a very interesting way to verify (4.73). However, the question of characterizing, for a particular domain, the minimal conditions on the damping γ for which (BLR) holds, is not easy. If

$$I \subset \Omega \text{ are two intervals of } \mathbf{R}\ , \quad \gamma(x) \geq 0\ , \forall x \in \Omega\ , \quad \gamma(x) > 0\ , \forall x \in I\ , \tag{4.74}$$

then (4.73) holds. Other approaches to prove the property (4.73) are given in [CFNS] and [Har89], for example (see also [FZ], [Zu90] and the references therein).

It remains to derive conditions on $\gamma(x)$, which will imply that $S(t)$ is point dissipative. From (4.72), we deduce that, for any $U_0 \in X$, $\omega(U_0)$ must be a subset of the bounded complete orbits of the system

$$\begin{aligned} u_{tt}(x,t) - \Delta u(x,t) &= f(u(x,t)) + g(x)\ , \quad x \in \Omega\backslash \mathrm{s}(\gamma)\ ,\ t \in \mathbf{R}\ , \\ u_t(x,t) &= 0\ , \quad x \in \mathrm{s}(\gamma)\ ,\ t \in \mathbf{R}\ , \\ u(x,t) &= 0\ ,\ x \in \partial\Omega\ ,\ t \in \mathbf{R}\ . \end{aligned} \tag{4.75}$$

We now distinguish the cases $n = 1$ and $n \geq 2$.

If $n = 1$, using the classical representation formula of the solution of a wave equation, we show that, if the condition (4.74) holds, any bounded complete orbit of (4.75) is an equilibrium point of (4.51). Thus, in this case, Proposition 4.37 implies that (4.51) has a compact global attractor ([HR93a]). Moreover, one shows that the orbits of (4.51) are convergent (see [HR93a] and [Ra95] for further examples of convergence in locally damped wave equations).

If $n \geq 2$, we remark that, for any bounded complete orbit $(u(t), u_t(t))$ of (4.75), $Df(u(t))$ belongs to $\mathcal{C}^0_b(\mathbf{R}, L^n(\Omega))$ and $w = u_t \in \mathcal{C}^1_b(\mathbf{R}, V') \cap \mathcal{C}^0(\mathbf{R}, L^2(\Omega))$ is a solution of the system

$$\begin{aligned} w_{tt}(x,t) - \Delta w(x,t) - Df(u(x,t))w(x,t) &= 0\ , \quad x \in \Omega\ ,\ t \in \mathbf{R}\ , \\ w(x,t) = w_t(x,t) &= 0\ , \quad x \in \mathrm{s}(\gamma)\ ,\ t \in \mathbf{R}\ . \end{aligned} \tag{4.76}$$

If the only solution $w(t) \in \mathcal{C}^1_b(\mathbf{R}, V') \cap \mathcal{C}^0(\mathbf{R}, L^2(\Omega))$ of (4.76) is $w = 0$, then the ω-limit set of any solution of (4.51) is an equilibrium point. We are thus led to the following unique continuation property **(u.c.p.)**:

(u.c.p.) Assume that w is a weak $L^2(\Omega \times (0,T))$ solution of the equation

$$w_{tt} - \Delta w + b(x,t)w \;=\; 0 \text{ in } \Omega \times (0,T)$$

where $T > \operatorname{diam}\Omega$ and $b \in L^\infty((0,T), L^n(\Omega))$.

Then, if w vanishes in some set $O \times (0,T)$, $O \subset \Omega$, w must be identically zero.

Ruiz [Ru] has shown that the (u.c.p.) property holds when O is a neighbourhood of $\partial\Omega$.

It follows from the above discussion and from Proposition 4.37 that, if $\partial\Omega \subset \mathrm{s}(\gamma)$ and $(n-2)\alpha < 2$, then the semigroup $S(t)$ has a compact global attractor in X. Feireisl and Zuazua [FZ] have generalized this existence result to the critical case $(n-2)\alpha = 2$. In their proof, they have used energy functionals arguments to show that $S(t)$ is bounded dissipative and the same splitting as in Part 3 of the proof of Theorem 4.31 to show that $S(t)$ is asymptotically smooth. We thus can state the following result (see [HR00] for the regularity results):

Theorem 4.38. *Assume that Ω is a bounded regular domain in $\mathbf{R}^n$. If the hypotheses (A.1), (A.2), (A.3) are satisfied and if, either the conditions $n = 1$ and (4.74) hold, or $\mathrm{s}(\gamma)$ is a neighbourhood of $\partial\Omega$, then (4.51) has a connected compact global attractor $\mathcal{A} = W^u(\mathcal{E}_P \times \{0\})$ in X. Moreover, the time and spatial regularity properties of the complete orbits in $\mathcal{A}$, given in Theorem 4.33, still hold and the property of finite number of determining modes remains true.*

Finally, we note that many other wellknown dissipative gradient systems, having a compact global attractor, could have been approached. Among them, we quote the strongly damped wave equation (see Remarks 4.36), the Cahn-Hilliard equation (see [Te] and the references therein), nonlinear diffusion systems ([Hal88]) etc ….

5. Further topics

So far, we have mainly studied equations, which have a gradient structure. The most famous and most studied non gradient dissipative system arising in PDE's is certainly the one generated by the Navier-Stokes equations on a bounded domain in space dimension two or three. As already shown by Ladyzenskaya in 1972 ([La72], [La73]), in space dimension two, this equation has a compact global attractor, which is of finite fractal dimension ([MP76], [FT79], [La82]). The associated semi-flow is a smooth function of the time variable for $t > 0$ (up to analyticity) and the global attractor

is composed of smooth functions in the spatial variable (see [FT79], [FT89], [FeTi98], [Te] and the references therein). Estimates of the fractal and Hausdorff dimensions of the attractor in terms of various physical parameters have been extensively studied (see [BV83], [CF85], [CFT85], [La87a], [EFT], [JTi93]). In the two-dimensional case, the Navier-Stokes equations have also the property of finite number of determining modes (see [FP67], [La72], [La87b], [CJTi97]). For further details and study on the Navier-Stokes equations, we refer to [BN] in this volume.
Among the wellknown evolutionary partial differential equations, which have smoothing properties in finite positive time and admit a compact global attractor, we should also mention the one-dimensional Kuramoto-Sivashinsky equation (see [NST], [CEES] for example), modeling pattern formation in thermohydraulics and also the propagation of a front flame, as well as the complex Ginzburg-Landau equation in space dimensions one or two, describing the finite amplitude evolution of instability waves. The complex Ginzburg-Landau equation is actually a strongly damped Schrödinger equation; in space dimensions one or two, it admits a compact global attractor of finite fractal dimension ([GhHe], [Te]).

To conclude this paper, we present the weakly damped Schrödinger equation, which is a system generated by a dispersive equation with weak damping.

A weakly damped Schrödinger equation.

In what follows, Ω denotes, either the whole space $\mathbf{R}^n$, $n = 1, 2$ or 3 or a bounded $\mathcal{C}^2$-polygonal domain in $\mathbf{R}^n$, when $n = 1, 2$. For $\gamma > 0$ a fixed constant, f a function in $L^2(\Omega)$ and $g \in C^1([0, +\infty), \mathbf{R})$ a function satisfying the hypotheses (H.1) and (H.2) below, we consider the weakly damped Schrödinger equation, which arises in plasma physics or in optical fibers models (see [NB], for instance):

$$\begin{aligned} iu_t + \Delta u + g(|u|^2)u + i\gamma u \;&=\; f\ , \quad \text{in } \Omega \times (0, +\infty)\ , \\ u(0) \;&=\; u_0\ , \quad \text{in } \Omega \end{aligned} \quad . \tag{5.1}$$

If $\Omega \neq \mathbf{R}^n$, we associate homogeneous Dirichlet boundary conditions to (5.1)

$$u \;=\; 0\ , \quad \text{on } \partial\Omega\ . \tag{5.2}$$

Of course, we could consider homogeneous Neumann boundary conditions or periodic conditions as well. We assume that $g \in C^1([0, +\infty), \mathbf{R})$ and $G(y) = \int_0^y g(s)\, ds$ satisfy the following conditions:

(H.1) there exist two constants $C_1 > 0$ and $\alpha_1 \in [0, 2/n)$ such that

$$\begin{aligned} G(y) &\leq C_1 y(1 + y^{\alpha_1}) \ , \quad y \geq 0 \ , \\ yg(y) - G(y) &\leq C_1 y(1 + y^{\alpha_1}) \ , \quad y \geq 0 \ , \end{aligned} \tag{5.3}$$

(H.2) in the case when $n = 2$ or 3, there exist two constants $C_2 > 0$ and $\alpha_2 \geq 0$, with $(n-2)\alpha_2 < 4$, such that, for any $(\xi, \xi') \in \mathbf{C}^2$,

$$|g(|\xi|^2)\xi - g(|\xi'|^2)\xi'| \leq C_2(1 + |\xi|^{\alpha_2} + |\xi'|^{\alpha_2})|\xi - \xi'| \ . \tag{5.4}$$

(H.3) in the case when Ω is a bounded subset of $\mathbf{R}^2$, there exists a positive constant C_3 such that

$$|g'(y)| \leq C_3 \ , \quad y \geq 0 \ . \tag{5.5}$$

Later we shall also impose the next additional conditions on g, which mainly require that the nonlinearity is subcritical:

(H.4) The function g is in $C^\infty([0, +\infty), \mathbf{R})$ and there exist two constants $C_4 > 0$ and $\alpha_1 \in (0, 2/n)$ such that

$$y|g'(y)| + |g(y)| \leq C_4 y^{\alpha_1} \ , \forall y \geq 0 \ , \tag{5.6}$$

Moreover, for $k \geq 2$, the derivatives $g^{(k)}$ are bounded.

As before, we denote by $A = -\Delta_{BC} u$ the unbounded operator on $H = L^2(\Omega)$, where Δ_{BC} is the Laplace operator with the corresponding boundary conditions. We set $V^2 = D(A)$, $V = D(A^{1/2})$ and we denote by V' the dual space of V.

Proposition 5.1. *Under the assumptions (H.1), (H.2) and (H.3), for any $u_0 \in V$, there exists a unique solution $u(t) \in C^0([0, +\infty), V)$ of (5.1) and (5.2). Moreover, $u(t) \in C^1([0, +\infty), V')$ and the mapping $S(t)u_0 = u(t)$ defines a continuous semigroup on V. If $u_0 \in V^2$, then $u(t)$ belongs to $C^0([0, +\infty), V^2) \cap C^1([0, +\infty), H)$.*
Furthermore, for any $t \geq 0$, the mapping $S(t)$ is continuous on the bounded sets of V for the topology of H.

Proof. In the case where $\Omega = \mathbf{R}^n$, mutatis mutandis, we can follow the proofs of [CH, Theorem 7.4.1 and Proposition 7.5.1]. These proofs use the well known Strichartz inequalities. The above results are shown in [Gh88a], when Ω is a bounded domain in $\mathbf{R}$.

For Ω a bounded domain in $\mathbf{R}^2$, the proofs can be found in [Ab1] (see also [Ab2]). The proof of the uniqueness of the solution in V is more delicate than in dimension 1 and requires the hypothesis (H.3). Indeed, in the case of a bounded domain, estimates similar to Strichartz inequalities are not yet known. The continuity of $S(t)$ on the bounded sets of V for the topology of H can be shown by arguing as in [CH, Proposition 7.4.2]. $\square$

Uniqueness of solutions of (5.1) is not known, when Ω is a bounded domain in $\mathbf{R}^3$. In [Gh88a], it was first proved that, if Ω is a bounded domain in $\mathbf{R}$, then $S(t)$ has a global weak attractor $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) in V (resp. $D(A)$) (see Remark 2.30). Using the functionals given below and applying Proposition 2.35, Abounouh (resp. Laurençot) have showed the existence of a compact global attractor $\mathcal{A}$ in V, when Ω is a bounded domain of $\mathbf{R}^2$ (resp. $\Omega = \mathbf{R}^n$).

We introduce the functionals Φ, Φ_0 and Ψ defined on V by

$$\begin{aligned}
\Phi(v) &= \|\nabla v\|_H^2 + \int_\Omega (-G(|v|^2) + 2\mathrm{Re}\,(f\overline{v}))dx \equiv \|\nabla v\|_H^2 + \Phi_0(v) \;, \\
\Psi(v) &= \int_\Omega (g(|v|^2)|v|^2 - G(|v|^2) + \mathrm{Re}\,(f\overline{v}))dx \;.
\end{aligned} \tag{5.7}$$

Obviously, due to the hypothesis (H.2), the functionals Φ_0 and Ψ are continuous on the bounded sets of V for the topology of H. Taking successively the inner product of (5.1) with $\overline{u}$ and $\overline{u}_t$, one shows (see [Ab2], [Gh88a] and [Lau]) that, for any $u_0 \in H^2(\Omega) \cap V$, $S(t)u_0 = u(t)$ satisfies, for $t \geq 0$,

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_H^2 + 2\gamma\|u(t)\|_H^2 &= 2\int_\Omega \mathrm{Im}\,(f\overline{u}(x,t))dx \;, \\
\frac{d}{dt}\Phi(u(t)) + 2\gamma\Phi(u(t)) &= 2\gamma\Psi(u(t))
\end{aligned} \quad . \tag{5.8}$$

which implies that, for $t \geq 0$,

$$\Phi(S(t)u_0) = \exp(-2\gamma t)\Phi(u_0) + 2\gamma \int_0^t e^{2\gamma(s-t)}\Psi(S(s)u_0)ds \;, \quad \forall u_0 \in V \;. \tag{5.9}$$

From the hypotheses (H.1), (H.2) and from the equalities (5.8), one deduces that $r > 0$ can be chosen so that the ball $B_V(0,r)$ is positively invariant under $S(t)$ and is an absorbing set for the semigroup $S(t)$. The above properties lead to the following result.

Theorem 5.2.

1. *Under the hypotheses (H.1), (H.2) and (H.3), the semigroup $S(t)$ has a connected,*

compact global attractor $\mathcal{A}$ in V.
2. Suppose that the condition (H.4) holds. If Ω is either the whole space $\mathbf{R}^n$, $n = 1, 2$ or a bounded interval of $\mathbf{R}$, the global attractor $\mathcal{A}$ of (5.1) and (5.2) is compact in $H^2(\Omega)$. Moreover, in the one-dimensional case, for any $u_0 \in \mathcal{A}$, the mapping $t \in \mathbf{R} \mapsto S(t)u_0$ is of class $\mathcal{C}^k$, for any $k \geq 0$ (resp. analytic if g is analytic). If in addition, $f \in H^k(\Omega)$, then $\mathcal{A}$ is bounded in $H^{k+2}(\Omega)$.

Proof. 1. The first statement is a direct consequence of Theorem 2.26, Proposition 2.20, if we show that $S(t)$ is asymptotically smooth in V. Since the mapping $S(t)$, for $t \geq 0$, and the functionals Φ_0 and Ψ are continuous on the bounded sets of V for the topology of H, we apply Proposition 2.35 with $X = V$, $Y = H$, $\mathcal{F}_0 = \Phi_0$, $\mathcal{F}_1(v) = 2\gamma\Psi(v) + 2\int_\Omega \mathrm{Im}\,(f\overline{v})dx$. If Ω is a bounded domain in $\mathbf{R}^n$, $n = 1, 2$, the condition (ii) of Proposition 2.35 is clearly satisfied, since the Sobolev imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. When $\Omega = \mathbf{R}^n$, $n = 1, 2$ or 3, the condition (ii) is proved in [Lau, Lemmas 2.6 , 2.7, 2.8], by using a splitting of the solutions like in [Fe94] or [Fe95].
2. The first part of the proof does not indicate if the compact global attractor is bounded or compact in a more regular space. When the hypothesis (H.4) holds, the boundedness of $\mathcal{A}$ in $H^2(\Omega)$ is shown by Goubet in [Go96] and [Go98], by using a splitting of $S(t)u$ into a low wavenumber part $P_N(S(t)u)$ and a high wavenumber part $Q_N(S(t)u)$. The low wavenumber part is obviously smooth and the high wavenumber part can be approximated asymptotically by the solution of an equation with zero initial data. The compactness in $H^2(\Omega)$ then follows from the fact that $\mathcal{A}$ is a bounded invariant set in $H^2(\Omega)$ and thus in contained in the compact global attractor of $S(t)$ in $H^2(\Omega)$. When Ω is bounded interval of $\mathbf{R}$, the boundedness of $\mathcal{A}$ in $H^{k+2}(\Omega)$ is proved in the same way in [Go96]. The $\mathcal{C}^k$-regularity (resp. the analyticity) of the map $t \in \mathbf{R} \mapsto S(t)u_0$, for any $k \geq 0$ and any $u_0 \in \mathcal{A}$, is shown in [HR00] as a consequence of a generalized version of Theorem 3.18 and Theorem 3.19. We note that, in this proof, the hypothesis (H.4) can be slightly relaxed. $\square$

Remarks.
1. In the one-dimensional case, under a relaxed version of the hypothesis (H.4), the system generated by (5.1) and (5.2) has the property of finite number of determining modes (see [OTi], [GR00] and [HR00]).
2. In the one-dimensional case, the global attractor of the equation (5.1) with periodic boundary conditions is regular in the same Gevrey class as g and f and thus is analytic

in the spatial variable (see [OTi], [HR00]).
3. Goubet [Go99a] has also proved the compactness of the global attractor of the equation (5.1) with periodic boundary conditions in the two-dimensional case. There the proof is more involved and uses spaces introduced by Bourgain.
4. From [Gh88a, Theorem 3.2 and Remark 3.1], it follows that in the one-dimensional case, $\mathcal{A}$ has finite fractal dimension. Adapting these proofs, one can certainly show that, in the other cases considered in Theorem 5.2, $\mathcal{A}$ has also finite fractal dimension.

Finally, we notice that the existence and regularity of the compact global attractor for other weakly damped dispersive equations like the weakly damped KdV and Zakharov equations are proved by using similar methods (see [Gh88b], [GoMo], [MRW] and [Go99b]).

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