Spectral invariants for Lagrangian intersection Floer theory

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Brussels–Cologne seminar
on symplectic and contact geometry

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Motivation

(M, ω) symplectic manifold

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Geometry of $\mathrm{Ham}(M, \omega)$
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Geometry of $\mathrm{Ham}(M, \omega; L_0)$ (for a given L_0)
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Spectral invariants

Background

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Extensions

- to the Lagrangian intersection settings, using the PSS morphism (Schwarz' approach)
- using spectral sequences due to Barraud–Cornea, definition of spectral invariants of higher order

Morse homology

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- generators : critical points of f,
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Compactification and gluing : $\partial \circ \partial = 0$

$$HM_*(L; f, g) := H(CM_*(L; f, g), \partial) \simeq H_*(L)$$

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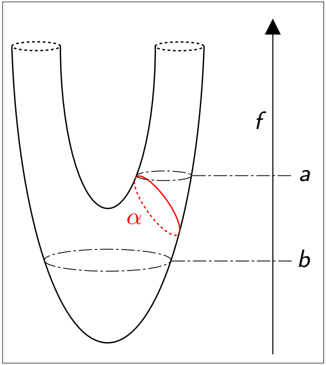

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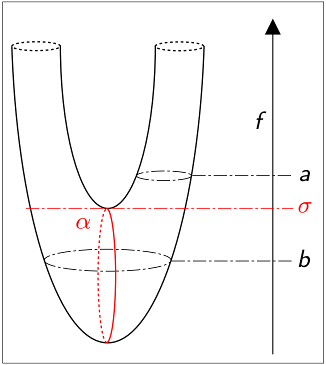


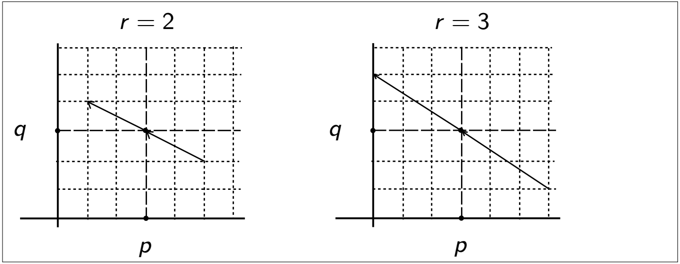
$$b \leq \sigma(\alpha) \leq a$$

$$\sigma(\alpha) = f(p_1)$$

Spectral sequence

A spectral sequence is a book

- each page is a bi-graded chain complex



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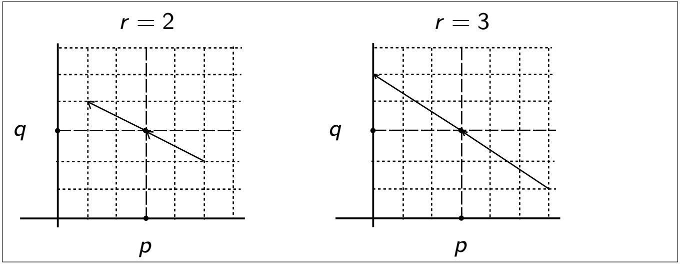

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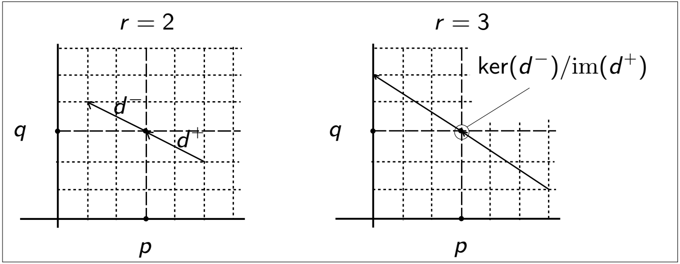
- page $r+1$ is the homology of page r



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Theorem

Any filtered chain complex induces a spectral sequence. Moreover, if the filtration is bounded, this spectral sequence converges to the homology of the initial complex.

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Theorem (Leray–Serre)

Any fibration (with connected fiber) over an arcwise, connected CW-complex induces a first quadrant spectral sequence such that

- *it converges to the homology of the total space,*
- *its second page is the homology of the base with local coefficients in the homology of the fiber.*

Barraud–Cornea spectral sequences

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- w embedded path whose image contains $\mathrm{Crit}(f)$
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Connecting manifolds seen as cubical chains

- *representing chain system* for $\overline{\mathcal{M}}(f,g)$:

$$a_{pq} \leftrightarrow \overline{\mathcal{M}}_{p,q}(f,g)$$

- cubical complex $\mathcal{R}_* := \mathcal{S}_*(\Omega'\widetilde{L})$

$\mathcal{R}_*$ is a differential module endowed with a product

Barraud–Cornea spectral sequences

Construction

The usual chain complex is enriched by the ring

$$(\mathcal{C}_*(L; f, g), d) = (\mathcal{R}_* \otimes CM_*(L; f, g), d)$$

$$d(a \otimes p) = \partial a \otimes p + \sum_q [(a \cdot a_{pq}) \otimes q]$$

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There is an obvious filtration (by the degree)

$$F_k\mathcal{C}_* = \mathcal{R}_* \otimes \langle \mathrm{Crit}_j(f) \mid j \leq k\rangle_{\mathbb{Z}_2} = \bigoplus_{j\leq k} \mathcal{R}_* \otimes CM_j(L; f, g)$$

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Definition

The spectral sequence of Barraud–Cornea *associated to L and to the Morse-Smale pair (f, g) is the spectral sequence induced by this filtration : $EM(L; f, g)$.*

Barraud–Cornea spectral sequences

Properties

Let L be simply-connected.

1. If the differential at page r is not trivial, there exist critical points p and q such that
   - $i_f(p) - i_f(q) \leq r$,
   - their connecting manifold is not empty.

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1. If the differential at page r is not trivial, there exist critical points p and q such that
   - $i_f(p) - i_f(q) \leq r$,
   - their connecting manifold is not empty.
2. $EM^r(L; f, g)$ is isomorphic to $E^r(L)$ $(r \geq 2)$. Thus
   - its second page is isomorphic to the tensor product $EM^2_{p,q}(L; f, g) \simeq H_q(\Omega L) \otimes H_p(L)$,
   - pages $r \geq 2$ do not depend on the Morse–Smale pair (f, g),
   - it converges to a trivial page.

Higher order spectral numbers

(Morse case)

homological / order 2

higher order

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Filtration of the usual complex by f: CM_*^ν

$\mathbb{Z}_2\langle p \in \mathrm{Crit} f \,|\, f(p) < \nu\rangle$

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Definition (homological spectral numbers)

For $0 \neq \alpha \in HM_*(L; f, g)$,

$$\sigma(\alpha) = \min\{\nu \mid \alpha \in \mathrm{im}(i_*^\nu)\}.$$

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Definition (higher order spectral numbers)

For $0 \neq \alpha \in EM_*^r(L; f, g)$,

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Example : $S^2 \times S^4$ with f, sum of “height” functions

$$p_6 = (\max(f_2), \max(f_4)),$$
$$p_4 = (\min(f_2), \max(f_4)),$$
$$p_2 = (\max(f_2), \min(f_4)),$$
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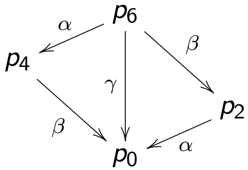

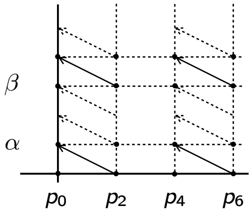

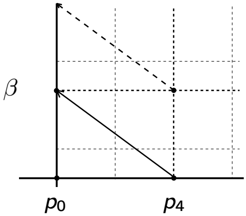

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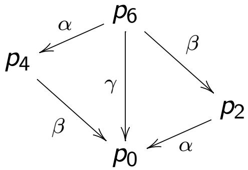

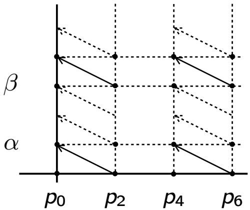

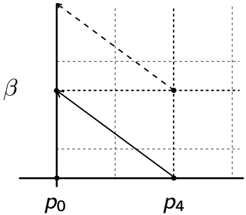

The spectral invariant associated to $\alpha \otimes p_4$ is
$$\sigma^2(\alpha \otimes p_4) = \sigma(p_4) = f(p_4).$$

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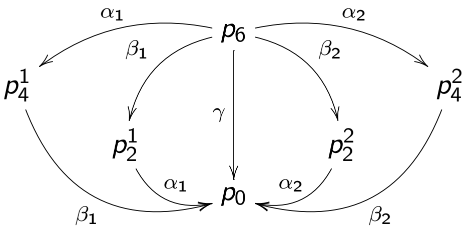
Example : $(S^2 \times S^4)_{[1]} \# (S^2 \times S^4)_{[2]}$, with $f = f_{[1]} \# f_{[2]}$



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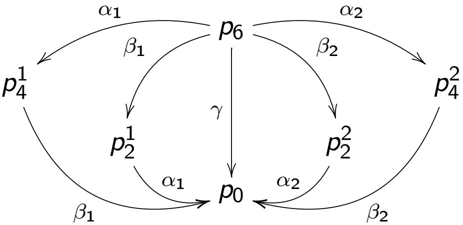

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- thus at page 3 : $[\alpha_1 \otimes p_4^1] = [\alpha_2 \otimes p_4^2] \neq 0$

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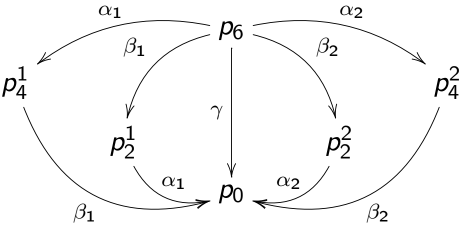

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Precise settings

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$$\mathcal{P}_\eta(L, L') = \{\gamma \in C^\infty(I, M) |\, \gamma(0) \in L,\, \gamma(1) \in L',\, [\gamma] = [\eta]\}$$

$$\mathcal{A}_{H,\eta}(\gamma) = -\int_{I\times I} \bar{\gamma}^* \omega + \int_I H(t, \gamma(t))\, dt$$

Lagrangian Floer homology

Lagrangian Floer homology

Chain complex $(CF_*(L, L'; \eta, H, J), \partial)$

- generators : orbits of the Hamiltonian vector field (in the homotopy class of η)
  $\mathrm{Crit}(\mathcal{A})$ or $\mathcal{I}(L, L'; \eta, H)$ or $\mathcal{I}(\eta, H)$,
- graduation : Maslov index $(\mu|_{\pi_2(M,L)} = 0)$,
- differential : counts Floer trajectories between orbits.

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$$HF_*(L, L'; \eta, H, J) := H(CF_*(L, L'; \eta, H, J), \partial)$$
$$\simeq HM_*(L; f, g) \qquad (L = L')$$

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$\mathbb{Z}_2\langle p \in \mathrm{Crit} f \,|\, f(p) < \nu\rangle$
$i_*^\nu : H(CM_*^\nu \to CM_*)$

Homology $H_*(L)$.

Floer theory

Filtration by $\mathcal{A}$: CF_*^ν
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Lagrangian spectral numbers

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For $0 \neq \alpha \in HM_*(L; f, g)$,

$$\sigma(\alpha) = \min\{\nu \,|\, \alpha \in \mathrm{im}(i_*^\nu)\}.$$

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They depend a priori on (f, g), (H, J), η.

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Definition ((absolute) Lagrangian spectral numbers)

For $0 \neq \alpha \in HM_*(L; f, g)$,

$$c(\alpha) = \sigma(\alpha) - \sigma(1).$$

They depend a priori on (f, g), (H, J).

Invariance of Lagrangian spectral numbers

Theorem

Lagrangian spectral numbers only depend on L and $(\phi_H^1)^{-1}(L)$. We define

$$c(\alpha; L, L') := c(\alpha; H, J) \ \textit{with} \ \phi_H^1(L') = L.$$

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Remark

So this is well-defined for two Hamiltonian isotopic, compact, transverse Lagrangians L and L'.

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Proof.

Etape 1 – Commutativity of the diagram

$$\begin{array}{ccc} HM_*(L; f, g) & \xrightarrow{\phi_f^H} & HF_*(L, L; H, J) \\ \Big\downarrow{\scriptstyle \phi_f^{H'}} & & \Big\downarrow{\scriptstyle b_H^{-1}} \\ HF_*(L, L; H', J') & \xrightarrow[b_{H'}^{-1}]{} & HF_*(L, L_0; 0, \widetilde{J}) \end{array}$$

Etape 2 – Technical lemma. □

Invariance of Lagrangian spectral numbers

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Let (H, J) and (H', J') regular such that

- $\phi_H^* J = \phi_{H'}^* J' =: \widetilde{J}$ *and*
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$(\eta_0 \in L \cap L_0,\ \eta := b_H(\eta_0)$ *et* $\eta' := b_{H'}(\eta_0))$

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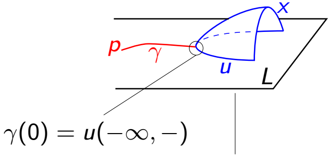

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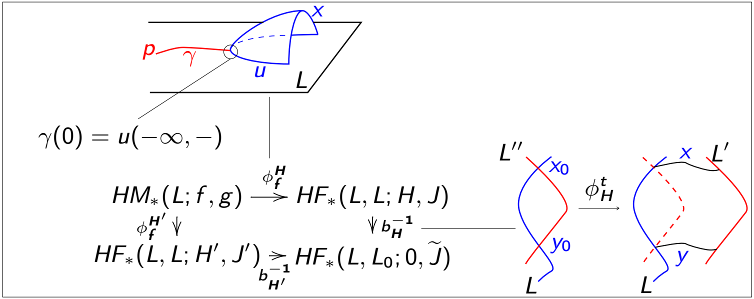

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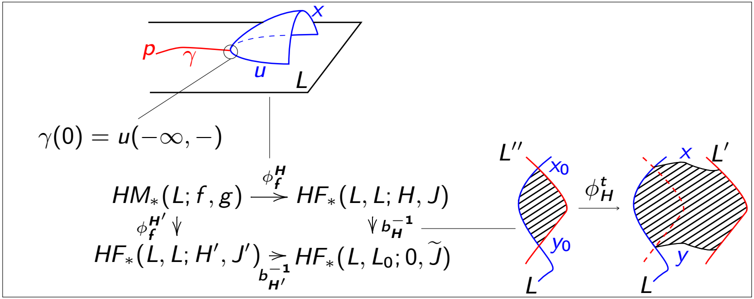

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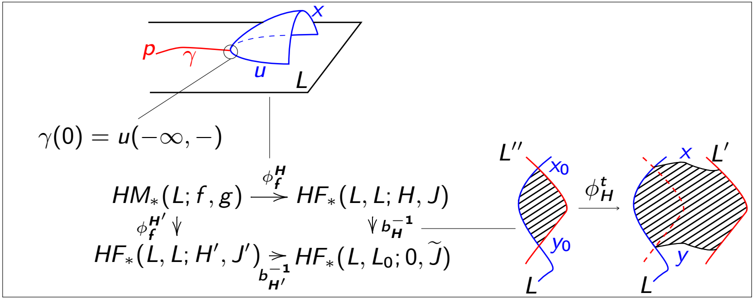

Proof.

$$\Phi = (\phi_f^{H'})^{-1} \circ b_{H'} \circ b_H^{-1} \circ \phi_f^H : HM_*(L; f, g) \to HM_*(L; f, g)$$

Algebraic structures on $HM_*(L; f, g)$ and $HF_*(L, L'; H, J)$
Φ preserve these structures, thus

$$\Phi(a) = \Phi(a \cdot [L]) = a \cdot \Phi([L]) = a \cdot [L] = a.$$

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The technical lemma

Let $E^+(H) := \int_I \sup H_t \, dt$ and $E^-(H) := \int_I \inf H_t \, dt$.

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$$E_-(H' - H) + a_{\eta,\eta'} \leq \sigma_L(\alpha; H', J', \eta') - \sigma_L(\alpha; H, J, \eta) \leq E_+(H' - H) + a_{\eta,\eta'}$$

with $a_{\eta,\eta'} := \int u^*\omega$ for any map u satisfying

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- *Independence on J of spectral numbers*

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Invariance of Lagrangian spectral numbers

Applications

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Corollary (of the invariance property)

- $|c(\alpha; L, L_0) - c(\alpha; L, L_1)| \leq \nabla(L_0, L_1)$
- *Hofer's distance for Lagrangians is non degenerate*
- *The set of Lagrangians, Hamiltonian isotopic to the "small" circle of the torus, endowed with Hofer's distance is of infinite diameter*

Spectral invariants of higher order

Definition : spectral sequence of Barraud–Cornea

Same procedure than in the Morse case

- the image of w contains $x(0)$ for any $x \in \mathcal{I}(\eta, H)$
- we consider $\widetilde{L} = L/\mathrm{im}(w)$, $\widetilde{M} = M/\mathrm{im}(w)$
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Definition

The spectral sequence of Barraud–Cornea *associated to L, L' and (H, J), $EF(L, L'; \eta, H, J)$, is the spectral sequence induced by the filtration (by the degree) of the complex $(\mathcal{C}_*(L, L'; \eta, H, J), d)$.*

Spectral sequence of Barraud–Cornea

Properties

Let L be simply-connected.

1. If the differential at page r is non trivial, there exist orbits x and y such that
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   And this isomorphism, restricted to homology classes, coincides with the PSS morphism.

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Definition

Let $\alpha \neq 0$ an element of $EM^{r}_{*,*}(L; f, g)$. Its associated relative and absolute, Lagrangian spectral numbers of higher order are

$$\sigma^{r}(\alpha; L, H, J, \eta) := \inf\{\nu \in \mathbb{R} | \Phi^{H}_{f}(\alpha) \in \mathrm{im}(i_{\nu})\}$$
$$c^{r}(\alpha; L, L') := \sigma^{r}(\alpha; L, H, J, \eta) - \sigma(1; L, H, J, \eta)$$

with 1 the generator of $HM_0(L; f, g)$ and H any Hamiltonian such that $\phi^{1}_{H}(L') = L$.

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Theorem (Floer)

There exists an identification between the complexes $CM_*(L; f, g)$ and $CF_*(L, \Gamma_{df}; 0, J_g)$.

Moreover, the "action" is preserved via this identification.

$L = (S^2 \times S^4)_{[1]} \# (S^2 \times S^4)_{[2]}$, $f_{[1]} \# f_{[2]}$ (perturbed):

$$c^3(\alpha_1 \otimes p_1; L, \Gamma_{df}) < c^2(\alpha \otimes p_1; L, \Gamma_{df})$$

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Proposition

To $0 \neq \alpha \in H_*(M)$ *corresponds* $\underline{\alpha} \in H_*(\Delta)$

$$c(\underline{\alpha}; \Delta, \Gamma_\phi) = \rho(\alpha; \phi) - \rho(1; \phi).$$

In particular, $c([\Delta]; \Delta, \Gamma_\phi) = \rho([M]; \phi) - \rho(1; \phi)$.

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Improvement of the classical bound

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There exists (Ostrover) : $\{\varphi_t\}$, $t \in [0,\infty)$ such that

$$d(\mathrm{id},\varphi_t) \to \infty \text{ for } t \to \infty \quad \text{and} \quad \nabla(\Gamma_{\mathrm{id}},\Gamma_{\varphi_t}) = c \text{ for all } t.$$

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Thus we get

$$\rho([M];\varphi_t) - \rho(1;\varphi_t) = c([\Delta];\Delta,\Gamma_{\varphi_t}) \leq \nabla(\Delta,\Gamma_{\varphi_t}) = c.$$

Main property

The geometric quantity $r(L, L')$

L and L' two compact, transverse Lagrangians and $x \in L \cap L'$. There exist $\varepsilon > 0$ and an embedding $e^x_\varepsilon : B(0,\varepsilon) \to M$ such that

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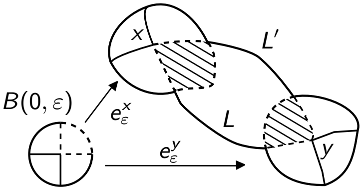

Main property

The geometric quantity $r(L, L')$

L and L' two compact, transverse Lagrangians and $x \in L \cap L'$. There exist $\varepsilon > 0$ and an embedding $e^x_\varepsilon : B(0,\varepsilon) \to M$ such that

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Definition

L and L' two such Lagrangians. $r(L, L')$ is defined as

$$\sup \left\{ \varepsilon > 0 \;\middle|\; \begin{array}{l} \forall x \in L \cap L',\ \exists\, e^x_\varepsilon \textit{ satisfying i. and ii.,} \\ \textit{such that } x \neq y \ \Rightarrow\ \operatorname{im} e^x_\varepsilon \cap \operatorname{im} e^y_\varepsilon = \emptyset \end{array} \right\}$$

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Remark

$0 < \#(L \cap L') < \infty$ *implies* $r(L, L') > 0$.

Main property

Theorem

If $d^r(\alpha) = \beta \neq 0$ in $E(L)$, then

$$c^r(\alpha; L, L') - c^r(\beta; L, L') \geq \frac{\pi r(L, L')^2}{2}$$

for all Lagrangian L' transverse, Hamiltonian isotopic to L.

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Corollary (order 2)

L $(r-1)$*–connected,* $\{x_i\}$ *a base of* $H_{r-1}(\Omega L)$.
If there exists $\alpha \in H_p(L)$ *such that*
$d^r(1 \otimes \alpha) = \sum x_i \otimes \beta_i \neq 0$, *then*

$$\forall i,\ c(\alpha; L, L') - c(\beta_i; L, L') \geq \frac{\pi r(L, L')^2}{2}$$

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Corollaries of the main property

Corollary

- $\alpha \in H_k(L)$, $\beta \in H_*(L)$, *with* $1 < k < n-1$ *and* $\alpha \cdot \beta \neq 0$ *:*

$$c(\alpha \cdot \beta; L, L') \leq c(\beta; L, L') - \frac{\pi r(L, L')^2}{2}.$$

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Corollary

L and L' transverse, Hamiltonian isotopic Lagrangians :

$$0 < \pi r(L, L')^2 \leq c([L]; L, L') \leq \nabla(L, L').$$

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Corollary

Let $\mathrm{cl}(L)$ *the cup-length of L, we get*

$$\nabla(L, L') \ge \mathrm{cl}(L) \cdot \frac{\pi r(L, L')^2}{2}$$

for all Lagrangian L' transverse, Hamiltonian isotopic to L.

(The end)

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