Persistence modules and barcodes in symplectic geometry

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Laboratoire de Mathématiques d'Orsay

Symplectic geometry

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ex. dH for $H : M \to \mathbb{R}$

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X_H : Hamiltonian vector field generated by H

Symplectic geometry

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- generators: critical points of f
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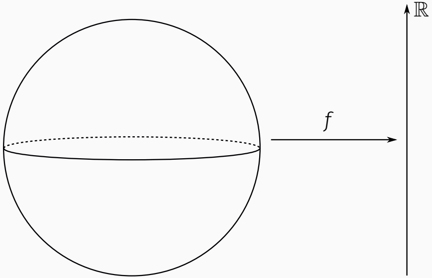

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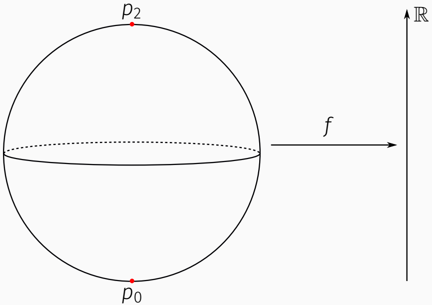

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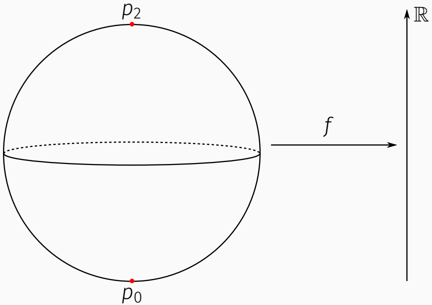

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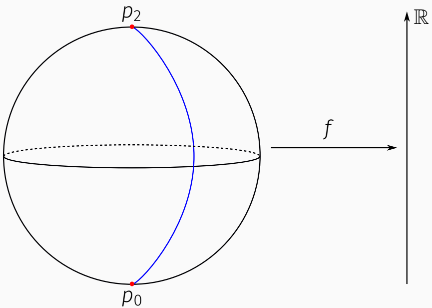

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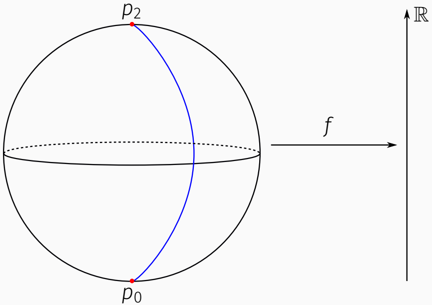

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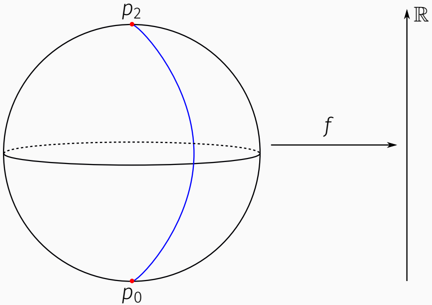

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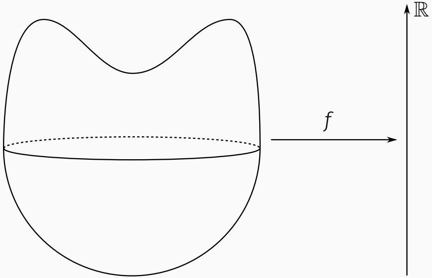

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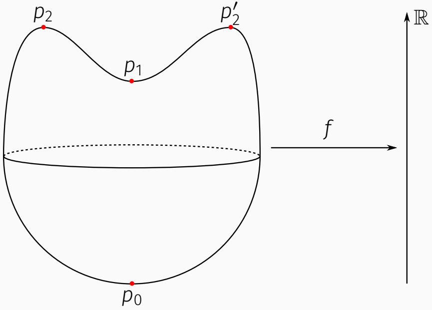

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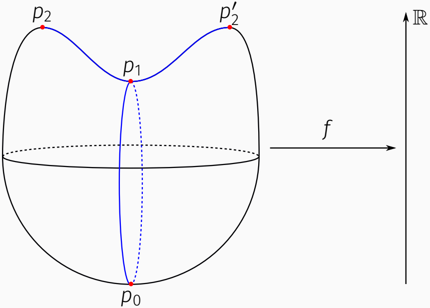

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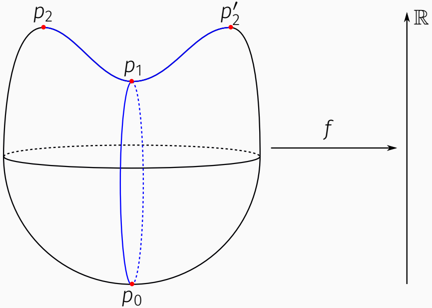

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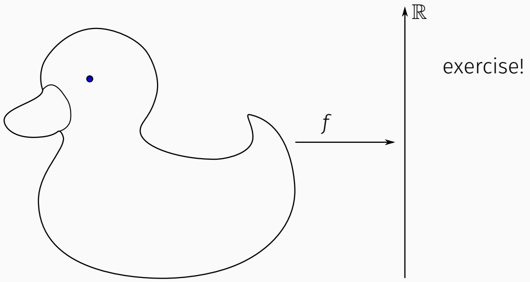

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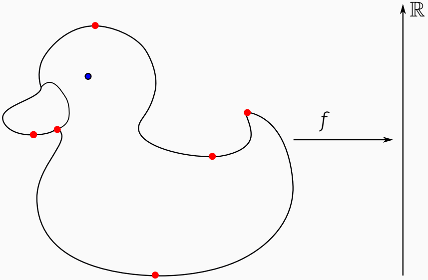

exercise!

Answer:
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$J \in \mathrm{End}(TM)$ s.t. $J^2 = -\mathrm{id}$

$\mathbb{R}^2 \simeq \mathbb{C}$: take $J = i$ and in general for $\mathbb{R}^{2n}$: take $\left(\begin{array}{c|c} 0 & -\mathrm{id} \\ \hline \mathrm{id} & 0 \end{array}\right)$

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$\rightsquigarrow$ Arnold!

Spectral invariants

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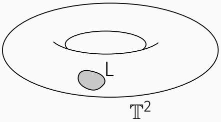

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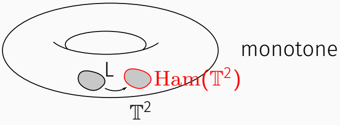

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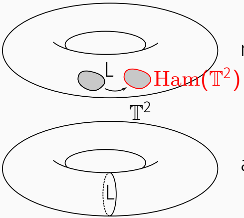

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If L is aspherical, $\mathrm{FH}_*(L) = \mathrm{H}_*(L)$.

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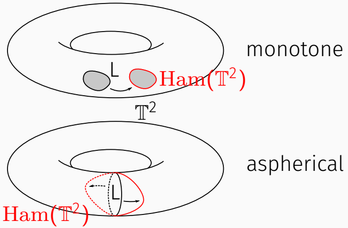
For all $\varphi \in \mathrm{Ham}(M, \omega)$, $\varphi(L) \cap L \neq \emptyset$.

Example
Embedded loop on a surface endowed with an area form



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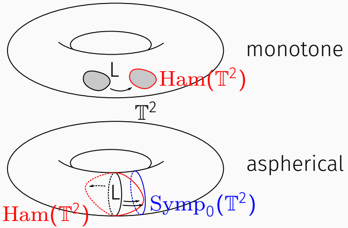

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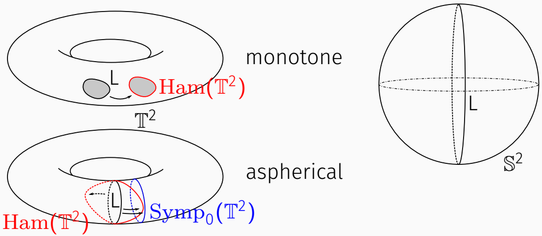

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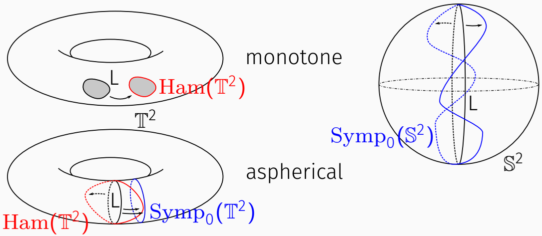

Spectral invariants: definition

Definition (Morse / finite dimension case)

Let $f\colon M \to \mathbb{R}$ be Morse and $\alpha \in \mathsf{H}_*(M)$, $\ell(\alpha, f)$ is defined by:
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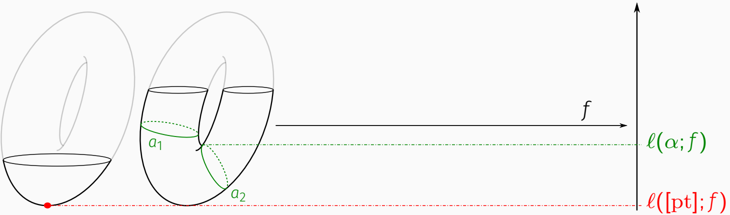

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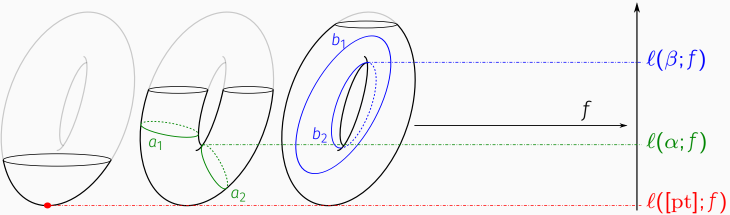

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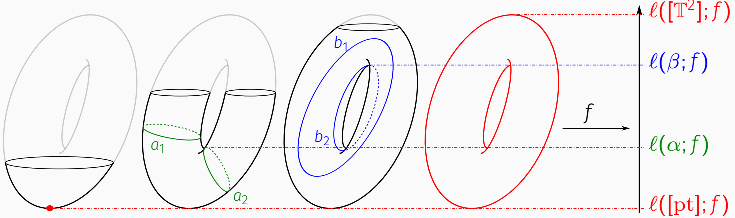

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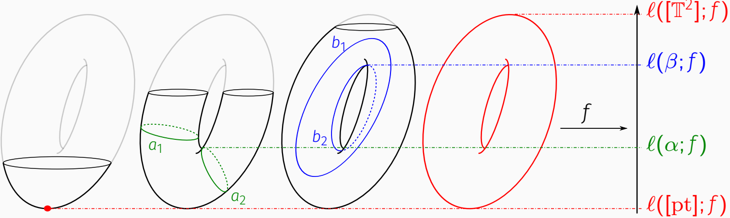

<u>Question.</u> Why am I here?

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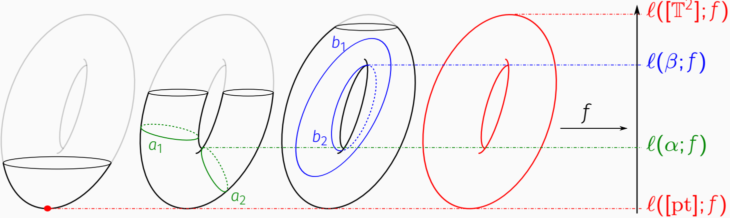

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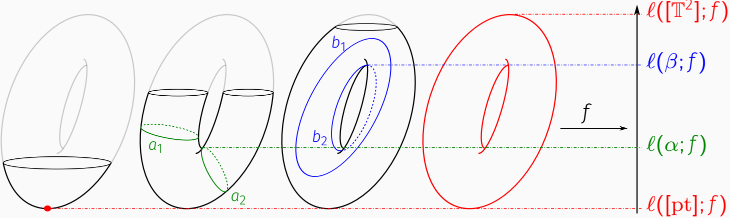

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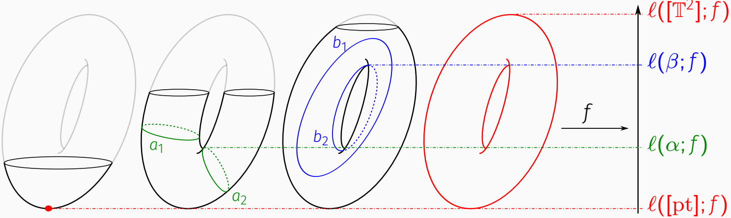

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The boundary depth $\beta(f)$ of a Morse function f is infimum of $\beta \geq 0$ s.t. for any λ:

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And .... that's all we had to play with until Dec 2014!