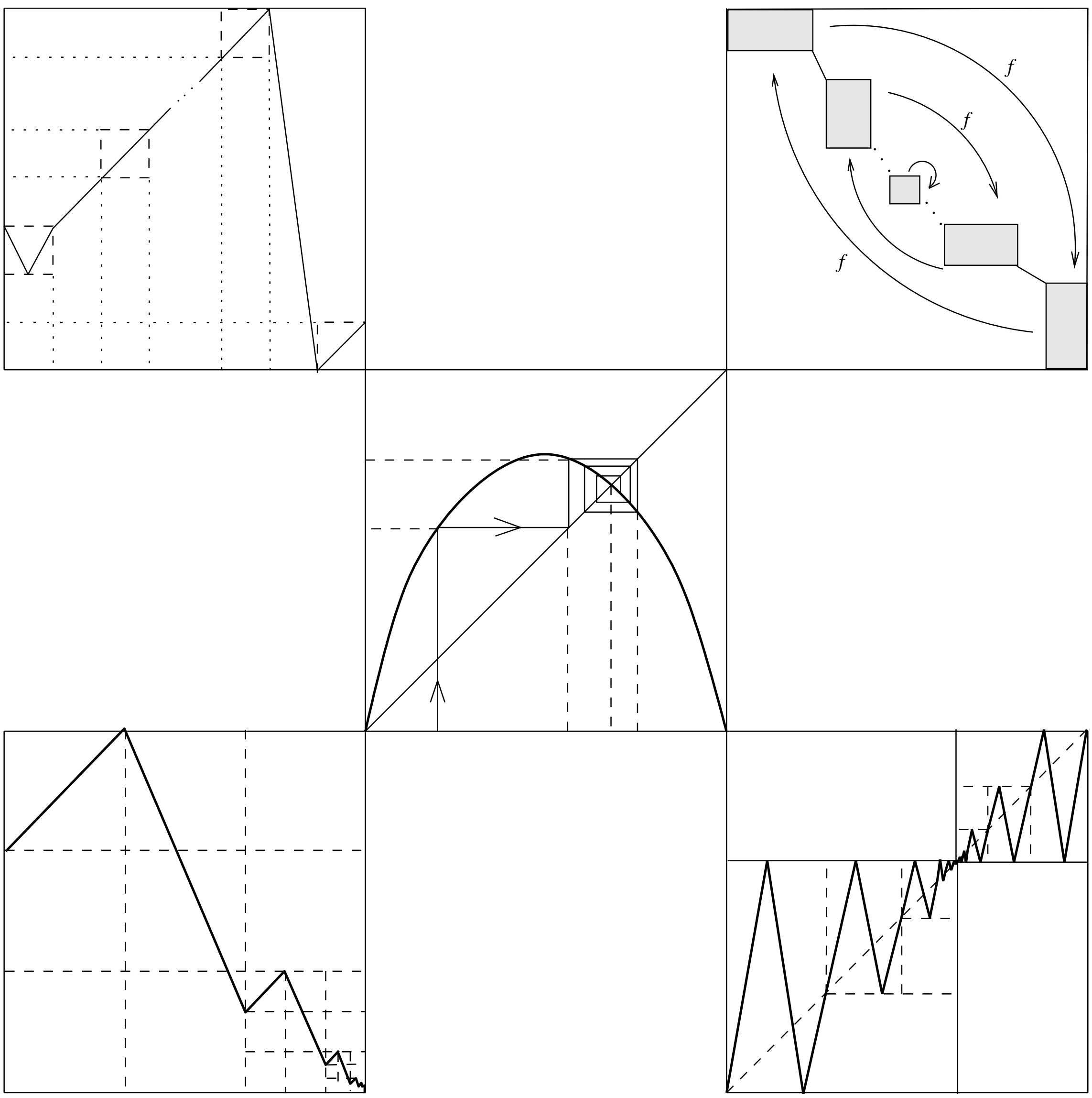
CHAOS ON THE INTERVAL
a survey of relationship between the various kinds of chaos for continuous interval maps

Sylvie Ruette



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Introduction

Generally speaking, a *dynamical system* is a space in which the points (which can be viewed as configurations) move along with time according to a given rule, usually not depending on time. Time can be either continuous (the motion of planets, fluid mechanics, etc) or discrete (the number of bees each year, etc). In the discrete case, the system is determined by a map $f\colon X \to X$, where X is the space, and the evolution is given by successive iterations of the transformation: starting from the point x at time 0, the point $f(x)$ represents the new position at time 1 and $f^n(x) = f \circ f \circ \cdots \circ f(x)$ (f iterated n times) is the position at time n.

A dynamical system ruled by a deterministic law can nevertheless be unpredictable. In particular, in the early 1960's, Lorenz underlined this phenomenon after realizing by chance that in his meteorological model, two very close initial values may lead to totally different results [**116, 117, 118**]; he discovered the so called "butterfly effect". This kind of behavior has also been exhibited in other dynamical systems. One of the first to be studied, among the simplest, is given by the map $f(x) = rx(1-x)$ acting on the interval $[0,1]$, and models the evolution of a population. If the parameter r is small enough, then all the trajectories converge to a fixed point – the population stabilizes. However, May showed that for larger values of r, the dynamics may become very complicated [**123**].

This book focuses on dynamical systems given by the iteration of a continuous map on an interval. These systems were broadly studied because they are simple but nevertheless exhibit complex behaviors. They also allow numerical simulations using a computer or a mere pocket calculator, which enabled the discovery of some chaotic phenomena. Moreover, the "most interesting" part of some higher-dimensional systems can be of lower dimension, which allows, in some cases, to boil down to systems in dimension one. This idea was used for instance to reduce the study of Lorenz flows in dimension 3 to a class of maps on the interval. However, continuous interval maps have many properties that are not generally found in other spaces. As a consequence, the study of one-dimensional dynamics is very rich but not representative of all systems.

In the 1960's, Sharkovsky began to study the structure of systems given by a continuous map on an interval, in particular the co-existence of periodic points of various periods, which is ruled by *Sharkovsky's order* [**153**]. Non Russian-speaking scientists were hardly aware of this striking result until a new proof of this theorem was given in English by Štefan in 1976 in a preprint [**165**] (published one year later in [**166**]). In 1975, in the paper "Period three implies chaos" [**113**], Li and Yorke proved that a continuous interval map with a periodic point of period 3 has periodic points of all periods – which is actually a part of Sharkovsky's Theorem eleven years earlier; they also proved that, for such a map f, there exists an uncountable set such that, if x, y are two distinct points in this set, then $f^n(x)$ and $f^n(y)$ are arbitrarily close for some n and are further than some fixed positive distance for other integers n tending to infinity; the term "chaos" was introduced in mathematics in this paper of Li and Yorke, where it was used in reference to this behavior.

Afterwards, various definitions of chaos were proposed. They do not coincide in general and none of them can be considered as the unique "good" definition of chaos. One may ask "What is chaos then?". It relies generally on the idea of unpredictability or instability, i.e., knowing the trajectory of one point is not enough to know what happens elsewhere. The map $f\colon X \to X$ is said to be *sensitive to initial conditions* if near every point x there exists a point y arbitrarily close to x such that the distance between $f^n(x)$ and $f^n(y)$ is greater than a given $\delta > 0$ for some n. *Chaos in the sense of Li-Yorke* (see above) asks for more instability, but only on a subset. For Devaney, chaos is seen as a mixing of unpredictability and regular behavior: a system is *chaotic in the sense of Devaney* if it is transitive, sensitive to initial conditions and has a dense set of periodic points [**74**]. Others put as a part of their definition that the entropy should be positive, which means that the number of different trajectories of length n, up to some approximation, grows exponentially fast.

In order to obtain something uniform, the system is often assumed to be transitive. Roughly speaking, this means that it cannot be decomposed into two parts with nonempty interiors that do not interact under the action of the transformation. This "basic" assumption actually has strong consequences for systems on one-dimensional spaces. For a continuous interval map, it implies most of the other notions linked to chaos: sensitivity to initial conditions, dense set of periodic points, positive entropy, chaos in the sense of Li-Yorke, etc. This leads us to search for (partial) converses: for instance, if the interval map f is sensitive to initial conditions then, for some integer n, the map f^n is transitive on a subinterval.

The study of periodic points has taken an important place in the works on interval maps. For these systems, chaotic properties not only imply existence of periodic points, but the possible periods also provide some information about the system. For instance, for a transitive interval map, there exist periodic points of all even periods, and an interval map has positive entropy if and only if there exists a periodic point whose period is not a power of 2. This kind of relationship is very typical of one-dimensional systems.

The aim of this book is not to collect all the results about continuous interval maps but to survey the relations between the various kinds of chaos and related notions for these systems. The papers on this topic are numerous but very scattered in the literature, sometimes little known or difficult to find, sometimes originally published in Russian (or Ukrainian, or Chinese), and sometimes without proof. Furthermore some results were found twice independently, which was often due to a lack of communication and language barriers, leading research to develop separately in English and Russian literature. This has complicated our task when attributing authorship; we want to apologize for possible errors or omissions when indicating who first proved the various results.

We adopt a topological point of view, i.e., we do not speak about invariant measures or ergodic properties. Moreover, we are interested in the set of continuous interval maps, not in particular families such as piecewise monotone, C^∞ or unimodal maps. We give complete proofs of the results concerning interval maps.

Many results for interval maps have been generalized to other one-dimensional systems. We briefly describe them in paragraphs called "Remarks on graph maps" at the end of the concerned sections. We indicate some main ideas and give the references. This subject is still in evolution, and the most recent works and references may be missing.

This book is addressed to both graduate students and researchers. We have tried to keep to the elementary level. The prerequisites are basic courses of real analysis and topology, and some linear algebra.

Contents of the book

In the **first Chapter**, we define some elementary notions and introduce some notation. Throughout this book, a continuous map $f\colon I \to I$ on a non degenerate compact interval I will be called an *interval map*. We also provide some basic results about ω-limit sets and tools to find periodic points.

In **Chapter 2**, we study the links between transitivity, topological mixing and sensitivity to initial conditions. We first prove that a transitive interval map has a dense set of periodic points. Then we show that transitivity is very close to the notion of topological mixing in the sense that for a transitive interval map $f\colon I \to I$, either f is topologically mixing, or the interval I is divided into two subintervals J, K which are swapped under the action of f and such that both $f^2|_J$ and $f^2|_K$ are topologically mixing. Furthermore, the notions of topological mixing, topological weak mixing and total transitivity are proved to be equivalent for interval maps.

Next we show that a transitive interval map is sensitive to initial conditions and, conversely, if the map is sensitive, then there exists a subinterval J such that $f^n|_J$ is transitive for some positive integer n.

Chapter 3 is devoted to periodic points. First we prove that topological mixing is equivalent to the specification property, which roughly means that any collection of pieces of orbits can be approximated by the orbit of a periodic point.

Next we show that, if the set of periodic points is dense for the interval map f, then there exists a non degenerate subinterval J such that either $f|_J$ or $f^2|_J$ is transitive provided that f^2 is not equal to the identity map.

Then we present Sharkovsky's Theorem, which says that there is a total order on $\mathbb{N}$ – called *Sharkovsky's order* – such that, if an interval map has a periodic point of period n, then it also has periodic points of period m for all integers m greater than n with respect to this order. The *type* of a map f is the minimal integer n for Sharkovsky's order such that f has a periodic point of period n; if there is no such integer n, then the set of periods is exactly $\{2^n \mid n \geq 0\}$ and the type is 2^∞. We build an interval map of type n for every $n \in \mathbb{N} \cup \{2^\infty\}$.

Next, we study the relation between the type of a map and the existence of horseshoes. Finally, we compute the type of transitive and topologically mixing interval maps.

In **Chapter 4**, we are concerned with topological entropy. A *horseshoe* for the interval map f is a family of two or more closed subintervals $J_1, \ldots, J_p$ with disjoint interiors such that $f(J_i) \supset J_1 \cup \cdots \cup J_p$ for all $1 \leq i \leq p$. We show that the existence of a horseshoe implies that the topological entropy is positive. Reciprocally, Misiurewicz's Theorem states that, if the entropy of the interval map f is positive, then f^n has a horseshoe for some positive integer n.

Next we show that an interval map has a homoclinic point if and only if it has positive topological entropy. For an interval map f, x is a *homoclinic point* if there exists a periodic point z different from x such that x is in the unstable manifold of z and z is a limit point of $(f^{np}(x))_{n\geq 0}$, where p is the period of z.

We then give some upper and lower bounds on the entropy, focusing on lower bounds for transitive and topologically mixing maps and lower bounds depending on the periods of periodic points (or, in other words, on the type of the map for Sharkovsky's order). In particular, an interval map has positive topological entropy if and only if it has a periodic point whose period is not a power of 2. The sharpness of these bounds is illustrated by some examples.

To conclude this chapter, we show that a topologically mixing interval map has a uniformly positive entropy; that is, every cover by two open non dense sets has positive topological entropy. Actually, this property is equivalent to topological mixing for interval maps.

Chapter 5 is devoted to chaos in the sense of Li-Yorke. Two points x, y form a *Li-Yorke pair of modulus* δ for the map f if

$$\limsup_{n\to+\infty} |f^n(x) - f^n(y)| \geq \delta \quad \text{and} \quad \liminf_{n\to+\infty} |f^n(x) - f^n(y)| = 0.$$

A *δ-scrambled set* is a set S such that every pair of distinct points in S is a Li-Yorke pair of modulus δ; the set S is *scrambled* if for every $x, y \in S$, $x \neq y$, (x, y) is a Li-Yorke pair (of modulus δ for some $\delta > 0$ depending on x, y). The map is *chaotic in the sense of Li-Yorke* if it has an uncountable scrambled set. We prove that an interval map of positive topological entropy admits a δ-scrambled Cantor set for some $\delta > 0$, and is thus chaotic in the sense of Li-Yorke. We also show that a topologically mixing map has a dense δ-scrambled set which is a countable union of Cantor sets.

Next, we study an equivalent condition for zero entropy interval maps to be chaotic in the sense of Li-Yorke, which implies the existence of a δ-scrambled Cantor set as in the positive entropy case. A zero entropy interval map that is chaotic in the sense of Li-Yorke is necessarily of type 2^∞ for Sharkovsky's order, but the converse is not true; we build two maps of type 2^∞ having an infinite ω-limit set, one being chaotic in the sense of Li-Yorke and the other not.

Then we state that the existence of one Li-Yorke pair for an interval map is enough to imply chaos in the sense of Li-Yorke.

Finally, we show that an interval map is chaotic in the sense of Li-Yorke if and only if it has positive topological sequence entropy.

In **Chapter 6**, we study some notions related to Li-Yorke pairs.

Generic chaos and *dense chaos* are somehow two-dimensional notions. A topological system $f\colon X \to X$ is generically (resp. densely) chaotic if the set of Li-Yorke pairs is residual (resp. dense) in $X \times X$. A transitive interval map is generically chaotic; conversely, a generically chaotic interval map has exactly one or two transitive subintervals. Dense chaos is strictly weaker than generic chaos: a densely interval map may have no transitive subinterval, as illustrated by an example. We show that, if f is a densely chaotic interval map, then f^2 has a horseshoe, which implies that f has a periodic point of period 6 and the topological entropy of f is at least $\frac{\log 2}{2}$.

Distributional chaos is based on a refinement of the conditions defining Li-Yorke pairs. We show that, for interval maps, distributional chaos is equivalent to positive topological entropy.

In **Chapter 7**, we focus on the existence of some kinds of chaotic subsystems and we relate them to the previous notions.

A system is said to be *chaotic in the sense of Devaney* if it is transitive, sensitive to initial conditions and has a dense set of periodic points. For an interval map, the existence of an invariant closed subset that is chaotic in the sense of Devaney is equivalent to positive topological entropy. We also show that an interval map has an invariant uncountable closed subset X on which f^n is topologically mixing for some $n \geq 1$ if and only if f has positive topological entropy.

Finally, we study the existence of an invariant closed subset on which the map is transitive and sensitive to initial conditions. We show that this property is implied by positive topological entropy and implies chaos in the sense of Li-Yorke. However these notions are distinct: there exist zero entropy interval maps with a transitive sensitive subsystem and interval maps with no transitive sensitive subsystem that are chaotic in the sense of Li-Yorke.

The **last chapter** is an appendix that recalls succinctly some background in topology.

The relations between the main notions studied in this book are summarized by the diagram in Figure 1.

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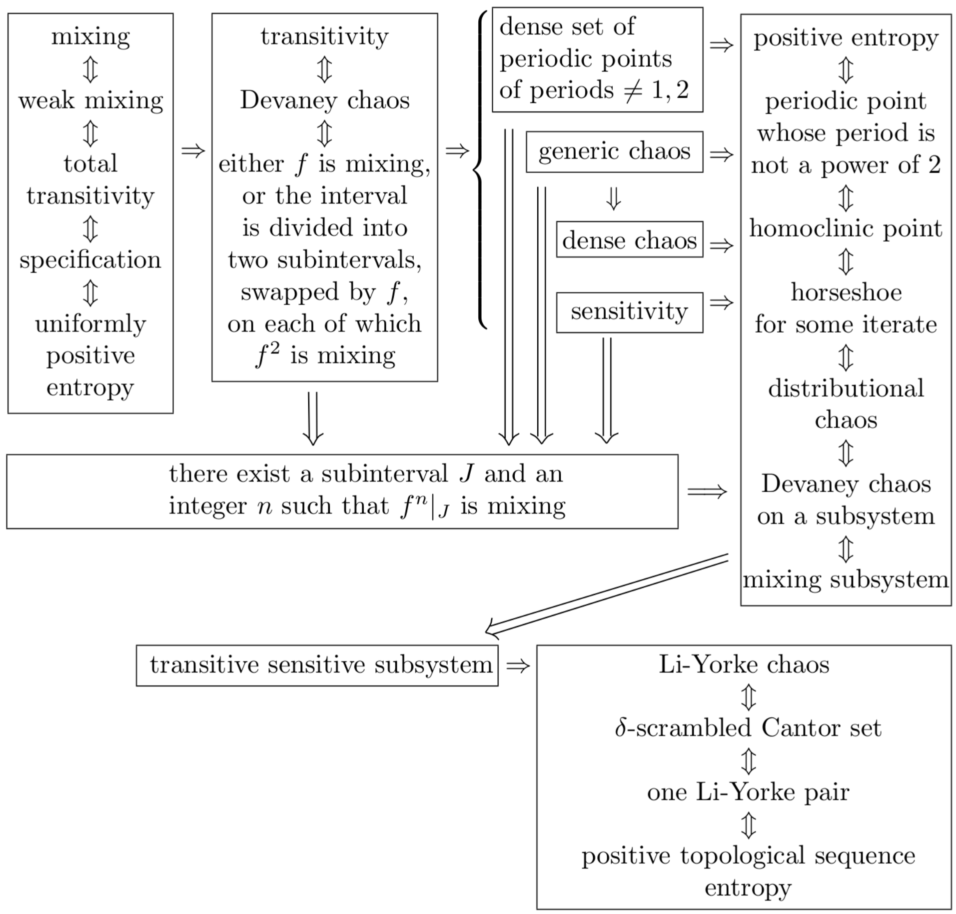


FIGURE 1. Diagram summarizing the relations between the main notions related to chaos for an interval map f.

CHAPTER 1

Notation and basic tools

1.1. General notation

1.1.1. Sets of numbers. The set of natural numbers (that is, positive integers) is denoted by $\mathbb{N}$. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote respectively the set of all integers, rational numbers, real numbers and complex numbers. The non negative integers and non negative real numbers are denoted respectively by $\mathbb{Z}^+$ and $\mathbb{R}^+$.

1.1.2. Interval of integers. The notation $[\![n, m]\!]$ denotes an interval of integers, that is, $[\![n, m]\!] := \{k \in \mathbb{Z} \mid n \leq k \leq m\}$.

We shall often deal with sets $X_1, \ldots, X_n$ that are cyclically permuted. The notation $X_{i+1 \bmod n}$ means X_{i+1} if $i \in [\![1, n-1]\!]$ and X_1 if $i = n$. More generally, if the set of indices $\mathcal{I}$ under consideration is $[\![1, n]\!]$ (resp. $[\![0, n-1]\!]$), then $i \bmod n$ denotes the integer $j \in \mathcal{I}$ such that $j \equiv i \bmod n$.

1.1.3. Cardinality of a set. If E is a finite set, $\#E$ denotes its cardinality, that is, the number of elements in E.

A set is *countable* if it can be written as $\{x_n \mid n \in \mathbb{N}\}$. A finite set is countable.

1.1.4. Notation of topology. The definitions of the topological notions used in this book are recalled in the appendix. Here we only give some notation.

Let X be a metric space and let Y be a subset of X. Then $\overline{Y}$, $\mathrm{Int}\,(Y)$, $\mathrm{Bd}(Y)$ denote respectively the closure, the interior and the boundary of Y.

Remark 1.1. When talking about topological notions (neighborhood, interior, etc), we always refer to the induced topology on the ambient space X. For instance, in Example 1.2 below, $[0, 1/2)$ is an open set since the ambient space is $[0, 1]$.

The distance on a metric space X is denoted by d. If $x \in X$ and $r > 0$, the open ball of center x and radius r is $B(x, r) := \{y \in X \mid d(x, y) < r\}$, and the closed ball of center x and radius r is $\overline{B}(x, r) := \{y \in X \mid d(x, y) \leq r\}$.

The diameter of a set $Y \subset X$ is $\mathrm{diam}(Y) := \sup\{d(y, y') \mid y, y' \in Y\}$. If Y is compact, then the supremum is reached.

1.1.5. Restriction of a map. Let $f\colon X \to X'$ be a map and $Y \subset X$. The restriction of f to Y, denoted by $f|_Y$, is the map $f|_Y\colon \begin{array}{ccc} Y & \to & X' \\ x & \mapsto & f(x) \end{array}$.

1.2. Topological dynamical systems, orbits, ω-limit sets

Our purpose is to study dynamical systems on intervals. However we prefer to give the notation in a broader context because most of the definitions have a meaning for any dynamical system, and a few properties will not be specific to the interval case.

1.2.1. Topological dynamical systems, invariant set. A *topological dynamical system* (X, f) is given by a continuous map $f\colon X \to X$, where X is a nonempty compact metric space. The evolution of the system is given by the successive iterations of the map. If $n \in \mathbb{N}$, the n-th iterate of f is denoted by f^n, that is,

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

By convention, f^0 is the identity map on X. We can think of n as time: starting from an initial position x at time 0, the point $f^n(x)$ represents the new position at time n.

EXAMPLE 1.2. Let $f\colon [0,1] \to [0,1]$ be the map defined by $f(x) = 3x(1-x)$. The successive iterates of x can be plotted on the graph of f, as illustrated in Figure 1; the diagonal $y = x$ is utilized to re-use the result of an iteration.

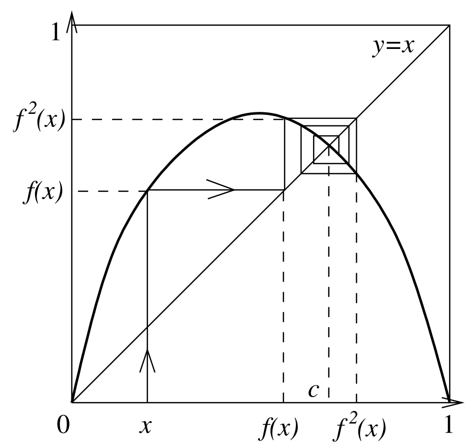


FIGURE 1. The first iterates of x plotted on the graph of f.

Let (X, f) be a topological dynamical system. An *invariant* (or *f-invariant*) set is a nonempty closed set $Y \subset X$ such that $f(Y) \subset Y$; it is *strongly invariant* if in addition $f(Y) = Y$. If Y is an invariant set, let $f|_Y$ denote the map f restricted to Y and *arriving in* Y, that is, $f|_Y\colon Y \to Y$. With this slight abuse of notation, $(Y, f|_Y)$ is a topological dynamical system, called a *subsystem* of (X, f), and we shall speak of the properties of $f|_Y$ (e.g., "$f|_Y$ is transitive").

1.2.2. Trajectory, orbit, periodic point. In the literature, the words *trajectory* and *orbit* often have the same meaning. However we prefer to follow the terminology of Block-Coppel [**41**] because it is convenient to make a distinction between two notions. In this book, when (X, f) is a topological dynamical system and x is a point in X, the *trajectory* of x is the infinite sequence $(f^n(x))_{n\geq 0}$ (there may be repetitions in the sequence) and the *orbit* of x is the set $\mathcal{O}_f(x) := \{f^n(x) \mid n \geq 0\}$. Similarly, if E is a subset of X, then $\mathcal{O}_f(E) := \bigcup_{n\geq 0} f^n(E)$.

A point x is *periodic* (for the map f) if there exists a positive integer n such that $f^n(x) = x$. The *period* of x is the least positive integer p such that $f^p(x) = x$.

It is easy to see that, if x is periodic of period p and $n \in \mathbb{N}$, then $f^n(x) = x$ if and only if n is a multiple of p; moreover $\mathcal{O}_f(x)$ is a finite set of p distinct points: $\mathcal{O}_f(x) = \{x, f(x), \ldots, f^{p-1}(x)\}$. If x is a periodic point of period p, then its orbit is called a *periodic orbit of period* p. Each point of a periodic orbit is a periodic point with the same period and the same orbit. If $f(x) = x$, then x is called a *fixed point*. Let

$$P_n(f) := \{x \in X \mid f^n(x) = x\};$$

this is the set of periodic points whose periods divide n.

A point x is *eventually periodic* if there exists an integer $n \geq 0$ such that $f^n(x)$ is periodic.

1.2.3. Omega-limit set. Let (X, f) be a topological dynamical system. The *ω-limit set* of a point $x \in X$, denoted by $\omega(x, f)$, is the set of all limit points of the trajectory of x, that is,

$$\omega(x, f) := \bigcap_{n \geq 0} \overline{\{f^k(x) \mid k \geq n\}}.$$

The *ω-limit set of the map* f is

$$\omega(f) := \bigcup_{x \in X} \omega(x, f).$$

LEMMA 1.3. *Let (X, f) be a topological dynamical system, $x \in X$ and $n \geq 1$. Then*

i) *$\omega(x, f)$ is a closed set, and it is strongly invariant,*
ii) *$\omega(f^n(x), f) = \omega(x, f)$,*
iii) *$\forall i \geq 0$, $\omega(f^i(x), f^n) = f^i(\omega(x, f^n))$,*
iv) *$\omega(x, f) = \bigcup_{i=0}^{n-1} \omega(f^i(x), f^n)$,*
v) *if $\omega(x, f)$ is infinite, then $\omega(f^i(x), f^n)$ is infinite for all $i \geq 0$,*
vi) *$f(\omega(f)) = \omega(f)$,*
vii) *$\omega(f^n) = \omega(f)$.*

PROOF. Assertions (i) to (iv) can be easily deduced from the definition. Assertion (vi) follows from (i), assertions (v) and (vii) follow from (iii)-(iv). $\square$

LEMMA 1.4. *Let (X, f) be a topological dynamical system and $x \in X$. If $\omega(x, f)$ is finite, then it is a periodic orbit.*

PROOF. Let F be a nonempty subset of $\omega(x, f)$ different from $\omega(x, f)$. We set $F' := \omega(x, f) \setminus F$. Both F, F' are finite and nonempty. Let U, U' be two open sets such that $F \subset U$, $F' \subset U'$, $\overline{U} \cap F' = \emptyset$ and $\overline{U'} \cap F = \emptyset$. Thus, for every large enough integer n, the point $f^n(x)$ belongs to $U \cup U'$. Moreover, there are infinitely many integers n such that $f^n(x) \in U$ and infinitely many n such that $f^n(x) \in U'$. Therefore, there exists an increasing sequence $(n_i)_{i \geq 0}$ such that, $\forall i \geq 0$, $f^{n_i}(x) \in U$ and $f^{n_i+1}(x) \in U'$. By compactness, the sequence $(f^{n_i}(x))_{i \geq 0}$ has a limit point $y \in \overline{U} \cap \omega(x, f) = F$. Since f is continuous, $f(y)$ is a limit point of $(f^{n_i+1}(x))_{i \geq 0}$, and hence $f(y) \in \omega(x, f) \cap \overline{U'} = F'$. Thus $f(F) \cap F' \neq \emptyset$, and so $\omega(x, f)$ contains no invariant subset except itself. This implies that f acts as a cyclic permutation on $\omega(x, f)$, that is, $\omega(x, f)$ is a periodic orbit. $\square$

1.2.4. Semi-conjugacy, conjugacy. Let (X, f) and (Y, g) be two topological dynamical systems. The system (Y, g) is said to be *(topologically) semi-conjugate* to (X, f) if there exists a continuous onto map $\varphi\colon X \to Y$ such that $\varphi \circ f = g \circ \varphi$. If in addition the map φ is a homeomorphism, (Y, g) is *(topologically) conjugate* to (X, f); conjugacy is an equivalence relation. Two conjugate dynamical systems share the same dynamical properties as long as topology is concerned (differential properties may not be preserved if φ is only assumed to be continuous).

1.3. Intervals, interval maps

1.3.1. Intervals, endpoints, length, non degenerate interval, inequalities between subsets of $\mathbb{R}$. The (real) intervals are exactly the connected sets of $\mathbb{R}$. An interval J is either the empty set or one of the following forms:

- $J = [a, b]$ with $a, b \in \mathbb{R}$, $a \leq b$ (if $a = b$, then $J = \{a\}$),
- $J = (a, b)$ with $a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R} \cup \{+\infty\}$, $a < b$,
- $J = [a, b)$ with $a \in \mathbb{R}, b \in \mathbb{R} \cup \{+\infty\}$, $a < b$,
- $J = (a, b]$ with $a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R}$, $a < b$.

Suppose that J is nonempty and bounded (i.e., when $a, b \in \mathbb{R}$). The *endpoints* of J are a and b; let ∂J denote the set $\{a, b\}$. The *length* of J, denoted by $|J|$, is equal to $b - a$.

An interval is *degenerate* if it is either empty or reduced to a single point, and it is *non degenerate* otherwise.

If $a, b \in \mathbb{R}$, let $\langle a, b\rangle$ denote the smallest interval containing $\{a, b\}$, that is, $\langle a, b\rangle = [a, b]$ if $a \leq b$ and $\langle a, b\rangle = [b, a]$ if $b \leq a$.

If X and Y are two nonempty subsets of $\mathbb{R}$, the notation $X < Y$ means that, $\forall x \in X, \forall y \in Y$, $x < y$ (in this case X and Y are disjoint) and $X \leq Y$ means that, $\forall x \in X, \forall y \in Y$, $x \leq y$ (X and Y may have a common point, equal to $\max X = \min Y$). We may also say that X is on the left of Y.

LEMMA 1.5. *Every open set $U \subset \mathbb{R}$ can be written as the union of countably many disjoint open intervals.*

PROOF. The connected components of U are disjoint nonempty open intervals, and every non degenerate interval contains a rational number, which implies that the connected components of U are countable. □

1.3.2. Interval maps, monotonicity, critical points, piecewise monotone and piecewise linear maps. We say that $f\colon I \to I$ is an *interval map* if I is a non degenerate compact interval and f is a continuous map.

When dealing with an interval map $f\colon I \to I$, we shall always refer to the ambient space. The topology is the induced topology on I; points and sets are implicitly points in I and subsets of I, and intervals are subintervals of I (and hence are bounded intervals).

REMARK 1.6. The fixed points of an interval map f can be easily seen on the graph of f. Indeed, x is a fixed point if and only if (x, x) is in the intersection of the graph $y = f(x)$ with the diagonal $y = x$. E.g., in Example 1.2, the map has two fixed points, 0 and c. Similarly, the points of $P_n(f)$ correspond to the intersection of the graph of f^n with $y = x$.

Let $f: I \to \mathbb{R}$ be a continuous map, where I is an interval, and let J be a non degenerate subinterval of I.

- The map f is increasing (resp. decreasing) on J if for all points $x, y \in J$, $x < y \Rightarrow f(x) < f(y)$ (resp. $f(x) > f(y)$).
- The map f is non decreasing (resp. non increasing) on J if for all $x, y \in J$, $x < y \Rightarrow f(x) \leq f(y)$ (resp. $f(x) \geq f(y)$).
- The map f is *monotone* (resp. *strictly monotone*) on J if f is either non decreasing or non increasing (resp. either increasing or decreasing) on J.

A *critical point of* f is a point $x \in I$ such that there exists no neighborhood of x on which f is strictly monotone. Notice that if f is differentiable, the set of critical points is included in the set of zeros of f'.

The map f is *piecewise monotone* if the interval I can be divided into finitely many subintervals on each of which f is monotone, that is, there exist points $a_0 = \min I < a_1 < \dots a_{n-1} < a_n = \max I$ such that f is monotone on $[a_i, a_{i+1}]$ for all $i \in [\![0, n-1]\!]$. The set of critical points of f is included in $\{a_1, \dots, a_{n-1}\}$. Conversely, if the set of critical points of f is finite, then f is piecewise monotone.

Remark 1.7. The critical points are also called *turning points*, especially when the map f is piecewise monotone.

Let $f: I \to \mathbb{R}$ be a continuous map, where $I := [a, b]$, $a < b$. The map f is *linear* if there exist $\alpha, \beta \in \mathbb{R}$ such that $f(x) = \alpha x + \beta$ for all $x \in [a, b]$. The *slope* of f is $\mathsf{slope}(f) := \alpha$. One has $\mathsf{slope}(f) = \frac{f(b)-f(a)}{b-a}$ and $|\mathsf{slope}(f)| = \frac{|f(I)|}{|I|}$.

f is *piecewise linear* if there exist $a_0 = \min I < a_1 < \dots < a_{n-1} < a_n = \max I$ such that f is linear on $[a_i, a_{i+1}]$ for all $i \in [\![0, n-1]\!]$. In particular, a piecewise linear map is piecewise monotone.

Most of our examples will be piecewise linear.

1.3.3. Rescaling. If two interval maps f and g are conjugate by an increasing linear homeomorphism, they have the same graph up to the action of a homothety or a translation. We call this action a *rescaling*. If g is conjugate to f by a decreasing linear homeomorphism, the graph of g is obtained from the one of f by a half-turn rotation and a rescaling. Not only are the maps f and g conjugate, but they have exactly the same properties (when the conjugacy is decreasing, it just reverses the order when order is involved in a property).

Remark 1.8. When dealing with interval maps, one may assume that the interval is $[0, 1]$. Indeed, if $f: [a, b] \to [a, b]$ is an interval map, let $\varphi: [0, 1] \to [a, b]$ be the linear homeomorphism defined by $\varphi(x) := a + (b-a)x$ and let $g := \varphi^{-1} \circ f \circ \varphi$. The maps $f: [a, b] \to [a, b]$ and $g: [0, 1] \to [0, 1]$ are conjugate, and g is a mere rescaling of f.

1.3.4. Periodic intervals. Let $f: I \to I$ be an interval map. If $J_1, \dots, J_p$ are disjoint non degenerate closed subintervals of I such that $f(J_i) = J_{i+1 \bmod p}$ for all $i \in [\![1, p]\!]$, then $(J_1, \dots, J_p)$ (as well as the set $C := J_1 \cup \dots \cup J_p$) is called a *cycle of intervals of period* p. Moreover, J_1 is called a *periodic interval of period* p.

1.3.5. Intermediate value theorem. The intermediate value theorem is fundamental and we shall use it constantly. For a convenience, we give several equivalent statements.

THEOREM 1.9 (intermediate value theorem). *Let $f\colon I \to \mathbb{R}$ be a continuous map, where I is a nonempty interval.*

- *Let J be a nonempty subinterval of I. Then $f(J)$ is also a nonempty interval.*
- *Let $x_1, x_2 \in I$ with $x_1 \leq x_2$. Then $f([x_1, x_2]) \supset \langle f(x_1), f(x_2)\rangle$. In particular, for every c between $f(x_1)$ and $f(x_2)$, there exists $x \in [x_1, x_2]$ such that $f(x) = c$.*

PROOF. The first assertion follows from the fact that the image of a connected set by a continuous map is connected (Theorem 8.74 in Appendix) and the image of a nonempty set is nonempty. The second assertion is a straightforward consequence of the first one with $J = [x_1, x_2]$. $\square$

Definition of graph maps. A *topological graph* is a compact connected metric space G containing a finite subset V such that $G \setminus V$ has finitely many connected components and every connected component of $G \setminus V$ is homeomorphic to $(0, 1)$. A topological graph is *non degenerate* if it contains more than one point. A *subgraph* of G is a closed connected subset of G; a subgraph is a topological graph. A *tree* is a topological graph containing no subset homeomorphic to a circle. A *branching point* is a point having no neighborhood homeomorphic to a real interval. An *endpoint* is a point having a neighborhood homeomorphic to the half-closed interval $[0, 1)$. The sets of branching points and endpoints are finite (they are included in V). If H is a subgraph of G, the set of endpoints of H is denoted by ∂H. A subset of G is called an *interval* (resp. a *circle*) if it is homeomorphic to an interval of the real line (resp. a circle of positive radius).

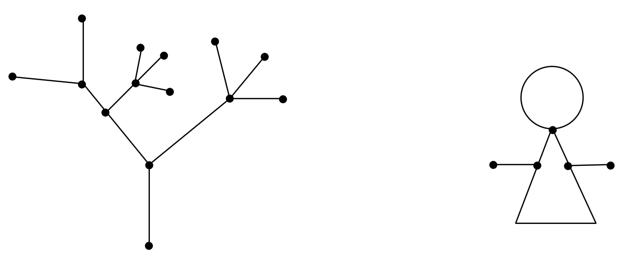


FIGURE 2. A tree (on the left) and a topological graph (on the right). The branching points and the endpoints are indicated by big dots.

A *graph* (resp. *tree*) *map* is a continuous map $f\colon X \to X$, where X is a non degenerate topological graph (resp. tree). If $G_1, \ldots, G_p$ are disjoint non degenerate subgraphs of X such that $f(G_i) = G_{i+1 \bmod p}$ for all $i \in [\![1, p]\!]$, then $(G_1, \ldots, G_p)$ is called a *cycle of graphs of period p*.

DEFINITION 1.10. Let $f\colon G \to G$ be a graph map. If $I \subset G$ is either a non degenerate interval or a circle, the map $f|_I$ is said to be *monotone* if it is locally monotone at every point $x \in I$, that is, there exists an open neighborhood U of x with respect to the topology of I such that:

- U contains $K(x)$, where $K(x) \subset I$ is the largest subinterval of I containing x on which f is constant,
- U and $f(U)$ are homeomorphic to intervals,

- $f|_U : U \to f(U)$, seen as a map between intervals, is monotone (more precisely, there exist intervals $J, J' \subset \mathbb{R}$ and homeomorphisms $h: U \to J$, $h': f(U) \to J'$ such that $h' \circ f|_U \circ h^{-1} : J \to J'$ is monotone).

Notice that, when G is a tree, the fact that $f|_I$ is monotone implies that $f(I)$ is necessarily an interval, whereas in general $f(I)$ may not be an interval (in particular, $f(I)$ may wrap around circles).

1.4. Chains of intervals and periodic points

The next lemma is a basic tool to prove the existence of fixed points. Below, Lemma 1.13 states the existence of periodic points when some intervals are nested under the action of f.

LEMMA 1.11. *Let $f: [a,b] \to \mathbb{R}$ be a continuous map. If $f([a,b]) \subset [a,b]$ or $f([a,b]) \supset [a,b]$, then f has a fixed point.*

PROOF. Let $g(x) := f(x) - x$. If $f([a,b]) \subset [a,b]$, then

$$g(a) = f(a) - a \geq a - a = 0 \text{ and } g(b) = f(b) - b \leq b - b = 0.$$

By the intermediate value theorem applied to g, there exists $c \in [a,b]$ with $g(c) = 0$. If $f([a,b]) \supset [a,b]$, there exist $x, y \in [a,b]$ such that $f(x) \leq a$ and $f(x) \geq b$. We then have

$$g(x) = f(x) - x \leq a - x \leq 0 \text{ and } g(y) = f(y) - y \geq b - y \geq 0.$$

Thus there exists $c \in [x,y]$ with $g(c) = 0$ by the intermediate value theorem. In both cases, c is a fixed point of f. □

DEFINITION 1.12 (covering, chain of intervals). Let f be an interval map.

- Let J, K be two nonempty closed intervals. Then J is said to *cover* K (for f) if $K \subset f(J)$. This is denoted by $J \underset{f}{\to} K$, or simply $J \to K$ if there is no ambiguity. If k is a positive integer, J *covers* K k *times* if J contains k closed subintervals with disjoint interiors such that each one covers K.
- Let $J_0, \ldots, J_n$ be nonempty closed interval such that J_{i-1} covers J_i for all $i \in [\![1, n]\!]$. Then $(J_0, J_1, \ldots, J_n)$ is called a *chain of intervals (for f)*. This is denoted by $J_0 \to J_1 \to \ldots \to J_n$.

LEMMA 1.13. *Let f be an interval map and $n \geq 1$.*

i) *Let $J_0, \ldots, J_n$ be nonempty intervals such that $J_i \subset f(J_{i-1})$ for all i in $[\![1, n]\!]$. Then there exists an interval $K \subset J_0$ such that $f^n(K) = J_n$, $f^n(\partial K) = \partial J_n$ and $f^i(K) \subset J_i$ for all $i \in [\![0, n]\!]$. If in addition $J_0, \ldots, J_n$ are closed (and so $(J_0, \ldots, J_n)$ is a chain of intervals), then K can be chosen to be closed.*
ii) *Let $(J_0, \ldots, J_n)$ be a chain of intervals such that $J_0 \subset J_n$. Then there exists $x \in J_0$ such that $f^n(x) = x$ and $f^i(x) \in J_i$ for all $i \in [\![0, n-1]\!]$.*
iii) *Suppose that, for every $i \in [\![1, p]\!]$, $(J_0^i, \ldots, J_n^i)$ is a chain of intervals and, for every pair (i,j) of distinct indices in $[\![1, p]\!]$, there exists $k \in [\![0, n]\!]$ such that J_k^i and J_k^j have disjoint interiors. Then there exist closed intervals $K_1, \ldots, K_p$ with pairwise disjoint interiors such that*

$$\forall i \in [\![1, p]\!],\ f^n(K_i) = J_n^i,\ f^n(\partial K_i) = \partial J_n^i$$
$$\textit{and} \quad \forall k \in [\![0, n]\!],\ \forall i \in [\![1, p]\!],\ f^k(K_i) \subset J_k^i.$$

Proof. We first prove by induction on n the following:

Fact 1. Let $J_0, \dots, J_n$ be nonempty intervals such that $J_i \subset f(J_{i-1})$ for all $i \in [\![1, n]\!]$. Then there exist intervals $K_n \subset K_{n-1} \subset \cdots \subset K_1 \subset J_0$ such that, for all $k \in [\![1, n]\!]$ and all $i \in [\![0, k]\!]$,

$$f^i(K_k) \subset J_i, f^k(K_k) = J_k, f^k(\partial K_k) = \partial J_k \text{ and } f^k(\mathrm{Int}\,(K_k)) = \mathrm{Int}\,(J_k)\,.$$

Moreover, if $J_0, \dots, J_n$ are closed, then $K_1, \dots, K_n$ can be chosen to be closed too.

• Case $n = 1$. We write $\overline{J_1} = [a, b]$. There exist $x, y \in \overline{J_0}$ such that $f(x) = a$ and $f(y) = b$. If a (resp. b) belongs to $f(J_0)$, we choose x (resp. y) in J_0. If a (resp. b) does not belong to $f(J_0)$, then it does not belong to J_1 either, and x (resp. y) is necessarily an endpoint of J_0. With no loss of generality, we may suppose that $x \leq y$ (the other case being symmetric). We define

$$y' := \min\{z \geq x \mid f(z) = b\}, \quad x' := \max\{z \leq y' \mid f(z) = a\}$$

and $K_1' := [x', y']$. Then $f(K_1') = \overline{J_1}$, $f(\{x', y'\}) = \{a, b\}$ and no other point in K_1' is mapped to a or b by f. If J_1 is closed, then $K_1 := K_1'$ is suitable. Otherwise, it is easy to check that K_1 can be chosen among $(x', y'), [x', y'), (x', y']$ in such a way that $f(K_1) = J_1$ and $K_1 \subset J_0$.

• Suppose that Fact 1 holds for n and consider nonempty intervals $J_0, \dots, J_n$, J_{n+1} such that $J_i \subset f(J_{i-1})$ for all $i \in [\![1, n+1]\!]$. Let $K_1, \dots, K_n$ be the intervals given by Fact 1 applied to $J_0, \dots, J_n$. Since $f^{n+1}(K_n) = f(J_n) \supset J_{n+1}$, we can apply the case $n = 1$ for the map $g := f^{n+1}$ and the two intervals K_n, J_{n+1}. We deduce that there exists an interval $K_{n+1} \subset K_n$, which is closed if $J_0, \dots, J_{n+1}$ are closed, and such that

$$f^{n+1}(K_{n+1}) = J_{n+1},$$
$$f^{n+1}(\partial K_{n+1}) = \partial J_{n+1} \text{ and } f^{n+1}(\mathrm{Int}\,(K_{n+1})) = \mathrm{Int}\,(J_{n+1})\,.$$

Moreover, $f^i(K_{n+1}) \subset J_i$ for all $i \in [\![0, n]\!]$ because $K_{n+1} \subset K_n$. This ends the proof of Fact 1, which trivially implies (i).

Let $(J_0, \dots, J_n)$ be a chain of intervals such that $J_0 \supset J_n$. Fact 1 implies that there exists a closed interval $K_n \subset J_0$ such that $f^n(K_n) = J_n$ and $f^i(K_n) \subset J_i$ for all $i \in [\![0, n]\!]$. Thus $f^n(K_n) \supset K_n$ and it is sufficient to apply Lemma 1.11 to $g := f^n|_{K_n}$ in order to find a point $x \in K_n$ such that $f^n(x) = x$. For all $i \in [\![0, n-1]\!]$, $f^i(x)$ obviously belongs to J_i. This proves (ii).

Let $(J_0^i, \dots, J_n^i)_{1 \leq i \leq p}$ be chains of intervals satisfying the assumptions of (iii). For every $i \in [\![1, p]\!]$, let $(K_1^i, \dots, K_n^i)$ be the closed intervals given by Fact 1 for $(J_0^i, \dots, J_n^i)$, and set $K_i := K_n^i$. We fix $i \neq j$ in $[\![1, p]\!]$. By assumption, there exists $k \in [\![0, n]\!]$ such that J_k^i and J_k^j have disjoint interiors. If $k = 0$, then K_i and K_j have trivially disjoint interiors because they are respectively included in J_0^i and J_0^j. From now on, we assume that $k \geq 1$. Suppose that $K_i^k \cap K_j^k \neq \emptyset$. The set $f^k(K_i^k \cap K_j^k)$ is included in $J_k^i \cap J_k^j$ and, by assumption, J_k^i and J_k^j have disjoint interiors. Therefore the intervals J_k^i and J_k^j have a common endpoint, say b, and $f^k(K_i^k \cap K_j^k) = \{b\}$. By definition of K_i^k, there is a unique point z in K_i^k such that $f^k(z) = b$, and the same holds for K_j^k. Hence $K_i^k \cap K_j^k$ contains at most one point. Since $K_i \subset K_i^k$ and $K_j \subset K_j^k$, the intervals K_i and K_j have disjoint interiors. This concludes the proof of (iii). $\square$

Definitions for graph maps. The notion of covering extends to graph maps provided Definition 1.12 is phrased differently. A modification is needed for two reasons:

- one may want to consider circles as "intervals" whose endpoints are equal,
- for a graph map f, it may occur that a compact interval I satisfies $f(I) \supset I$ but contains no fixed point, as illustrated in Figure 3.

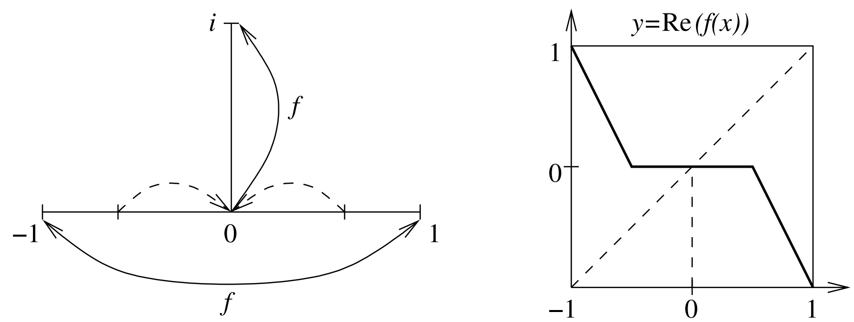


FIGURE 3. Let $f\colon X \to X$ be a continuous map, where X is the tree $[-1,1] \cup i[0,1] \subset \mathbb{C}$ (on the left), and f is such that $f(-1) = 1$, $f(1) = -1$, $f(0) = i$ and f is one-to-one on $[-1,0]$ and $[0,1]$ (the definition of f on $i[0,1]$ does not matter). Set $I := [-1,1]$. It is clear that $f(I) \supset I$. Nevertheless f has no fixed point in I. On the right is represented the real part of $f|_I$; the constant interval corresponds to the points $x \in I$ such that $f(x) \in i[0,1]$.

DEFINITION 1.14. Let $f\colon G \to G$ be a graph map and let J, K be two non degenerate intervals in G. Then J is said to *cover* K if there exists a subinterval $J' \subset J$ such that $f(J') = K$ and $f(\partial J') = \partial K$. If $J_0, J_1, \ldots, J_n$ are intervals in X such that J_{i-1} covers J_i for all $i \in [\![1, n]\!]$, then $(\overline{J_0}, \ldots, \overline{J_n})$ is a *chain of intervals* (this is a slight abuse of notation since, if $\overline{J_i}$ is a circle, it is necessary to remember the endpoint of J_i).

Using this definition, Lemma 1.13(ii)-(iii) remains valid for graph maps. In particular, if $(\overline{J_0}, \ldots, \overline{J_{n-1}}, \overline{J_0})$ is a chain of intervals for a graph map f, then there exists a point $x \in \overline{J_0}$ such that $f^n(x) = x$ and $f^i(x) \in \overline{J_i}$ for all $i \in [\![1, n-1]\!]$.

A variant, called *positive covering*, has been introduced in [**16**]. Positive covering does not imply covering, but implies the same conclusions concerning periodic points. We do not state the definition because it will not be needed in this book. See [**16, 17**] for the details.

1.5. Directed graphs

A (finite) directed graph G is a pair (V, A) where V, A are finite sets and there exist two maps $i, f\colon A \to V$. The elements of V are the *vertices* of G and the elements of A are the *arrows* of G. An arrow $a \in A$ goes from its *initial vertex* $u = i(a)$ to its *final vertex* $v = f(a)$. The arrow a is also denoted by $u \xrightarrow{a} v$. A directed graph is often given by a picture, as in Example 1.15. If $V = \{v_1, \ldots, v_p\}$, the *adjacency matrix* of G is the matrix $M = (m_{ij})_{1 \le i,j \le p}$, where m_{ij} is equal to the number of arrows from v_i to v_j. Conversely, if $M = (m_{ij})_{1\le i,j\le p}$ is a matrix such that $m_{ij} \in \mathbb{Z}^+$ for all $i, j \in [\![1, p]\!]$, one can build a directed graph whose adjacency matrix is M: it has p vertices $\{v_1, \dots, v_p\}$ and there are m_{ij} arrows from v_i to v_j for all $i, j \in [\![1, p]\!]$.

A directed graph is *simple* if, for every pair of vertices (u, v), there is at most one arrow from u to v. In this case, an arrow $u \xrightarrow{a} v$ is simply denoted by $u \to v$ since there is no ambiguity. A directed graph is simple if and only if all the coefficients of its adjacency matrix belong to $\{0, 1\}$.

There are several, equivalent norms for matrices. We shall use the following one: if $M = (m_{ij})_{1\le i,j\le p}$, we set $\|M\| := \sum_{1\le i,j\le p} |m_{ij}|$.

EXAMPLE 1.15. Figure 4 represents a directed graph with three vertices v_1, v_2, v_3. Its adjacency matrix is $\begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

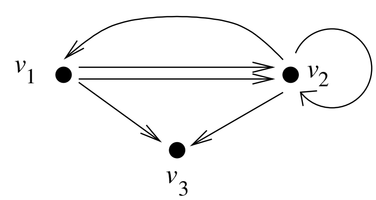


FIGURE 4. An example of a directed graph.

Let G be a directed graph. A *path of length n from u_0 to u_n* is a sequence

$$u_0 \xrightarrow{a_1} u_1 \xrightarrow{a_2} u_2 \xrightarrow{a_3} \cdots u_{n-1} \xrightarrow{a_n} u_n,$$

where $u_0, \dots, u_n$ are vertices of G and $u_i \xrightarrow{a_i} u_{i+1}$ is an arrow in G for all i in $[\![0, n-1]\!]$. Such a path is called a *cycle* if $u_0 = u_n$.

If $\mathcal{A} := A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} A_n$ and $\mathcal{B} := B_0 \xrightarrow{b_1} B_1 \xrightarrow{b_2} \cdots \xrightarrow{b_m} B_m$ are two paths such that $A_n = B_0$, the *concatenation* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{AB}$, is the path

$$A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} A_n \xrightarrow{b_1} B_1 \xrightarrow{b_2} \cdots \xrightarrow{b_m} B_m.$$

If $\mathcal{A}$, $\mathcal{B}$ are of respective lengths n, m, then $\mathcal{AB}$ is of length $n + m$.

A cycle is *primitive* if it is not the repetition of a shorter cycle, that is, it cannot be written $\underbrace{\mathcal{A}\mathcal{A}\cdots\mathcal{A}}_{n \text{ times}}$ where $\mathcal{A}$ is a cycle and $n \ge 2$.

A straightforward computation leads to the following result.

PROPOSITION 1.16. *Let G be a directed graph and let $\{v_1, \dots, v_p\}$ denote its set of vertices. Let M be its adjacency matrix. For every $n \in \mathbb{N}$, let $M^n = (m^n_{ij})_{1\le i,j\le p}$. Then, $\forall n \ge 1$, $\forall i, j \in [\![1, p]\!]$, the number of paths of length n from v_i to v_j is equal to m^n_{ij}. As a consequence, the number of paths of length n in G is equal to*

$$\|M^n\| = \sum_{1\le i,j\le p} m^n_{ij}.$$

CHAPTER 2

Links between transitivity, mixing and sensitivity

2.1. Transitivity and mixing

We are going to see that, for interval maps, the properties of total transitivity, topological weak mixing and topological mixing coincide, contrary to the general case. Moreover the notions of transitivity and topological mixing are very close. Indeed, if f is a transitive interval map which is not topologically mixing, then the interval can be divided into two invariant subintervals and f^2 is topologically mixing on each of them. We shall also give some properties equivalent to topological mixing for interval maps. The results of this section are classical (see, e.g., [**41**]).

2.1.1. Definitions.

Definition 2.1 (transitivity, transitive set). Let (X, f) be a topological dynamical system. The map f is *transitive* if, for all nonempty open sets U, V in X, there exists $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$ (or, equivalently, $U \cap f^{-n}(V) \neq \emptyset$).

A *transitive set* is an invariant set $E \subset X$ such that $f|_E \colon E \to E$ is transitive.

Definition 2.2 (total transitivity). Let (X, f) be a topological dynamical system. The map f is *totally transitive* if f^n is transitive for all $n \geq 1$,

The next result provides an equivalent definition of transitivity when the space has no isolated point (see e.g. [**73**]). Lemma 2.4 states two easy properties of transitive interval maps.

Proposition 2.3. *Let (X, f) be a topological dynamical system.*

i) *If f is transitive, there exists a dense G_δ-set of points whose orbit is dense in X. If a point x has a dense orbit, then $\omega(x, f) = X$ and the orbit of $f^n(x)$ is dense in X for all $n \geq 0$. Moreover, either X is finite, or X has no isolated point.*
ii) *If there exists a point whose orbit is dense in X and if X has no isolated point, then f is transitive.*
iii) *If there exists a point x such that $\omega(x, f) = X$, then f is transitive.*

In particular, if X has no isolated point, then f is transitive iff there is a point of dense orbit iff there is a point $x \in X$ such that $\omega(x, f) = X$.

Proof. Assume first that f is transitive. Let U be a nonempty open set. By transitivity, for every nonempty open set V, there exists $n \geq 0$ such that $f^{-n}(U) \cap V \neq \emptyset$. In other words, $\bigcup_{n\geq 0} f^{-n}(U)$ is dense in X. Since X is a compact metric space, there exists a countable basis of nonempty open sets, say $(U_k)_{k\geq 0}$. For all $k \geq 0$, the set $\bigcup_{n\geq 0} f^{-n}(U_k)$ is a dense open set by transitivity. Let

$$G := \bigcap_{k\geq 0} \bigcup_{n\geq 0} f^{-n}(U_k).$$

Then G is a dense G_δ-set and, if $x \in G$, then $f^n(x)$ enters any set U_k for some n, which means that $\mathcal{O}_f(x)$ is dense in X.

Assume that x_0 is an isolated point and set $U := \{x_0\}$. Since U is a nonempty open set, $U \cap G \neq \emptyset$, that is, x_0 has a dense orbit. We set $V_0 := f^{-1}(U)$; this is an open set. Suppose that V_0 is empty. Then $f^{-n}(U) = \emptyset$ for all $n \geq 1$. The space X is not reduced to $\{x_0\}$ (otherwise we would have $f(x_0) = x_0$ and $V_0 = \{x_0\}$), and thus there exists a nonempty open set V not containing x_0. This implies that $f^{-n}(U) \cap V = \emptyset$ for $n = 0$ and for all $n \geq 1$, which contradict the transitivity. Therefore, V_0 is a nonempty open set. By transitivity, there exists $n \geq 0$ such that $f^n(U) \cap V_0 \neq \emptyset$. This implies that $f^n(x_0) \in V_0$ and $f^{n+1}(x_0) = x_0$. Therefore, the point x_0 is periodic. Since x_0 has a dense orbit, this implies that X is finite and equal to $\mathcal{O}_f(x_0)$. In this case, f acts as a cyclic permutation on X and it is clear that for every point $x \in X$, $X = \mathcal{O}_f(x) = \omega(x_0, f)$.

By refutation, if f is transitive and X is infinite, then X has no isolated point.

Assume now that there exists a point x whose orbit is dense in X and that X has no isolated point. Let U, V be two nonempty open sets in X. There exists an integer $n \geq 0$ such that $f^n(x) \in U$. The set $V \setminus \{x, f(x), \dots, f^n(x)\}$ is open, and it is nonempty because X has no isolated point. Thus there exists $m \geq 0$ such that $f^m(x) \in V \setminus \{x, f(x), \dots, f^n(x)\}$. It follows that $m > n$ and $f^m(x) = f^{m-n}(f^n(x)) \in f^{m-n}(U) \cap V$, and thus $f^{m-n}U \cap V \neq \emptyset$. We deduce that f is transitive, which is (ii). Moreover, we have proved that, for all nonempty open sets V, for all $n \geq 0$, there exists $m > n$ such that $f^m(x) \in V$. This implies that the orbit of $f^n(x)$ is dense for all $n \geq 0$, and hence $\omega(x, f) = X$. This ends the proof of (i).

Finally, assume that $\omega(x, f) = X$ for some point $x \in X$. This implies that every nonempty open set contains some point $f^n(x)$ with n arbitrarily large. Let U, V be two nonempty open sets in X. Then there exist integers $n_2 > n_1 \geq 0$ such that $f^{n_1}(x) \in U$ and $f^{n_2}(x) \in V$. Then $n_2 - n_1 > 0$ and $f^{n_2-n_1}(U)$ contains the point $f^{n_2-n_1}(f^{n_1}(x)) = f^{n_2}(x)$. Thus $f^{n_2-n_1}(U) \cap V \neq \emptyset$. This implies that f is transitive, which is (iii). □

LEMMA 2.4. *Let $f\colon I \to I$ be a transitive interval map.*

i) *The image of a non degenerate interval is a non degenerate interval.*
ii) *The map f is onto.*

PROOF. Let J be a non degenerate interval. Since J is connected, $f(J)$ is also connected, that is, it is an interval. Suppose that $f(J)$ is reduced to a single point; we write $f(J) = \{y\}$. By Proposition 2.3, there exists a point $x \in J$ whose orbit is dense, and $y = f(x)$ also has a dense orbit. Thus there exists $n \geq 0$ such that $f^n(y) \in J$. This implies that $y = f^{n+1}(y)$, that is, y is a periodic point. But this is impossible because the orbit of y is dense in I. We deduce that the interval $f(J)$ is not degenerate and thus (i) holds.

By Proposition 2.3, there exists a point x such that $I = \overline{\{f^n(x) \mid n \geq 1\}}$. Notice that $\{f^n(x) \mid n \geq 1\} \subset f(I)$. Since I is compact, $f(I)$ is compact too, and hence $I \subset f(I)$, which implies that $f(I) = I$. This is (ii). □

DEFINITION 2.5 (mixing, weak mixing). Let (X, f) be a topological dynamical system. The map f is *topologically mixing* if, for all nonempty open sets U, V in

X, there exists an integer $N \geq 0$ such that, $\forall n \geq N$, $f^n(U) \cap V \neq \emptyset$. The map f is *topologically weakly mixing* if $f \times f$ is transitive, where $f \times f$ is the map

$$\begin{array}{rcl} X \times X & \to & X \times X \\ (x,y) & \mapsto & (f(x), f(y)) \end{array}$$

It is well known that topological mixing implies topological weak mixing (see, e.g., [**73**]). Moreover, topological weak mixing implies total transitivity. This is a folklore result. It can be proved using the following result, due to Furstenberg [**82**].

PROPOSITION 2.6. *Let (X, f) be a topological dynamical system. If f is topologically weakly mixing, then the product system $(X^n, \underbrace{f \times \cdots \times f}_{n \text{ times}})$ is transitive for all integers $n \geq 1$.*

PROOF. For all open sets U, V in X, we define

$$N(U, V) := \{n \geq 0 \mid U \cap f^{-n}(V) \neq \emptyset\}.$$

Let U_1, U_2, V_1, V_2 be nonempty open sets in X. Since $f \times f$ is transitive, there exists an integer $n \geq 0$ such that $(U_1 \times V_1) \cap (f \times f)^{-n}(U_2 \times V_2) \neq \emptyset$, that is, $U_1 \cap f^{-n}(U_2) \neq \emptyset$ and $V_1 \cap f^{-n}(V_2) \neq \emptyset$. We first remark that this implies

$$\forall\, U_1, U_2 \text{ nonempty open sets, } N(U_1, U_2) \neq \emptyset. \tag{2.1}$$

Now we are going to show that there exist nonempty open sets U, V such that $N(U, V) \subset N(U_1, V_1) \cap N(U_2, V_2)$. We set $U := U_1 \cap f^{-n}(U_2)$ and $V := V_1 \cap f^{-n}(V_2)$. These sets are open, and we have shown that they are not empty. Let $k \in N(U, V)$. This integer exists by (2.1) and satisfies $U_1 \cap f^{-n}(U_2) \cap f^{-k}(V_1) \cap f^{-n-k}(V_2) \neq \emptyset$. This implies that $U_1 \cap f^{-k}(V_1) \neq \emptyset$ and $U_2 \cap f^{-k}(V_2) \neq \emptyset$, and thus $N(U, V) \subset N(U_1, V_1) \cap N(U_2, V_2)$. Then, by a straightforward induction, we see that, for all nonempty open sets $U_1, \ldots, U_n, V_1 \ldots, V_n$, there exist nonempty open sets U, V such that

$$N(U, V) \subset N(U_1, V_1) \cap N(U_2, V_2) \cap \cdots \cap N(U_n, V_n).$$

Combined with (2.1), this implies that $(X^n, f \times \cdots \times f)$ is transitive. $\square$

THEOREM 2.7. *Let (X, f) be a topological dynamical system. If f is topologically mixing, then it is topologically weakly mixing. If f is topologically weakly mixing, then f^n is topologically weakly mixing for all $n \geq 1$ and f is totally transitive.*

PROOF. First we assume that f is topologically mixing. Let W_1, W_2 be two nonempty open sets in $X \times X$. There exist nonempty open sets U, U', V, V' in X such that $U \times U' \subset W_1$ and $V \times V' \subset W_2$. Since f is topologically mixing, there exists $N \geq 0$ such that, $\forall n \geq N$, $f^n(U) \cap V \neq \emptyset$ and $f^n(U') \cap V' \neq \emptyset$. Hence $f^N(W_1) \cap W_2 \neq \emptyset$. We deduce that f is topologically weakly mixing.

From now on, we assume that f is topologically weakly mixing and we fix $n \geq 1$. Let U, U', V, V' be nonempty open sets in X. We define

$$W := U \times f^{-1}(U) \times \cdots \times f^{-(n-1)}(U) \times V \times f^{-1}(V) \times \cdots \times f^{-(n-1)}(V)$$

and

$$W' := \underbrace{U' \times \cdots \times U'}_{n \text{ times}} \times \underbrace{V' \times \cdots \times V'}_{n \text{ times}}.$$

The sets W, W' are open in X^{2n}. Moreover, $(X^{2n}, f\times\cdots\times f)$ is transitive by Proposition 2.6. Thus there exists $k\geq 0$ such that $f^{-k}(W)\cap W'\neq\emptyset$. This implies that $f^{-(k+i)}(U)\cap U'\neq\emptyset$ and $f^{-(k+i)}(V)\cap V'\neq\emptyset$ for all $i\in[\![0,n-1]\!]$. We choose $i\in[\![0,n-1]\!]$ such that $k+i$ is a multiple of n; we write $k+i=np$. We deduce that $(f\times f)^{-np}(U\times V)\cap(U'\times V')\neq\emptyset$. Therefore, f^n is topologically weakly mixing. This trivially implies that f^n is transitive. □

Here is an equivalent definition of mixing for interval maps.

Proposition 2.8. *An interval map $f\colon [a,b]\to[a,b]$ is topologically mixing if and only if for all $\varepsilon>0$ and all non degenerate intervals $J\subset[a,b]$, there exists an integer N such that $f^n(J)\supset[a+\varepsilon,b-\varepsilon]$ for all $n\geq N$.*

Proof. Suppose first that f is topologically mixing. Let $\varepsilon>0$. Let $U_1:=(a,a+\varepsilon)$ and $U_2:=(b-\varepsilon,b)$. If J is a nonempty open interval, there exists N_1 such that $f^n(J)\cap U_1\neq\emptyset$ for all $n\geq N_1$ because f is topologically mixing. Similarly, there exists N_2 such that $f^n(J)\cap U_2\neq\emptyset$ for all $n\geq N_2$. Therefore, for all $n\geq\max\{N_1,N_2\}$, $f^n(J)$ meets both U_1 and U_2, which implies that $f^n(J)\supset[a+\varepsilon,b-\varepsilon]$ by connectedness. If J is a non degenerate subinterval, the same result holds by considering the nonempty open interval $\mathrm{Int}\,(J)$.

Suppose now that, for every $\varepsilon>0$ and every non degenerate interval $J\subset[a,b]$, there exists an integer N such that $f^n(J)\supset[a+\varepsilon,b-\varepsilon]$ for all $n\geq N$ Let U,V be two nonempty open sets in $[a,b]$. We choose two nonempty open subintervals J,K such that $J\subset U$, $K\subset V$ and neither a nor b is an endpoint of K. There exists $\varepsilon>0$ such that $K\subset[a+\varepsilon,b-\varepsilon]$. By assumption, there exists N such that $f^n(J)\supset[a+\varepsilon,b-\varepsilon]\supset K$ for all $n\geq N$. This implies that $f^n(U)\cap V\neq\emptyset$ for all $n\geq N$. We conclude that f is topologically mixing. □

2.1.2. A basic example of mixing map. In the sequel, we shall need to show that several interval maps are transitive or mixing. In some simple cases, this can be done by using Lemmas 2.10 and 2.11, combined together.

Recall that the definition of critical points is given page 5.

Definition 2.9. Let f be an interval map and $\lambda>1$. Suppose that f has finitely or countably many critical points. The map f is called λ-*expanding* if, for every subinterval $[x,y]$ on which f is monotone, $|f(y)-f(x)|\geq\lambda|x-y|$.

Lemma 2.10. *Let $f\colon I\to I$ be a λ-expanding interval map with $\lambda>N$, where N is a positive integer. Then, for every non degenerate subinterval J, there exists an integer $n\geq 0$ such that $f^n(J)$ contains at least N distinct critical points.*

Proof. Let C_f be the set of critical points of f. We set $\alpha:=\lambda/N>1$. Consider a nonempty open subinterval J. If J contains exactly k distinct critical points with $k\in[\![0,N-1]\!]$, then $J\setminus C_f$ has $k+1$ connected components, say $J_0,\ldots,J_k$, and $|J_0|+\cdots+|J_k|=|J|$. By the pigeonhole principle, there exists $i\in[\![0,k]\!]$ such that $|J_i|\geq\frac{|J|}{k+1}\geq\frac{|J|}{N}$. Since J_i contains no critical point, the map $f|_{J_i}$ is monotone. Hence $|f(J_i)|\geq\lambda|J_i|$ and

$$|f(J)|\geq|f(J_i)|\geq\lambda|J_i|\geq\alpha|J|. \tag{2.2}$$

Suppose that, for all $n\geq 0$, $f^n(J)$ contains strictly less than N distinct critical points. Then $|f^n(J)|\geq\alpha^n|J|$ for all $n\geq 0$ by (2.2). But this is impossible because $|f^n(J)|$ is bounded by $|I|$ whereas $\alpha^n|J|$ goes to infinity when $n\to+\infty$. This is sufficient to conclude the proof because any non degenerate interval contains a nonempty open interval. □

LEMMA 2.11. *Let $f: I \to I$ be an interval map, $\lambda > 1$ and $a, b \in I$ with $a < b$. Suppose that $f(a) = a$ and*

$$\forall x \in [a,b],\ f(x) - f(a) \geq \lambda(x-a).$$

Then, for all $\varepsilon > 0$, there exists $n \geq 0$ such that $f^n([a, a+\varepsilon]) \supset [a,b]$.

PROOF. Let $\varepsilon \geq 0$. If $\varepsilon > b-a$, then $f^0([a,a+\varepsilon]) \supset [a,b]$. Suppose on the contrary that $a+\varepsilon \leq b$. Then $f(a+\varepsilon) - f(a) \geq \lambda\varepsilon$ by assumption, and hence $f([a,a+\varepsilon]) \supset [a, a+\lambda\varepsilon]$ by the intermediate value theorem (recall that $f(a)=a$). A straightforward induction on n shows that $f^n([a,a+\varepsilon]) \supset [a, a+\lambda^n\varepsilon]$ as long as $a + \lambda^{n-1}\varepsilon \leq b$. Since $\lambda > 1$, there exists an integer $n \geq 1$ such that $a + \lambda^{n-1}\varepsilon \leq b < a + \lambda^n\varepsilon$. Hence $f^n([a,a+\varepsilon]) \supset [a,b]$. □

REMARK 2.12. We shall use Lemmas 2.10 and 2.11 for piecewise linear maps (as in Example 2.13 below) or for maps $f: I \to I$ such that the interval I can be divided into countably many subintervals on each of which f is linear. In these situations, f is λ-expanding if and only if the absolute value of the slope of f is greater than or equal to λ on each interval on which f is linear.

In Lemma 2.11, the assumption $f(x) - f(a) \geq \lambda(x-a)$ is verified as soon as $f|_{[a,b]}$ is linear of slope greater than or equal to λ.

EXAMPLE 2.13. We are going to exhibit a family of topologically mixing interval maps. These maps are piecewise linear, and the absolute value of their slope is constant. These maps are basic examples; they will be reused later to build other examples.

We fix an integer $p \geq 2$. We define the map $T_p: [0,1] \to [0,1]$ by:

$$\forall 0 \leq k \leq \frac{p-1}{2},\ \forall x \in \left[\frac{2k}{p}, \frac{2k+1}{p}\right], \qquad T_p(x) := px - 2k,$$

$$\forall 0 \leq k \leq \frac{p-2}{2},\ \forall x \in \left[\frac{2k+1}{p}, \frac{2k+2}{p}\right], \qquad T_p(x) := -px + 2k + 2.$$

The slope of T_p is either p or $-p$ on each interval of monotonicity. More precisely,

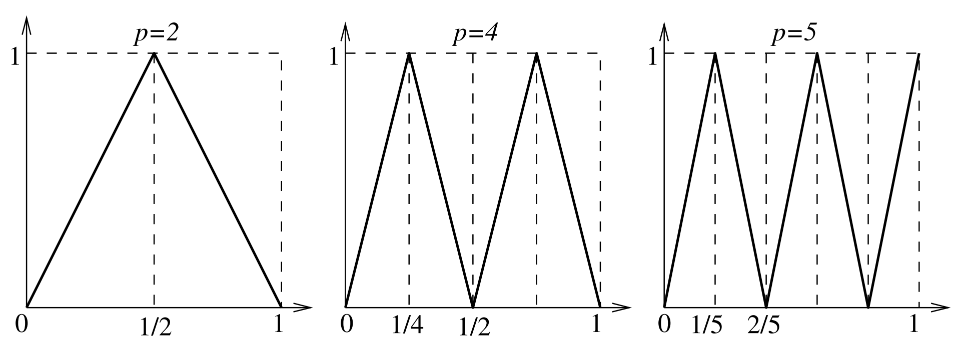


FIGURE 1. Mixing maps T_p of slope $\pm p$, for $p=2$, $p=4$ and $p=5$. The map T_2 (on the left) is called the tent map.

starting from the fixed point 0, the slope is alternatively $p, -p, p, \ldots, (-1)^{p-1}p$, and the image of each interval of monotonicity is $[0,1]$. See Figure 1 for the graph of T_p. The map T_2 is the so called *tent map*.

Let J be a non degenerate interval in $[0,1]$. The image of a non degenerate interval by T_p is obviously non degenerate, so $T_p^n(J)$ is a non degenerate interval for all $n \geq 0$. By Lemma 2.10, there exists n such that $T_p^n(J)$ contains $p-1$ distinct critical points. If $p \geq 3$, $T_p^n(J)$ contains at least one critical point whose image is 0, and thus $0 \in T^{n+2}(J)$ because 0 is a fixed point. If $p = 2$, $T_p^n(J)$ contains the unique critical point $1/2$, and $T_p^2(1/2) = T_p(1) = 0$. In both cases, $T_p^{n+2}(J)$ is a non degenerate interval containing 0. Applying Lemma 2.11 with $a = 0$ and $b = \frac{1}{p}$ (T_p is of slope $\lambda = p$ on $[0, \frac{1}{p}]$), we deduce that there exists an integer $m \geq 0$ such that $T_p^{n+m+2}(J) \supset \left[0, \frac{1}{p}\right]$, which implies that $T_p^{n+m+3}(J) \supset [0,1]$, and hence $T_p^k(J) = [0,1]$ for all $k \geq n+m+3$. Conclusion: T_p is topologically mixing.

2.1.3. Transitivity implies denseness of the set of periodic points. Proposition 2.15 below states that the set of periodic points of a transitive interval map is dense. This is a consequence of a result of Sharkovsky [**154**]. We are going to follow the proof of [**41**]; see also [**26**] for a different proof. This result will be needed in the next section; a stronger theorem will be given in Chapter 3.

LEMMA 2.14. *Let $f\colon I \to I$ be an interval map, $x, y \in I$ and $n, m \in \mathbb{N}$. Let J be a subinterval of I containing no periodic point and suppose that x, y, $f^m(x)$, $f^n(y)$ belong to J. If $x < f^m(x)$ then $y < f^n(y)$.*

PROOF. We set $g := f^m$. We first prove by induction on k that $g^k(x) > x$ for all $k \geq 1$. By assumption, the statement holds for $k = 1$. Suppose that $g^i(x) > x$ for all $i \in [\![1, k-1]\!]$ and that $g^k(x) \leq x$. We write

$$\{g^i(x) \mid i \in [\![0, k-1]\!]\} = \{x_0 \leq x_1 \leq \cdots \leq x_{k-1}\}.$$

We have $x_0 = x$ and $x_1 \neq x$ because x is not periodic. It follows that

$$g^k(x) \leq x = x_0 < x_1 \leq \cdots \leq x_{k-1}.$$

Let j be the integer in $[\![1, k-1]\!]$ such that $x_1 = g^j(x)$. By the intermediate value theorem,

$$g^{k-j}([x_0, x_1]) \supset [g^k(x), g^{k-j}(x)] \supset [x_0, x_1].$$

Therefore, by Lemma 1.11, g has a periodic point in $[x_0, x_1]$. But $[x_0, x_1] \subset J$ because $x_1 = \min\{g^i(x) \mid i \in [\![1, k-1]\!]\} \leq g(x)$. This leads to a contradiction since J contains no periodic point for $g = f^m$. We deduce that $g^k(x) > x$ for all $k \geq 1$ and the induction is over.

Suppose that $f^n(y) < y$. The same argument as above (with reverse order) shows that $f^{kn}(y) < y$ for all $k \geq 1$. Hence

$$y > f^{mn}(y) \quad \text{and} \quad x < f^{mn}(x).$$

The map $t \mapsto f^{mn}(t) - t$ is continuous on the interval $\langle x, y\rangle$ (recall that $\langle x, y\rangle$ is either $[x, y]$ or $[y, x]$ depending on the order of x, y). Thus, by the intermediate value theorem, there exists a point $z \in \langle x, y\rangle$ such that $f^{mn}(z) = z$. This leads to a contradiction because $\langle x, y\rangle \subset J$. Thus we conclude that $y < f^n(y)$ (equality is not possible because y is not periodic). $\square$

PROPOSITION 2.15. *If $f\colon I \to I$ is a transitive interval map, then the set of periodic points is dense in I.*

Proof. Suppose that there exist $a, b \in I$, with $a < b$, such that (a, b) contains no periodic point. Since f is transitive, there exists a point $x \in (a, b)$ with a dense orbit (Proposition 2.3). Thus there exist integers $m > 0$ and $0 < p < q$ such that $x < f^m(x) < b$ and $a < f^q(x) < f^p(x) < x$. We set $y := f^p(x)$. We then have

$$a < f^{q-p}(y) < y < x < f^m(x) < b.$$

But this is impossible by Lemma 2.14 applied to $J = (a, b)$. This concludes the proof. □

2.1.4. Transitivity, total transitivity and mixing. The next proposition states that, if an interval map f is transitive then, either f is totally transitive, or the interval can be divided into two subintervals on each of which f^2 is totally transitive. Then Proposition 2.17 states that total transitivity implies mixing. These two results were proved by Barge and Martin [**26, 28**]. Blokh also showed the same results a little earlier, but in an unpublished paper [**45**]. We are going to follow the ideas of the proof of Barge and Martin. Blokh's proof, which is different, can be found in [**55**].

Proposition 2.16. *Let $f\colon [a, b] \to [a, b]$ be a transitive interval map. Then one of the following cases holds:*

i) *The map f is totally transitive.*
ii) *There exists $c \in (a, b)$ such that $f([a, c]) = [c, b]$, $f([c, b]) = [a, c]$, and both maps $f^2|_{[a,c]}$ and $f^2|_{[c,b]}$ are totally transitive. Moreover, c is the unique fixed point of f.*

Proof. Since f is transitive, there exists a point $x_0 \in [a, b]$ such that $\omega(x_0, f) = [a, b]$ by Proposition 2.3. We fix an integer $n \geq 1$ and we set $W_i^n := \omega(f^i(x_0), f^n)$ for all $i \in [\![0, n-1]\!]$. We have $[a, b] = W_0^n \cup \cdots \cup W_{n-1}^n$ by Lemma 1.3(iv), which implies that at least one of the sets $W_1^n, \ldots, W_{n-1}^n$ has a nonempty interior by the Baire category theorem (Corollary 8.52). Moreover, according to Lemma 1.3(ii)-(iii),

$$\forall i \in [\![0, n-2]\!],\ f(W_i^n) = W_{i+1}^n \quad \text{and} \quad f(W_{n-1}^n) = W_0^n. \tag{2.3}$$

Thus, all the sets $W_0^n, \ldots, W_{n-1}^n$ have nonempty interiors by Lemma 2.4(i).

Suppose that $\mathrm{Int}\,(W_i^n) \cap \mathrm{Int}\,\big(W_j^n\big) \neq \emptyset$. Since the set $\mathrm{Int}\,(W_i^n) \cap \mathrm{Int}\,\big(W_j^n\big)$ is open and included in $W_i^n = \omega(f^i(x_0), f^n)$, there exists $k \geq 0$ such that $f^{kn+i}(x_0)$ belongs to $\mathrm{Int}\,(W_i^n) \cap \mathrm{Int}\,\big(W_j^n\big)$. Moreover, $f^n(W_j^n) = W_j^n$ (Lemma 1.3(i)), and thus $f^{k'n+i}(x_0) \in W_j^n$ for all $k' \geq k$. This implies that $W_i^n \subset W_j^n$. The same argument shows that $W_j^n \subset W_i^n$. Therefore

$$\text{if } \mathrm{Int}\,(W_i^n) \cap \mathrm{Int}\,\big(W_j^n\big) \neq \emptyset, \text{ then } W_i^n = W_j^n. \tag{2.4}$$

Let $\mathcal{E}_n$ be the collection of all connected components of the sets $(\mathrm{Int}\,(W_i^n))_{0 \leq i \leq n-1}$. The elements of $\mathcal{E}_n$ are open intervals and, by (2.4), two different elements of $\mathcal{E}_n$ are disjoint. For every $C \in \mathcal{E}_n$, the closed interval $f(\overline{C})$ is non degenerate by Lemma 2.4, and is contained in W_i^n for some $i \in [\![0, n-1]\!]$. Thus, by connectedness, there exists $C' \in \mathcal{E}_n$ such that $f(\overline{C}) \subset \overline{C'}$. Moreover, the orbit of x_0 enters infinitely many times every element of $\mathcal{E}_n$, which implies that, for all $C, C' \in \mathcal{E}_n$, there exists $k \geq 1$ such that $f^k(\overline{C}) \cap \overline{C'} \neq \emptyset$, and hence $f^k(\overline{C}) \subset \overline{C'}$. It follows that $\mathcal{E}_n$ is finite and the closures of its elements are cyclically permuted under the action of f. Thus we can write $\mathcal{E}_n = \{C_1, \ldots, C_{p_n}\}$ for some integer $p_n \geq 1$, the C_i's satisfying

$$\forall i \in [\![1, p_n - 1]\!],\ f(\overline{C_i}) \subset \overline{C_{i+1}} \quad \text{and} \quad f(\overline{C_{p_n}}) \subset \overline{C_1}.$$

The fact that the orbit of x_0 is dense implies that $C_1 \cup \cdots \cup C_{p_n}$ is dense too. Since $C_1, \ldots, C_{p_n}$ are disjoint open intervals, we deduce that $C_1 \cup \cdots \cup C_{p_n}$ is equal to $[a, b]$ deprived of finitely many points, which are the endpoints of $C_1, \ldots, C_{p_n}$.

If $p_n = 1$, then $W_0^n = \cdots = W_{n-1}^n$, and thus $\omega(x_0, f^n) = [a, b]$. Therefore, if $\mathcal{E}_n$ has a single element for every integer $n \geq 1$, then f is totally transitive and we are in case (i) of the proposition. From now on, we suppose that, for a given n, the number p_n of elements of $\mathcal{E}_n$ is greater than 1. We are going to show that $p_n = 2$. Let $c \in [a, b]$ be a fixed point of f (such a point exists by Lemma 1.11). If there exists $C \in \mathcal{E}_n$ with $c \in C$, then $f(\overline{C}) = \overline{C}$. Similarly, if c is an endpoint of $[a, b]$, then there is a unique $C \in \mathcal{E}_n$ such that $c \in \overline{C}$, and thus $f(\overline{C}) = \overline{C}$. In both cases, this leads to a contradiction because $\overline{C_1}, \ldots, \overline{C_{p_n}}$ are cyclically permuted and $p_n \geq 2$. We deduce that c belongs to (a, b) and is a common endpoint of two distinct elements of $\mathcal{E}_n$, say C and C'. The fact that c is a fixed point implies that the only possibility for permuting cyclically $\overline{C_1}, \ldots, \overline{C_{p_n}}$ is that $p_n = 2$, $\mathcal{E}_n = \{C, C'\}$, $f(\overline{C}) = \overline{C'}$ and $f(\overline{C'}) = \overline{C}$. We thus have $\mathcal{E}_n = \{[a, c), (c, b]\}$ and

$$f([a, c]) = [c, b], \quad f([c, b]) = [a, c]. \tag{2.5}$$

This implies that c is the unique fixed point of f. Let $\mathcal{N} := \{i \in [\![0, n-1]\!] \mid C \subset W_i^n\}$ and $\mathcal{N}' := \{i \in [\![0, n-1]\!] \mid C' \subset W_i^n\}$. The sets $\mathcal{N}, \mathcal{N}'$ are nonempty and their union is $[\![0, n-1]\!]$ by definition of $\mathcal{E}_n$. We cannot have $C \cup C' \subset W_i^n$ for some $i \in [\![0, n-1]\!]$; otherwise the connected set $\overline{C \cup C'}$ would be included in W_i^n, which would contradict the fact that C, C' are distinct elements of $\mathcal{E}_n$. This implies that $\mathcal{N}, \mathcal{N}'$ are disjoint. Since $W_0^n, \ldots, W_{n-1}^n$ are cyclically permuted by f according to (2.3), a set W_i^n with $i \in \mathcal{N}$ (resp. $i \in \mathcal{N}'$) is sent to a set W_j^n with $j \in \mathcal{N}'$ (resp. $j \in \mathcal{N}$). This implies that $\mathcal{N}, \mathcal{N}'$ have the same number of elements, and that the integer n is necessarily even. So $\omega(x_0, f^n) \subset \omega(x_0, f^2)$ by Lemma 1.3(iv). Combining this with (2.5), we see that $\{W_0^2, W_1^2\} = \{[a, c], [c, b]\}$. Therefore both maps $f^2|_{[a,c]}$ and $f^2|_{[c,b]}$ are transitive. If $f^2|_{[a,c]}$ is not totally transitive, then the same argument as above, applied to the map $f^2|_{[a,c]}$, shows that $f^2|_{[a,c]}$ has a unique fixed point, which belongs to (a, c). But this is impossible because c is already a fixed point of $f^2|_{[a,c]}$. We conclude that $f^2|_{[a,c]}$ is totally transitive, and so is $f^2|_{[c,b]}$ for similar reasons, and we are in case (ii) of the proposition. □

PROPOSITION 2.17. *Let $f \colon I \to I$ be an interval map. If f is totally transitive, then it is topologically mixing.*

PROOF. We write $I = [a, b]$. Let J be a non degenerate subinterval of I and $\varepsilon > 0$. According to Proposition 2.15, the periodic points are dense in I. Thus, there exist periodic points x, x_1, x_2 with $x \in J$, $x_1 \in (a, a + \varepsilon)$ and $x_2 \in (b - \varepsilon, b)$. Moreover, x_1 and x_2 can be chosen in such a way that their orbits are included in (a, b) because there is at most one periodic orbit containing a (resp. b). We set

$$\forall i \in \{1, 2\}, \ y_i := \min\{f^n(x_i) \mid n \geq 0\} \text{ and } z_i := \max\{f^n(x_i) \mid n \geq 0\}.$$

Then $y_1 \in (a, x_1] \subset (a, a + \varepsilon)$, $z_2 \in [x_2, b) \subset (b - \varepsilon, b)$ and $y_2, z_1 \in (a, b)$. Let k be a common multiple of the periods of x, y_1 and y_2. We set $g := f^k$ and

$$K := \bigcup_{n=0}^{+\infty} g^n(J).$$

The point $x \in J$ is fixed under the action of g and thus $g^n(J)$ contains x for all $n \geq 0$. This implies that K is an interval. Moreover K is dense in $[a, b]$ because g

is transitive, and hence $K \supset (a,b)$. It follows that $y_1, y_2, z_1, z_2 \in K$. For $i = 1, 2$, let p_i and q_i be non negative integers such that $y_i \in g^{p_i}(J)$ and $z_i \in g^{q_i}(J)$. We set $N := \max\{p_1, p_2, q_1, q_2\}$. Since y_1, y_2, z_1, z_2 are fixed points of g, they belong to $g^N(J)$ and thus, by the intermediate value theorem, $[y_i, z_i] \subset g^N(J) = f^{kN}(J)$ for $i = 1, 2$. According to the definition of y_i, z_i, the interval $[y_i, z_i]$ contains the whole orbit of x_i. A trivial induction shows that $[y_i, z_i] \subset f^n([y_i, z_i])$ for all $n \geq 0$. Therefore,

$$\forall n \geq kN,\ [y_1, z_1] \cup [y_2, z_2] \subset f^n(J).$$

Since $y_1 < a + \varepsilon$ and $z_2 > b - \varepsilon$, the fact that $f^n(J)$ is connected implies that $[a - \varepsilon, b + \varepsilon] \subset f^n(J)$ for all $n \geq kN$. We conclude that f is topologically mixing, using Proposition 2.8. □

Corollary 2.18. *Let $f: I \to I$ be a transitive interval map. Then f is topologically mixing if and only if it has a periodic point of odd period greater than* 1.

Proof. We write $I = [a, b]$. Suppose first that f is topologically mixing. The set of fixed points of f is closed, and it has an empty interior (otherwise, it would contradict the mixing assumption). Thus we can choose a non degenerate closed subinterval $J \subset (a, b)$ such that J contains no fixed point. Since f is topologically mixing, there exists an integer N such that $f^n(J) \supset J$ for all $n \geq N$ (Proposition 2.8). We choose an odd integer $n \geq N$. Applying Lemma 1.11, we obtain a point $x \in J$ such that $f^n(x) = x$. The period of x is odd because it divides n, and it is greater than 1 because J contains no fixed point.

Suppose now that f is transitive but not totally transitive. We are in case (ii) of Proposition 2.16: there exists a fixed point $c \in (a, b)$ such that $f([a, c]) = [c, b]$ and $f([c, b]) = [a, c]$. Consequently, every periodic point has an even period, except c. By refutation, a transitive map with a periodic point of odd period different from 1 is totally transitive, and thus topologically mixing by Proposition 2.17. □

2.1.5. Transitivity vs. mixing – summary theorems. The next two theorems sum up the results 2.7, 2.8, 2.16, 2.17 and 2.18. The first one is about the difference between transitivity and mixing. The second one states several properties equivalent to mixing.

Theorem 2.19. *Let $f: [a, b] \to [a, b]$ be a transitive interval map. Then one of the following cases holds:*

- *The map f is topologically mixing.*
- *There exists $c \in (a, b)$ such that $f([a, c]) = [c, b]$, $f([c, b]) = [a, c]$, and both maps $f^2|_{[a,c]}, f^2|_{[c,b]}$ are topologically mixing. In addition, c is the unique fixed point of f.*

Theorem 2.20. *Let $f: [a, b] \to [a, b]$ be an interval map. The following properties are equivalent:*

- *f is transitive and has a periodic point of odd period different from* 1.
- *f^2 is transitive.*
- *f is totally transitive.*
- *f is topologically weakly mixing.*
- *f is topologically mixing.*
- *For all $\varepsilon > 0$ and all non degenerate intervals J, there exists an integer N such that $f^n(J) \supset [a + \varepsilon, b - \varepsilon]$ for all $n \geq N$.*

Example 2.21. We give an example of a transitive, non topologically mixing interval map. The map $S\colon [-1,1] \to [-1,1]$, represented in Figure 2, is defined by:

$$\begin{cases} \forall x \in [-1, -\frac{1}{2}], & S(x) := 2x + 2, \\ \forall x \in [-\frac{1}{2}, 0], & S(x) := -2x, \\ \forall x \in [0, 1], & S(x) := -x. \end{cases}$$

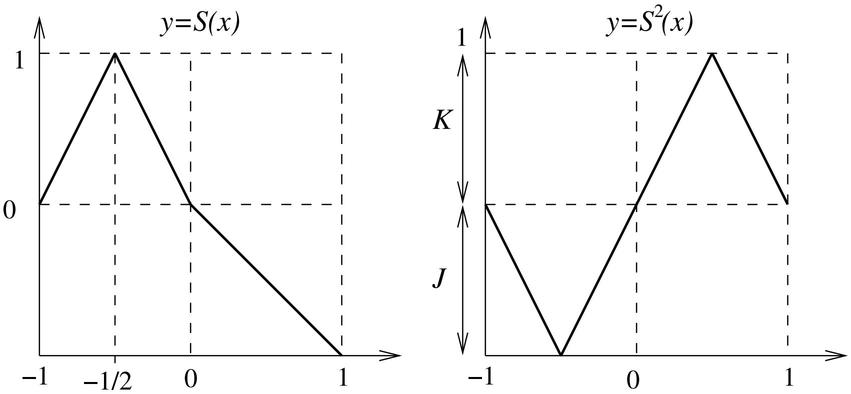


Figure 2. The map S is transitive but not topologically mixing because S^2 is not transitive.

We set $J := [-1, 0]$ and $K := [0, 1]$. We have $S(J) = K$ and $S(K) = J$, which implies that S is not topologically mixing. Since $S^2|_K$ is equal to the tent map T_2 defined in Example 2.13, the map $S^2|_K$ is topologically mixing and, for every non degenerate subinterval $U \subset K$, there exists $n \geq 0$ such that $S^{2n}(U) = K$. The map $S^2|_J$ is similar to $S^2|_K$ except that its graph is upside down. Therefore, if U is a nonempty open set, then

- either $U \cap J$ contains a non degenerate interval, and there exists $n \geq 0$ such that $S^{2n}(U) \supset J$,
- or $U \cap K$ contains a non degenerate interval, and there exists $n \geq 0$ such that $S^{2n}(U) \supset K$.

In both cases, there exists $n \geq 0$ such that $S^n(U) \cup S^{n+1}(U) = [-1, 1]$, which implies that S is transitive.

Remarks on graph maps. Rotations are important examples because they exhibit behaviors that cannot appear for interval maps.

Definition 2.22. Let $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. The *rotation* of angle $\alpha \in \mathbb{R}$ is the circle map:

$$\begin{array}{rccl} R_\alpha\colon & \mathbb{S} & \longrightarrow & \mathbb{S} \\ & x & \longmapsto & x + \alpha \bmod 1 \end{array}$$

If $\alpha = \frac{p}{q}$ with $p \in \mathbb{Z}, q \in \mathbb{N}$ and $\gcd(p, q) = 1$, it is clear that all points in $\mathbb{S}$ are periodic of period q, and R_α^q is the identity map. On the contrary, if $\alpha \notin \mathbb{Q}$ (in this case, R_α is called an *irrational rotation*), one can show that R_α is totally transitive but not topologically weakly mixing, and there is no periodic point. Consequently, if one wants to generalize the results concerning transitive interval maps, the case of irrational rotations must be excluded. We shall see that the results of this section extend fairly well to transitive graph maps, irrational rotations being the only exceptions up to conjugacy.

The next result is due to Blokh [**48**] (see [**49**, Theorem S, p. 506] for a statement in English).

THEOREM 2.23. *A transitive graph map with no periodic point is conjugate to an irrational rotation on the circle.*

For graph maps, transitivity is still close to total transitivity, which is equivalent to topological mixing unless for irrational rotations. The next two theorems generalize Propositions 2.16 and 2.17; they are due to Blokh [**48**] (see [**52**] for a statement in English).

THEOREM 2.24. *Let $f\colon G\to G$ be a totally transitive graph map. If f is not conjugate to an irrational rotation, it is topologically mixing.*

THEOREM 2.25. *Let $f\colon G\to G$ be a transitive graph map. Then there exist a cycle of graphs $(G_1,\dots,G_p)$ such that $f^p|_{G_i}$ is totally transitive for all $i\in[\![1,p]\!]$.*

Alsedà, del Río and Rodríguez proved that a splitting close to the preceding theorem holds in a broader situation [**4**]. More precisely, if (X,f) is a topological dynamical system where X is locally connected, then, either f is totally transitive, or there exist $k\geq 2$ and closed subsets $X_1,\dots,X_k$ with disjoint interiors, whose union is X, such that $f(X_i)=X_{i+1 \bmod k}$ and $f^k|_{X_i}$ is transitive for all $i\in[\![1,k]\!]$. Then they showed that a graph map has a splitting of maximal cardinality (bounded by combinatorial data of the graph), which implies Theorem 2.25.

2.2. Accessible endpoints and mixing

An interval map $f\colon[a,b]\to[a,b]$ is topologically mixing if the iterates of every non degenerate interval J eventually cover "almost all" $[a,b]$ (to be precise, if $f^n(J)\supset[a+\varepsilon,b-\varepsilon]$ for all $\varepsilon>0$ and all large enough n). When do the iterates of every non degenerate interval eventually cover the whole interval $[a,b]$? Blokh showed that this property holds if and only if the two endpoints of $[a,b]$ are *accessible* (see Definition 2.29 below). Proposition 2.30 and Lemma 2.32 about (non) accessible points are stated in [**45**] (see [**55**] for a published paper).

DEFINITION 2.26 (locally eventually onto). A topological dynamical system (X,f) is *locally eventually onto* (or *leo*) if, for every nonempty open set $U\subset X$, there exists an integer N such that $f^n(U)=X$ for all $n\geq N$.

A dynamical system with this property is trivially topologically mixing.

REMARK 2.27. In the literature, the name *topologically exact* is synonymous to locally eventually onto.

Alternative definitions of locally eventually onto appear in the literature. They are equivalent according to the next lemma.

LEMMA 2.28. *Let (X,f) be a topological dynamical system. The following two properties are equivalent:*

i) *(X,f) is locally eventually onto.*
ii) *For every nonempty open set U, there exists $n\geq 0$ such that $f^n(U)=X$.*

An interval map $f\colon I\to I$ is locally eventually onto if and only if

iii) $\forall \varepsilon > 0,\ \exists M \geq 0, \forall J$ *subinterval of* $I,\ |J| > \varepsilon \Rightarrow \forall n \geq M,\ f^n(J) = I$.

PROOF. The implication (i)⇒(ii) is trivial. Suppose that (ii) holds and let U be a nonempty open set. There exists an integer n such that $f^n(U) = X$. This implies that f is onto and thus, $\forall m \geq n$, $f^m(U) = f^{m-n}(X) = X$. Hence (ii)⇒(i).

Let $f\colon I \to I$ be an interval map. If (iii) holds, then f is locally eventually onto because every nonempty open set U contains an interval J with $|J| > 0$. Now we assume that f is locally eventually onto. Let $\varepsilon > 0$. We write $I = [a, b]$ and we choose an integer $k \geq 1$ such that $\frac{b-a}{k} < \frac{\varepsilon}{2}$. For all $i \in [\![0, k-1]\!]$, we set

$$J_i := \left(a + \frac{i}{k}(b-a), a + \frac{i+1}{k}(b-a)\right).$$

For every $i \in [\![0, k-1]\!]$, we choose an integer N_i such that $f^n(J_i) = I$ for all $n \geq N_i$, and we set $M := \max\{N_i \mid i \in [\![0, k-1]\!]\}$. If J is a subinterval of I with $|J| > \varepsilon$, then J contains J_i for some $i \in [\![0, k-1]\!]$. Thus $f^n(J) = I$ for all $n \geq M$, that is, (iii) holds. □

DEFINITION 2.29 (accessible endpoint). Let $f\colon [a, b] \to [a, b]$ be an interval map. The endpoint a (resp. b) is *accessible* if there exist $x \in (a, b)$ and $n \geq 1$ such that $f^n(x) = a$ (resp. $f^n(x) = b$).

PROPOSITION 2.30. *Let $f\colon [a, b] \to [a, b]$ be a topologically mixing interval map. Then f is locally eventually onto if and only if both a and b are accessible.*

More precisely, for every $\varepsilon > 0$ and every non degenerate subinterval $J \subset (a, b)$, there exists N such that $f^n(J)$ contains $[a, b-\varepsilon]$ (resp. $[a+\varepsilon, b]$) for all $n \geq N$ if and only if a (resp. b) is accessible.

PROOF. We show the second part of the proposition; the first statement follows trivially.

First we suppose that a is accessible. Let $x_0 \in (a, b)$ and $n_0 \geq 1$ be such that $f^{n_0}(x_0) = a$. Let $\varepsilon > 0$ be such that $x_0 \in [a+\varepsilon, b-\varepsilon]$. Let J be a non degenerate subinterval in $[a, b]$. Since f is topologically mixing, there exists an integer $N \geq 0$ such that $f^n(J) \supset [a+\varepsilon, b-\varepsilon]$ for all $n \geq N$. Since $x_0 \in [a+\varepsilon, b-\varepsilon]$, the intermediate value theorem implies that $f^{n+n_0}(J) \supset [a, b-\varepsilon]$ for all $n \geq N$. Conversely, if J is a subinterval containing neither a nor b and such that $a \in f^n(J)$ for some integer $n \geq 1$, then the point a is accessible by definition. This shows that a is accessible if and only if, for every $\varepsilon > 0$ and every non degenerate subinterval $J \subset (a, b)$, there exists N such that $f^n(J)$ contains $[a, b-\varepsilon]$ for all $n \geq N$. The case of the endpoint b is similar. □

REMARK 2.31. An interval map $f\colon I \to I$ is called *strongly transitive* if, for every non degenerate subinterval J, there exists $N \geq 0$ such that $\bigcup_{n=0}^{N} f^n(J) = I$. This definition is due to Parry [**141**]. This notion is very close to the property of being locally eventually onto. Indeed, if a topologically mixing map is strongly transitive, then it is locally eventually onto. Using Theorem 2.19, one can reduce the transitive case to the mixing one and sees that a transitive interval map is strongly transitive if and only if the two endpoints of I are accessible. In this case, for every non degenerate subinterval J, there exists an integer $n \geq 0$ such that $f^n(J) \cup f^{n+1}(J) = I$.

The next lemma specifies the behavior of a mixing map near a non accessible endpoint. Roughly speaking, a mixing map has infinitely many oscillations in a neighborhood of a non accessible endpoint.

LEMMA 2.32. *Let $f\colon [a,b] \to [a,b]$ be a topologically mixing interval map.*

i) *If a (resp. b) is the unique non accessible endpoint, then it is a fixed point. If both a and b are non accessible then, either $f(a) = a$ and $f(b) = b$, or $f(a) = b$ and $f(b) = a$.*

ii) *If a (resp. b) is a fixed non accessible point, then there exists a decreasing (resp. increasing) sequence of fixed points $(x_n)_{n\geq 0}$ converging to a (resp. b). Moreover, for all $n \geq 0$, $f|_{[x_{n+1},x_n]}$ is not monotone.*

PROOF. i) If a is not accessible, then $a \notin f((a,b))$. Since f is topologically mixing, it is onto (Lemma 2.4(ii)). Thus, either $f(a) = a$, or $f(b) = a$. If b is accessible and $f(b) = a$, then a is accessible too. Therefore, if a is the only non accessible endpoint, then $f(a) = a$. Similarly, if b is the only non accessible endpoint, then $f(b) = b$. If both a and b are non accessible then, either $f(a) = a$ and $f(b) = b$, or $f(a) = b$ and $f(b) = a$.

ii) Assume that a is not accessible and that $f(a) = a$ (the case of b is symmetric). By definition, $a \notin f((a,b))$. According to (i), if b is not accessible, then $f(b) = b$. If b is accessible, then $f(b) \neq a$. In both cases, $a \notin f((a,b])$. Let $\varepsilon \in (0, b-a)$. By transitivity, $f([a,a+\varepsilon]) \not\subset [a,a+\varepsilon]$. Thus there exists $y \in (a,a+\varepsilon]$ such that $f(y) \geq a+\varepsilon$. In particular, y satisfies $f(y) \geq y$. Suppose that $f(x) \geq x$ for all $x \in [a,y]$. We set

$$z := \min\{y, \min(f([y,b]))\}.$$

Then $z > a$, $f([z,b]) = f([z,y]) \cup f([y,b])$, and both $f([z,y])$ and $f([y,b])$ are included in $[z,b]$ by definition of z. Hence $f([z,b]) \subset [z,b]$. But this contradicts the transitivity of f. We deduce that there exists $x \in [a,y]$ such that $f(x) < x$. Thus $f([x,y]) \supset [f(x),f(y)] \supset [x,y]$, and necessarily $x \neq a$. By Lemma 1.11, there is a fixed point in $[x,y] \subset (a,a+\varepsilon]$. Since ε can be chosen arbitrarily small, this implies that there exists a decreasing sequence of fixed points $(x_n)_{n\geq 0}$ with $\lim_{n\to+\infty} x_n = a$. Moreover, $f|_{[x_{n+1},x_n]}$ is not monotone, otherwise we would have $f([x_{n+1},x_n]) = [x_{n+1},x_n]$, which would contradict the transitivity. $\square$

REMARK 2.33. In Lemma 2.32, notice that, if a is a non accessible endpoint which is not fixed, then $f^2(a) = a$ by (i), so statement (ii) holds for the map f^2.

The next result states that the kind of behavior described in Lemma 2.32(ii) is impossible if f is piecewise monotone or C^1. The piecewise monotone case can be found in [**70**] (more precisely, Coven and Mulvey proved in [**70**] that a transitive piecewise monotone map is strongly transitive; see Remark 2.31 for the relation between locally eventually onto and strong transitivity). Recall that f is piecewise monotone if the interval can be divided into finitely many subintervals on each of which f is monotone (see page 5).

PROPOSITION 2.34. *Let $f\colon [a,b] \to [a,b]$ be a topologically mixing interval map. If f is piecewise monotone or C^1, then the two endpoints a,b are accessible, and thus f is locally eventually onto.*

PROOF. Suppose that a is not accessible. Then $f^2(a) = a$ by Lemma 2.32(i). We set $g := f^2$. The map g is topologically mixing because f is topologically mixing. If f is C^1, then g is C^1 too. The case $g'(a) < 0$ is impossible because $g(a) = a$. If $g'(a) = 0$, then there exists $c \in (a, b)$ such that $g(x) < x$ for all $x \in (a, c)$, which is impossible because g is transitive. Thus $g'(a) > 0$ and g is increasing in a neighborhood of a. Similarly, if f is piecewise monotone, then g is increasing in a neighborhood of a. In both cases, there exists $c \in (a, b)$ such that $g|_{[a,c]}$ is increasing. But, according to Lemma 2.32(ii), there exist two distinct points $x < y$ in (a, c) such that $g|_{[x,y]}$ is not monotone, a contradiction. The case when b is not accessible is similar. We conclude that both a, b are accessible, and thus f is locally eventually onto by Proposition 2.30. □

Remark 2.35. Proposition 2.34 remains valid under the assumption that the mixing map f is monotone (or C^1) in a neighborhood of the two endpoints.

Example 2.36. We give an example of an interval map $f : [0, 1] \to [0, 1]$ that is topologically mixing but not locally eventually onto. This example appears in [**28**] to illustrate another property.

Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of points in $(0, 1)$ such that $a_n < a_{n+1}$ for all $n \in \mathbb{Z}$, and

$$\lim_{n\to-\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to+\infty} a_n = 1.$$

For all $n \in \mathbb{Z}$, we set $I_n := [a_n, a_{n+1}]$ and we define $f_n : I_n \to I_{n-1} \cup I_n \cup I_{n+1}$ by $f_n(a_n) := a_n$, $f_n(a_{n+1}) =: a_{n+1}$,

$$f_n\left(\frac{2a_n + a_{n+1}}{3}\right) := a_{n+2}, \quad f_n\left(\frac{a_n + 2a_{n+1}}{3}\right) := a_{n-1},$$

and f_n is linear between the points where it has already been defined. Then we define the map $f : [0, 1] \to [0, 1]$ (see Figure 3) by

$$f(0) := 0, \quad f(1) := 1,$$
$$\forall n \in \mathbb{Z},\ \forall x \in I_n,\ f(x) := f_n(x).$$

It is easy to check that f is continuous and that the points 0 and 1 are not accessible. We are going to show that f is topologically mixing. Let J be a non degenerate subinterval of $[0, 1]$. Since f is 3-expanding, we can apply Lemma 2.10: there exists $n \geq 0$ such that $f^n(J)$ contains two distinct critical points. This implies that $f^{n+1}(J)$ contains I_k for some $k \in \mathbb{Z}$. Moreover, it is easy to see that

$$\forall k \in \mathbb{Z},\ \forall m \geq 0,\ f^m(I_k) \supset [a_{k-m}, a_{k+m+1}].$$

Since, for a given $k \in \mathbb{Z}$, the lengths of $[0, a_{k-m}]$ and $[a_{k+m+1}, 1]$ tend to 0 when m goes to infinity, we deduce that, for all $\varepsilon > 0$, there exists M such that $f^m(J)$ contains $[\varepsilon, 1 - \varepsilon]$ for all $m \geq M$. Hence f is topologically mixing.

Remarks on graph maps. Generalizing the results of this section to graph maps poses no difficulty provided the notion of non accessible points is extended to points that may not be endpoints. If $f : G \to G$ is a topologically mixing graph map, a point $x \in G$ is *accessible* if, for every nonempty open set $U \subset G$, $x \in \bigcup_{n\geq 0} f^n(U)$. For graph maps, Lemma 2.32(i) becomes: every non accessible point is periodic and its orbit is included in the set of non accessible points. We leave to the reader the "translation" of the other statements.

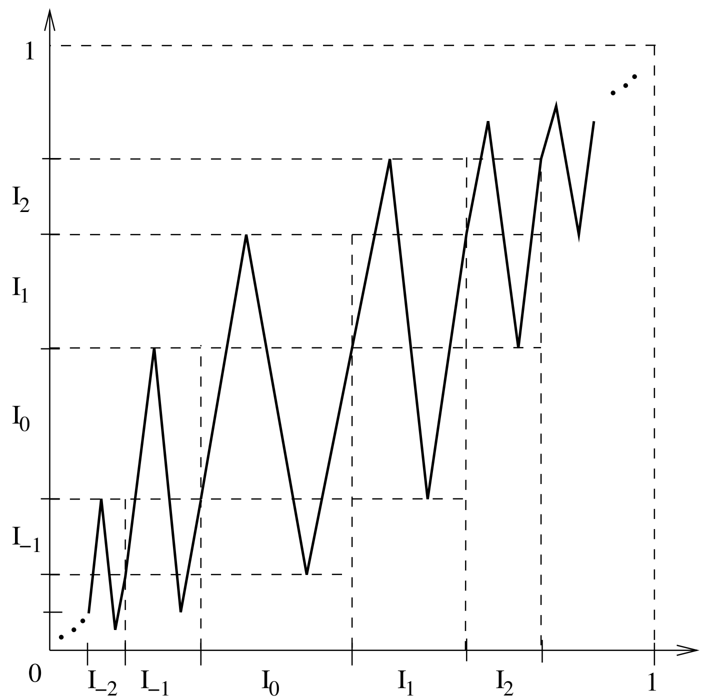


FIGURE 3. A topologically mixing map on $[0, 1]$ whose two endpoints are not accessible. For every non degenerate interval $J \subset (0, 1)$ and every $n \geq 0$, $f^n(J) \neq [0, 1]$.

The map of Example 2.36 can be seen as a circle map by gluing together the endpoints 0 and 1, and one gets a topologically mixing circle map with a fixed non accessible point (which is obviously not an endpoint).

2.3. Sensitivity to initial conditions

Roughly speaking, sensitivity to initial conditions means that there exist arbitrarily close points with divergent trajectories. We are going to see that, for interval maps, transitivity implies sensitivity and, conversely, sensitivity implies that an iterate of the map is transitive on a subinterval. This shows that the notions of transitivity and sensibility are closely related on the interval. The first implication is quite natural in view of the fact that transitivity is very close to mixing. The second implication may seem unexpected.

2.3.1. Definitions. A point x is ε-stable if the trajectories of all points in a neighborhood of x follow the trajectory of x up to ε, otherwise it is ε-unstable. The terminology "sensitivity to initial conditions" was first introduced by Guckenheimer [**86**] to mean that the ε-unstable points have a positive Lebesgue measure for some $\varepsilon > 0$. We would rather follow the definition of Devaney [**74**].

DEFINITION 2.37 (unstable point, sensitivity to initial conditions). Let (X, f) be a topological dynamical system and $\varepsilon > 0$. A point $x \in X$ is *ε-unstable (in the sense of Lyapunov)* if, for every neighborhood U of x, there exists $y \in U$ and $n \geq 0$

such that $d(f^n(x), f^n(y)) \geq \varepsilon$. The set of ε-unstable points is denoted by $U_\varepsilon(f)$. A point is *unstable* if it is ε-unstable for some $\varepsilon > 0$.

The map f is *ε-sensitive to initial conditions* (or more briefly *ε-sensitive*) if $U_\varepsilon(f) = X$. It is *sensitive to initial conditions* if it is ε-sensitive for some $\varepsilon > 0$.

The next lemma states some basic properties of the sets $U_\varepsilon(f)$. The last assertion gives an equivalent definition for sensibility.

LEMMA 2.38. *Let (X, f) be a topological dynamical system and $\varepsilon > 0$. The following properties hold:*

i) $\forall n \geq 1$, $U_\varepsilon(f^n) \subset U_\varepsilon(f)$.
ii) $\forall n \geq 1$, $\exists \delta > 0$, $U_\varepsilon(f) \subset U_\delta(f^n)$.
iii) $f(U_\varepsilon(f)) \subset U_\varepsilon(f)$.
iv) $\overline{U_\varepsilon(f)} \subset U_{\varepsilon/2}(f)$.
v) *If V is open and $V \cap U_\varepsilon(f) \neq \emptyset$, then there exists $n \geq 0$ such that* $\mathrm{diam}(f^n(V)) \geq \varepsilon$.
vi) *f is sensitive if and only if there exists $\delta > 0$ such that, for all nonempty open sets V, there exists $n \geq 0$ such that* $\mathrm{diam}(f^n(V)) \geq \delta$.

PROOF. i) Trivial.

ii) The map f is uniformly continuous because X is compact. Thus

$$\exists \delta > 0,\ d(x, y) < \delta \Rightarrow \forall i \in [\![0, n-1]\!],\ d(f^i(x), f^i(y)) < \varepsilon. \tag{2.6}$$

Let $x \notin U_\delta(f^n)$. Then there exists a neighborhood U of x such that, for all $y \in U$, $\forall k \geq 0$, $d(f^{kn}(x), f^{kn}(y)) < \delta$. Then (2.6) implies $d(f^{kn+i}(x), f^{kn+i}(y)) < \varepsilon$ for all $k \geq 0$ and all $i \in [\![0, n-1]\!]$. We deduce that $x \notin U_\varepsilon(f)$. This shows that $U_\varepsilon(f) \subset U_\delta(f^n)$.

iii) Let $x \in U_\varepsilon(f)$ and let V be a neighborhood of x. We first show that

$$\text{there are infinitely many } n \in \mathbb{N} \text{ such that } \exists y \in V,\ d(f^n(x), f^n(y)) \geq \varepsilon. \tag{2.7}$$

Suppose on the contrary that there exists n_0 such that, if $d(f^n(x), f^n(y)) \geq \varepsilon$ for some $y \in V$ and $n \geq 0$, then $n \leq n_0$. By the continuity of the maps $f, f^2, \ldots, f^{n_0}$, there exists $\delta > 0$ such that

$$\forall y \in X,\ d(x, y) < \delta \Rightarrow \forall k \in [\![0, n_0]\!],\ d(f^k(x), f^k(y)) < \varepsilon. \tag{2.8}$$

The set $W := V \cap B(x, \delta)$ is a neighborhood of x. Let $y \in W$ and $n \geq 0$. If $n \leq n_0$, then $d(f^n(x), f^n(y)) < \varepsilon$ by (2.8). If $n > n_0$, then $d(f^n(x), f^n(y)) < \varepsilon$ according to the choice of n_0. This contradicts the fact that x is ε-unstable. Hence (2.7) holds.

Now we consider an open set V containing $f(x)$. Since $U := f^{-1}(V)$ is open and contains x, what precedes implies that there exist $y \in U$ and $n \geq 2$ such that $d(f^n(x), f^n(y)) \geq \varepsilon$. We set $z := f(y)$. Then z belongs to V and $d(f^{n-1}(f(x)), f^{n-1}(z)) \geq \varepsilon$. Thus $f(x) \in U_\varepsilon(f)$.

iv) We fix $x \in \overline{U_\varepsilon(f)}$. Let V be an open set containing x. There exists a point $y \in U_\varepsilon(f) \cap V$, and thus, by definition, there exist $z \in V$ and $n \geq 0$ such that $d(f^n(y), f^n(z)) \geq \varepsilon$. By the triangular inequality, we have either $d(f^n(x), f^n(y)) \geq \varepsilon/2$, or $d(f^n(x), f^n(z)) \geq \varepsilon/2$. We deduce that $x \in U_{\varepsilon/2}(f)$.

v) Let V be an open set such that $V \cap U_\varepsilon(f) \neq \emptyset$. By definition, there exist $x \in V \cap U_\varepsilon(f)$, $y \in V$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) \geq \varepsilon$, that is, $\mathrm{diam}(f^n(V)) \geq \varepsilon$.

vi) First we assume that f is ε-sensitive, that is, $U_\varepsilon(f) = X$. By (v), for every nonempty open set V, there exists $n \geq 0$ such that $\mathrm{diam}(f^n(V)) \geq \varepsilon$.

Now we suppose that there exists $\delta > 0$ such that, for every nonempty open set V, there exists $n \geq 0$ such that $\mathrm{diam}(f^n(V)) \geq \delta$. We fix $\varepsilon \in (0, \delta/2)$. Let $x \in X$. Let V be an open set containing x and let $n \geq 0$ be such that $\mathrm{diam}(f^n(V)) \geq \delta$. Thus there exist two points $y, z \in V$ such that $d(f^n(y), f^n(z)) \geq \delta \geq 2\varepsilon$. The triangular inequality implies that, either $(f^n(x), f^n(y)) \geq \varepsilon$, or $(f^n(x), f^n(z)) \geq \varepsilon$. Hence $x \in U_\varepsilon(f)$, and the map f is ε-sensitive. $\square$

2.3.2. Sensitivity and transitivity. Barge and Martin proved that, for a transitive interval map, every point x is ε-unstable for some ε depending on x [**26**]. We give a different proof, which additionally shows that the constant of instability can be taken uniform for all points x.

PROPOSITION 2.39. *Let $f : I \to I$ be an interval map.*

- *If f is topologically mixing, then f is δ-sensitive for all $\delta \in (0, \frac{|I|}{2})$.*
- *If f is transitive, then f is δ-sensitive for all $\delta \in (0, \frac{|I|}{4})$.*

PROOF. We write $I = [a, b]$. First we assume that f is topologically mixing. Let $\varepsilon \in (0, \frac{|I|}{2})$, $x \in [a, b]$ and U be a neighborhood of x. By Theorem 2.20, there exists $n \geq 0$ such that $f^n(U) \supset [a + \varepsilon, b - \varepsilon]$. Therefore, there exist y, z in U such that $f^n(y) = a + \varepsilon$ and $f^n(z) = b - \varepsilon$. This implies that

$$\max\{|f^n(x) - f^n(y)|, |f^n(x) - f^n(z)|\} \geq \frac{b - a - 2\varepsilon}{2} = \frac{|I|}{2} - \varepsilon.$$

Consequently, x is δ-unstable, where $\delta := \frac{|I|}{2} - \varepsilon$. Since ε is arbitrary, the map f is δ-sensitive for every $\delta \in (0, \frac{|I|}{2})$.

Now we suppose that f is transitive but not topologically mixing. According to Theorem 2.19, there exists $c \in (a, b)$ such that $f([a, c]) = [c, b]$, $f([c, b]) = [a, c]$ and both maps $f^2|_{[a,c]}$ and $f^2|_{[c,b]}$ are topologically mixing. What precedes implies that $f^2|_{[a,c]}$ is δ-sensitive for all $\delta \in (0, \frac{c-a}{2})$. Therefore, according to Lemma 2.38(i)-(iii), we have $[a, c] \subset U_\delta(f)$ and $f([a, c]) = [c, b] \subset U_\delta(f)$. Thus f is δ-sensitive for all $\delta \in (0, \frac{c-a}{2})$. Similarly, f is δ-sensitive for all $\delta \in (0, \frac{b-c}{2})$. Finally, we conclude that f is δ-sensitive for all $\delta \in (0, \frac{|I|}{4})$ because $\max\{c - a, b - c\} \geq \frac{|I|}{2}$. $\square$

The converse of Proposition 2.39 is obviously false. However, somewhat surprisingly, a partial converse holds: the instability on a subinterval implies the existence of a transitive cycle of intervals. This result is due to Blokh; it is stated without proof in [**44**] (we do not know any reference for the proof).

PROPOSITION 2.40. *Let f be an interval map. Suppose that, for some $\varepsilon > 0$, the set of ε-unstable points $U_\varepsilon(f)$ has a nonempty interior. Then there exists a cycle of intervals $(J_1, \ldots, J_p)$ such that $f|_{J_1 \cup \cdots \cup J_p}$ is transitive. Moreover, $J_1 \cup \cdots \cup J_p \subset \overline{U_\varepsilon(f)}$ and there exists $i \in [\![1, p]\!]$ such that $|J_i| \geq \varepsilon$.*

PROOF. We consider the family of sets

$$\mathcal{F} := \{Y \subset \overline{U_\varepsilon(f)} \mid Y \text{ closed},\ f(Y) \subset Y,\ \mathrm{Int}\,(Y) \neq \emptyset\}.$$

By assumption, there exists a nonempty open interval $K \subset U_\varepsilon(f)$. Moreover, $f^n(K) \subset U_\varepsilon(f)$ for all $n \geq 0$ by Lemma 2.38(iii). The set $\overline{\bigcup_{n \geq 0} f^n(K)}$ is thus an element of $\mathcal{F}$, and hence $\mathcal{F} \neq \emptyset$. Let Y belong to $\mathcal{F}$ and let J be a non degenerate interval included in Y. Since $\mathrm{Int}\,(J) \cap U_\varepsilon(f) \neq \emptyset$, there exists $n \geq 0$ such that $|f^n(J)| \geq \varepsilon$ by Lemma 2.38(v). This implies that

$$\text{every } Y \in \mathcal{F} \text{ has a connected component } C \text{ with } |C| \geq \varepsilon. \tag{2.9}$$

We endow $\mathcal{F}$ with the partial order given by inclusion. We are first going to show that every totally ordered family of elements of $\mathcal{F}$ admits a lower bound in $\mathcal{F}$. Let $(Y_\lambda)_{\lambda\in\Lambda}$ be a family of elements of $\mathcal{F}$ which is totally ordered (that is, all the elements of Λ are comparable and $Y_\lambda \subset Y_{\lambda'}$ if $\lambda \leq \lambda'$). We set

$$Y := \bigcap_{\lambda\in\Lambda} Y_\lambda.$$

Then Y is a closed set, $f(Y) \subset Y$ and $Y \subset \overline{U_\varepsilon(f)}$. Moreover, each Y_λ has a finite non zero number of connected components of length at least ε, and thus so has Y. Therefore, $\mathrm{Int}\,(Y) \neq \emptyset$, so $Y \in \mathcal{F}$ and Y is a lower bound for $(Y_\lambda)_{\lambda\in\Lambda}$. Zorn's Lemma then implies that $\mathcal{F}$ admits at least one minimal element, say Z.

We now turn to prove that $f|_Z$ is transitive. The set Z has finitely or countably many non degenerate connected components, and at least one of them has a length greater than or equal to ε by (2.9). Let $(I_i)_{i\geq 1}$ be the (finite or infinite) family of all non degenerate connected components of Z, where $I_1, \ldots, I_k$ are the connected components of length at least ε (for some $k \geq 1$). Let $i \geq 1$. Since $\mathrm{Int}\,(I_i) \cap U_\varepsilon(f) \neq \emptyset$, there exists $n_i \geq 1$ such that $|f^{n_i}(I_i)| \geq \varepsilon$ by Lemma 2.38(v). Therefore, there exists $\tau_i \in [\![1, k]\!]$ such that $f^{n_i}(I_i) \subset I_{\tau_i}$. Since $[\![1, k]\!]$ is finite, this implies that there exist integers $j \in [\![1, k]\!]$ and $m \geq 1$ such that $f^m(I_j) \subset I_j$. The set $Z' := \bigcup_{n=0}^m f^n(I_j)$ obviously belongs to $\mathcal{F}$. Thus $Z' = Z$ by minimality, that is, Z has finitely many connected components which are cyclically mapped under f. We call $J_1, \ldots, J_p$ the connected components of Z, labeled in such a way that $f(J_i) \subset J_{i+1}$ for all $i \in [\![1, p-1]\!]$ and $f(J_p) \subset J_1$. By minimality of Z, these inclusions are actually equalities, that is, $(J_1, \ldots, J_p)$ is a cycle of intervals (note that $J_1, \ldots, J_k$ are closed). If $f|_Z$ is not transitive, there exist two open sets U, V such that

$$U \cap Z \neq \emptyset,\ V \cap Z \neq \emptyset \quad \text{and} \quad \forall n \geq 0,\ f^n(U \cap Z) \cap (V \cap Z) = \emptyset.$$

Since Z is the union of finitely many non degenerate intervals, there exists a nonempty open interval $J \subset U \cap Z$. We set

$$X := \overline{\bigcup_{n\geq 0} f^n(J)}.$$

Then X belongs to $\mathcal{F}$ and $X \subset Z$, but $X \cap V = \emptyset$, and thus $X \neq Z$. This contradicts the fact that Z is minimal. We conclude that $f|_Z$ is transitive. $\square$

Example 2.41. For a given $\varepsilon > 0$, the number of transitive cycles of intervals $(J_1, \ldots, J_p)$ given by Proposition 2.40 is finite because one of these intervals has a length at least ε and two different cycles have disjoint interiors by transitivity. Nevertheless, infinitely many transitive cycles of intervals can coexist if their constants of sensitivity tend to 0, as illustrated in Figure 4.

Example 2.42. Even if there is $\varepsilon > 0$ such that all points are ε-unstable, the union of all transitive cycles of intervals is not necessarily dense. In order to illustrate this fact, we are going to build a sensitive interval map which admits a transitive cycle of $p+1$ intervals and no other transitive cycle of intervals.

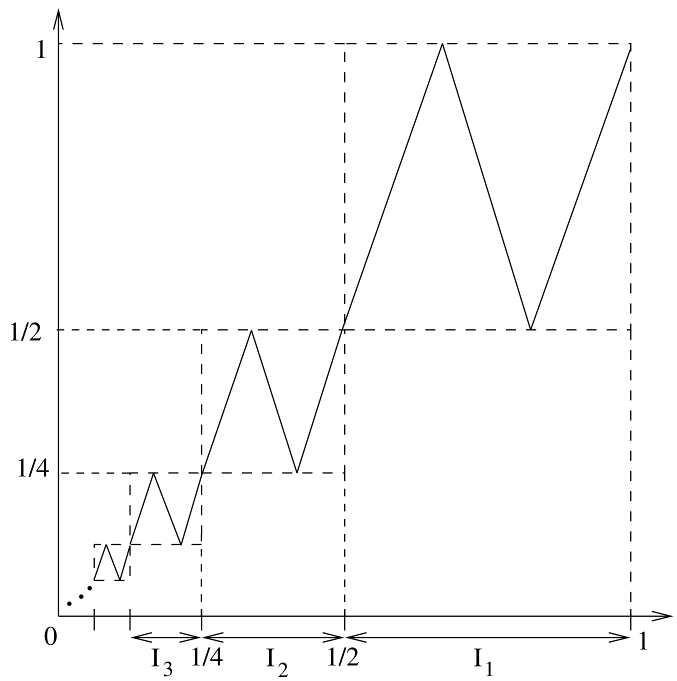


FIGURE 4. An interval map f with infinitely many transitive subintervals $(I_n)_{n\geq 1}$, where $I_n := \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$. The map $f|_{I_n}$ is equal to the map T_3 of Example 2.13, up to a rescaling. It is easy to show that $f|_{I_n}$ is ε_n-sensitive with $\varepsilon_n := \frac{1}{2^{n+1}}$ for all $n \geq 1$.

We fix an integer $p \geq 1$ and we set

$$\forall i \in [\![0, 2p+1]\!],\ x_i := \frac{i}{2p+1} \quad \text{and} \quad \forall i \in [\![0, 2p]\!],\ J_i := [x_i, x_{i+1}].$$

We define the continuous map $f\colon [0,1] \to [0,1]$ by

$$f(x_0) := x_3, \quad f\left(\frac{x_0+x_1}{2}\right) := x_2, \quad f(x_1) := x_3,$$
$$f(x_{2p-1}) := x_{2p+1}, \quad f(x_{2p}) := x_0, \quad f(x_{2p+1}) := x_1,$$

and f is linear between the points where it has already been defined (see Figure 5). Note that f is of slope 1 on $[x_1, x_{2p-1}]$ and $f(x_i) = x_{i+2}$ for all $i \in [\![1, 2p-1]\!]$. It is trivial to see that $(J_0, J_2, \ldots, J_{2i}, \ldots, J_{2p})$ is a cycle of intervals. Moreover, $f^{p+1}|_{J_0}$ is the map T_2 defined in Example 2.13 except that its graph is upside down (and rescaled). Since T_2 is topologically mixing, so is $f^{p+1}|_{J_0}$. By Proposition 2.39, $f^{p+1}|_{J_0}$ is δ-sensitive for $\delta := \frac{1}{2(2p+1)}$. Therefore, $C := J_0 \cup J_2 \cup \cdots \cup J_{2p}$ is a transitive cycle of $p+1$ intervals and $C \subset U_\delta(f)$.

We now turn to show that f is $\frac{\delta}{4}$-sensitive. Let $x \in [0,1]$. Suppose that there exists $n \geq 0$ such that $f^n(x)$ belongs to C. The form of C implies that there exists $\varepsilon_0 > 0$ such that either $f^n([x, x+\varepsilon_0]) \subset C$ or $f^n([x-\varepsilon_0, x]) \subset C$. Let $\varepsilon \in (0, \varepsilon_0]$. The image of a non degenerate interval is non degenerate and $C \subset U_\delta(f)$. Therefore, there exist two distinct points $y, z \in [x-\varepsilon, x+\varepsilon]$ and an integer $k \geq n$ such that

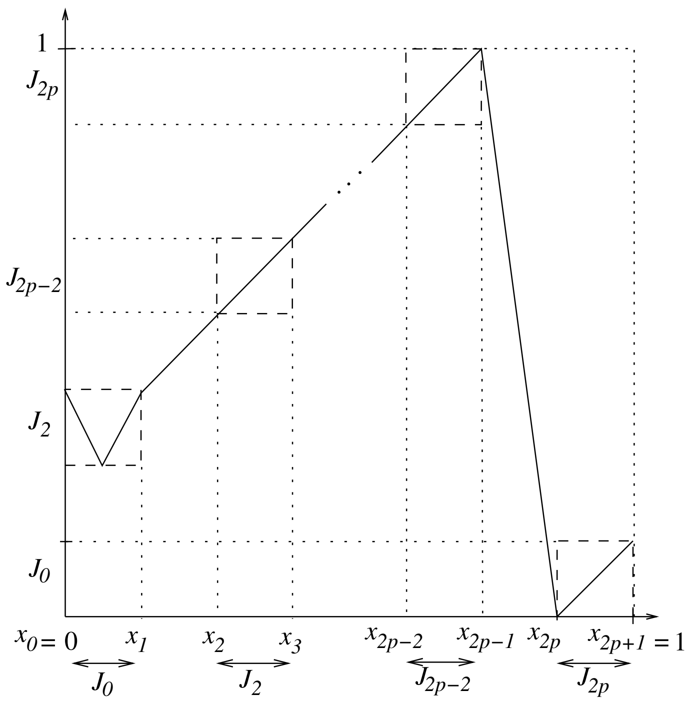


FIGURE 5. For this interval map, all points are ε-unstable with $\varepsilon := \frac{1}{8(2p+1)}$, but the system admits a single transitive cycle of intervals $(J_0, J_2, \dots, J_{2p-2}, J_{2p})$, where $J_i := \left[\frac{i}{2p+1}, \frac{i+1}{2p+1}\right]$.

$|f^k(y) - f^k(z)| \geq \delta$. By the triangular inequality, either $|f^k(x) - f^k(y)| \geq \frac{\delta}{2}$ or $|f^k(x) - f^k(z)| \geq \frac{\delta}{2}$. In other words:

$$\bigcup_{n\geq 0} f^{-n}(C) \subset U_{\frac{\delta}{2}}(f). \tag{2.10}$$

It remains to consider the points whose orbit does not meet C. These points are included in the set

$$\begin{aligned} X &:= \{x \in [0,1] \mid \forall n \geq 0, f^n(x) \in J_1 \cup J_3 \cup \cdots \cup J_{2p-1}\} \\ &= \bigcap_{n\geq 0} f^{-n}\left(\bigcup_{i=0}^{p-1} J_{2i+1}\right). \end{aligned}$$

We are going to show that X is a Cantor set (see Section 8.6 in Appendix for the definition of a Cantor set). Note that

$$\forall i \in [\![0, p-2]\!],\ f|_{J_{2i+1}} \colon J_{2i+1} \to J_{2i+3} \text{ is a linear homeomorphism.} \tag{2.11}$$

This fact implies that it is enough to focus on the set $X \cap J_{2p-1}$. The map $f|_{J_{2p-1}} \colon J_{2p-1} \to [0,1]$ is a linear homeomorphism, and thus the set of points $x \in J_{2p-1}$ such that $f(x) \in J_1 \cup J_3 \cup \cdots \cup J_{2p-1}$ is the union of p disjoint closed subintervals of equal lengths, which are $K_i := J_{2p-1} \cap f^{-1}(J_{2i+1})$, $0 \leq i \leq p-1$.

Using (2.11), we can see that the map $f^{p-i}|_{K_i}\colon K_i \to [0,1]$ is a linear homeomorphism for every $i \in [\![0, p-1]\!]$, and thus $\{x \in K_i \mid f^{p-i+1}(x) \in J_1 \cup J_3 \cup \cdots \cup J_{2p-1}\}$ is the union of p disjoint closed subintervals of equal lengths. Applying this argument inductively, we can show that X is a Cantor set. In particular, the interior of X is empty, and thus $\overline{[0,1] \setminus X} = [0,1]$. According to (2.10), $[0,1] \setminus X$ is included in $U_{\frac{\delta}{2}}(f)$. Then f is $\frac{\delta}{4}$-sensitive by Lemma 2.38(iv).

Finally, we show that C is the only transitive cycle of intervals of f. Suppose on the contrary that there exists another transitive cycle of intervals C'. By transitivity, the interiors of C and C' are disjoint, and thus there exists a non degenerate subinterval $J \subset [0,1]$ such that $f^n(J) \cap \mathrm{Int}\,(C) = \emptyset$ for all $n \geq 0$. This exactly means that $J \subset X$, but this is impossible because X is a Cantor set.

Remarks on graph maps and general dynamical systems. The first part of the proof of Proposition 2.39 can be easily adapted to show the following result.

PROPOSITION 2.43. *Let (X, f) be a topologically mixing dynamical system. Then f is δ-sensitive for every $\delta \in \left(0, \frac{\mathrm{diam}(X)}{2}\right)$.*

On the other hand, the next result is proved in [**25**] and [**83**].

THEOREM 2.44. *Let (X, f) be a transitive dynamical system having a dense set of periodic points. Then f is sensitive to initial conditions provided X is infinite.*

For interval maps, Proposition 2.39 can be derived from Proposition 2.43 using the fact that, on the interval, transitivity is close to mixing; or it can be seen as a particular case of Theorem 2.44 since a transitive interval map has a dense set of periodic points by Proposition 2.15.

The next theorem is a straightforward consequence of a result of Blokh [**48**] (see [**52**, Theorem 1] for a statement in English). Alsedà, Kolyada, Llibre and Snoha showed a related result in a broader context: if (X, f) is a transitive topological dynamical system and if X has a disconnecting interval (that is, a subset $I \subset X$ homeomorphic to $(0,1)$ such that, $\forall x \in I$, $X \setminus \{x\}$ is not connected), then the set of periodic points is dense [**10**, Theorem 1.1].

THEOREM 2.45. *Let $f\colon G \to G$ be a transitive graph map. If f is not conjugate to an irrational rotation, the set of periodic points is dense.*

Together with Theorem 2.23, this implies that for a transitive graph map f, either f is conjugate to an irrational rotation and has no periodic point, or f has a dense set of periodic points.

The previous theorem, combined with Theorem 2.44, implies the next result.

COROLLARY 2.46. *Let $f\colon G \to G$ be a transitive graph map. If f is not conjugate to an irrational rotation, it is sensitive to initial conditions.*

It is not difficult to extend the proof of Proposition 2.40 to graph maps, which leads to the following result.

PROPOSITION 2.47. *Let $f\colon G \to G$ be a graph map. Suppose that, for some $\varepsilon > 0$, the set of ε-unstable points $U_\varepsilon(f)$ has a nonempty interior. Then there exists a cycle of graphs $(G_1, \ldots, G_p)$ such that $f|_{G_1 \cup \cdots \cup G_p}$ is transitive. Moreover, $G_1 \cup \cdots \cup G_p \subset \overline{U_\varepsilon(f)}$ and there exists $i \in [\![1, p]\!]$ such that* $\mathrm{diam}(G_i) \geq \varepsilon$.

Since a rotation is not sensitive, Theorems 2.24 and 2.25 imply that, in the previous proposition, each subgraph G_i can be decomposed in subgraphs $(H_i)_{1\leq i\leq k_i}$ that are cyclically mapped under f^p and are topologically mixing for f^{pk_i}. A similar result is stated in [**100**, Theorem 4.4].

CHAPTER 3

Periodic points

3.1. Specification

We saw that, for a transitive interval map, the set of periodic points is dense (Proposition 2.15). We are going to see that if in addition the map is mixing, then it satisfies the *specification property*, which roughly means that one can approximate any finite collection of pieces of trajectories by a periodic orbit provided enough time is left to pass from a piece of trajectory to another. This result was stated by Blokh, without proof in [**47**]; see [**55**] for the proof.

Specification is a strong property. In particular, a topological dynamical system (X, f) with the specification property is topologically mixing [**73**, (21.3)]. Therefore, specification and topological mixing are equivalent for interval maps.

DEFINITION 3.1 (specification). Let (X, f) be a topological dynamical system. The map f has the *specification property* if the following property holds: for all $\varepsilon > 0$, there exists an integer $N \geq 1$ such that, for all $p \geq 1$, for all points $x_1, \dots, x_p \in X$ and all positive integers m_i, n_i, $i = 1, \dots, p$, satisfying

$$m_1 \leq n_1 < m_2 \leq n_2 < \cdots < m_p \leq n_p \quad \text{and} \quad \forall i \in [\![2, p]\!],\ m_i - n_{i-1} \geq N,$$

then, for all integers $q \geq N + n_p - m_1$, there exists a point $x \in X$ such that

$$f^q(x) = x \quad \text{and} \quad \forall i \in [\![1, p]\!],\ \forall k \in [\![m_i, n_i]\!],\ d(f^k(x), f^k(x_i)) \leq \varepsilon.$$

We first state two lemmas.

LEMMA 3.2. *Let $f\colon I \to I$ be an interval map and $0 < \varepsilon < \frac{|I|}{2}$. For all $x \in I$ and all integers $n \geq 0$, there exist closed subintervals $J_0, \dots, J_n$ in I such that:*

- *$\forall i \in [\![0, n-1]\!]$, $f(J_i) = J_{i+1}$,*
- *$\forall i \in [\![0, n]\!]$, $f^i(x) \in J_i$ and $J_i \subset [f^i(x) - \varepsilon, f^i(x) + \varepsilon]$,*
- *there exists $i \in [\![0, n]\!]$ such that J_i contains either $f^i(x) - \varepsilon$ or $f^i(y) + \varepsilon$.*

Moreover, if $x + \varepsilon \in I$ (resp. $x - \varepsilon \in I$), then J_0 can be chosen in such a way that J_0 is included in $[x, x + \varepsilon]$ (resp. $[x - \varepsilon, x]$).

PROOF. We fix $x \in I$ and we set $x_k := f^k(x)$ for all $k \geq 0$. We show the lemma by induction on n.

• Case $n = 0$: since $\varepsilon < \frac{|I|}{2}$, the interval I contains either $x - \varepsilon$ or $x + \varepsilon$. We can set $J_0 := [x, x + \varepsilon]$ if $x + \varepsilon \in I$, or $J_0 := [x, x + \varepsilon]$ if $x - \varepsilon \in I$. The interval J_0 is suitable.

• Suppose that the lemma is true at rank $n - 1$, and let $J_0, \dots, J_{n-1}$ be the subintervals given by the lemma. If $f(J_{n-1}) \subset [x_n - \varepsilon, x_n + \varepsilon]$, we set $J_n := f(I_{n-1})$ and the intervals $(J_0, \dots, J_n)$ are suitable. From now on, we suppose that $f(J_{n-1})$ is not included in $[x_n - \varepsilon, x_n + \varepsilon]$. Thus, by connectedness, $f(J_{n-1})$ contains either

$x_n-\varepsilon$ or $x_n+\varepsilon$. We may assume that $J_0\subset[x,x+\varepsilon]$, the case when $J_0\subset[x-\varepsilon,x]$ being similar. According to the assumption on $f(J_{n-1})=f^n(J_0)$, we can define

$$y:=\min\{z\in J_0\mid f^n(z)\in\{x_n-\varepsilon,x_n+\varepsilon\}\}.$$

It follows that $f^n([x,y])$ equals either $[x_n-\varepsilon,x_n]$ or $[x_n,x_n+\varepsilon]$. We set $J_0':=[x,y]$ and $J_i':=f^i(J_0')$ for all $i\in[\![1,n]\!]$. The intervals $(J_0',\ldots,J_n')$ are suitable because $J_i'\subset J_i\subset[x_i-\varepsilon,x_i+\varepsilon]$ for all $i\in[\![0,n-1]\!]$ and J_n' contains x_n and one of the points $x_n-\varepsilon$, $x_n+\varepsilon$. This ends the induction. □

Lemma 3.3. *Let $f\colon[a,b]\to[a,b]$ be a topologically mixing interval map and $0<\varepsilon<\frac{b-a}{2}$. Suppose that the endpoint a (resp. b) is fixed and non accessible. Then there exists $\delta\in(0,\varepsilon)$ such that, for all $x\in[a,a+\delta]$ (resp. $x\in[b-\delta,b]$) and all $n\geq 0$, there exist closed subintervals $J_0,\ldots,J_n$ satisfying:*

- *$J_0\subset[a+\delta,b-\delta]$,*
- *$\forall i\in[\![0,n-1]\!]$, $f(J_i)=J_{i+1}$,*
- *$\forall i\in[\![0,n]\!]$, $J_i\subset[f^i(x)-\varepsilon,f^i(x)+\varepsilon]$,*
- *there exists $i\in[\![0,n]\!]$ such that $|J_i|\geq\frac{\varepsilon}{4}$.*

Proof. We prove the lemma when a is a non accessible fixed point. The proof for b is similar. If both endpoints are fixed and non accessible, the same δ can be chosen for a and b by taking the minimum of the values found for a and b respectively. By continuity, there exists $\eta>0$ such that

$$\forall y\in[a,a+\eta],\ f(y)<a+\varepsilon. \tag{3.1}$$

By transitivity, $f([a,a+\varepsilon])$ is not included in $[a,a+\varepsilon]$; that is, there exists z in $[a,a+\varepsilon]$ such that $f(z)\geq a+\varepsilon$. In fact, $z\in(a+\eta,a+\varepsilon]$ by (3.1). According to Lemma 2.32(ii), there exists a fixed point c in the interval $(a,a+\min\{\eta,\frac{\varepsilon}{2}\})$. We set $\delta:=c-a\in(0,\frac{\varepsilon}{2}]$ and $K:=[c,a+\varepsilon]$. The interval K contains both c and z, and hence $f(K)\supset K$ by the intermediate value theorem. Notice that $a+\varepsilon<b-\delta$ because $\delta\leq\frac{\varepsilon}{2}<\frac{b-a}{4}$.

We fix $x\in[a,a+\delta]=[a,c]$ and $n\geq 0$. We set $x_k:=f^k(x)$ for all $k\geq 0$. Let $m\in[\![0,n]\!]$ be the greatest integer such that $x_0,\ldots,x_m\in[a,c]$. Notice that $K\subset[x_i-\varepsilon,x_i+\varepsilon]$ for all $i\in[\![0,m]\!]$. Applying Lemma 1.13(i) to the chain of intervals $(K,\ldots,K)$ with $m+1$ times K, we see that there exist closed subintervals $J_0,\ldots,J_m$ such that $J_m=K$ and $J_i\subset K$, $f(J_i)=J_{i+1}$ for all $i\in[\![0,m-1]\!]$. If $m=n$, then the proof is over because the length of K is $a+\varepsilon-c>\varepsilon/2$. If $m<n$, then $x_{m+1}>c$ according to the choice of m, and $x_{m+1}=f(x_m)<a+\varepsilon$ by (3.1) (recall that $c<a+\eta$). Hence $x_{m+1}\in K$. The interval K contains either $x_{m+1}-\frac{\varepsilon}{4}$ or $x_{m+1}+\frac{\varepsilon}{4}$ because $|K|\geq\frac{\varepsilon}{2}$. Applying Lemma 3.2 to x_{m+1}, $\frac{\varepsilon}{4}$ and $n-m+1$ (instead of x, ε and n respectively), we see that there exist closed intervals $J'_{m+1},\ldots,J'_n$ such that

- $J'_{m+1}\subset K$,
- $\forall i\in[\![m+1,n-1]\!]$, $f(J_i')=J_{i+1}$,
- $\forall i\in[\![m+1,n]\!]$, $x_i\in J_i'$ and $J_i'\subset[x_i-\frac{\varepsilon}{4},x_i+\frac{\varepsilon}{4}]$
- there exists $i\in[\![m+1,n]\!]$ such that J_i' contains either $x_i-\frac{\varepsilon}{4}$ or $x_i+\frac{\varepsilon}{4}$, and thus the length of J_i' is at least $\frac{\varepsilon}{4}$.

It follows that $(J_0,\ldots,J_m=K,J'_{m+1})$ is a chain of intervals. Therefore, according to Lemma 1.13(i), there exist $J_0',\ldots,J_m'$, subintervals of $J_0,\ldots,J_m$ respectively, such that $f(J'_m) = J'_{m+1}$ and $f(J'_i) = f(J'_{i+1})$ for all $i \in [\![0, m-1]\!]$. The sequence $(J'_0, \dots, J'_n)$ satisfies the required properties. $\square$

THEOREM 3.4. *A topologically mixing interval map $f: I \to I$ has the specification property.*

PROOF. If f^2 has the specification property, then so has f by continuity. Moreover, if f is topologically mixing, then so is f^2 by Theorem 2.20. Therefore, it is equivalent to prove the theorem for f or for f^2. Then, in view of Lemma 2.32, we can assume that the non accessible endpoints (if any) are fixed, by considering f^2 instead of f if necessary.

Let $0 < \varepsilon < \frac{|I|}{4}$. We write $I = [a, b]$. If both a and b are accessible, we define $I_0 := [a, b]$. Otherwise, let $0 < \delta < \varepsilon$ be given by Lemma 3.3 and

$$\begin{aligned} I_0 &:= [a+\delta, b] \quad \text{if } a \text{ is the only non accessible endpoint,} \\ I_0 &:= [a, b-\delta] \quad \text{if } b \text{ is the only non accessible endpoint,} \\ I_0 &:= [a+\delta, b-\delta] \quad \text{if both } a \text{ and } b \text{ are non accessible.} \end{aligned}$$

We fix a positive integer p such that $\frac{b-a}{p} < \frac{\varepsilon}{8}$, and we define

$$\forall k \in [\![0, p-1]\!],\ A_k := \left(a + \frac{k(b-a)}{p}, a + \frac{(k+1)(b-a)}{p}\right).$$

According to Proposition 2.30, for every $k \in [\![0, p-1]\!]$, there exists an integer N_k such that $f^n(A_k) \supset I_0$ for all $n \geq N_k$. We set $N := \max\{N_0, \dots, N_{p-1}\}$. Let $J_0, \dots, J_k$ be intervals such that $f(J_i) = J_{i+1}$ for all $i \in [\![0, k-1]\!]$. Then, according to the definition of N,

$$\exists\, i \in [\![0, k]\!],\ |J_i| \geq \varepsilon/4, \Longrightarrow \forall n \geq N,\ f^n(J_k) \supset I_0 \tag{3.2}$$

because the assumption $|J_i| \geq \frac{\varepsilon}{4}$ implies that $A_j \subset I_i$ for some $j \in [\![0, p-1]\!]$.

FACT 1. *Let $x \in I$ and $n \geq 0$. There exist closed intervals $J_0, \dots, J_n$ such that:*

i) $J_0 \subset I_0$,
ii) $\forall i \in [\![0, n]\!],\ J_i \subset [f^i(x) - \varepsilon, f^i(x) + \varepsilon]$,
iii) $\forall i \in [\![0, n-1]\!],\ f(J_i) = J_{i+1}$,
iv) *there exists $i \in [\![0, n]\!]$ such that $|J_i| \geq \varepsilon/4$.*

We split the proof of the fact depending on $x \in I_0$ or not. If $x \in I_0$, let $J_0, \dots, J_n$ denote the intervals given by Lemma 3.2. They satisfy (ii)-(iv). Moreover, $|I_0| \geq 2\varepsilon$ by definition. This implies that either $[x-\varepsilon, x] \subset I_0$ or $[x, x+\varepsilon] \subset I_0$, and thus J_0 can be chosen to be a subinterval of I_0 (still by Lemma 3.2), which is (i). If a is not accessible and if $x \in [a, a+\delta]$, then Lemma 3.3 gives the suitable subintervals. The same conclusion holds if $x \in [b-\delta, b]$ and if b is non accessible.

FACT 2. *Let $x_1, \dots, x_p$ be points in I and let $m_1 \leq n_1 < m_2 \leq n_2 < \cdots < m_p \leq n_p$ be integers satisfying $m_{i+1} - n_i \geq N$ for all $i \in [\![1, p-1]\!]$. Then there exist closed intervals $(J_i)_{m_1 \leq i \leq n_p}$ such that*

- $J_{m_1} \subset I_0$,
- $\forall i \in [\![m_1, n_p - 1]\!],\ f(J_i) = J_{i+1}$,
- $\forall k \in [\![1, p]\!], \forall i \in [\![m_k, n_k]\!],\ J_i \subset [f^i(x_k) - \varepsilon, f^i(x_k) + \varepsilon]$,
- $\forall n \geq N,\ f^n(J_{n_p}) \supset I_0$.

We prove Fact 2 by induction on p.

• Case $p = 1$: we apply Fact 1 to $x := f^{m_1}(x_1)$ and $n := n_1 - m_1$. The last condition is satisfied because of (3.2).

• Suppose that Fact 2 holds at rank $p-1$ and let $J_{m_1}, \dots, J_{n_{p-1}}$ be the intervals given by Fact 2. We apply Fact 1 with $x := f^{m_p}(x_p)$ and $n := n_p - m_p$ and we call the resulting intervals $J'_{m_p}, \dots, J'_{n_p}$. Then $f^n(J'_{n_p}) \supset I_0$ for all $n \geq N$ by (3.2). We set $J_i := f^{i-n_{p-1}}(J_{n_{p-1}})$ for all $i \in [\![n_{p-1}+1, m_p]\!]$. By assumption, $m_p - n_{p-1} \geq N$, and thus $J_{m_p} = f^{m_p - n_{p-1}}(J_{n_{p-1}}) \supset I_0$ by (3.2). Therefore $(J_{m_1}, \dots, J_{m_p-1}, J'_{m_p})$ is a chain of intervals because $J'_{m_p} \subset I_0$ by construction. By Lemma 1.13(i), there exist subintervals $J'_i \subset J_i$ such that $f(J'_i) = f(J'_{i+1})$ for all $i \in [\![m_1, m_p - 1]\!]$. It follows that the sequence $J'_{m_1}, \dots, J'_{n_p}$ satisfies Fact 2. This concludes the induction.

It is now easy to prove that f has the specification property. Let $x_1, \dots, x_p$ be points in I, let $m_1 \leq n_1 < m_2 \leq n_2 < \cdots < m_p \leq n_p$ be integers satisfying

$$\forall i \in [\![1, p-1]\!],\ m_{i+1} - n_i \geq N \text{ and } q \geq n_p - m_1 + N.$$

Let $J_{m_1}, \dots, J_{n_p}$ be the intervals given by Fact 2. Then $f^n(J_{n_p})$ contains I_0 for all $n \geq N$, so $f^q(J_{m_1}) = f^{q-n_p+m_1}(J_{n_p}) \supset I_0 \supset J_{m_1}$. By Lemma 1.11, there exists $x \in J_{m_1}$ such that $f^q(x) = x$. We set $y := f^{q-m_1}(x)$ in order to have $f^{m_1}(y) = x \in J_{m_1}$. Then $f^q(y) = y$ and

$$\forall k \in [\![1, p]\!],\ \forall i \in [\![m_k, n_k]\!],\ f^i(y) \in J_i \subset [f^i(x_k) - \varepsilon, f^i(x_k) + \varepsilon].$$

This is exactly the specification property. $\square$

Remarks on graph maps. Theorem 3.4 was extended to graph maps, by Blokh [**48**]; see [**52**] for a statement in English.

THEOREM 3.5. *A topologically mixing graph map has the specification property.*

3.2. Periodic points and transitivity

We recall Proposition 2.15: for a transitive interval map, the set of periodic points is dense. The converse is obviously false, but one may ask the following question: if the set of periodic points is dense, does there exist a transitive cycle of intervals? The identity and the map $f(x) = 1 - x$ on $[0, 1]$ give counter-examples. Therefore one has to consider only interval maps such that f^2 is different from the identity. With this restriction, the answer is positive. This is a result of Blokh, which is stated without proof in [**44**]. The same result was proved independently by Barge and Martin [**27**], but their proof relies on complicated notions. We give a more basic proof here.

We start with a lemma. Then Proposition 3.7 states that, if the set of periodic points of f is dense, all the points outside $P_2(f) := \{x \mid f^2(x) = x\}$ are unstable. This result makes a link with the results on sensitivity from the previous chapter and will allow us to conclude.

LEMMA 3.6. *Let f be an interval map such that the set of periodic points is dense. Then, for every non degenerated interval J, the set $\bigcup_{n \geq 0} f^n(J)$ has at most two connected components.*

PROOF. Let J be a non degenerate interval. By assumption, there exists a periodic point in J, say x. Let p denote the period of x. For all $i \in [\![0, p-1]\!]$ and all $n \geq 0$, $f^{np+i}(J)$ contains $f^i(x)$. Thus the set $\bigcup_{n\geq 0} f^n(J)$ is invariant and has at most p connected components; we call them $J_1, \ldots, J_q$ (with $q \in [\![1, p]\!]$) in such a way that $J_1 < J_2 < \cdots < J_q$. By the intermediate value theorem, for every $i \in [\![1, q]\!]$ there exists $\sigma(i) \in [\![1, q]\!]$ such that $f(J_i) \subset J_{\sigma(i)}$. Let i_0 be the integer such that $J \subset J_{i_0}$. For every $i \in [\![1, q]\!]$, there exists $n \geq 0$ such that $f^n(J) \subset J_i$, that is, $\sigma^n(i_0) = i$. This implies that the orbit of i_0 under σ is the whole set, and hence σ is necessarily a cyclic permutation of $\{1, \ldots, q\}$. We want to show that $q = 1$ or 2. From now on, we assume that $q \geq 2$.

If there exists $i \in [\![1, q-1]\!]$ such that $|\sigma(i) - \sigma(i+1)| \geq 2$, we choose an integer k strictly between $\sigma(i)$ and $\sigma(i+1)$. Let $a := \sup J_i$ and $b := \inf J_{i+1}$. Then $f(a) \in \overline{J_{\sigma(i)}}$ and $f(b) \in \overline{J_{\sigma(i+1)}}$, which implies that $f((a, b))$ contains J_k by the intermediate value theorem and that (a, b) is not empty. Let $V \subset (a, b)$ be a nonempty open interval such that $f(V) \subset J_k$. It follows that

$$\forall n \geq 1,\ f^n(V) \subset \mathcal{O}_f(J_k) \subset J_1 \cup \cdots \cup J_q.$$

This implies that, $\forall n \geq 1$, $f^n(V) \cap V = \emptyset$, but this contradicts the assumption that V contains periodic points. We deduce that

$$\forall i \in [\![1, q-1]\!],\ |\sigma(i) - \sigma(i+1)| = 1.$$

If $\sigma(2) - \sigma(1) = 1$, we obtain from place to place: $\sigma(k) = \sigma(1) + k - 1$. Since σ is a cyclic permutation of length $q \geq 2$, we have $\sigma(1) \geq 2$, and thus $\sigma(q) \geq q + 1$, which is impossible. We deduce that $\sigma(2) - \sigma(1) = -1$ and $\sigma(q) = \sigma(1) - q + 1$. The only possibility is $\sigma(1) = q$ and $\sigma(q) = 1$ because $\sigma(q) \geq 1$. Since σ is a cycle of length q, we must have $q = 2$. This concludes the proof. $\square$

PROPOSITION 3.7. *Let $f\colon I \to I$ be an interval map such that the set of periodic points is dense. For every point x such that $f^2(x) \neq x$, there exists $\varepsilon > 0$ (depending on x) such that x is ε-unstable.*

PROOF. If U is a nonempty open interval, the set $\mathcal{O}_f(U) := \bigcup_{n\geq 0} f^n(U)$ has one or two connected components according to Lemma 3.6. We fix a point x such that $f^2(x) \neq x$. If there exists an open interval U_0 containing x such that $\mathcal{O}_f(U_0)$ has two connected components, we call them J_1 and J_2 in such a way that $U_0 \subset J_1$, and we set $g := f^2$. In this situation, we necessarily have $f(J_1) \subset J_2$ and $f(J_2) \subset J_1$. Moreover, for every nonempty open interval $U \subset U_0$, we see that $U \subset J_1$, $f(U) \subset J_2$ and $\mathcal{O}_f(U) \subset J_1 \cup J_2$, and hence $\mathcal{O}_f(U)$ has two connected components too. On the other hand, if $\mathcal{O}_f(U)$ is a connected set for every open interval U containing x, we set $g := f$ and $U_0 := I$. With this notation, for every open subinterval $U \subset U_0$ containing x, the set $\mathcal{O}_g(U)$ is connected. The two points

$$a := \inf_{n\geq 0} \mathcal{O}_g(x) \quad \text{and} \quad b := \sup_{n\geq 0} \mathcal{O}_g(x)$$

are distinct because $g(x) \neq x$ by assumption.

First we suppose that $\mathcal{O}_g(x)$ is not dense in $[a, b]$, which means that there exist $z \in (a, b)$ and $\varepsilon > 0$ such that $(z - \varepsilon, z + \varepsilon) \subset (a, b) \setminus \mathcal{O}_g(x)$. Let $U \subset U_0$ be an open interval containing x. The set $\mathcal{O}_g(U)$ is connected and contains $\mathcal{O}_g(x)$, and thus it contains (a, b) too. In particular, there exist $y \in U$ and $k \geq 0$ such that $g^k(y) = z$, and hence

$$|g^k(x) - g^k(y)| \geq \inf_{n\geq 0} |g^n(x) - z| \geq \varepsilon.$$

We deduce that the point x is ε-unstable.

Now we suppose that $\mathcal{O}_g(x)$ is dense in $[a, b]$. This implies that $g([a, b]) = [a, b]$ and that $g|_{[a,b]}$ is transitive. Then, by Proposition 2.39, the map $g|_{[a,b]}$ is ε'-sensitive for every $\varepsilon' \in \left(0, \frac{b-a}{4}\right)$, In particular, the point x is ε'-unstable. $\square$

PROPOSITION 3.8. *Let $f\colon I \to I$ be an interval map such that the set of periodic points is dense. Suppose that f^2 is different from the identity map. Then at least one of the following holds:*

- *there exists a non degenerate closed interval J such that $f(J) = J$ and $f|_J$ is transitive,*
- *there exist two disjoint non degenerate closed intervals J_1, J_2 such that $f(J_1) = J_2$, $f(J_2) = J_1$ and $f|_{J_1\cup J_2}$ is transitive.*

PROOF. Recall that $P_2(f) := \{x \in I \mid f^2(x) = x\}$. This is a closed set and, by assumption, the open set $I \setminus P_2(f)$ is not empty. By Proposition 3.7, all the points of $I \setminus P_2(f)$ are unstable, and thus

$$I \setminus P_2(f) \subset \bigcup_{n=1}^{+\infty} U_{\frac{1}{n}}(f) \subset \bigcup_{n=1}^{+\infty} \overline{U_{\frac{1}{n}}(f)}.$$

By the Baire category theorem, there exists $n \geq 1$ such that $\overline{U_{\frac{1}{n}}(f)}$ has a nonempty interior. It follows that $U_{\frac{1}{2n}}(f)$ has a nonempty interior too, because $\overline{U_{\frac{1}{n}(f)}} \subset U_{\frac{1}{2n}}(f)$ by Lemma 2.38(iii). Then, by Proposition 2.40, there exists a cycle of intervals $(J_1, \ldots, J_p)$ such that $f|_{J_1\cup\cdots J_p}$ is transitive. Finally, $p = 1$ or 2 by Lemma 3.6. $\square$

Proposition 3.8 makes possible a decomposition of the interval into transitive components, as stated in the next theorem and illustrated in Figure 1.

THEOREM 3.9. *Let $f\colon I \to I$ be an interval map such that the set of periodic points is dense. Then there exists a finite (possibly empty) or countable family of sets $\mathcal{E}$ such that:*

i) *$\forall C \in \mathcal{E}$, the set C is either a non degenerate closed interval or the union of two disjoint non degenerate closed intervals,*
ii) *$\forall C \in \mathcal{E}$, C is invariant and $f|_C$ is transitive,*
iii) *the sets in $\mathcal{E}$ have pairwise disjoint interiors,*
iv) *$I \setminus \bigcup_{C\in\mathcal{E}} C \subset P_2(f)$.*

PROOF. We define

$$\mathcal{E} := \{C \subset I \mid C \text{ cycle of intervals, } f|_C \text{ transitive}\}.$$

By Lemma 3.6, every element in $\mathcal{E}$ has at most two connected components, and thus it satisfies (i) and (ii). Moreover,

$$\text{if } J \text{ is a connected component of } C \in \mathcal{E}, \text{ then } f^2(J) \subset J. \tag{3.3}$$

Let $C, C' \in \mathcal{E}$ and $V := \operatorname{Int}(C) \cap \operatorname{Int}(C')$. If $V \neq \emptyset$, then, by transitivity,

$$\overline{\bigcup_{n\geq 0} f^n(V)} = C = C'.$$

Therefore, two different elements of $\mathcal{E}$ have nonempty disjoint interiors, which is (iii). This implies that $\mathcal{E}$ is at most countable because for every $C \in \mathcal{E}$, $\mathrm{Int}\,(C)$ contains a rational number r_C, and $r_C \neq r_{C'}$ if $C \neq C'$. It remains to prove (iv). We set

$$X_0 := \bigcup_{C \in \mathcal{E}} C.$$

We are going to show first that $I \setminus \overline{X_0} \subset P_2(f)$, and second that $\overline{X_0} \setminus X_0 \subset P_2(f)$; these two facts clearly imply (iv). We set $Y := I \setminus \overline{X_0}$; this is an open set. If $f(Y) \cap \mathrm{Int}\left(\overline{X_0}\right) \neq \emptyset$, then there exists a non degenerate subinterval $J \subset Y$ such that $f(J) \subset \mathrm{Int}\left(\overline{X_0}\right)$. Since $f(\overline{X_0}) \subset \overline{X_0}$, this implies that $f^n(J) \cap J = \emptyset$ for all $n \geq 1$. But this contradicts the fact that J contains periodic points. We deduce that $f(Y) \subset I \setminus \mathrm{Int}\left(\overline{X_0}\right) = \overline{Y}$, and thus

$$f(\overline{Y}) \subset \overline{Y}. \tag{3.4}$$

Suppose that

$$Y \setminus P_2(f) \neq \emptyset \tag{3.5}$$

and let K' be a connected component of Y such that $K' \setminus P_2(f) \neq \emptyset$. Since Y is open, K' is an open interval. Moreover, there exists $n \geq 1$ such that $f^n(K') \cap K' \neq \emptyset$ because the set of periodic points is dense by assumption. Let K be the connected component of $\overline{Y}$ containing K'. Then $f^n(K) \cap K \neq \emptyset$ and $f^n(K)$ is included in a connected component of $\overline{Y}$ by (3.4). So $f^n(K) \subset K$. We consider the interval map

$$g := f^n|_K \colon K \to K.$$

By Proposition 3.7, all points in $K \setminus P_2(f)$ are unstable for f. Therefore, all points in $K \setminus P_2(f)$ are unstable for g by Lemma 2.38(i). Thus

$$K' \setminus P_2(f) \subset K \setminus P_2(f) \subset \bigcup_{k=1}^{+\infty} U_{\frac{1}{k}}(g) \subset \bigcup_{k=1}^{+\infty} \overline{U_{\frac{1}{k}}(g)},$$

where $K' \setminus P_2(f)$ is a nonempty open set. Then we use the same argument as in the proof of Proposition 3.8: using the Baire category theorem, we find k such that $\mathrm{Int}\left(\overline{U_{\frac{1}{k}}(g)}\right) \neq \emptyset$, so $\mathrm{Int}\left(U_{\frac{1}{2k}}(g)\right) \neq \emptyset$ by Lemma 2.38(iv). Then, according to Proposition 2.40, there exists a transitive cycle of intervals $C \subset K$ for g. It is straightforward to see that the set $C' := C \cup f(C) \cup \cdots \cup f^{n-1}(C)$ is a finite union of non degenerate closed intervals and that $f|_{C'}$ is transitive. Hence $C' \subset X_0$ and $C' \cap \mathrm{Int}\,(X_0) \neq \emptyset$ because $\mathrm{Int}\,(C') \neq \emptyset$. On the other hand, $C' \subset \overline{Y}$ by (3.4), and thus $C' \subset I \setminus \mathrm{Int}\,(X_0)$, which leads to a contradiction. Therefore, (3.5) does not hold, that is, $Y = I \setminus \overline{X_0} \subset P_2(f)$.

Now we are going to show that $\overline{X_0} \setminus X_0 \subset P_2(f)$. We define

$$\mathcal{A} := \{J \subset I \mid \exists C \in \mathcal{E}, J \text{ is a connected component of } C\}.$$

We fix $x \in \overline{X_0} \setminus X_0$. Let $(x_n)_{n\geq 0}$ be a monotone sequence in X_0 converging to x. For all $n \geq 0$, let $J_n \in \mathcal{A}$ be such that $x_n \in J_n$. If there exists n_0 such that $J_n = J_{n_0}$ for all $n \geq n_0$, then $\lim_{n\to+\infty} x_n$ belongs to the closed set J_{n_0}, and thus $x \in X_0$, which is impossible. Thus the sequence $(J_n)_{n\geq 0}$ is not eventually constant, which implies that $\lim_{n\to+\infty} |J_n| = 0$ because two distinct elements in $\mathcal{A}$ have disjoint interiors. Let $\varepsilon > 0$. By continuity, there exists $0 < \alpha < \varepsilon$ such that

$$\forall y \in I,\ |x - y| < \alpha \Rightarrow |f^2(x) - f^2(y)| < \varepsilon.$$

We choose n such that $|x - x_n| < \alpha$ and $|J_n| < \varepsilon$. Then $f^2(x_n) \in J_n$ by (3.3), so

$$|x - f^2(x)| \le |x - x_n| + |x_n - f^2(x_n)| + |f^2(x_n) - f^2(x)| < 3\varepsilon.$$

Since this is true for all $\varepsilon > 0$, we deduce that $f^2(x) = x$. In other words, $\overline{X_0} \setminus X_0 \subset P_2(f)$. This concludes the proof. $\square$

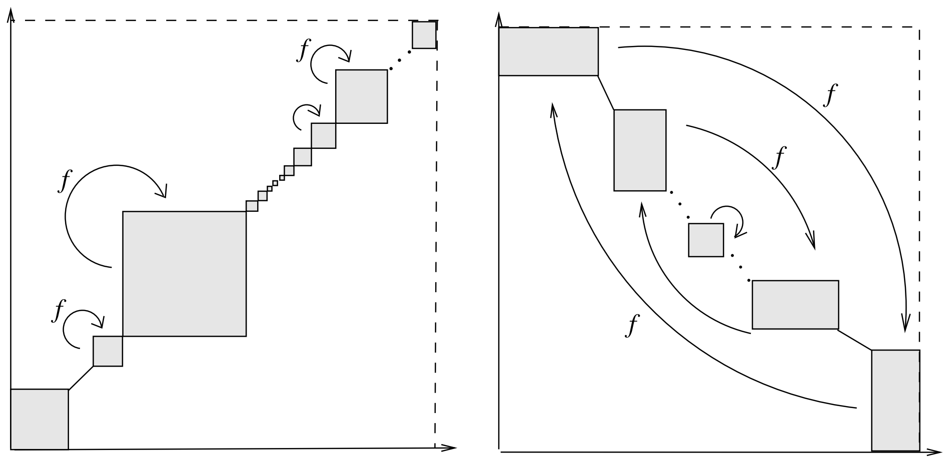


FIGURE 1. Decomposition into transitive components of a map f when the set of periodic points is dense: two cases. The gray areas represent the transitive components, whereas the part of the graph of f made of black lines is the set $P_2(f)$. The points where transitive components accumulate on both sides are not unstable, neither are the points in $\mathrm{Int}\,(P_2(f))$.

Theorem 3.9 can be more precise. Let $\mathcal{E}$ be the family of transitive components given by Theorem 3.9 and $C \in \mathcal{E}$. We write $I = [a, b]$. Let J be a connected component of C and suppose that J has an endpoint $c \notin \{a, b\}$. Theorem 3.9 implies that either c is the endpoint of a connected component J' of another element of $\mathcal{E}$, or $c \in P_2(f)$. In the first case, using the facts that $f^2(J) = J$ and $f^2(J') = J'$, we deduce that $f^2(c) = c$. Therefore, all the endpoints of the connected components, except maybe a and b, belong to $P_2(f)$. In particular, if a connected component is made of two disjoint intervals J, K, then Theorem 2.19 implies that $f^2|_J$ and $f^2|_K$ are topologically mixing. Now suppose that C is an interval and that there exists a fixed point z outside C (in particular, such a fixed point exists when there exist other transitive intervals). We may assume that $z < C$, the other case being symmetric. Then it can be shown that the decomposition given by Theorem 3.9 implies that $f([a, \min C]) = [a, \min C]$. Thus $\min C$ is a fixed point and $f|_C$ is topologically mixing by Theorem 2.19.

Figure 1 illustrates what kind of decomposition can exist when there are several transitive components. On the left side, all transitive components are intervals and f is topologically mixing on each of them. On the right side, there is only one transitive interval in the middle, f may or may not be topologically mixing on this interval (this middle transitive interval may not exist), and f^2 is topologically mixing on every connected component of the transitive components made of two intervals.

Remarks on graph maps. Theorem 3.9 was first extended to tree maps by Roe [**144**], then to graph maps by Yokoi [**173**].

THEOREM 3.10. *Let $f\colon G \to G$ be a graph map such that the set of periodic points is dense. Then there exist a positive integer N and a finite (possibly empty) or countable family of subgraphs $\mathcal{E}$ with disjoint interiors such that*

- *$\forall H \in \mathcal{E}$, the set H is f^N-invariant and $f^N|_H$ is topologically mixing,*
- *$G \setminus \bigcup_{H\in\mathcal{E}} X \subset P_N(f)$.*

If G is a tree with e endpoints, then one can take $N = \gcd(2, 3, \dots, e)$.

For topological graphs that are not trees, the integer N in the preceding theorem can be arbitrarily large. For example, the rational rotation $R_{\frac{1}{n}}$ gives a system in which all points are periodic of period n.

3.3. Sharkovsky's Theorem, Sharkovsky's order and type

Sharkovsky's Theorem states that, for an interval map, the presence of a periodic point with a given period implies the existence of other periods determined by so-called *Sharkovsky's order* [**153**].

DEFINITION 3.11. *Sharkovsky's order* is the total ordering on $\mathbb{N}$ defined by:

$$3 \lhd 5 \lhd 7 \lhd 9 \lhd \cdots \lhd 2\cdot 3 \lhd 2\cdot 5 \lhd \cdots \lhd 2^2\cdot 3 \lhd 2^2\cdot 5 \lhd \cdots \lhd 2^3 \lhd 2^2 \lhd 2 \lhd 1$$

(first, all odd integers $n > 1$, then 2 times the odd integers $n > 1$, then successively $2^2\times$, $2^3\times$, ..., $2^k\times$ ... the odd integers $n > 1$, and finally all the powers of 2 by decreasing order).

$a \rhd b$ means $b \lhd a$. The notation $\unlhd, \unrhd$ will denote the order with possible equality.

REMARK 3.12. In Sharkovsky's order, 3 is the minimum (as above) in some papers whereas it is the maximum in other ones (i.e. all inequalities are reversed). The ordering above is the same as in Sharkovsky's original paper [**153**], but there is no consensus in the literature. Even in Sharkovsky's papers, both orderings appear. In addition, the symbol for the inequalities varies much: one can find $\lhd$, $\prec$, $<_{sh}$, $\vdash$, $\ll$. The spelling of "Sharkovsky" varies much too.

THEOREM 3.13 (Sharkovsky). *If an interval map f has a periodic point of period n, then, for all integers $m \rhd n$, f has periodic points of period m.*

This striking result is one of the first about the dynamics on the interval and, more generally, one of the earliest results pointing out the existence of "complicated" behavior in some dynamical systems.

The original paper of Sharkovsky, in 1964, was in Russian [**153**]. The first proof in English, different from Sharkovsky's, is due to Štefan in 1976 [**165**] (published in [**166**]). In the meantime, Li and Yorke, unaware of the work of Sharkovsky, re-proved a particular case, namely that the existence of a periodic point of period 3 implies that there are periodic points of all periods [**113**]. This illustrates the lack of communication between Russian and English literatures. Later, several proofs of Sharkovsky's theorem were given [**36, 139, 41**]. The presentation we are going to give derives from the one of Block, Guckenheimer, Misiurewicz and Young [**36**]. We shall first introduce the notion of a *graph of a periodic orbit* and its main properties; then we shall prove Theorem 3.13 in Subsection 3.3.2.

3.3.1. Graph of a periodic orbit. We are going to associate a directed graph to a periodic orbit, and show that the existence of other periodic points can be read in this graph.

Recall that $\langle a, b\rangle$ denotes $[a, b]$ or $[b, a]$ depending on $a \leq b$ or $b \leq a$.

Definition 3.14. Let f be an interval map and x a periodic point of period $n \geq 2$. Let $x_1 < \cdots < x_n$ denote the ordered points in $\{x, f(x), \ldots, f^{n-1}(x)\}$ and let $I_j := [x_j, x_{j+1}]$ for all $j \in [\![1, n-1]\!]$. The *graph of the periodic orbit of* x is the directed graph whose vertices are $I_1, \ldots, I_{n-1}$ and

$$\forall j, k \in [\![1, n-1]\!], \text{ there is an arrow } I_j \to I_k \text{ iff } I_k \subset \langle f(x_j), f(x_{j+1})\rangle.$$

In this graph, a *fundamental* cycle is a cycle of length n, say $J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$, such that there exists a point $c \in \mathcal{O}_f(x)$ with the property that $f^k(c)$ is an endpoint of J_k for all $k \in [\![0, n-1]\!]$.

It is important to notice that if $I_i \to I_j$ is an arrow in the graph of a periodic orbit, then I_i covers I_j. Therefore, a cycle in this graph is a chain of intervals, starting and ending with the same interval.

Recall that a cycle is *primitive* if it is not the repetition of a shorter cycle.

Lemma 3.15. *In the graph of a periodic orbit, there exists a unique fundamental cycle (up to cyclic permutation). In this cycle, each vertex of the graph appears at most twice and one of them appears exactly twice. The fundamental cycle can be decomposed into two shorter primitive cycles.*

Proof. We consider a periodic orbit of period $n \geq 2$ composed of the points $x_1 < \cdots < x_n$ and we set $I_j := [x_j, x_{j+1}]$ for all $j \in [\![1, n-1]\!]$. We fix $i \in [\![1, n-1]\!]$ and $c \in \{x_i, x_{i+1}\}$. We are going to show by induction on k that

$$\text{(3.6)} \quad \begin{array}{l} \text{there is a unique sequence of intervals } (J_k)_{k\geq 0}\text{, which are vertices} \\ \text{of the graph of the periodic orbit, such that} \\ J_0 = I_i \text{ and } \forall k \geq 0,\ f^k(c) \in \partial J_k \text{ and } J_k \to J_{k+1}. \end{array}$$

Suppose that $J_{k-1} = [a, b]$ is already defined. The interval J_k must satisfy

$$\text{(3.7)} \qquad J_k \subset \langle f(a), f(b)\rangle \text{ and } f^k(c) \text{ is an endpoint of } J_k.$$

According to the induction hypothesis for J_{k-1}, the points $f^{k-1}(c)$ belong to $\{a, b\}$. Thus either $f(a)$ or $f(b)$ is equal to $f^k(c)$, and (3.7) determines uniquely $J_k \in \{I_1, \ldots, I_{n-1}\}$. This ends the induction.

From now on, let $(J_k)_{k\geq 0}$ denote the sequence defined above starting with $J_0 := I_1$ and $c := x_1$. Since $f^n(x_1) = x_1$ and $x_1 < x_i$ for all $i \in [\![2, n]\!]$, the interval J_n is necessarily equal to J_0. Therefore, $J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$ is a fundamental cycle. Now, we are going to prove the uniqueness of the fundamental cycle. Let $K_0 \to K_1 \to \cdots \to K_n = K_0$ be a fundamental cycle and d a point of the periodic orbit such that $f^i(d)$ is an endpoint of K_i for all $i \in [\![0, n-1]\!]$. Since $d \in \{x_1, \ldots, x_n\}$, there exists $k \in [\![0, n-1]\!]$ such that $d = f^k(x_1)$. Thus $f^{n-k}(d) = f^n(x_1) = x_1$ is an endpoint of K_{n-k}, so $K_{n-k} = J_0$. Then (3.6) implies that

$$(K_0, K_1, \ldots, K_{n-1}, K_0) = (J_k, J_{k+1}, \ldots, J_{n-1}, J_0, \ldots, J_k),$$

that is, the fundamental cycle is unique up to cyclic permutation.

For every $k \in [\![1, n-1]\!]$, there exist two distinct integers $i, j \in [\![0, n-1]\!]$ such that $I_k = [f^i(x_1), f^j(x_1)]$. Consequently, J_i and J_j are the only two intervals of the fundamental cycle that may be equal to I_k. This implies that every vertex appears at most twice in the fundamental cycle. Moreover, there are only $n-1$ vertices in the graph and the fundamental cycle is of length n. Thus, one of the vertices of the graph appears at least twice in the fundamental cycle. Finally, if we cut the fundamental cycle at a vertex I_k appearing twice, we obtain two cycles which are primitive because each of them contains I_k only once. ☐

The next lemma is originally due to Štefan [**166**]. We follow the proof of [**36**]. This is a key tool for finding other periods when one periodic orbit is known.

LEMMA 3.16. *Let f be an interval map and x a periodic point. If the graph G of the periodic orbit of x contains a primitive cycle $J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$ of length n, then there exists a periodic point y of period n such that $f^k(y) \in J_k$ for all $k \in [\![0, n-1]\!]$.*

PROOF. By Lemma 1.13(ii), there exists y such that $f^n(y) = y$ and $f^k(y) \in J_k$ for all $k \in [\![0, n-1]\!]$. Let p be the period of y, which is a divisor of n. Suppose that $f^k(y)$ does not belong to $\mathcal{O}_f(x)$ for all $k \in [\![0, n-1]\!]$. Then J_k is the unique vertex of G containing $f^k(y)$ for all $k \in [\![0, n-1]\!]$. This implies that $p = n$, otherwise the cycle would not be primitive. On the contrary, suppose that there exists $k \in [\![0, n-1]\!]$ such that $f^k(y) \in \mathcal{O}_f(x)$. Then $y = f^{n-k}(f^k(y))$ belongs to the orbit of x, and thus x is of period p. Moreover $f^k(y)$ is an endpoint of J_k for all $k \in [\![0, n-1]\!]$, which implies that $J_0 \to J_1 \to \cdots J_{n-1} \to J_0$ is equal to either the fundamental cycle or a repetition of it. Finally, $p = n$ because this cycle is primitive. ☐

The next lemma describes the graph of a periodic orbit whose period is the smallest odd period greater than 1. Such a periodic orbit is called a *Štefan cycle*, and so is any periodic orbit with the same graph as in Figure 2.

LEMMA 3.17. *Let f be an interval map having a periodic point of odd period different from 1. Let p be the smallest odd period greater than 1, and x a periodic point of period p. Let c denote the median point of the orbit of x (that is, $c \in \mathcal{O}_f(x)$ and $\mathcal{O}_f(x)$ contains $(p-1)/2$ points less than c and $(p-1)/2$ greater than c). If $c < f(c)$, the points of its orbit are ordered as follows:*

$$f^{p-1}(c) < f^{p-3}(c) < \cdots < f^2(c) < c < f(c) < f^3(c) < \cdots < f^{p-2}(c).$$

If $c > f(c)$, the reverse order holds. Moreover, the graph of this periodic orbit is the one represented in Figure 2.

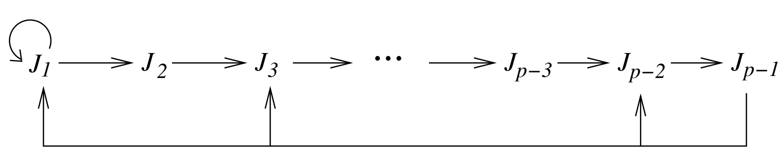


FIGURE 2. Graph of a periodic orbit of minimal odd period $p > 1$.

PROOF. By Lemma 3.15, the graph of the periodic orbit of c contains a fundamental cycle that can be split into two primitive cycles. One of these primitive cycles is of odd length because the fundamental cycle is of odd length p. According to Lemma 3.16 and because of the minimality of p, this length is necessarily equal to 1. Therefore, the fundamental cycle can be written as $J_1 \to J_1 \to J_2 \to \cdots \to J_{p-1} \to J_1$. Moreover $J_i \neq J_1$ for all $i \in [\![2, p-1]\!]$ because each vertex appears at most twice by Lemma 3.15. If $J_i = J_j$ for some i, j with $2 \le i < j \le p-1$, then the two cycles

$$J_1 \to J_2 \to \cdots \to J_i = J_j \to J_{j+1} \to \cdots \to J_{p-1} \to J_1$$

and

$$J_1 \to J_1 \to J_2 \to \cdots \to J_i = J_j \to J_{j+1} \to \cdots \to J_{p-1} \to J_1$$

are of respective lengths $p+i-j-1$ and $p+i-j$. These lengths are in $[\![1, p-1]\!]$, and one of them is odd. But then, using Lemma 3.16, we get a contradiction with the choice of p. Therefore, we have $J_i \neq J_j$ for all $i, j \in [\![2, p-1]\!]$ with $i < j$. which implies that $(J_1, J_2, \ldots, J_{p-1})$ is a permutation of the $p-1$ vertices of the graph of the orbit of x. Similarly, if $J_i \to J_k$ for some $i, k \in [\![1, p-1]\!]$ with $k > i+1$, or if $J_i \to J_1$ for some $i \in [\![2, p-2]\!]$, there exists a primitive cycle of odd length (the cycle $J_1 \to J_1$ may be added if necessary to get an odd length) with a length in $[\![2, p-1]\!]$, which leads to a contradiction again by Lemma 3.16.

Let $x_1 < \cdots < x_p$ be the ordered points of $\mathcal{O}_f(x)$. We set $I_j := [x_j, x_{j+1}]$ for all $j \in [\![1, p-1]\!]$. Let $k \in [\![1, p-1]\!]$ be the integer such that $J_1 = I_k$. We have shown that the vertex J_1 is only directed to J_1 and J_2. This implies that the intervals J_1 and J_2 have a common endpoint, and thus J_2 is equal to I_{k-1} or I_{k+1}. Since $f(x_j) \neq x_j$ for all $j \in [\![1, p]\!]$, it is easy to check that we are in one of the following two cases:

- $J_2 = I_{k-1}$, $x_{k+1} = f(x_k)$ and $x_{k-1} = f^2(x_k)$,
- $J_2 = I_{k+1}$, $x_k = f(x_{k+1})$ and $x_{k+2} = f^2(x_{k+1})$.

We assume that we are in the first case, the second one being symmetric and leading to the reverse order. We encourage the reader to redraw the points of Figure 3 step by step when reading the proof. We set $c := x_k$. If $p = 3$, then the

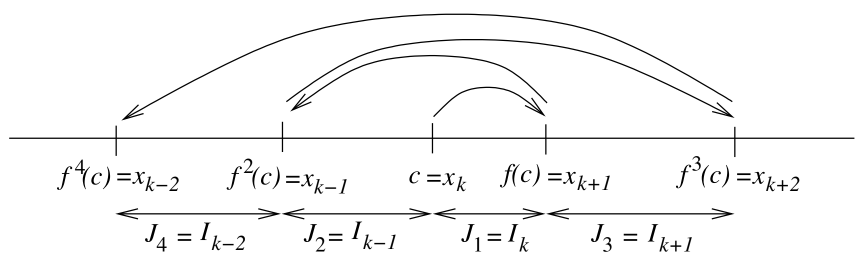


FIGURE 3. Position of the first iterates of c.

proof is complete. If $p > 3$, then $f^3(c) > c$, otherwise there would be an arrow $J_2 \to J_1$. Hence $f^3(c) = x_i$ for some $i > k+1$. Since there is an arrow $J_2 \to J_3$ and no arrow $J_2 \to J_j$ for all $j > 3$, the only possibility is that $f^3(c) = x_{k+2}$ and $J_3 = [f(c), f^3(c)] = I_{k+1}$. If $f^4(c) > f^2(c)$, then necessarily $f^4(c) > f^3(c) = x_{k+2}$. But this implies that $J_3 \to J_1$, which is impossible, and hence $f^4(c) < f^2(c)$. Since there is an arrow $J_3 \to J_4$ and no arrow $J_3 \to J_j$ for all $j > 4$, the only possibility is that $f^4(c) = x_{k-2}$ and $J_4 = [f^4(c), f^2(c)] = I_{k-2}$. We can go on in this way, and finally we find that the points are ordered as follows:

$$f^{p-1}(c) < f^{p-3}(c) < \cdots < f^2(c) < c < f(c) < f^3(c) < \cdots < f^{p-2}(c).$$

The point c is the median point of the orbit, and we are in the case $c < f(c)$. The order of these points enables us to check that the graph of the periodic orbit is the one represented in Figure 2. $\square$

In the proof of Sharkovsky's Theorem, we shall need the next elementary lemma.

LEMMA 3.18. *Let f be an interval map and let x be a point.*

i) *If x is a periodic point of period n for f, then x is periodic of period n/d for f^k, where $d := \gcd(n,k)$.*
ii) *If x is a periodic point of period m for f^k, then there exists d a divisor of k satisfying $\gcd(m,d) = 1$ and such that x is periodic of period $m\frac{k}{d}$ for f.*

PROOF. i) Let x be a periodic point of period n for f. We set $d := \gcd(n,k)$ and $k' := k/d$. We have $f^{k\frac{n}{d}}(x) = f^{k'n}(x) = x$. Let $m \geq 1$ be an integer such that $f^{km}(x) = x$. This implies that km is a multiple of n, say $km = pn$ for some $p \in \mathbb{N}$. Then $m = \frac{pn}{k} = \frac{pn}{k'd}$. Since $\gcd(n,k') = 1$, the quantity $\frac{p}{k'}$ must be an integer, and thus $\frac{n}{d}$ divides m. We conclude that x is periodic of period $\frac{n}{d}$ for f^k.

ii) Let x be a periodic point of period m for f^k, and let n be the period of x for f. Then n divides km because $f^{km}(x) = x$. Let d be the integer such that $km = dn$. We set $p := \gcd(m,d)$. Then $\frac{m}{p}$ and $\frac{d}{p}$ are integers and $k\frac{m}{p} = \frac{d}{p}n$. This implies that $f^{k\frac{m}{p}}(x) = f^{n\frac{d}{p}}(x) = x$, and thus $p = 1$ because x is of period m for f^k. Since $n = \frac{km}{d}$ and $p = \gcd(m,d) = 1$, we deduce that d divides k. $\square$

3.3.2. Proof of Sharkovsky's Theorem.

PROOF OF THEOREM 3.13. We first deal with the existence of periodic points of period 1 or 2. By Lemma 1.11, f has a fixed point. We are going to show:

(3.8) f has a periodic point of period $p > 1 \Rightarrow f$ has a periodic point of period 2.

Let n denote the least period greater than 1 and suppose that $n \geq 3$. According to Lemma 3.15, the fundamental cycle of a periodic point of period n can be split into two shorter primitive cycles, and thus one of them is of length m with $m \in [\![2, n-1]\!]$. But then Lemma 3.16 implies that there exists a periodic point of period m, which contradicts the definition of n. Therefore $n = 2$ and (3.8) is proved.

Second, we show that if f has a periodic point of period p,

$$p > 1,\ p \text{ odd} \Rightarrow \forall n \trianglerighteq p,\ f \text{ has a periodic point of period } n. \tag{3.9}$$

According to the definition of Sharkovsky's order, it is sufficient to prove (3.9) when p is the smallest odd period greater than 1. For such a p, the graph of a periodic point of period p is given by Lemma 3.17. We keep the same notation as in Figure 2. If n is even and $n \in [\![2, p-1]\!]$, then

$$J_{p-n} \to J_{p-n+1} \to \cdots \to J_{p-1} \to J_{p-n}$$

is a primitive cycle of length n. If n is greater than p, we add $n - p$ times the cycle $J_1 \to J_1$ at the end of the fundamental cycle in order to obtain a primitive cycle of length n. Then, for all even integers $n \geq 2$ and all odd integers $n \geq p$, there exists a periodic point of period n by Lemma 3.16. This proves (3.9).

We now turn to the general case. Assume that f has a periodic point x of period $n = 2^d q$, where $q \geq 1$ is an odd integer. We want to show that, for all $m \neq 1$ with $m \triangleright n$, f has a periodic point of period m. We split the proof into three cases.

i) Case $q = 1$ and $m = 2^e$ for some $0 < e < d$. According to Lemma 3.18(i), the point x is of period $2^{d-e+1} > 1$ for $g := f^{\frac{m}{2}}$. Thus g has a periodic point y of period 2 by (3.8), and y is periodic of period $m = 2^e$ for f by Lemma 3.18(ii).

ii) Case $q > 1$ and $m = 2^d r$ for some $r \geq 2$, r even. By Lemma 3.18(i), the point x is of period q for $g := f^{2^d}$. Since q is odd and greater than 1, g has a periodic point y of period r according to (3.9). Then y is periodic of period $m = 2^d r$ for f by Lemma 3.18(ii).

iii) Case $q > 1$ and $m = 2^d r$ for some $r > q$, r odd. By Lemma 3.18(i), the point x is of period q for $g := f^{2^d}$. Since q is odd and greater than 1, g has a periodic point y of period r by (3.9). According to Lemma 3.18(ii), there exists an integer $e \in [\![1, d]\!]$ such that y is of period $2^e r$ for f. If $e = d$, then f has a periodic point of period m. Otherwise, we set $r' := 2^{d-e} r$. The map f has a periodic point of period $2^e r$ with r odd, and the integer m can be written as $m = 2^e r'$ with r' even. Then the case (ii) above implies that f has a periodic point of period m.

This concludes the proof. □

3.3.3. Interval maps of all types. Because of the structure of Sharkovsky's order, Theorem 3.13 implies that the set of periods of an interval map is of the form either $\{m \in \mathbb{N} \mid m \trianglerighteq n\}$ for some $n \in \mathbb{N}$ or $\{2^k \mid k \geq 0\}$. This motivates the next definition.

Definition 3.19. Let $n \in \mathbb{N} \cup \{2^\infty\}$. An interval map f is *of type n (for Sharkovsky's order)* if the periods of the periodic points of f form exactly the set $\{m \in \mathbb{N} \mid m \trianglerighteq n\}$, where the notation $\{m \in \mathbb{N} \mid m \trianglerighteq 2^\infty\}$ stands for $\{2^k \mid k \geq 0\}$.

Every interval map has a type. Conversely, there exist maps of all types. This result was shown by Sharkovsky in [**153**] for integer types and in [**155**] for type 2^∞. We are going to exhibit interval maps of all types. Some of these examples will be referred to in other chapters. We first state a lemma, which is a partial converse of Lemma 3.16.

Lemma 3.20. *Let f be an interval map. Let $\{x_1 < \cdots < x_n\}$ be a periodic orbit of period $n > 1$, and let G denote the graph of this periodic orbit. Suppose that $f|_{[x_i, x_{i+1}]}$ is monotone for all $i \in [\![1, n-1]\!]$. If f has a periodic point of period m in $[x_1, x_n]$, then*

- *either G contains a primitive cycle of length m,*
- *or m is even and G contains a primitive cycle of length $m/2$.*

Proof. Let $y \in [x_1, x_n]$ be a periodic point of period m. If $y \in \{x_1, \ldots, x_n\}$, then $n = m$ and G contains a primitive cycle of length m by Lemma 3.15. From now on, we suppose that $y \notin \{x_1, \ldots, x_n\}$. We show by induction that for all $k \geq 0$ there is a unique vertex J_k in G such that $f^k(y) \in J_k$, and in addition $J_k \to J_{k+1}$.
• There is a unique vertex J_0 containing y because $y \in [x_1, x_n] \setminus \{x_1, \ldots, x_n\}$.
• Suppose that $J_k = [x_i, x_{i+1}]$ is already defined. Since f is monotone on J_k, the point $f^{k+1}(y)$ belongs to $f(J_k) = \langle f(x_i), f(x_{i+1}) \rangle$. Since $\langle f(x_i), f(x_{i+1}) \rangle$ is a nonempty union of vertices of G, this implies that there exists a vertex J_{k+1} in G such that $f^{k+1}(y) \in J_{k+1}$ and $J_k \to J_{k+1}$. The vertex J_{k+1} is unique because $f^{k+1}(y) \notin \{x_1, \ldots, x_n\}$. This concludes the induction.

Since $f^m(y) = y$, we have $J_m = J_0$, and thus $J_0 \to \cdots J_{m-1} \to J_0$ is a cycle in G. This cycle is a multiple of a primitive cycle of length p for some p dividing m. Therefore $J_p = J_0$ and $f^{kp}(y) \in J_0$ for all $k \geq 0$. Since $f^p|_{J_0}$ is monotone, the set $J := f^{-p}(J_0) \cap J_0$ is an interval and $f^{2p}|_J$ is non decreasing. Moreover, $f^{2kp}(y)$ belongs to J for all $k \geq 0$. If $y \leq f^{2p}(y)$, a straightforward induction leads to:

$$y \leq f^{2p}(y) \leq f^{4p}(y) \leq \cdots \leq f^{2m}(y).$$

The reverse inequalities hold if $y \geq f^{2p}(y)$. In both cases, the fact that $y = f^{2m}(y)$ implies $y = f^{2p}(y)$. We deduce that m divides $2p$. Since m is a multiple of p, this implies that $m = p$ or $m = 2p$. $\square$

EXAMPLE 3.21. We fix $n \geq 1$. We are going to build a map f_p of odd type $p = 2n + 1 > 1$. The map $f_p \colon [0, 2n] \to [0, 2n]$, represented in Figure 4, is defined

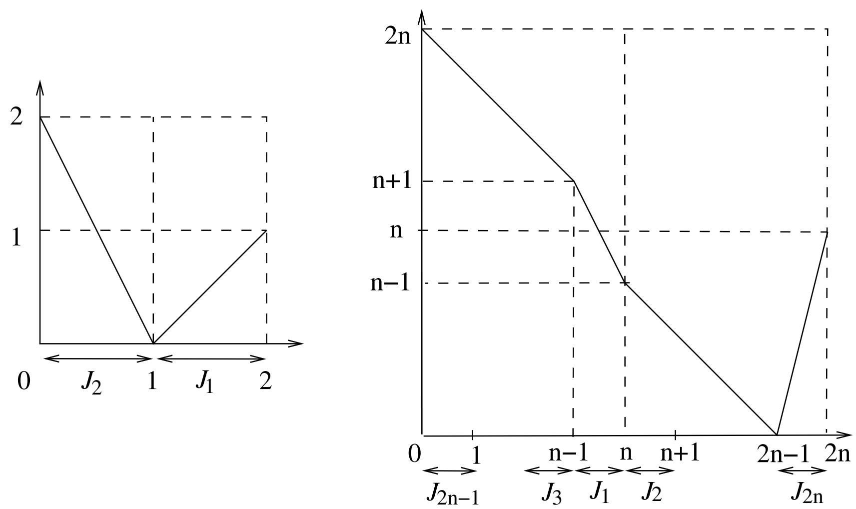


FIGURE 4. On the left: an interval map of type 3. On the right: an interval map of odd type $p = 2n + 1 > 3$.

as follows: it is linear on $[0, n-1]$, $[n-1, n]$, $[n, 2n-1]$ and $[2n-1, 2n]$, and

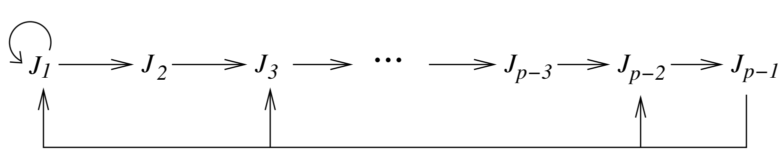
$$f_p(0) := 2n,\ f_p(n-1) := n+1,\ f_p(n) := n-1,\ f_p(2n-1) := 0,\ f_p(2n) := n.$$

Notice that $n = 1$ is a particular case because $0 = n - 1$ and $n = 2n - 1$. This map satisfies:

$$\forall k \in [\![1, n]\!],\ f_p(n-k) = n+k,$$
$$\forall k \in [\![0, n-1]\!],\ f_p(n+k) = n-k-1.$$

It follows that $f_p^{2k-1}(n) = n - k$ and $f_p^{2k}(n) = n + k$ for all $k \in [\![1, n]\!]$. Thus $f_p^{2n+1}(n) = n$, and the point n is periodic of period $p = 2n + 1$.

We set $J_{2k-1} := [n-k, n-k+1]$ and $J_{2k} := [n+k-1, n+k]$ for all $k \in [\![1, n]\!]$. It is easy to check that the graph of the periodic point n is the one given in Lemma 3.17, that is:



This graph does not contain any primitive cycle of odd length $m \in [\![2, p-1]\!]$. Thus, by Lemma 3.20, f_p has no periodic point of odd period $m \in [\![2, p-1]\!]$. This means that f_p is of type p.

For further reference, we are going to show that f_p is topologically mixing. The transitivity of f_p was shown by Block and Coven [**34**]. We are first going to show that, for every subinterval J in $[0, 2n]$,

$$\exists i \in [\![1, 2n]\!],\ J \subset J_i \Rightarrow \exists k \geq 1,\ |f_p^k(J)| \geq 2|J|. \tag{3.10}$$

If $J \subset J_1$, then $|f_p(J)| = 2|J|$ because $\mathsf{slope}(f_p|_{J_1}) = -2$. If there exists $i \in [\![2, 2n]\!]$ such that $J \subset J_i$, then $f_p^{2n-i}(J) \subset J_{2n}$. For all $i \in [\![2, 2n-1]\!]$, $\mathsf{slope}(f_p|_{J_i}) = -1$ and $\mathsf{slope}(f_p|_{J_{2n}}) = n$. It follows that $|f_p^{2n-i+1}(J)| = n|f_p^{2n-i}(J)| = n|J|$. If $n \geq 2$, then $|f_p^{2n-i+1}(J)| \geq 2|J|$. If $n = 1$, then $f_p(J_{2n}) = J_1$, and thus $f_p^{2n-i+1}(J) \subset J_1$ and $|f_p^{2n-i+2}(J)| \geq 2|J|$. In both cases, there exists an integer $k \geq 1$ such that $|f_p^k(J)| \geq 2|J|$. This proves (3.10).

Let J be a non degenerate subinterval of $[0, 2n]$. If, for all integers k, the interval $f_p^k(J)$ does not meet $\{0, 1, \ldots, 2n\}$, then (3.10) implies that the length of $f_p^k(J)$ grows to infinity, which is impossible. Thus there exists an integer $k \geq 0$ such that $f_p^k(J)$ contains one of the points $0, 1, \ldots, 2n$. Since $\{0, 1, \ldots, 2n\}$ is a periodic orbit, k can be chosen such that $0 \in f_p^k(J)$. Moreover, $f_p^k(J)$ is a non degenerate interval according to the definition of f_p. The point 0 is fixed for $g := (f_p)^p$ and g is of slope $4n > 1$ on $\left[0, \frac{1}{4n}\right]$. Thus, by Lemma 2.11, there exists $i \geq 0$ such that $g^i(f^k(J)) \supset \left[0, \frac{1}{4n}\right]$, and hence $g^{i+1}(f^k(J)) \supset [0, 1]$. Moreover, for every $i \in [\![1, p-1]\!]$,

$$J_{p-2} \to J_{p-1} \to \underbrace{J_1 \to \cdots \to J_1}_{p-1-i \text{ arrows}} \to J_2 \cdots \to J_i$$

is a path of length p from $J_{p-2} = [0, 1]$ to J_i in the graph of the periodic point n. This implies that $f_p^p([0, 1]) \supset \bigcup_{i=1}^{p-1} J_i = [0, 2n]$. Therefore, $f_p^{p(i+2)+k}(J) = [0, 2n]$. We conclude that f_p is topologically mixing.

Example 3.22. We are going to build interval maps of type n for all integers $n \in \mathbb{N}$, following the construction in [**166**]. We start with the definition of the so-called *square root* of a map. If $f\colon [0, b] \to [0, b]$ is an interval map and $\delta \in [0, b]$, the *square root* of f (more precisely, one realization of the square root of f) is the continuous map $g\colon [0, 2b + \delta] \to [0, 2b + \delta]$ defined by:

- $\forall x \in [0, b]$, $g(x) := f(x) + (b + \delta)$,
- $\forall x \in [b + \delta, 2b + \delta]$, $g(x) := x - (b + \delta)$,
- g is linear on $[b, b + \delta]$.

The map g is not well defined if $\delta = 0$ and $g(b) > 0$. The value chosen for δ is usually $\delta = 0$ if $g(b) = 0$, and $\delta = b$ otherwise. This construction is represented in Figure 5 with $\delta = b$. This map satisfies:

$$\forall x \in [0, b],\ g^2(x) = f(x), \tag{3.11}$$

$$g([0, b]) \subset [b + \delta, 2b + \delta] \text{ and } g([b + \delta, 2b + \delta]) = [0, b]. \tag{3.12}$$

It is clear that g has a unique fixed point c, and that $c \in [b, b + \delta]$. Moreover, $\lambda := \mathsf{slope}(g|_{[b, b+\delta]}) < -1$. Thus, if $x, g(x), \ldots, g^k(x)$ belong to $[b, b + \delta]$, then $|g^k(x) - c| \geq |\lambda|^k |x - c|$. This implies that, for all $x \in [b, b + \delta] \setminus \{c\}$, there exists $k \geq 0$ such that $g^k(x) \in [0, b] \cup [b + \delta, 2b + \delta]$. Thus, by (3.12), all periodic orbits

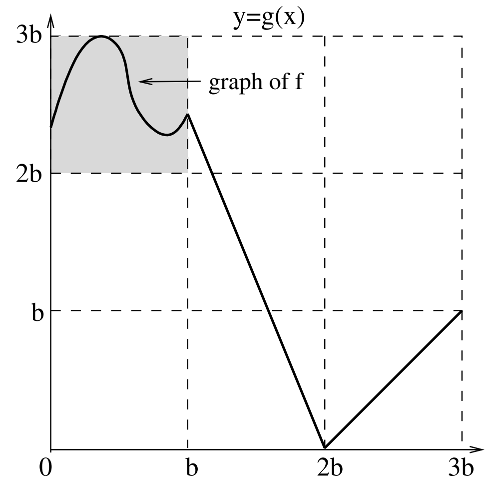


FIGURE 5. The map g is the square root of f. If f is of type n, g is of type $2n$.

of g, except c, have at least one point in $[0, b]$ and are of even period. Moreover, (3.11) implies that a point $x \in [0, b]$ is a periodic point of period $2m$ for g if and only if it is a periodic point of period m for f. We deduce that, if f is of type n, then g is of type $2n$.

With this procedure, it is possible to build an interval map of type n for every positive integer n. We write $n = 2^d q$ with $d \geq 0$ and q odd. If $q = 1$, we start with a constant map $f\colon [0, 1] \to [0, 1]$, which is of type 1. If $q > 1$, we start with the interval map f_q of type q defined in Example 3.21. Then we build the square root of f, then the square root of the square root, etc. At step d, we get an interval map of type $n = 2^d q$.

EXAMPLE 3.23. We are going to build an interval map of type 2^∞. We follow [**72**]; see also [**63**]. For all $n \geq 0$, we set

$$I_n := \left[1 - \frac{1}{3^n}, 1 - \frac{2}{3^{n+1}}\right].$$

For every $n \geq 0$, let $f_n\colon I_n \to I_n$ be the map of type 2^n built in Example 3.22 and rescaled (i.e., conjugate by an increasing linear homeomorphism) to fit into I_n. Then the continuous map $f\colon [0, 1] \to [0, 1]$, illustrated in Figure 6, is defined by:

- $\forall x \in I_n$, $f(x) := f_n(x)$,
- $f(1) := 1$,
- $\forall n \geq 0$, f is linear on $\left[1 - \frac{2}{3^{n+1}}, 1 - \frac{1}{3^{n+1}}\right]$.

It is obvious that the only periodic points of f in $\left[1 - \frac{2}{3^{n+1}}, 1 - \frac{1}{3^{n+1}}\right]$ are fixed, and x is a periodic point of period $p > 1$ for f if and only if there is some $n \geq 0$ such that x is a periodic point of period p for f_n. Therefore the type of f is 2^∞.

We remark that, for all $x \in [0, 1]$, the ω-limit set of x is a periodic orbit of period 2^n for some integer $n \geq 0$. This is not always the case for maps of type 2^∞. In [**72**], there is another example of a map of type 2^∞ with an infinite ω-limit set. We shall see such a map in depth in Example 5.56.

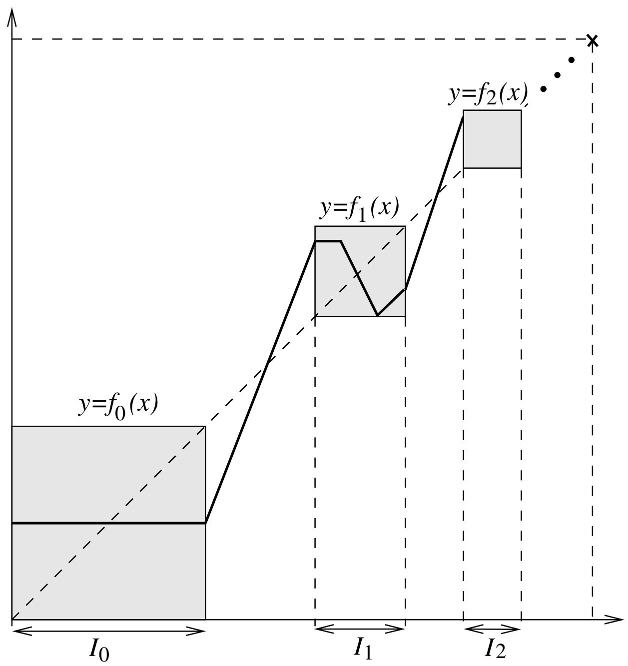


FIGURE 6. Each map f_n is of type 2^n, the whole map is of type 2^∞.

REMARK 3.24. There is a completely different way of proving that all types are realized. It consists of the study of a one-parameter family of interval maps that exhibits all possible types. The most famous family is the *logistic* family $f_\lambda(x) = \lambda x(1-x)$, where $x \in [0,1]$ and $\lambda \in [0,4]$. For every $n \in \mathbb{N} \cup \{2^\infty\}$, there exists $\lambda_n \in [0,4]$ such that f_{λ_n} is of type n; the map $f_{\lambda_{2^\infty}}$ is called the *Feigenbaum map*. More generally, every "typical" family of smooth unimodal maps exhibits all possible types; an interval map $f\colon [0,1] \to [0,1]$ is *unimodal* if $f(0) = f(1) = 0$ and there is $c \in (0,1)$ such that $f|_{[0,c]}$ is increasing and $f|_{[c,1]}$ is decreasing. The proofs are non-constructive and rely on quite sophisticate tools like the kneading theory. The study of such families of interval maps is out of the scope of this book. See [**71, 67, 124, 85, 98, 101**].

In [**14**, Section 2.2], Alsedà, Llibre and Misiurewicz gave a short proof consisting of showing that the family of truncated tent maps exhibits all types. The truncated tent maps are defined as $g_\lambda(x) = \min(T_2(x), \lambda)$, where $x \in [0,1]$, $\lambda \in [0,1]$ and T_2 is the tent map defined in Example 2.13. The proof is non constructive, as for families of smooth unimodal maps, but is much simpler.

Remarks on graph maps. Sharkovsky's Theorem 3.13 has motivated a lot of work aimed at finding characterizations of the set of periods for more general one-dimensional spaces.

One of the lines of generalization of Sharkovsky's Theorem consists of characterizing the possible sets of periods of tree maps. The first remarkable results in this line are due to Alsedà, Llibre and Misiurewicz [**13**] and Baldwin [**21**]. In [**13**] the characterization of the set of periods of the maps on the 3-star with a fixed branching point, in terms of three linear orderings, was obtained, whereas in [**21**] the characterization of the set of periods of all dynamical systems on n-stars is given (an n-star is a tree made of n segments glued together by one of their endpoints at a single point, e.g., $S_n := \{z \in \mathbb{C} \mid z^n \in [0,1]\}$). Further extensions were given by Baldwin and Llibre [**22**] for tree maps such that all the branching points are fixed, then by Bernhardt [**29**] for tree maps such that all the branching points are periodic. Finally, Alsedà, Juher and Mumbrú overcame the general case of tree maps [**6, 7, 8, 9**]. They showed that the set of periods of a tree map is the union of finitely many terminal segments of the orders of Baldwin and of a finite set (for every integer $p \geq 2$, the p-order of Baldwin is a partial ordering on $\mathbb{N}$, coinciding with Sharkovsky's order for $p = 2$). The precise statement is quite complicated; we refer to [**8**, Theorem 1.1].

Another direction is to consider topological graphs which are not trees, the circle being the simplest one. Circle maps display a new feature: the set of periods depend on the degree of the map and, in the case of degree 1, on the rotation interval.

Consider a circle map $f\colon \mathbb{S} \to \mathbb{S}$, where $\mathbb{S} := \mathbb{R}/\mathbb{Z}$, and a lifting of f, that is, a continuous map $F\colon \mathbb{R} \to \mathbb{R}$ such that $\pi \circ F = f \circ \pi$, where $\pi\colon \mathbb{R} \to \mathbb{S}$ denotes the canonical projection (F is uniquely defined up to the addition of an integer). The *degree* of f (or F) is the integer $d \in \mathbb{Z}$ such that $F(x+1) = F(x) + d$ for all $x \in \mathbb{R}$.

The characterization of the sets of periods for circle maps of degree different from 1 is simpler than the one for the case of degree 1. The case of degree different from $1, -1$ and part of the degree -1 case are due to Block, Guckenheimer, Misiurewicz and Young [**36**]. See also [**14**, Section 4.7].

THEOREM 3.25. *Let $f\colon \mathbb{S} \to \mathbb{S}$ be a circle map of degree $d \neq 1$.*

- *If $|d| \geq 2$, $d \neq -2$, then the set of periods of f is $\mathbb{N}$.*
- *If $d = -2$, then the set of period of f is either $\mathbb{N}$ or $\mathbb{N} \setminus \{2\}$.*
- *If $d \in \{0, -1\}$, then there exists $s \in \mathbb{N} \cup \{2^\infty\}$ such that the set of periods of f is $\{m \in \mathbb{N} \mid m \trianglerighteq s\}$.*

Moreover, all cases are realized by some circle maps.

The characterization of the sets of periods of circle maps of degree 1 is due to Misiurewicz [**128**] and uses as a key tool the rotation theory. The reader can refer to [**14**] for an exposition of rotation theory for (non invertible) circle maps of degree 1. The sets of periods of circle maps of degree 1 contain the set of all denominators of all rational numbers (not necessarily written in irreducible form) in the interior of an interval of the real line. As a consequence, these sets of periods cannot be expressed in terms of a finite collection of orderings.

THEOREM 3.26. *Let $f\colon \mathbb{S} \to \mathbb{S}$ be a circle map of degree 1, and let $[a,b]$ be the rotation interval of a lifting of f. Then there exist $s_a, s_b \in \mathbb{N} \cup \{2^\infty\}$ such that the set of periods of f is equal to $S(a, s_a) \cup M(a,b) \cup S(b, s_b)$, where*

- *$M(a,b) := \{q \in \mathbb{N} \mid \exists p \in \mathbb{Z},\ \frac{p}{q} \in (a,b)\}$.*
- *$S(x,s) := \emptyset$ if $a \in \mathbb{R} \setminus \mathbb{Q}$ and $S(x,s) := \{nq \mid n \trianglerighteq s\}$ if $a = \frac{p}{q}$ with $p \in \mathbb{N}$, $q \in \mathbb{Z}$ and $\gcd(p,q) = 1$.*

Moreover, all cases are realized.

Finding a characterization of the sets of periods of graphs maps when the graph is neither a tree nor the circle is a big challenge and in general it is not known what the sets of periods may look like. Only two cases have been studied: the graph shaped like σ [**108**], and the graph shaped like 8 [**115**], with the restriction, in both cases, that the map fixes the branching point. This assumption greatly facilitates the study (similarly, tree maps fixing all the branching points were dealt with first).

A rotation theory has been developed by Alsedà and the author for maps of degree 1 on graphs containing a single loop [**16**]. It leads to results similar to, but weaker than, the ones obtained from the rotation theory for circle maps. They give information about the periods, but this is far from leading to a characterization of the sets of periods, even in the simplest case of the graph σ [**18**].

3.4. Relations between types and horseshoes

If an interval is mapped across itself twice, the effect on the dynamics is similar to Smale's horseshoe for two-dimensional homeomorphisms [**161**]. This leads to the following definition. The name *horseshoe* for interval maps was given by Misiurewicz [**125**], but the notion was introduced much earlier by Sharkovsky under the name of *L-scheme* [**153**].

Definition 3.27 (horseshoe). Let f be an interval map. If $J_1, \ldots, J_n$ are non degenerate closed intervals with pairwise disjoint interiors such that $J_1 \cup \cdots \cup J_n \subset f(J_i)$ for all $i \in [\![1, n]\!]$, then $(J_1, \ldots, J_n)$ is called an *n-horseshoe*, or simply a *horseshoe* if $n = 2$. If in addition the intervals are disjoint, $(J_1, \ldots, J_n)$ is called a *strict* n-horseshoe.

Remark 3.28. The definition of horseshoe slightly varies in the literature. For some authors, a horseshoe is made of disjoint closed subintervals, or is a partition of an interval into subintervals such that the image of every subinterval contains the whole interval (thus, the subintervals forming the horseshoe are disjoint but not closed). The definition above follows [**14**]. For Block and Coppel, an interval map with a horseshoe is called *turbulent* [**40**]; this terminology was suggested by Lasota and Yorke [**107**]. However, *turbulent* may refer to a point with an infinite ω-limit set (see e.g. [**72**]), and so this word might be confusing.

Sometimes it will be useful to boil down to a particular form of a horseshoe, as given by the next lemma.

Lemma 3.29. *Let f be an interval map and (J, K) a horseshoe. Then there exist points u, v, w such that $f(u) = u$, $f(v) = w$, $f(w) = u$ and, either $u < v < w$, or $u > v > w$. Note that $\langle u, v\rangle, \langle v, w\rangle$ form a horseshoe.*

Proof. We assume $J \le K$. Let $a, b \in J$ be such that $f(a) = \min J$ and $f(b) = \max K$. We have $\langle a, b\rangle \subset J$ and $f(\langle a, b\rangle) \supset J \cup K$.

• **Case 1:** $a < b$**.** Let $u \in [a, b]$ be a fixed point (u exists by Lemma 1.11); $u \neq b$ because $f(b) \notin [a, b]$, hence $u \notin K$. Since $f(K) \supset J \supset [a, b]$, there is $w \in K$ such that $f(w) = u$ (by the intermediate value theorem). Moreover $f([u, b]) \supset [u, \max K] \supset K$, thus there is $v \in [u, b]$ such that $f(v) = w$. We have $u < v$ by definition (note that $u = v$ is impossible because $u \notin K$ and $f(v) \in K$) and $v \le w$ because $v \in J$ and $w \in K$. Moreover $v \neq w$ because $f(w) = u < v \le w = f(v)$.

• **Case 2:** $b < a$**.** Let $u \in K$ be a fixed point; $u > a$ because $f(a) \notin K$. Let $w \in [b, a]$ be such that $f(w) = u$. Since $f([w, a]) \supset [\min J, u] \supset J$, there is $v \in [w, a]$ such that $f(v) = w$. We have $w \le v \le u$, and equalities $v = u$, $w = v$ are not possible because $v \le a < u$ and $f(w) = u > v \ge w = f(v)$. $\square$

The next lemma is a straightforward consequence of Lemma 1.13(iii).

LEMMA 3.30. *Let f be an interval map and $(J_1, \dots, J_p)$ a p-horseshoe for f. Then, for all $n \geq 1$,*

i) *$\forall i, j \in [\![1, p]\!]$, J_i covers p^{n-1} times J_j for f^n,*
ii) *f^n has a p^n-horseshoe.*

We shall see in Section 4.2 that horseshoes are intimately related to entropy, but what interests us now is the relationship between horseshoes and the periods of periodic points. We first show that a map with a horseshoe is of type 3. This result appears as part of a proof due to Sharkovsky [**153**, Lemma 4]. It was also stated by Block and Coppel [**40**].

PROPOSITION 3.31. *An interval map f with a horseshoe has periodic points of all periods.*

PROOF. Let (J, K) be a horseshoe for f. First, we assume that J and K are disjoint. Let $n \geq 1$. Applying Lemma 1.13(ii) to the chain of intervals $(I_0, \dots, I_n)$ with $I_i := K$ for all $i \in [\![1, n-1]\!]$ and $I_0 = I_n := J$, we see there exists a periodic point $x \in J$ such that $f^n(x) = x$ and $f^k(x) \in K$ for all $k \in [\![1, n-1]\!]$. The fact that J and K are disjoint implies that the period of x is exactly n.

Now we assume that J and K have a common endpoint. We write $J = [a, b]$ and $K = [b, c]$ (we may suppose with no loss of generality that J is on the left of K). If b is a fixed point, we set

$$d := \min\{x \geq b \mid f(x) \in \{a, c\}\}.$$

It follows that $d > b$ and the image of $[b, d)$ contains neither a nor c. Thus $f([d, c])$ contains a and c because $[a, c] \subset f([b, c])$, so $[a, c] \subset f([d, c])$ by connectedness. We deduce that $(J, [d, c])$ is a strict horseshoe. The first part of the proof implies that f has periodic points of all periods.

Suppose now that b is not a fixed point. Applying Lemma 1.13(ii) to the chain of intervals (J, K, K, J), we see that there exists a periodic point $x \in J$ such that $f^3(x) = x$, $f(x) \in K$ and $f^2(x) \in K$. The period of x divides 3, and thus it is equal to 1 or 3. If x is a fixed point, then $x \in J \cap K = \{b\}$, which is impossible because b is not fixed. Thus x is of period 3. Then f has periodic points of all periods according to Sharkovsky's Theorem 3.13. □

An interval map with a periodic point of period 3 may have no horseshoe. Such a map will be built in Example 4.61. However, if f has a periodic point of odd period greater than 1, then f^2 has a horseshoe. This result was underlying in a paper of Block [**33**] and was stated by Osikawa and Oono in [**139**]; see also [**40**].

PROPOSITION 3.32. *Let $f \colon I \to I$ be an interval map. If f has a periodic point of odd period greater than 1, then there exist two intervals J, K containing no endpoint of I and such that (J, K) is a strict horseshoe for f^2.*

PROOF. Let p be the least odd integer different from 1 such that f has a periodic point of period p and let x be a periodic point of period p. According to Lemma 3.17, there exists a point x_0 in the orbit of x such that the points $x_i := f^i(x_0), 0 \leq i \leq p-1$, are ordered as:

$$x_{p-1} < x_{p-3} < \cdots < x_2 < x_0 < x_1 < \cdots < x_{p-2}$$

or in the reverse order. Suppose that the order above holds, the other case being symmetric.

The interval $f([x_0, x_1])$ contains $[x_2, x_0]$, which implies that there exists d in (x_0, x_1) such that $f(d) = x_0$, and hence $d < f^2(d) = x_1$. Since $f^2([x_{p-1}, x_{p-3}])$ contains $[x_{p-1}, x_1]$, there exists a point $a \in (x_{p-1}, x_{p-3})$ such that $f^2(a) > d$. Then $f^2([a, x_{p-3}]) \supset [x_{p-1}, d]$, and thus there exists $b \in (a, x_{p-3})$ such that $f^2(b) < a$. Similarly, there exists $c \in (x_{p-3}, d)$ such that $f^2(c) < a$ because $f^2([x_{p-3}, d]) \supset [x_{p-1}, f^2(d)] \supset [x_{p-1}, d]$. Then $J := [a, b]$ and $K := [c, d]$ are disjoint intervals and form a horseshoe for f^2. Finally, J and K do not contain any endpoint of I because $x_{p-1} < a$ and $d < x_1$. $\square$

We end this section with two small results, related to horseshoes and periodic points of odd period; both will be referred to later. The first one states that, if f has no horseshoe, every orbit splits into two sets U and D with $U \leq D$ such that all points in U (resp. D) are going "up" (resp. "down") under the action of f. The second one is a tool to prove the existence of periodic points when only partial information on the location of the points is known.

The next result, already implicit in a paper of Sharkovsky [**153**, proof of Lemma 4], was proved by Li, Misiurewicz, Pianigiani and Yorke under a slightly weaker assumption [**112**, Corollary 3.2].

LEMMA 3.33. *Let f be an interval map with no horseshoe, and let x_0 be a point. Let $U(x_0) := \{x \in \mathcal{O}_f(x_0) \mid f(x) \geq x\}$ and $D(x_0) := \{x \in \mathcal{O}_f(x_0) \mid f(x) \leq x\}$. If these two sets are nonempty, then $\sup U(x_0) \leq \inf D(x_0)$ and there exists a fixed point $z \in [\sup U(x_0), \inf D(x_0)]$.*

PROOF. We set $x_n := f^n(x_0)$ for all $n \geq 0$. Let n, m be integers such that $x_n \in U(x_0)$ and $x_m \in D(x_0)$. We are going to show that $x_n \leq x_m$. Suppose on the contrary that $x_n > x_m$. We assume that $m > n$, the case $m < n$ being similar. The point x_n is not fixed because $x_m = f^{m-n}(x_n) > x_n$. Thus, according to the definition of $U(x_0)$ and $D(x_0)$, we have

$$f(x_m) \leq x_m < x_n < f(x_n). \tag{3.13}$$

By continuity, there exists a fixed point in $[x_m, x_n]$. Let y be the maximal fixed point in $[x_m, x_n]$. Then $y < x_n$ and, since $f(x_n) > x_n$,

$$\forall x \in (y, x_n],\ f(x) > x. \tag{3.14}$$

By (3.13), there exists an integer $k \in [\![n+1, m]\!]$ such that $x_i > y$ for all $i \in [\![n, k]\!]$ and $x_{k+1} \leq y$. We show by induction on i that $x_i < x_k$ for all $i \in [\![n, k]\!]$.

- Case $i = n$: since $f(x_k) = x_{k+1} \leq y < x_k$, the point x_k does not belong to $(y, x_n]$ by (3.14), and thus $x_n < x_k$.

- Suppose that $x_i < x_k$ for some $i \in [\![n, k-1]\!]$. If $x_{i+1} \geq x_k$, then $([y, x_i], [x_i, x_k])$ is a horseshoe for f, which is a contradiction. Hence $x_{i+1} < x_k$.

For $i = k$, the induction statement is that $x_k < x_k$, which is absurd. Hence $x_n \leq x_m$. We deduce that $\sup U(x_0) \leq \inf D(x_0)$. Moreover, the definitions of $U(x_0), D(x_0)$ imply that $f(\sup(U(x_0))) \geq \sup U(x_0)$ and $f(\inf(D(x_0))) \leq \inf D(x_0)$. Thus there exists a fixed point $z \in [\sup U(x_0), \inf D(x_0)]$ by continuity. $\square$

The next result was shown by Li, Misiurewicz, Pianigiani and Yorke [**111**].

PROPOSITION 3.34. *Let f be an interval map and let x be a point. Let $p \geq 3$ be an odd integer and suppose that either $f^p(x) \leq x < f(x)$ or $f^p(x) \geq x > f(x)$. Then f has a periodic point of period p.*

PROOF. We assume that $f^p(x) \leq x < f(x)$, the case with reverse inequalities being symmetric. We also assume that f has no horseshoe, otherwise f has periodic points of all periods by Proposition 3.31.

We set $x_n := f^n(x)$ for all $n \geq 0$. We define the sets

$$U := \{x_n \mid x_{n+1} \geq x_n,\ n \in [\![0,p]\!]\} \quad \text{and} \quad D := \{x_n \mid x_{n+1} \leq x_n,\ n \in [\![0,p]\!]\}.$$

By assumption, $x_p \leq x_0 < x_1$, which implies that $x_0 \in U$ and

$$\text{there exists } j \in [\![1,p-1]\!] \text{ such that } x_{j+1} < x_j, \tag{3.15}$$

so $x_j \in D$. Since U and D are not empty, Lemma 3.33 implies that

$$\max U \leq \min D \text{ and there exists a fixed point } z \in [\max U, \min D].$$

If $x_i = z$ for some $i \in [\![0,p]\!]$, then $x_p = z \geq \max U \geq x_0$. Since $x_p \leq x_0$, this implies $x_0 = z$, which is a contradiction because x_0 is not a fixed point. We deduce that $\max U < z < \min D$, and thus

$$x_p \leq x_0 \leq \max U < z < \min D \leq x_j.$$

We claim that there exists $k \in [\![0,p-1]\!]$ such that

$$\text{either } x_k, x_{k+1} \in U \quad \text{or} \quad x_k, x_{k+1} \in D.$$

Otherwise, all the points x_i with even index $i \in [\![0,p]\!]$ would be in U (because $x_0 \in U$) and all the points x_i with odd index $i \in [\![0,p]\!]$ would be in D, and thus $x_0 < z < x_p$ because p is odd. This would contradict the assumption that $x_p \leq x_0$. Therefore the claim holds, which implies that

$$\text{either } x_k \leq x_{k+1} < z \quad \text{or} \quad z < x_{k+1} < x_k. \tag{3.16}$$

We assume that the case $x_k \leq x_{k+1} < z$ holds in (3.16), the other case being symmetric. We set $J_k := [x_k, \max U]$ and $J_i := \langle x_i, z\rangle$ for all $i \in [\![0,p]\!]$ with $i \neq k$. Then $f(J_k) \supset [x_{k+1}, \min D] \supset [x_{k+1}, z]$ and $f(J_i) \supset \langle x_{i+1}, z\rangle$ for all $i \in [\![0,p]\!]$ with $i \neq k$. Then $(J_0, \ldots, J_p)$ is a chain of intervals. Moreover, we have $J_0 \subset J_p$. Thus there exists $y \in J_0$ such that $f^p(y) = y$ by Lemma 1.13(ii). Let q be the period of y; this is a divisor of p. If $q = 1$, then $y \in J_0 \cap J_j = [x_0, z] \cap [z, x_j]$ (recall that j is such that $x_j > z$ by (3.15)), and hence $y = z$. But this is not possible because $y \in J_k = [x_k, \max U]$, with $\max U < z$. We deduce that $q > 1$. Since p is odd, then q is odd too, and $1 < q \leq p$. Then Sharkovsky's Theorem 3.13 gives the conclusion. □

3.5. Types of transitive and mixing maps

We saw that a mixing interval map has a periodic point of odd period greater than 1 (Theorem 2.20). Moreover, Example 3.21 shows that, for every odd $q > 1$, there exists a mixing map of type q. If an interval map f is transitive but not mixing, then, according to Theorem 2.19, there exists a subinterval J such that $f^2|_J$ is mixing, and thus f^2 is of type q for some odd $q > 1$. Actually, q is always equal to 3 in this case, which implies that f is of type 6. This result was proved by Block and Coven [**34**]; it is also a consequence of a result of Blokh [**42**]. We start with a lemma, stated in [**34**].

LEMMA 3.35. *Let $f\colon [a,b] \to [a,b]$ be a transitive interval map. If f has no horseshoe, then it has a unique fixed point. Moreover, this fixed point is neither a nor b.*

PROOF. Suppose that f is transitive and has at least two fixed points. Then Theorem 2.19 implies that f is topologically mixing. The set of fixed points $P_1(f)$ has an empty interior by transitivity, and it is a closed set. This implies that there exist two points $x_1 < x_2$ in $P_1(f)$ such that $(x_1, x_2) \cap P_1(f) = \emptyset$ and thus, either

$$\forall x \in (x_1, x_2),\ f(x) < x,$$

or

$$\forall x \in (x_1, x_2),\ f(x) > x. \tag{3.17}$$

We assume that (3.17) holds, the other case being symmetric. If

$$\forall x \in (x_1, b],\ f(x) > x_1,$$

then the interval $[x_1, b]$ is invariant, which is impossible by transitivity except if $x_1 = a$. In this case, a is a non accessible endpoint because $a \notin f((a, b])$, and thus there exists a sequence of fixed points that tend to a by Lemma 2.32. But this contradicts the choice of x_1 and x_2. We deduce that there exists $t \in (x_1, b]$ such that $f(t) \leq x_1$. Actually, t belongs to $[x_2, b]$ because of (3.17). Since $f(x_2) = x_2 > x_1$, there exists $z \in [x_2, t]$ such that $f(z) = x_1$ by the intermediate value theorem. Thus we can define

$$z := \min\{x \in [x_1, b] \mid f(x) = x_1\}.$$

Actually $z \in [x_2, b]$ because of (3.17). If $f(x) \neq z$ for all $x \in (x_1, z)$, then $f(x) < z$ for all $x \in (x_1, z)$ (because $f(x_1) = x_1 < z$), and the minimality of z implies that $f(x) > x_1$ for all $x \in (x_1, z)$. Thus the non degenerate interval $[x_1, z]$ is invariant and $z \notin f([x_1, z])$, which is impossible because f is transitive. We deduce that there exists $y \in (x_1, z)$ such that $f(y) = z$. If we set $J := [x_1, y]$ and $K := [y, z]$, then (J, K) is a horseshoe.

If f is transitive and has no horseshoe, what precedes implies that f has at most one fixed point. Thus f has a unique fixed point according to Lemma 1.11. If a (resp. b) is the unique fixed point of f, then $f(x) < x$ for all $x \in (a, b]$ (resp. $f(x) > x$ for all $x \in [a, b)$), and thus f is not onto. This is impossible because f is transitive, so we conclude that the unique fixed point of f is neither a nor b. $\square$

PROPOSITION 3.36. *Let $f\colon I \to I$ be a transitive interval map. Then f^2 has a horseshoe and f has a periodic point of period 6. Moreover,*

- *if f is topologically mixing, then it is of type p for some odd $p > 1$,*
- *if f is transitive but not mixing, then it is of type 6.*

PROOF. If f is topologically mixing, it has a periodic point of odd period $q > 1$ by Theorem 2.20. Sharkovsky's Theorem 3.13 implies that the type of f is an odd integer p in $[\![3, q]\!]$ and f has a periodic point of period 6. Moreover, f^2 has a horseshoe by Proposition 3.32.

If f is transitive but not topologically mixing, then it has no periodic point of odd period greater than 1 by Theorem 2.20, and thus the type of f is at least 6 for Sharkovsky's order. Moreover, according to Theorem 2.19, there exists a fixed point $c \in I$ which is not an endpoint of I and such that, if we set $J := [\min I, c]$ and $K := [c, \max I]$, the subintervals J, K are invariant under f^2, and both maps $f^2|_J$, $f^2|_K$ are topologically mixing. Then $f^2|_J$ is transitive and has a fixed endpoint, and thus it has a horseshoe according to Lemma 3.35. Therefore f^2 has a periodic point of period 3 by Proposition 3.31. The period of this point for f cannot be an odd integer, and thus it is equal to 6. We conclude that the type of f is 6. $\square$

CHAPTER 4

Topological entropy

4.1. Definitions

The notion of topological entropy for a dynamical system was introduced by Adler, Konheim and McAndrew [**1**]. Topological entropy is a conjugacy invariant. The aim of this first section is to recall briefly the definitions and introduce the notation used in the sequel, without entering into details. The readers who are not familiar with topological entropy can refer to [**169**] or [**73**].

4.1.1. Definition with open covers. Let (X, f) be a topological dynamical system. A *finite cover* is a collection of sets $\mathcal{C} = \{C_1, \ldots, C_p\}$ such that $C_1 \cup \cdots \cup C_p = X$. It is an *open cover* if in addition the sets $C_1, \ldots, C_p$ are open. A *partition* is a cover made of pairwise disjoint sets. The topological entropy is usually defined for open covers only. Nevertheless we give the definition for any finite cover because we shall sometimes deal with the entropy of covers composed of intervals which are not open.

Let $\mathcal{C} = \{C_1, \ldots, C_p\}$ and $\mathcal{D} = \{D_1, \ldots, D_q\}$ be two covers. The cover $\mathcal{C} \vee \mathcal{D}$ is defined by

$$\mathcal{C} \vee \mathcal{D} := \{C_i \cap D_j \mid i \in [\![1, p]\!],\ j \in [\![1, q]\!],\ C_i \cap D_j \neq \emptyset\}.$$

We say that $\mathcal{D}$ is *finer* than $\mathcal{C}$, and we write $\mathcal{C} \prec \mathcal{D}$, if every element of $\mathcal{D}$ is included in an element of $\mathcal{C}$. Let $N(\mathcal{C})$ denote the minimal cardinality of a subcover of $\mathcal{C}$, that is,

$$N(\mathcal{C}) := \min\{n \mid \exists i_1, \ldots, i_n \in [\![1, p]\!]\},\ X = C_{i_1} \cup \cdots \cup C_{i_n}\}.$$

Then, for all integers $n \geq 1$, we define

$$N_n(\mathcal{C}, f) := N\left(\mathcal{C} \vee f^{-1}(\mathcal{C}) \vee \cdots \vee f^{-(n-1)}(\mathcal{C})\right).$$

If there is no ambiguity on the map, $\mathcal{C}^n$ will denote $\mathcal{C} \vee f^{-1}(\mathcal{C}) \vee \cdots \vee f^{-(n-1)}(\mathcal{C})$. Note that $N(\mathcal{C}) \leq \#\mathcal{C}$. Moreover, if $\mathcal{P}$ is a partition (not containing the empty set), then $\mathcal{P}^n$ is a partition too, and $N(\mathcal{P}^n) = \#(\mathcal{P}^n)$ for all $n \geq 1$.

LEMMA 4.1. *Let $(a_n)_{n\geq 1}$ be a sub-additive sequence, that is, $a_{n+k} \leq a_n + a_k$ for all $n \geq 1$ and all $k \geq 1$. Then $\lim_{n\to+\infty} \frac{1}{n} a_n$ exists and is equal to $\inf_{n\geq 1} \frac{1}{n} a_n$.*

PROOF. The inequality

$$\liminf_{n\to+\infty} \frac{1}{n} a_n \geq \inf_{n\geq 1} \frac{1}{n} a_n \tag{4.1}$$

is obvious. Let k be a positive integer. For every positive integer n, there exist integers q, r such that $n = qk + r$ and $r \in [\![0, k-1]\!]$. The sub-additivity implies that $a_n \leq qa_k + a_r$, and thus $\limsup_{n\to+\infty} \frac{1}{n} a_n \leq \frac{1}{k} a_k$. Therefore,

$$\limsup_{n\to+\infty} \frac{1}{n} a_n \leq \inf_{k\geq 1} \frac{1}{k} a_k, \tag{4.2}$$

and the lemma follows from (4.1) and (4.2). □

It is easy to show that, for all finite covers $\mathcal{C}$, the sequence $\left(\frac{1}{n}\log N_n(\mathcal{C}, f)\right)_{n\geq 1}$ is sub-additive. Thus Lemma 4.1 can be used to define the *topological entropy* of the cover $\mathcal{C}$ by:

$$h_{top}(\mathcal{C}, f) := \lim_{n\to+\infty} \frac{\log N_n(\mathcal{C}, f)}{n} = \inf_{n\geq 1} \frac{\log N_n(\mathcal{C}, f)}{n}.$$

The next lemma follows straightforwardly from the definitions.

Lemma 4.2. *Let (X, f) be a topological dynamical system. If $\mathcal{C}$ and $\mathcal{D}$ are two finite covers such that $\mathcal{C} \prec \mathcal{D}$, then $h_{top}(\mathcal{C}, T) \leq h_{top}(\mathcal{D}, T)$.*

The *topological entropy* of a dynamical system (X, f) is defined by

$$h_{top}(f) := \sup\{h_{top}(\mathcal{U}, f) \mid \mathcal{U} \text{ finite open cover of } X\}.$$

The topological entropy is a non negative number (it may be infinite). It satisfies the following properties:

Proposition 4.3. *Let (X, f) be a topological dynamical system.*

- *For all integers $n \geq 1$, $h_{top}(f^n) = nh_{top}(f)$.*
- *If Y is an invariant subset of X, then $h_{top}(f|_Y) \leq h_{top}(f)$.*
- *if (Y, g) is a topological dynamical system that is conjugate to (X, f), then $h_{top}(f) = h_{top}(g)$.*

When dealing with entropy in the sequel, we shall often use that $h_{top}(f^n) = nh_{top}(f)$, without referring systematically to Proposition 4.3.

4.1.2. Definition with Bowen's formula. The topological entropy can be computed with *Bowen's formula*. The following notions were introduced in [**59**].

Let X be a metric space with a distance d, and let $f\colon X \to X$ be a continuous map. Let $\varepsilon > 0$ and $n \geq 1$. The *Bowen ball* of center x, radius ε and order n is defined by

$$\begin{aligned} B_n(x, \varepsilon) &:= \{y \in X \mid d(f^k(x), f^k(y)) \leq \varepsilon,\ k \in [\![0, n-1]\!]\} \\ &= \bigcap_{i=0}^{n-1} f^{-i}(\overline{B}(f^i(x), \varepsilon)). \end{aligned}$$

Let $E \subset X$. The set E is (n, ε)*-separated* if for all distinct points x, y in E, there exists $k \in [\![0, n-1]\!]$ such that $d(f^k(x), f^k(y)) > \varepsilon$. The maximal cardinality of an (n, ε)-separated set is denoted by $s_n(f, \varepsilon)$. The set E is an (n, ε)*-spanning set* if $X \subset \bigcup_{x\in E} B_n(x, \varepsilon)$. The minimal cardinality of an (n, ε)-spanning set is denoted by $r_n(f, \varepsilon)$.

Lemma 4.4. *Let (X, f) be a topological dynamical system, $\varepsilon > 0$ and $n \in \mathbb{N}$.*

i) *If $0 < \varepsilon' < \varepsilon$, then $s_n(f, \varepsilon') \geq s_n(f, \varepsilon)$ and $r_n(f, \varepsilon') \geq r_n(f, \varepsilon)$.*
ii) *$r_n(f, \varepsilon) \leq s_n(f, \varepsilon) \leq r_n(f, \frac{\varepsilon}{2})$.*

PROOF. (i) Obvious.
(ii) Let E be an (n,ε)-separated set of maximal cardinality $s_n(f,\varepsilon)$. By maximality, for every $y \in X \setminus E$, $E \cup \{y\}$ is not (n,ε)-separated, that is, $y \in \bigcup_{x\in E} B_n(x,\varepsilon)$. Moreover, E is clearly included in $\bigcup_{x\in E} B_n(x,\varepsilon)$. This means that E is an (n,ε)-spanning set, and so $r_n(f,\varepsilon) \leq s_n(f,\varepsilon)$. Let F be an $(n,\frac{\varepsilon}{2})$-spanning set of cardinality $r_n(f,\frac{\varepsilon}{2})$. For every $x \in X$, there exists $y(x) \in F$ such that $x \in B_n(y(x),\frac{\varepsilon}{2})$. If x_1, x_2 are two distinct points in E, then $y(x_1) \neq y(x_2)$ (otherwise this would imply that $d(f^i(x_1), f^i(x_2)) < \varepsilon$ for all $i \in [\![0, n-1]\!]$). Thus $\#E \leq \#F$, that is, $s_n(f,\varepsilon) \leq r_n(f,\frac{\varepsilon}{2})$. $\square$

The next result is due to Bowen [**60**]; see also [**146**].

THEOREM 4.5 (Bowen's formula). *Let (X, f) be a topological dynamical system. Then*

$$h_{top}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log s_n(f,\varepsilon) = \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log r_n(f,\varepsilon).$$

PROOF. First, the limits

$$\lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log s_n(f,\varepsilon) \quad \text{and} \quad \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log r_n(f,\varepsilon)$$

exist by Lemma 4.4(i), and they are equal by Lemma 4.4(ii). Let h denote the value of these limits. We are going to show that $h_{top}(f) = h$.

Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Let E be an (n,ε)-separated set of cardinality $s_n(f,\varepsilon)$. Let $\mathcal{U}$ be a finite open cover such that the diameter of all elements of $\mathcal{U}$ is less than ε (such a cover exists because X is compact). Two distinct points in E are in distinct elements of $\mathcal{U}^n$, so $s_n(f,\varepsilon) \leq N_n(\mathcal{U})$. This implies that

$$\limsup_{n\to+\infty} \frac{1}{n} \log s_n(f,\varepsilon) \leq h_{top}(\mathcal{U}, f),$$

and so $h \leq h_{top}(f)$.

Let $\mathcal{V}$ be a finite open cover and let $\delta > 0$ be a Lebesgue number for $\mathcal{V}$, that is, for all $x \in X$, there exists $V \in \mathcal{V}$ such that $B(x,\delta) \subset V$. Let $\varepsilon \in (0,\delta)$ and let F be an (n,ε)-spanning set of cardinality $r_n(f,\varepsilon)$. For all $y \in F$ and all $k \in [\![0, n-1]\!]$, there exists $V_{y,k} \in \mathcal{V}$ such that $B(f^k(y),\delta) \subset V_{y,k}$. Let $x \in X$. By definition of F, there exists $y \in F$ such that $x \in B_n(y,\varepsilon)$, hence $f^k(x) \in \overline{B}(f^k(y),\varepsilon) \subset B(f^k(y),\delta)$ for all $k \in [\![0, n-1]\!]$. Thus

$$x \in \bigvee_{k=0}^{n-1} f^{-k}(V_{y,k}).$$

This implies that $\mathcal{V}' := \{\bigvee_{k=0}^{n-1} f^{-k}(V_{y,k}) \mid y \in F\}$ is a subcover of $\mathcal{V}^n$, and so $N_n(\mathcal{V}) \leq N(\mathcal{V}') \leq \#F = r_n(f,\varepsilon)$. This implies that

$$h_{top}(\mathcal{V}, f) \leq \limsup_{n\to+\infty} \frac{1}{n} \log r_n(f,\varepsilon),$$

and so $h_{top}(f) \leq h$. Finally, we get $h_{top}(f) = h$. $\square$

4.2. Entropy and horseshoes

4.2.1. Horseshoes imply positive entropy. Recall that $(J_1, \dots, J_p)$ is a p-horseshoe for the interval map f if $J_1, \dots, J_p$ are non degenerate closed intervals with pairwise disjoint interiors such that $J_1 \cup \dots \cup J_p \subset f(J_i)$ for all $i \in [\![1, p]\!]$.

The next proposition appears under this form belatedly in the literature (e.g., [**41**, Proposition VIII.8]). However it basically follows from the computations of Adler and McAndrew in [**2**].

Proposition 4.6. *Let $f\colon I \to I$ be an interval map. If f has a p-horseshoe, then $h_{top}(f) \geq \log p$.*

Proof. We first suppose that f has a strict p-horseshoe, say $(J_1, \dots, J_p)$. There exist disjoint open sets $U_1, \dots, U_p$ in I such that $J_i \subset U_i$ for all $i \in [\![1, p]\!]$. Let $U_{p+1} := I \setminus \bigcup_{i=1}^{p} J_i$. Then $\mathcal{U} := (U_1, \dots, U_p, U_{p+1})$ is an open cover of I and $U_{p+1} \cap J_i = \emptyset$ for all $i \in [\![1, p]\!]$. Let $n \geq 1$. For all n-tuples $(i_0, \dots, i_{n-1}) \in [\![1, p]\!]^n$, we set

$$J_{i_0,\dots,i_{n-1}} := \{x \in I \mid \forall k \in [\![0, n-1]\!],\ f^k(x) \in J_{i_k}\}.$$

Since $(J_1, \dots, J_p)$ is a p-horseshoe, the set $J_{i_0,\dots,i_{n-1}}$ is not empty by Lemma 1.13(i). Moreover, it is contained in a unique element of $\mathcal{U}^n$, namely

$$U_{i_0} \cap f^{-1}(U_{i_1}) \cap \dots \cap f^{-(n-1)}(U_{i_{n-1}}).$$

Thus $N_n(\mathcal{U}, f) \geq p^n$ for all integers $n \geq 1$, so

$$h_{top}(f) \geq h_{top}(\mathcal{U}, f) = \lim_{n \to +\infty} \frac{N_n(\mathcal{U}, f)}{n} \geq \log p.$$

We now turn to the general case, i.e., f has a p-horseshoe. Let $n \geq 1$. According to Lemma 3.30, f^n has a p^n-horseshoe. We number the p^n intervals of this horseshoe from left to right in I and we consider only the intervals whose number is odd. In this way, we obtain a strict $\left\lceil \frac{p^n}{2} \right\rceil$-horseshoe. Then, applying the first part of the proof to f^n, we get

$$h_{top}(f^n) \geq \log\left(\frac{p^n}{2}\right).$$

By Proposition 4.3, we have $h_{top}(f) = \frac{1}{n} h_{top}(f^n)$, and thus $h_{top}(f) \geq \log p - \frac{\log 2}{n}$. Finally, $h_{top}(f) \geq \log p$ by taking the limit when n goes to infinity. □

4.2.2. Misiurewicz's Theorem. Misiurewicz's Theorem states that the existence of horseshoes is necessary to have positive entropy. This theorem was first proved for piecewise monotone maps by Misiurewicz and Szlenk [**131, 132**], then Misiurewicz generalized the result for all continuous interval maps [**125, 127**]. There is no significant difference between the piecewise monotone case and the general case.

Theorem 4.7 (Misiurewicz). *Let $f\colon I \to I$ be an interval map of positive topological entropy. For every $\lambda < h_{top}(f)$ and every N, there exist intervals $J_1, \dots, J_p$ and a positive integer $n \geq N$ such that $(J_1, \dots, J_p)$ is a strict p-horseshoe for f^n and $\frac{\log p}{n} \geq \lambda$.*

We are first going to state three technical lemmas about limits of sequences, then we shall prove Theorem 4.7.

LEMMA 4.8. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of positive numbers. Then*

$$\limsup_{n\to+\infty} \frac{1}{n}\log(a_n+b_n) = \max\left\{\limsup_{n\to+\infty}\frac{1}{n}\log a_n, \limsup_{n\to+\infty}\frac{1}{n}\log b_n\right\}.$$

The same result holds for finitely many sequences of positive numbers:

$$\limsup_{n\to+\infty} \frac{1}{n}\log(a_n^1+\cdots+a_n^k) = \max\left\{\limsup_{n\to+\infty}\frac{1}{n}\log a_n^i \mid i\in[\![1,k]\!]\right\}.$$

PROOF. We show the lemma for two sequences, the general case follows by a straightforward induction. We set

$$L := \max\left\{\limsup_{n\to+\infty}\frac{1}{n}\log a_n,\ \limsup_{n\to+\infty}\frac{1}{n}\log b_n\right\}.$$

Since $a_n+b_n\geq a_n$ and $a_n+b_n\geq b_n$, it is obvious that

$$\limsup_{n\to+\infty}\frac{1}{n}\log(a_n+b_n)\geq L.$$

Conversely, for every $\varepsilon>0$, there exists an integer n_0 such that, for all $n\geq n_0$, $a_n\leq e^{(L+\varepsilon)n}$ and $b_n\leq e^{(L+\varepsilon)n}$. This implies that

$$\forall n\geq n_0,\ \frac{1}{n}\log(a_n+b_n)\leq L+\varepsilon+\frac{\log 2}{n}.$$

To conclude, we first take the limsup when $n\to+\infty$, then we let ε tend to zero. ☐

LEMMA 4.9. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$ be two sequences of real numbers. Then*

$$\limsup_{n\to+\infty}\frac{1}{n}\log\sum_{k=1}^{n}\exp(a_k+b_{n-k})\leq\max\left\{\limsup_{n\to+\infty}\frac{a_n}{n},\ \limsup_{n\to+\infty}\frac{b_n}{n}\right\}.$$

PROOF. We set

$$L := \max\left\{\limsup_{n\to+\infty}\frac{a_n}{n},\ \limsup_{n\to+\infty}\frac{b_n}{n}\right\}.$$

We assume that $L<+\infty$, otherwise there is nothing to prove. For every $\varepsilon>0$, there exists an integer n_0 such that

$$\forall n\geq n_0,\quad \frac{a_n}{n}\leq L+\varepsilon\quad\text{and}\quad\frac{b_n}{n}\leq L+\varepsilon. \tag{4.3}$$

We set $M:=\max\left\{0,\frac{a_n}{n},\frac{b_n}{n}\mid n\in[\![1,n_0-1]\!]\right\}$. Let n,k be two integers such that $n\geq 2n_0$ and $k\in[\![1,n]\!]$. Necessarily, they satisfy either $k\geq n_0$ or $n-k\geq n_0$. We split into three cases.

- If $k\geq n_0$ and $n-k\geq n_0$, then by (4.3):
$$a_k+b_{n-k}\leq k(L+\varepsilon)+(n-k)(L+\varepsilon)=n(L+\varepsilon).$$
- If $k\geq n_0$ and $n-k<n_0$, then by (4.3) and the definition of M:
$$a_k+b_{n-k}\leq k(L+\varepsilon)+(n-k)M\leq n(L+\varepsilon)+n_0M.$$
- If $k<n_0$ and $n-k\geq n_0$, then by (4.3) and the definition of M:
$$a_k+b_{n-k}\leq kM+(n-k)(L+\varepsilon)\leq n_0M+n(L+\varepsilon).$$

In the three cases, we have $a_k + b_{n-k} \leq n(L+\varepsilon) + n_0 M$, and thus

$$\forall n \geq 2n_0,\ \frac{1}{n} \log \sum_{k=1}^{n} \exp(a_k + b_{n-k}) \leq \frac{1}{n} \log n + L + \varepsilon + \frac{n_0 M}{n}.$$

We first take the limsup when $n \to +\infty$; then we let ε tend to zero, and we get

$$\limsup_{n \to +\infty} \frac{1}{n} \log \sum_{k=1}^{n} \exp(a_k + b_{n-k}) \leq L,$$

which proves the lemma. □

LEMMA 4.10. *Let $(a_n)_{n \geq 1}$ be a sequence of real numbers and $\alpha, \beta \in \mathbb{R}$. Suppose that there exists $C > 0$ such that $a_{n+1} \leq a_n + C$ for all $n \geq 1$, and that*

$$0 < \alpha < \beta < \limsup_{n \to +\infty} \frac{a_n}{n}.$$

Then, for all integers N, there exists $n \geq N$ such that $a_n \geq \beta n$ and $a_{n+1} \geq a_n + \alpha$.

PROOF. We suppose that the lemma is false, that is, there exists N such that

$$\forall n \geq N,\ a_n \geq \beta n \Longrightarrow a_{n+1} < a_n + \alpha. \tag{4.4}$$

If $a_n \geq \beta n$ for all $n \geq N$, then $a_{n+N} < a_N + \alpha n$ for all $n \geq 1$ by (4.4), which implies that

$$\limsup_{n \to +\infty} \frac{a_n}{n} \leq \alpha.$$

But this contradicts the assumption on α. Thus there exists an integer $n \geq N$ such that $a_n < \beta n$. We set $N_0 := \min\{n \geq N \mid a_n < \beta n\}$. Suppose that $a_p < \beta p$ for some integer $p \geq N$ and that r is a positive integer such that $a_n \geq \beta n$ for all $n \in [\![p+1, p+r]\!]$. We are going to show that r is bounded by a constant independent from p. By (4.4), we have $a_{p+r} \leq a_{p+1} + \alpha(r-1)$. Since $a_{p+1} \leq a_p + C$ and $a_p < \beta p$, we have

$$\beta(p+r) \leq a_{p+r} \leq \beta p + C + \alpha(r-1). \tag{4.5}$$

This implies that $\beta r \leq C + \alpha(r-1)$, and thus $r \leq M$ if we set

$$M := \frac{C-\alpha}{\beta-\alpha}.$$

Notice that $M > 0$ because $a_n \leq a_1 + (n-1)C$ for all $n \geq 1$, and hence $\limsup_{n \to +\infty} \frac{a_n}{n} \leq C$, which implies that $C - \alpha > 0$; moreover, $\beta - \alpha > 0$ by assumption. Let n be an integer greater than $N_0 + M$. We consider two cases.

• If $a_n \geq \beta n$, what precedes implies that there exists an integer $p \in [n-M, n)$ such that $a_p < \beta p$ and $a_i \geq \beta i$ for all $i \in [\![p+1, n]\!]$, Then, by (4.5), we have

$$a_n \leq \beta p + C + \alpha(M-1) \leq \beta n + C + \alpha M.$$

• If $a_n < \beta n$, the inequality $a_n \leq \beta n + C + \alpha M$ trivially holds.

Since the inequality $a_n \leq \beta n + C + \alpha M$ holds in the two cases, we have

$$\limsup_{n \to +\infty} \frac{a_n}{n} \leq \beta.$$

But this contradicts the assumption on β. We conclude that the lemma is true. □

Proof of Theorem 4.7. First we are going to prove the theorem under the extra assumptions that $h_{top}(f) > \log 3$ and $\log 3 < \lambda < h_{top}(f)$. We choose λ' such that $\lambda < \lambda' < h_{top}(f)$. According to the definition of topological entropy, there exists a finite open cover $\mathcal{U}$ such that $h_{top}(\mathcal{U}, f) > \lambda$'. We choose a partition $\mathcal{P}$ consisting of finitely many disjoint non degenerate intervals such that $\mathcal{P}$ is finer than $\mathcal{U}$. Then $h_{top}(\mathcal{P}, f) \geq h_{top}(\mathcal{U}, f)$ by Lemma 4.2, so $h_{top}(\mathcal{P}, f) > \lambda'$. We have $N(\mathcal{P}^n) = \#(\mathcal{P}^n)$ because $\mathcal{P}^n$ is a partition, and thus

$$h_{top}(\mathcal{P}, f) = \lim_{n\to+\infty} \frac{1}{n} \log \# \left(\mathcal{P}^n\right).$$

If $\mathcal{Q}$ is a family of subsets of I, we define, for all $n \geq 1$ and all $A \in \mathcal{Q}$,

$$\mathcal{Q}^n := \left\{(A_0, \ldots, A_{n-1}) \mid \forall i \in [\![0, n-1]\!], A_i \in \mathcal{Q} \text{ and } \bigcap_{i=0}^{n-1} f^{-i}(A_i) \neq \emptyset\right\}$$
$$\text{and} \quad \mathcal{Q}^n|A := \{(A_0, \ldots, A_{n-1}) \in \mathcal{Q}^n \mid A_0 = A\}.$$

We have $\#(\mathcal{P}^n) = \sum_{A\in\mathcal{P}} \#(\mathcal{P}^n|A)$. Thus, by Lemma 4.8, there exists $A \in \mathcal{P}$ such that

$$h_{top}(\mathcal{P}, f) = \limsup_{n\to+\infty} \frac{1}{n} \log \# \left(\mathcal{P}^n|A\right). \tag{4.6}$$

Let $\mathcal{F}$ be the family of $A \in \mathcal{P}$ satisfying (4.6). We claim that:

$$\forall A \in \mathcal{F},\ h_{top}(\mathcal{P}, f) = \limsup_{n\to+\infty} \frac{1}{n} \log \# \left(\mathcal{F}^n|A\right). \tag{4.7}$$

Proof of (4.7). The inequality $\geq$ is straightforward. We are going to prove the reverse inequality. We fix $A \in \mathcal{F}$. Let $(A_0, \ldots, A_{n-1}) \in \mathcal{P}^n|A$ and let k be the greatest integer in $[\![1, n]\!]$ such that $A_i \in \mathcal{F}$ for all $i \in [\![0, k-1]\!]$. Then $(A_0, \ldots, A_{k-1}) \in \mathcal{F}^k|A$ and, if $k < n$, $(A_k, \ldots, A_{n-1}) \in \mathcal{P}^{n-k}|B$ for some $B \in \mathcal{P}\backslash\mathcal{F}$. Thus

$$\#(\mathcal{P}^n|A) \leq \sum_{k=1}^{n-1} \left(\#(\mathcal{F}^k|A) \sum_{B\in\mathcal{P}\backslash\mathcal{F}} \#(\mathcal{P}^{n-k}|B)\right) + \#(\mathcal{F}^n|A). \tag{4.8}$$

We set $b_0 := 0$ and

$$\forall n \geq 1,\ a_n := \log \#(\mathcal{F}^n|A) \text{ and } b_n := \log \sum_{B\in\mathcal{P}\backslash\mathcal{F}} \#(\mathcal{P}^n|B).$$

Then (4.8) can be rewritten as

$$\#(\mathcal{P}^n|A) \leq \sum_{k=1}^{n} \exp(a_k + b_{n-k}).$$

Inserting this inequality in (4.6), we get

$$h_{top}(\mathcal{P}, f) \leq \limsup_{n\to+\infty} \frac{1}{n} \log \left(\sum_{k=1}^{n} \exp(a_k + b_{n-k})\right).$$

Thus, by Lemma 4.9,

$$h_{top}(\mathcal{P}, f) \leq \max\left\{\limsup_{n\to+\infty} \frac{a_n}{n}, \limsup_{n\to+\infty} \frac{b_n}{n}\right\}. \tag{4.9}$$

According to the definition of $\mathcal{F}$, we have

$$\forall B \in \mathcal{P} \setminus \mathcal{F}, \quad \limsup_{n\to+\infty} \frac{1}{n} \log \#(\mathcal{P}^n|B) < h_{top}(\mathcal{P}, f),$$

and thus $\limsup_{n\to+\infty} \frac{b_n}{n} < h_{top}(\mathcal{P}, f)$ by Lemma 4.8. Finally, in view of (4.9), we have $h_{top}(\mathcal{P}, f) \leq \limsup_{n\to+\infty} \frac{a_n}{n}$. This concludes the proof of (4.7). □

Let $A_0, \ldots, A_{n-1} \in \mathcal{F}$. We set $A'_0 := A_0$ and $A'_i := A_i \cap f(A'_{i-1})$ for all $i \in [\![1, n-1]\!]$. We claim that:

$$\begin{aligned} A'_{n-1} &= f^{n-1}\left(\{x_0 \mid \forall i \in [\![0, n-1]\!], f^i(x_0) \in A_i\}\right) \\ &= f^{n-1}\left(\bigcap_{i=0}^{n-1} f^{-i}(A_i)\right). \end{aligned} \tag{4.10}$$

Indeed,

$$\begin{aligned} & x_{n-1} \in A'_{n-1} \\ &\quad \Leftrightarrow \quad \exists x_{n-2} \in A'_{n-2},\ f(x_{n-2}) = x_{n-1} \in A_{n-1}, \\ &\quad \Leftrightarrow \quad \exists x_{n-3} \in A'_{n-3},\ f(x_{n-3}) = x_{n-2} \in A_{n-2} \text{ and } f^2(x_{n-3}) = f(x_{n-2}) = x_{n-1}, \\ &\quad\ \vdots \\ &\quad \Leftrightarrow \quad \exists x_0 \in A'_0,\ f(x_0) = x_1 \in A_1, f^2(x_0) = x_2 \in A_2, \ldots f^{n-1}(x_0) = x_{n-1} \in A_{n-1}. \end{aligned}$$

Therefore, (4.10) holds, which implies, according to the definition of $\mathcal{F}^n$:

$$(A_0, \ldots, A_{n-1}) \in \mathcal{F}^n \Longleftrightarrow A'_{n-1} \neq \emptyset. \tag{4.11}$$

If $A'_{n-1} \neq \emptyset$, then A'_i is nonempty and $A'_i \subset f(A'_{i-1})$ for all $i \in [\![i, n-1]\!]$. Thus, by Lemma 1.13, for every $(A_0, \ldots, A_{n-1}) \in \mathcal{F}^n$, there exists a nonempty interval $J_{A_0 \ldots A_{n-1}}$ such that

$$f^{n-1}(J_{A_0 \ldots A_{n-1}}) = A'_{n-1} \quad \text{and} \tag{4.12}$$

$$\forall i \in [\![0, n-1]\!],\ f^i(J_{A_0 \ldots A_{n-1}}) \subset A'_i \subset A_i. \tag{4.13}$$

Moreover, $(J_{A_0 \ldots A_{n-1}})_{(A_0, \ldots, A_{n-1}) \in \mathcal{F}^n}$ is a family of pairwise disjoint intervals. If $(A_0, \ldots, A_{n-1}) \in \mathcal{F}^n$ and $A_n \in \mathcal{F}$, then $f^n(J_{A_0 \ldots A_{n-1}}) \cap A_n = f\left(A'_{n-1}\right) \cap A_n = A'_n$, and thus, according to (4.11),

$$f^n(J_{A_0 \ldots A_{n-1}}) \cap A_n \neq \emptyset \Longleftrightarrow (A_0, \ldots, A_n) \in \mathcal{F}^{n+1}.$$

Therefore

$$\#(\mathcal{F}^{n+1}|A) = \sum_{\substack{(A_0, \ldots, A_{n-1}) \in \mathcal{F}^n \\ A_0 = A}} \#\{B \in \mathcal{F} \mid f^n(J_{A_0 \ldots A_{n-1}}) \cap B \neq \emptyset\}. \tag{4.14}$$

For all $A, B \in \mathcal{F}$, we set

$$c(A, B, n) := \#\{(A_0, \ldots, A_{n-1}) \in \mathcal{F}^n \mid A_0 = A,\ f^n(J_{A_0 \ldots A_{n-1}}) \supset B\}.$$

We shall need the following result:

$$\forall A, B, C \in \mathcal{F},\ \forall n, m \geq 1,\ c(A, B, n)c(B, C, m) \leq c(A, C, m+n). \tag{4.15}$$

Proof of (4.15). For all $A, B \in \mathcal{F}$ and all $n \geq 1$, we set

$$\mathcal{C}(A,B,n) := \{(A_0,\ldots,A_{n-1}) \in \mathcal{F}^n \mid A_0 = A,\ f^n(J_{A_0\ldots A_{n-1}}) \supset B\}.$$

Let $(A_0,\ldots,A_{n-1}) \in \mathcal{C}(A,B,n)$ and $(B_0,\ldots,B_{m-1}) \in \mathcal{C}(B,C,m)$. We are going to show that $(A_0,\ldots,A_{n-1},B_0,\ldots,B_{m-1}) \in \mathcal{C}(A,C,n+m)$. The set $f^n(J_{A_0\ldots A_{n-1}})$ contains B by definition, and $J_{B_0\ldots B_{m-1}} \subset B_0 = B$ by (4.13). This implies (by Lemma 1.13(i)) that there exists a nonempty interval $K \subset J_{A_0\ldots A_{n-1}}$ such that $f^n(K) = J_{B_0\ldots B_{m-1}}$. Moreover, by (4.13), this interval satisfies:

$$\forall i \in [\![0,n-1]\!],\ f^i(K) \subset A_i \text{ and } \forall j \in [\![0,m-1]\!],\ f^{n-1+j}(K) = f^j(J_{B_0\ldots B_{m-1}}) \subset B_j.$$

Consequently,

$$K \subset \bigcap_{i=0}^{n-1} f^{-i}(A_i) \cap \bigcap_{i=n}^{n+m-1} f^{-i}(B_{i-n}).$$

This implies the following facts. First, $(A_0,\ldots,A_{n-1},B_0,\ldots,B_{m-1}) \in \mathcal{F}^{n+m}$ by (4.11)+(4.10), using the fact that $K \neq \emptyset$. Second, the set $f^{n+m-1}(K)$ is included in $f^{m+n-1}(J_{A_0\ldots A_{n-1}B_0\ldots B_{m-1}})$ by combining (4.12) and (4.10). Then, since $f^{n+m}(K) = f^m(J_{B_0\ldots B_{m-1}})$, we have $f^{n+m}(K) \supset C$ by definition of $\mathcal{C}(B,C,m)$, so $f^{n+m}(J_{A_0\ldots A_{n-1}B_0\ldots B_{m-1}}) \supset C$. We conclude that $(A_0,\ldots,A_{n-1},B_0,\ldots,B_{m-1}) \in \mathcal{C}(A,C,n+m)$. This clearly implies (4.15). □

We fix $A \in \mathcal{F}$. We have

$$\begin{aligned}
&\sum_{B\in\mathcal{F}} c(A,B,n)\\
&\quad = \#\left\{((A_0,\ldots,A_{n-1}),B) \in \mathcal{F}^n \times \mathcal{F} \mid A_0 = A, f^n(J_{A_0\ldots A_{n-1}}) \supset B\right\}\\
&\quad = \sum_{\substack{(A_0,\ldots,A_{n-1}) \in \mathcal{F}^n\\ A_0 = A}} \#\{B \in \mathcal{F} \mid f^n(J_{A_0\ldots A_{n-1}}) \supset B\}.
\end{aligned}$$

Consider $(A_0,\ldots,A_{n-1}) \in \mathcal{F}^n$. If $f^n(J_{A_0\ldots A_{n-1}})$ meets k intervals of $\mathcal{F}$, then $f^n(J_{A_0\ldots A_{n-1}})$ contains at least $k-2$ of them because $f^n(J_{A_0\ldots A_{n-1}})$ is an interval. Therefore

$$\sum_{B\in\mathcal{F}} c(A,B,n) \geq \sum_{\substack{(A_0,\ldots,A_{n-1}) \in \mathcal{F}^n\\ A_0 = A}} \left(\#\{B \in \mathcal{F} \mid f^n(J_{A_0\ldots A_{n-1}}) \cap B \neq \emptyset\} - 2\right).$$

Combining this inequality with (4.14), we get:

$$\sum_{B\in\mathcal{F}} c(A,B,n) \geq \#(\mathcal{F}^{n+1}|A) - 2\#(\mathcal{F}^n|A). \tag{4.16}$$

We set $a'_n := \log\#(\mathcal{F}^n|A)$ for all $n \geq 1$. According to (4.7), we have

$$h_{top}(\mathcal{P},f) = \limsup_{n\to+\infty} \frac{a'_n}{n}.$$

Moreover, $a'_{n+1} \leq a'_n + \log\#\mathcal{F}$ for all $n \geq 1$. Therefore, we can apply Lemma 4.10 with $\alpha = \log 3$, $\beta = \lambda'$ and $C = \log\#\mathcal{F}$, and we see that, for all integers N,

$$\exists n \geq N,\ \#(\mathcal{F}^n|A) > e^{\lambda' n} \text{ and } \#(\mathcal{F}^{n+1}|A) \geq 3\#(\mathcal{F}^n|A). \tag{4.17}$$

For an integer n satisfying (4.17), we inject these inequalities in (4.16) and we get

$$\sum_{B\in\mathcal{F}} c(A,B,n) \geq 3\#(\mathcal{F}^n|A) - 2\#(\mathcal{F}^n|A) = \#(\mathcal{F}^n|A) \geq e^{\lambda' n}.$$

Therefore

$$\limsup_{n\to+\infty}\frac{1}{n}\log\sum_{B\in\mathcal{F}}c(A,B,n)\geq\lambda'.$$

According to Lemma 4.8, for all $A\in\mathcal{F}$ there exists $B=\varphi(A)\in\mathcal{F}$ such that

$$\limsup_{n\to+\infty}\frac{1}{n}\log c(A,\varphi(A),n)\geq\lambda'>\lambda.$$

Since $\mathcal{F}$ is finite, the map $\varphi\colon\mathcal{F}\to\mathcal{F}$ has a periodic point, that is, there exist $A_0\in\mathcal{F}$ and $p\in\mathbb{N}$ such that $\varphi^p(A_0)=A_0$. Using the preceding inequality with $A=\varphi^i(A_0)$, $i=0,\dots,p-1$, we have

$$\forall i\in[\![0,p-1]\!],\ \forall N_i\geq 1,\ \exists n_i\geq N_i,\ c(\varphi^i(A_0),\varphi^{i+1}(A_0),n_i)\geq e^{n_i\lambda}. \tag{4.18}$$

For every $i\in[\![0,p-1]\!]$, let N_i be a positive integer and let $n_i\geq N_i$ be given by (4.18). We set $n:=\sum_{i=0}^{p-1}n_i$ and $k:=c(A_0,A_0,n)$. We have

$$\begin{aligned}k=c(A_0,A_0,n)\ &\geq\ \prod_{i=0}^{p-1}c(\varphi^i(A_0),\varphi^{i+1}(A_0),n_i)\quad\text{by (4.15)}\\ &\geq\ \prod_{i=0}^{p-1}e^{n_i\lambda}=e^{n\lambda}\quad\text{by (4.18).}\end{aligned}$$

According to the definition of $c(A_0,A_0,n)$, this means that there exist k disjoint intervals $J_1,\dots,J_k\subset A_0$ such that $f^n(J_i)\supset A_0$ for all $i\in[\![1,k]\!]$. Thus $(\overline{J_1},\dots,\overline{J_k})$ is a k-horseshoe for f^n, and $\frac{1}{n}\log k\geq\lambda$.

This result implies the theorem in the general case in the following way. Suppose that $h_{top}(f)>0$ and $0<\lambda<h_{top}(f)$. We choose λ'' such that $\lambda<\lambda''<h_{top}(f)$; then we choose an integer q such that $q\lambda''>\log 3$ and $q(\lambda''-\lambda)\geq\log 2$. By Proposition 4.3, $h_{top}(f^q)=qh_{top}(f)>q\lambda''$. Therefore, applying what precedes to f^q, we obtain that, for every integer N, there exist positive integers n,k with $n\geq N$ and a k-horseshoe $(J_1,\dots,J_k)$ for f^{nq} such that $\frac{1}{n}\log k\geq q\lambda''$. We order the intervals of this horseshoe such that $J_1\leq J_2\cdots\leq J_k$ and we set $k':=\lceil\frac{k}{2}\rceil$. Then the intervals J_i with odd indices are pairwise disjoint and form a k'-horseshoe for f^{nq}, and we have

$$\frac{\log k'}{qn}\geq\frac{\log k-\log 2}{qn}\geq\lambda''-\frac{\log 2}{qn}\geq\lambda.$$

This ends the proof of the theorem. $\square$

Remarks on graph maps. The notion of a horseshoe can be extended to graph maps in the following way.

Definition 4.11. Let $f\colon G\to G$ be a graph map. Let I be a closed interval of G containing no branching point except maybe its endpoints, and let $J_1,\dots,J_n$ be non degenerate closed subintervals of I with pairwise disjoint interiors such that $f(J_i)=I$ for all $i\in[\![1,n]\!]$. Then $(J_1,\dots,J_n)$ is called an *n-horseshoe* for f. If in addition the intervals are disjoint, $(J_1,\dots,J_n)$ is called a *strict* n-horseshoe.

Llibre and Misiurewicz proved that, with this definition, Proposition 4.6 and Theorem 4.7 remain valid for graph maps [**114**].

THEOREM 4.12. *Let f be a graph map. If f has an n-horseshoe, then $h_{top}(f) \geq \log n$. Conversely, if $h_{top}(f) > 0$, then for all $\lambda < h_{top}(f)$ and all $N \geq 1$, there exist integers $n \geq N$, $p \geq 1$ and a strict p-horseshoe for f^n such that $\frac{\log p}{n} \geq \lambda$.*

4.3. Homoclinic points

The notion of a homoclinic point for a diffeomorphism of a smooth manifold was introduced by Poincaré; this is a point belonging to both the stable and the unstable manifolds of a hyperbolic point (see, e.g., [**161**]). In [**33**], Block defined unstable manifolds and homoclinic points for an interval map.

DEFINITION 4.13. Let f be an interval map. Let z be a periodic point and let p denote its period. Let $\mathcal{V}(z)$ denote the family of neighborhoods of z. The *unstable manifold* of z is the set

$$W^u(z, f^p) := \bigcap_{V \in \mathcal{V}(z)} \bigcup_{n \geq 1} f^{np}(V).$$

Equivalently, $x \in W^u(z, f^p)$ if and only if there exist sequences of points $(x_k)_{k \geq 0}$ and of positive integers $(n_k)_{k \geq 0}$ such that $\lim_{k \to +\infty} x_k = z$ and $\forall k \geq 0$, $f^{pn_k}(x_k) = x$.

DEFINITION 4.14. Let f be an interval map. A point x is a *homoclinic point* if there exists a periodic point z such that:

i) $x \neq z$,
ii) $x \in W^u(z, f^p)$, where p is the period of z,
iii) $z \in \omega(x, f^p)$.

The point x is an *eventually periodic homoclinic point* if it satisfies the above conditions (i) and (ii) and if there exists a positive integer k such that $f^{kp}(x) = z$ (this condition is trivially stronger than (iii)).

REMARK 4.15. The notion of a homoclinic point introduced by Block in [**33**] corresponds to what is called an eventually periodic homoclinic point in the above definition. In [**41**], homoclinic points are called *homoclinic in the sense of Poincaré* to make a distinction with the previous kind of homoclinic point, which is more restrictive. We rather follow the terminology of [**103, 120**].

4.3.1. Preliminary results about unstable manifolds.

We shall need a few results about unstable manifolds. We state them for fixed points. In view of the definition, there is no loss of generality since a periodic point of period p for f is a fixed point for f^p.

The next easy lemma states that an unstable manifold is connected and invariant. The other three lemmas of the section are more technical.

LEMMA 4.16. *Let $f: I \to I$ be an interval map and let z be a fixed point. Then*

i) *$W^u(z, f)$ is an interval containing z (it may be reduced to $\{z\}$),*
ii) *$f(W^u(z, f)) \subset W^u(z, f)$.*

PROOF. For every $\varepsilon > 0$, the set $\bigcup_{n \geq 1} f^n((z - \varepsilon, z + \varepsilon) \cap I)$ is an interval containing z because it is a union of intervals containing the fixed point z. It follows straightforwardly from the definition that

$$W^u(z, f) = \bigcap_{\varepsilon > 0} \bigcup_{n \geq 1} f^n((z - \varepsilon, z + \varepsilon) \cap I),$$

which is an intersection of intervals containing z. Thus $W^u(z,f)$ is also an interval containing z, which gives (i).

Let $x \in W^u(z,f)$. This means that for all $V \in \mathcal{V}(z)$, there exists $n \geq 1$ such that $x \in f^n(V)$. Thus $f(x) \in f^{n+1}(V)$. This implies that $f(x) \in W^u(z,f)$, which proves (ii). □

LEMMA 4.17. *Let f be an interval map. Let z_1, z_2 be two fixed points with $z_1 < z_2$ and such that there is no fixed point in (z_1, z_2). Then there exists $i \in \{1,2\}$ such that $(z_1, z_2) \subset W^u(z_i, f)$.*

PROOF. The assumption that (z_1, z_2) contains no fixed point implies that

$$\text{either} \quad \forall x \in (z_1, z_2),\ f(x) > x, \tag{4.19}$$

$$\text{or} \quad \forall x \in (z_1, z_2),\ f(x) < x. \tag{4.20}$$

We assume that (4.19) holds and we are going to show that $(z_1, z_2) \subset W^u(z_1, f)$. If (4.20) holds, then by symmetry we have $(z_1, z_2) \subset W^u(z_2, f)$.

Let $y \in (z_1, z_2)$. Let V be a neighborhood of z_1 and $x \in (z_1, y) \cap V$. We set $\delta := \min\{f(t) - t \mid t \in [x,y]\}$. By compactness, (4.19) implies $\delta > 0$. If $t \in [x,y]$, then

$$f([z_1, t]) \supset [z_1, t+\delta]. \tag{4.21}$$

We define a sequence $(b_n)_{n\geq 0}$ by

- $b_0 := x$,
- if $b_n \leq z_2$, $b_{n+1} := \max f([z_1, b_n])$; if $b_n > z_2$, the sequence is not defined for greater indices.

By (4.19), the sequence $(b_n)_{n\geq 0}$ is increasing and $b_n = \max f^n([z_1, x])$. According to (4.21), if $b_n \in [x,y]$, then $b_{n+1} \geq b_n + \delta$, so $b_{n+1} \geq b_0 + (n+1)\delta$ by induction. Since $[x,y]$ is bounded, this implies that there exists n_0 such that $b_{n_0} \geq y$. Thus $y \in f^{n_0}([z_1, x]) \subset f^{n_0}(V)$. This implies that $y \in W^u(z_1, f)$, and we conclude that $(z_1, z_2) \subset W^u(z_1, f)$. □

LEMMA 4.18. *Let $f\colon I \to I$ be an interval map. Let z be a fixed point and let y be a point such that $y \neq z$ and $y \in W^u(z,f)$. Then for every neighborhood V of z, there exist $y' \in V \cap W^u(z,f)$ and an integer $n \geq 1$ such that $f^n(y') = y$.*

PROOF. Suppose that the result is false, that is, there exists a neighborhood V of z such that

$$\forall n \geq 1,\ y \notin f^n(V \cap W^u(z,f)). \tag{4.22}$$

We can assume that V is an interval. We also assume that $y > z$, the case $y < z$ being symmetric.

Since $y \in W^u(z,f)$ and according to the definition of an unstable manifold, (4.22) implies:

$$V \cap W^u(z,f) \text{ is not a neighborhood of } z. \tag{4.23}$$

By Lemma 4.16(i), $W^u(f,z)$ is an interval containing $[z,y]$. Thus $V \cap W^u(z,f)$ is also an interval containing z, and (4.23) implies:

$$z = \min\left(V \cap W^u(z,f)\right) = \min W^u(z,f) \quad \text{and} \quad z \neq \min I. \tag{4.24}$$

Let $b \in V \cap (z, y)$. Since $f(z) = z$ and $z \neq \min I$, there exists a point c such that

$$c \in V,\ c < z \quad \text{and} \quad \forall x \in [c, z],\ f(x) < b. \tag{4.25}$$

By (4.24), $c \notin W^u(z, f)$. Thus, by definition of $W^u(z, f)$, there exists $d \in (c, z)$ such that

$$\forall n \geq 1,\ c \notin f^n([d, z]). \tag{4.26}$$

We set $d_n := \min f^n([d, z])$ for all $n \geq 0$. Then $d_n \leq z$ and (4.26) implies that $d_n > c$ for all $n \geq 0$. We show by induction on n that

$$\forall n \geq 0,\ f^n([d, z]) \subset [d_n, z] \cup \bigcup_{i=0}^{n-1} f^i([z, b]). \tag{4.27}$$

- (4.27) is satisfied for $n = 0$.
- Suppose that (4.27) holds for n. Then $f([d_n, z]) \subset [d_{n+1}, b) = [d_{n+1}, z] \cup [z, b)$ using (4.25) and the fact that $d_n \in (c, z]$. Then

$$\begin{aligned} f^{n+1}([d, z]) &\subset f([d_n, z]) \cup \bigcup_{i=1}^{n} f^i([z, b]) \quad \text{by the induction hypothesis,} \\ &\subset [d_{n+1}, z] \cup \bigcup_{i=0}^{n} f^i([z, b]) \quad \text{by what precedes.} \end{aligned}$$

This is (4.27) for $n + 1$. This proves that (4.27) holds for all $n \geq 0$. Moreover, according to (4.22), $y \notin f^n([z, b])$ for any $n \geq 1$ because $[z, b] \subset V \cap W^u(z, f)$. By (4.27), $y \notin f^n([d, z])$ for all $n \geq 1$ (recall that $b < y$). Thus $y \notin f^n((d, b))$ for any $n \geq 1$. But this contradicts the fact that $y \in W^u(z, f)$ because (d, b) is a neighborhood of z. This concludes the proof. □

LEMMA 4.19. *Let f be an interval map. Let z be a fixed point and let y be a point such that $y \in W^u(z, f)$ and $y > z$. Then there exists $x \in W^u(z, f)$ such that $f(x) = y$ and $x < y$.*

PROOF. We prove the lemma by refutation. Suppose that

$$\forall x \in W^u(z, f),\ x < y \Rightarrow f(x) < y. \tag{4.28}$$

Then a straightforward induction, using the fact that $f(W^u(z, f)) \subset W^u(z, f)$ (by Lemma 4.16), gives:

$$\forall x \in W^u(z, f),\ x < y \Rightarrow \forall n \geq 0, f^n(x) \in W^u(z, f) \text{ and } f^n(x) < y. \tag{4.29}$$

Let V be a neighborhood of z such that $\sup V < y$. According to Lemma 4.18, there exist $x \in W^u(z, f) \cap V$ and $n \geq 1$ such that $f^n(x) = y$. The fact that $x \in V$ implies $x < y$. But this contradicts (4.29). We deduce that (4.28) does not hold, that is, there exists $x_0 \in W^u(z, f)$ such that $x_0 < y$ and $f(x_0) \geq y$. Since $f(z) = z$, the continuity of f implies that there exists $x \in \langle z, x_0 \rangle$ such that $f(x) = y$. Then $x < y$. Moreover, $W^u(z, f)$ is an interval (by Lemma 4.16) and it contains z and x_0, and so $W^u(z, f)$ contains x too. □

4.3.2. Homoclinic points and horseshoes. In [**33**], Block showed that an interval map f has an eventually periodic homoclinic point if and only if f has a periodic point whose period is not a power of 2. As we shall show in Theorem 4.58, f has a periodic point whose period is not a power of 2 if and only if f has positive entropy, which is also equivalent to the fact that f^n has a horseshoe for some n (note that this theorem is posterior to [**33**]). We are going to show a result very close to Block's: f has an eventually periodic homoclinic point if and only if some iterate of f has a horseshoe. Moreover, the integer n such that f^n has a horseshoe and the period of the eventually periodic homoclinic point are related.

The next result is a variant of [**33**, Theorem 5].

PROPOSITION 4.20. *Let f be an interval map having a horseshoe. Then there exist points x, z such that $x \neq z$, $f(z) = z$, $f(x) = z$ and $x \in W^u(z, f)$. In particular, x is an eventually periodic homoclinic point.*

PROOF. According to Lemma 3.29, there exist points a, b, c such that $f(a) = f(c) = a$, $f(b) = c$ and, either $a < b < c$, or $a > b > c$. We assume that $a < b < c$, the other case being symmetric. Then $([a, b], [b, c])$ is a horseshoe; in particular there exist fixed points in $[a, b]$ and in $[b, c]$. We set

$$z_1 := \max\{x \in [a, b] \mid f(x) = x\} \quad \text{and} \quad z_2 := \min\{x \in [b, c] \mid f(x) = x\}.$$

There exist $x_1 \in [b, c]$ and $x_2 \in [a, b]$ such that $f(x_i) = z_i$ for $i \in \{1, 2\}$. Since b is not a fixed point, we have $z_1 < b < z_2$ and $x_i \neq z_i$ for $i \in \{1, 2\}$. Moreover, there is no fixed point in (z_1, z_2). Therefore, according to Lemma 4.17, there exists $i \in \{1, 2\}$ such that $(z_1, z_2) \subset W^u(z_i, f)$, and hence $b \in W^u(z_i, f)$. By Lemma 4.16, the points $c = f(b)$ and $a = f(c)$ belong to $W^u(z_i, f)$ too, and thus $[a, c] \subset W^u(z_i, f)$ because $W^u(z_i, f)$ is an interval. Since $x_1, x_2 \in [a, c] \subset W^u(z_i, f)$, we conclude that the proposition holds for $z := z_i$ and $x := x_i$. □

The next proposition is [**41**, Theorem III.16], which is more precise than the original result of Block [**33**, Theorem A2].

PROPOSITION 4.21. *Let f be an interval map. Let y be an eventually periodic homoclinic point with respect to a fixed point (that is, there exist a point z and a positive integer k such that $y \neq z$, $f(z) = z$, $y \in W^u(z, f)$ and $f^k(y) = z$). Then f^2 has a horseshoe.*

PROOF. Let $k \geq 1$ be the minimal integer such that $f^k(y) = z$. We set $y' := f^{k-1}(y)$. Then $y' \in W^u(z, f)$ by Lemma 4.16, $y' \neq z$ because of the choice of k, and $f(y') = z$. We assume $y' > z$, the case $y' < z$ being symmetric.

If there exists $x \in (z, y')$ such that $f(x) = y'$, then $([z, x], [x, y'])$ is a horseshoe for f, and for f^2 too. From now on, we assume that:

$$\forall x \in (z, y'),\ f(x) \neq y'. \tag{4.30}$$

According to Lemma 4.19, there exists $x \in W^u(z, f)$ such that $x < y'$ and $f(x) = y'$. We set $w := \max\{x \leq y' \mid f(x) = y'\}$. Then $w \in W^u(z, f)$ because $W^u(z, f)$ is an interval by Lemma 4.16. Moreover, $w < z$ by (4.30) (notice that $w \notin \{y', z\}$ because $f(z) = f(y') = z \neq y'$). Since $f(y') = z < y'$, the definition of w and the continuity of f imply that

$$\forall x \in (w, y'),\ f(x) < y'. \tag{4.31}$$

Suppose that $f(x) > w$ for all $x \in (w, y')$. Combined with (4.31), this implies $f((w, y')) \subset (w, y')$. Thus

$$\forall x \in (w, y'), \forall n \geq 0,\ f^n(x) \neq w.$$

But this contradicts Lemma 4.18 because (w, y') is a neighborhood of z and w is in $W^u(z, f)$. We deduce that there exists $x \in (w, y')$ such that $f(x) \leq w$. Since $f(z) = z > w$, the continuity of f implies that there exists $v \in (w, y')$ such that $f(v) = w$ and $v \neq z$. If $v \in (w, z)$, then $([w, v], [v, z])$ is a horseshoe for f. If $v \in (z, y')$, it is easy to check that $[z, v], [v, y']$ form a horseshoe for f^2. This concludes the proof. □

REMARK 4.22. Propositions 4.20 and 4.21 can be restated for some iterate of f:

- If f^n has a horseshoe, then there exists an eventually periodic homoclinic point with respect to a periodic point whose period divides n.
- If f has an eventually periodic homoclinic point with respect to a periodic point of period p, then f^{2p} has a horseshoe.

It seems that the next result was first stated by Block and Coppel [**41**, Proposition VI.35]. We give a different proof.

PROPOSITION 4.23. *Let f be an interval map having a homoclinic point. Then there exists a positive integer n such that f^n has a horseshoe.*

PROOF. Let y be a homoclinic point with respect to the periodic point z and let p be the period of z. Then $y \in W^u(z, f^p)$ and $z \in \omega(y, f^p)$. An induction using Lemma 4.18 shows that there exist a sequence of points $(y_n)_{n\geq 0}$ and a sequence of positive integers $(k_n)_{n\geq 1}$ such that

- $y_0 := y$,
- $\forall n \geq 1$, $y_n \in W^u(z, f^p)$ and $f^{pk_n}(y_n) = y_{n-1}$,
- $\lim\limits_{n\to+\infty} y_n = z$.

This implies that $y_n \neq z$ for all $n \geq 0$ (because $y \neq z$). We assume that there are infinitely many integers n such that $y_n > z$ (otherwise, there are infinitely many integers n such that $y_n < z$ and the arguments are symmetric). Thus there exist $n' > n \geq 0$ such that $z < y_{n'} < y_n$. We set $x_0 := y_n$, $x_1 := y_{n'}$, $m := \sum_{i=n+1}^{n'} k_i$ and $g := f^{mp}$. In this way, $z < x_1 < x_0$ and $g(x_1) = x_0$.

The point z belongs to $\omega(y, f^p) = \omega(x_1, f^p)$. Since z is a fixed point for f^p, Lemma 1.3 implies that z belongs to $\omega(x_1, f^{pn})$ for all $n \geq 1$, in particular $z \in \omega(x_1, g)$. Thus there exists $j \geq 1$ such that $g^j(x_1) < x_1$ (by choosing $g^j(x_1)$ close enough to z). This implies that there exists $k \in [\![2, j]\!]$ such that

$$g^k(x_1) < x_1 \quad \text{and} \quad \forall i \in [\![1, k-1]\!],\ g^i(x_1) \geq x_1$$

(notice that $k = 1$ is not possible because $g(x_1) = x_0$). We deduce that

$$x_1 \in [g^k(x_1), g^{k-1}(x_1)]. \tag{4.32}$$

The interval $g^{k-1}([x_1, x_0])$ contains the points $g^{k-1}(x_0) = g^k(x_1)$ and $g^{k-1}(x_1)$. Thus $g^{k-1}([x_1, x_0])$ also contains x_1 by (4.32). This implies that $g^k([x_1, x_0])$ contains $g(x_1)$ and $g^k(x_1)$ with $g(x_1) = x_0$ and $g^k(x_1) < x_1$, so

$$g^k([x_1, x_0]) \supset [x_1, x_0]. \tag{4.33}$$

On the other hand, $g([z, x_1]) \supset [z, x_0] \supset [z, x_1]$. Thus there exists $x_2 \in (z, x_1)$ such that $g(x_2) = x_1$, and we have

$$g([x_2, x_1]) \supset [x_1, x_0]. \tag{4.34}$$

As above, since $z \in \omega(x_1, g^k)$ and $z < x_2$, there exists $j \geq 1$ such that $g^{kj}(x_1) < x_2$. Then $g^{kj}([x_1, x_0])$ contains $g^{kj}(x_1) < x_2$ and it also contains x_0 by (4.33). Thus

$$g^{kj}([x_1, x_0]) \supset [x_2, x_0] = [x_2, x_1] \cup [x_1, x_0]. \tag{4.35}$$

Let $J := [x_2, x_1]$ and $K := [x_1, x_0]$. The coverings given by (4.34) and (4.35) are represented in Figure 1.

$$\begin{tikzcd}
J \arrow[r, bend left, "g"] & K \arrow[l, bend left, "g^{kj}"] \arrow[loop right, "g^{kj}"]
\end{tikzcd}$$

Figure 1. The coverings between the intervals $J := [x_2, x_1]$ and $K := [x_1, x_0]$.

If we consider the following chains of coverings:

$$J \xrightarrow{g} K \xrightarrow{g^{kj}} K \xrightarrow{g^{kj}} J, \qquad J \xrightarrow{g} K \xrightarrow{g^{kj}} K \xrightarrow{g^{kj}} K,$$
$$K \xrightarrow{g^{kj}} J \xrightarrow{g} K \xrightarrow{g^{kj}} J, \qquad K \xrightarrow{g^{kj}} J \xrightarrow{g} K \xrightarrow{g^{kj}} K,$$

we see that J, K form a horseshoe for $g^{1+2kj} = f^{p(1+2kj)}$. $\square$

According to Propositions 4.20 and 4.23, the existence of a homoclinic point implies that f^n has a horseshoe for some n; and if f^n has a horseshoe, then f has an eventually periodic homoclinic point. This leads to the following theorem.

Theorem 4.24. *Let f be an interval map. The following are equivalent:*

i) $h_{top}(f) > 0$,
ii) *f has an eventually periodic homoclinic point,*
iii) *f has a homoclinic point.*

Proof. The implication (ii)⇒(iii) is trivial. According to Misiurewicz's Theorem 4.7, the topological entropy of f is positive if and only if there exists $n \geq 1$ such that f^n has a horseshoe. Then the implications (i)⇒(ii) and (iii)⇒(i) follow straightforwardly from Propositions 4.20 and 4.23 respectively. $\square$

Remarks on graph maps. The notions of an unstable manifold and a homoclinic point can be extended with no change to graph maps. In view of the definition of horseshoe for graph maps, it is natural to think that Proposition 4.20 can be generalized to graph maps. Indeed, Makhrova proved that a tree map of positive entropy has a homoclinic point [**120**, Corollary 1.2]; and Kočan, Kornecká-Kurková and Málek showed the same result for graph maps [**103**, Theorem 1]. Recall that a graph map f has positive topological entropy if and only if f^n has a horseshoe for some n by Theorem 4.12.

THEOREM 4.25. *Let $f: G \to G$ be a graph map of positive topological entropy. Then f has an eventually periodic homoclinic point.*

The converse of Theorem 4.25 holds for tree maps [**120**, Corollary 1.2] but not for graph maps [**103**, Example 3].

THEOREM 4.26. *Let $f: T \to T$ be a tree map. If f has a homoclinic point, then* $h_{top}(f) > 0$.

PROPOSITION 4.27. *There exists a circle map $f: \mathbb{S} \to \mathbb{S}$ of zero topological entropy having an eventually periodic homoclinic point.*

4.4. Upper bounds for entropy of Lipschitz and piecewise monotone maps

An interval map can have an infinite topological entropy, as illustrated in Example 4.28. However Lipschitz (in particular C^1) interval maps and piecewise monotone interval maps have finite topological entropy.

EXAMPLE 4.28. We choose an increasing sequence $(a_n)_{n\geq 0}$ with $a_0 := 0$ and $\lim_{n\to+\infty} a_n = 1$. Let $a_{-1} := 0$. We set $I_n := [a_{n-1}, a_n]$ for all $n \geq 1$. We consider a continuous map $f: [0,1] \to [0,1]$ which is rather similar to the map of Example 2.36 but with $2n+1$ linear pieces in I_n. This map is represented in Figure 2. More precisely, f is defined by

$$\forall n \geq 0,\ f_n(a_n) := a_n, \quad f(1) := 1$$

$$\forall n \geq 1,\ \forall i \in [\![1, 2n]\!],\ f\left(a_{n-1} + i \cdot \frac{a_n - a_{n-1}}{2n+1}\right) := \begin{cases} a_{n-2} & \text{if } i \text{ odd,} \\ a_{n+1} & \text{if } i \text{ even,} \end{cases}$$

and f is linear between these points.

The map f clearly has a $(2n+1)$-horseshoe in I_n for every $n \geq 1$, Therefore, $h_{top}(f) = +\infty$ by Proposition 4.6. Moreover, the same arguments as in Example 2.36 show that f is topologically mixing.

The next result is a particular case of [**73**, Proposition (14.20)], which states that, if (X, f) is a topological dynamical system with $X \subset \mathbb{R}^d$ and if f is λ-Lipschitz for some $\lambda \geq 1$, then $h_{top}(f) \leq d \log \lambda$.

PROPOSITION 4.29. *Let $f: I \to I$ be an interval map and $\lambda \geq 1$. If f is λ-Lipschitz, then $h_{top}(f) \leq \log \lambda$.*

PROOF. Let $\varepsilon > 0$ and $n \geq 1$. Let $E = \{x_1 < x_2 < \cdots < x_s\}$ be an (n, ε)-separated set of cardinality $s := s_n(f, \varepsilon)$. For every $i \in [\![1, s-1]\!]$, there exists $k \in [\![0, n-1]\!]$ such that $|f^k(x_{i+1}) - f^k(x_i)| > \varepsilon$. Since f is λ-Lipschitz with $\lambda \geq 1$,

$$|f^k(x_{i+1}) - f^k(x_i)| \leq \lambda^k |x_{i+1} - x_i| \leq \lambda^n |x_{i+1} - x_i|.$$

Thus $x_{i+1} - x_i \geq \lambda^{-n}\varepsilon$ and $x_s - x_1 \geq (s-1)\lambda^{-n}\varepsilon$. Since $x_s - x_1 \leq |I|$, this implies that

$$s \leq \frac{|I|}{\varepsilon}\lambda^n + 1.$$

Finally, $h_{top}(f) \leq \log \lambda$ by Bowen's formula (Theorem 4.5). □

In [**132**], Misiurewicz and Szlenk showed that the topological entropy of a piecewise monotone interval map f is equal to the exponential growth rate of the minimal number c_n of monotone subintervals for f^n. Furthermore, $h_{top}(f)$ is less

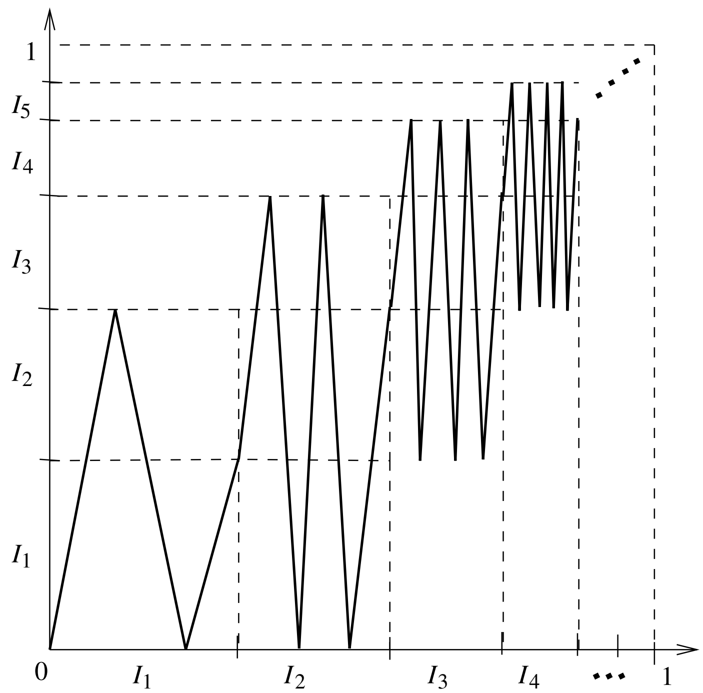


FIGURE 2. This map is topologically mixing and its topological entropy is infinite.

than or equal to $\frac{1}{n}\log c_n$ for all $n \geq 1$, which may be useful to estimate the entropy of a given map since we may not know c_n for all n. We first state two lemmas before proving this result.

DEFINITION 4.30. Let f be a piecewise monotone map. A *monotone cover (resp. partition) for f^n* is a cover (resp. partition) $\mathcal{C}$ such that, for all $C \in \mathcal{C}$, C is an interval and $f^n|_C$ is monotone.

LEMMA 4.31. *Let f be an interval map. If $\mathcal{A}$ and $\mathcal{B}$ are monotone covers for f^n and f^k respectively, then $\mathcal{A} \vee f^{-n}(\mathcal{B})$ is a monotone cover for f^{n+k}. In particular, if $\mathcal{A}$ is a monotone cover for f, then $\mathcal{A}^n$ is a monotone cover for f^n for all $n \geq 1$.*

PROOF. Let $J \in \mathcal{A}$ and $K \in \mathcal{B}$. We set $g := f^n|_J$. Since g is monotone, the set $g^{-1}(K) = J \cap f^{-n}(K)$ is an interval. Moreover, $f^{n+k}|_{J\cap f^{-n}(K)} = f^k|_K \circ g|_{g^{-1}(K)}$ is monotone as a composition of two monotone maps. This implies that $\mathcal{A} \vee f^{-n}(\mathcal{B})$ is a monotone cover for f^{n+k}. The second assertion of the lemma trivially follows from the first one. □

The next result is stated in [**132**, Remark 1].

PROPOSITION 4.32. *Let $f\colon I \to I$ be a piecewise monotone interval map and $\mathcal{A}$ a monotone cover. Then $h_{top}(f) = h_{top}(\mathcal{A}, f)$.*

PROOF. Let $\mathcal{U}$ be an open cover, and let $\mathcal{V}$ be the open cover composed of the connected components of the elements of $\mathcal{U}$. We fix an integer $n \geq 1$ and an element $A \in \mathcal{A}^n$. We set $\mathcal{V}^n \cap A := \{V \cap A \mid V \in \mathcal{V}^n\}$. For all $i \in [\![0, n-1]\!]$, A

is a subinterval of an element of $\mathcal{A}^i$, and thus $f^i|_A$ is monotone because $\mathcal{A}^i$ is a monotone cover for f^i by Lemma 4.31. Thus, for every $U \in \mathcal{V}$, $A \cap f^{-i}(U)$ is an interval (note that this interval may not be open because A is not assumed to be open). This implies that all elements of $\mathcal{V}^n \cap A$ are subintervals of A. Moreover, their endpoints are in the set

$$\bigcup_{i=0}^{n-1} \bigcup_{U \in \mathcal{V}} \partial A \cap f^{-i}(U).$$

Therefore the number of endpoints of the elements of $\mathcal{V}^n \cap A$ is at most $2n\#\mathcal{V}$. Since a nonempty interval is determined by its two endpoints and by its type (open, close, half-open, half-close), we deduce that $\#(\mathcal{V}^n \cap A) \leq 4(2n\#\mathcal{V})^2 = (4n\#\mathcal{V})^2$. Let $\widetilde{\mathcal{V}}_n$ (resp. $\widetilde{\mathcal{A}}_n$) be a subcover of minimal cardinality of $\mathcal{V}^n$ (resp. $\mathcal{A}^n$). We then have

$$\#\widetilde{\mathcal{V}}_n \leq \sum_{A \in \widetilde{\mathcal{A}}_n} \#(\widetilde{\mathcal{V}}_n \cap A) \leq \#(\widetilde{\mathcal{A}}_n)(4n\#\mathcal{V})^2.$$

It follows that

$$\log N_n(\mathcal{V}, f) \leq \log N_n(\mathcal{A}, f) + 2\log(4n\#\mathcal{V}).$$

Dividing by n and taking the limit when n goes to infinity, we get

$$h_{top}(\mathcal{V}, f) \leq h_{top}(\mathcal{A}, f).$$

Moreover, $h_{top}(\mathcal{U}, f) \leq h_{top}(\mathcal{V}, f)$ because $\mathcal{V} \prec \mathcal{U}$. Since what precedes is valid for all open covers $\mathcal{U}$, we deduce that $h_{top}(f) \leq h_{top}(\mathcal{A}, f)$. It remains to show the reverse inequality.

We fix $n \geq 1$. Let $\mathcal{B} := \mathcal{A} \vee f^{-1}\mathcal{A} \vee \cdots \vee f^{-(n-1)}\mathcal{A}$. From now on, we work with the map $g := f^n$ and the iterated covers (like $\mathcal{B}^k$) will be relative to g. By Lemma 4.31, $\mathcal{B}$ is a monotone cover for g. Let $\varepsilon > 0$ be such that

$$\varepsilon < \min\{|B| \mid B \in \mathcal{B}, B \text{ non degenerate}\}.$$

Let $E := \bigcup_{B \in \mathcal{B}} \partial B$ be the set of endpoints of $\mathcal{B}$. We define the open cover

$$\mathcal{U} := \{\mathrm{Int}\,(B) \mid B \in \mathcal{B}\} \cup \{(x-\varepsilon, x+\varepsilon) \cap I \mid x \in E\}.$$

For every $x \in E$, the interval $(x-\varepsilon, x+\varepsilon)$ meets at most three elements of $\mathcal{B}$ (one of them may be reduced to $\{x\}$) because of the choice of ε. Thus, for all $U \in \mathcal{U}$, $\#\{B \in \mathcal{B} \mid U \cap B \neq \emptyset\} \leq 3$. This implies that

$$\forall\, k \geq 1, \forall\, V \in \mathcal{U}^k,\ \#\{B \in \mathcal{B}^k \mid V \cap B \neq \emptyset\} \leq 3^k.$$

Consequently, if $\widetilde{\mathcal{U}}_k$ is a subcover of minimal cardinality of $\mathcal{U}^k$, we have

$$N_k(\mathcal{B}, g) \leq \sum_{V \in \widetilde{\mathcal{U}}_k} \#\{B \in \mathcal{B}^k \mid V \cap B \neq \emptyset\} \leq 3^k \#\widetilde{\mathcal{U}}_k = 3^k N_k(\mathcal{U}, g).$$

Dividing by nk and taking the limit when k goes to infinity, we get

$$\frac{1}{n} h_{top}(\mathcal{B}, g) \leq \frac{1}{n} h_{top}(\mathcal{U}, g) + \frac{\log 3}{n} \leq \frac{1}{n} h_{top}(g) + \frac{\log 3}{n}. \tag{4.36}$$

Since $\frac{1}{n} h_{top}(\mathcal{B}, f^n) = h_{top}(\mathcal{A}, f)$ and $\frac{1}{n} h_{top}(g) = h_{top}(f)$, we deduce from (4.36) that $h_{top}(\mathcal{A}, f) \leq h_{top}(f) + \frac{\log 3}{n}$. Finally, taking the limit when n goes to infinity, we conclude that $h_{top}(\mathcal{A}, f) \leq h_{top}(f)$. $\square$

Proposition 4.33. *Let f be a piecewise monotone interval map and, for all $n \geq 1$, let c_n be the minimal cardinality of a monotone partition for f^n. Then*

$$h_{top}(f) = \lim_{n\to+\infty} \frac{1}{n} \log c_n = \inf_{n\geq 1} \frac{1}{n} \log c_n.$$

Proof. For every $n \geq 1$, let $\mathcal{A}_n$ be a monotone partition for f^n with minimal cardinality, that is, $\#\mathcal{A}_n = c_n$. By Lemma 4.31, $\mathcal{A}_n \vee f^{-n}(\mathcal{A}_k)$ is a monotone partition for f^{n+k}, and thus, by definition of c_{n+k},

$$c_{n+k} \leq \#(\mathcal{A}_n \vee f^{-n}(\mathcal{A}_k)) \leq \#\mathcal{A}_n \cdot \#\mathcal{A}_k = c_n \cdot c_k.$$

This means that the sequence $(\log c_n)_{n\geq 1}$ is sub-additive. Thus, by Lemma 4.1, $\lim_{n\to+\infty} \frac{1}{n} \log c_n$ exists and is equal to $\inf_{n\geq 1} \frac{1}{n} \log c_n$. Applying Proposition 4.32 to f^n and $\mathcal{A}_n$, we get $h_{top}(f^n) = h_{top}(\mathcal{A}_n, f^n)$. Since $h_{top}(\mathcal{A}_n, f^n) \leq \log N(\mathcal{A}^n) = \log \#\mathcal{A}^n$ (see Section 4.1.1), we have $h_{top}(f^n) \leq \log \#\mathcal{A}_n = \log c_n$. Consequently,

$$h_{top}(f) \leq \lim_{n\to+\infty} \frac{1}{n} \log c_n.$$

It remains to show the reverse inequality.

We fix $n \geq 1$. From now on, we work with the map $g := f^n$, and $(\mathcal{A}_n)^k$ will denote the iterated partition relative to g. By Lemma 4.31, $(\mathcal{A}_n)^k$ is a monotone partition for g^k, so $c_{nk} \leq N_k(\mathcal{A}_n, g)$. Dividing by nk and taking the limit when k goes to infinity, we deduce that

$$\lim_{k\to+\infty} \frac{1}{nk} \log c_{nk} \leq \frac{1}{n} h_{top}(\mathcal{A}_n, g).$$

According to Proposition 4.32, $h_{top}(\mathcal{A}_n, g) = h_{top}(g)$. Thus

$$\lim_{m\to+\infty} \frac{1}{m} \log c_m = \lim_{k\to+\infty} \frac{1}{nk} \log c_{nk} \leq \frac{1}{n} h_{top}(f^n) = h_{top}(f).$$

This concludes the proof. □

Remark 4.34. The bounds of Propositions 4.29 and 4.33 are optimal since they can be reached: the map T_p in Example 2.13 is p-Lipschitz and has a p-horseshoe, and thus $h_{top}(T_p) = \log p$ by Propositions 4.6 and 4.29. Moreover, it can be easily computed that, for all $n \geq 1$, the minimal cardinality of a monotone partition for T_p^n is $c_n = p^n$. Therefore, the inequality $h_{top}(T_p) \leq \frac{1}{n} \log c_n$ given by Proposition 4.33 is an equality for all n in this example.

4.5. Graph associated to a family of intervals

4.5.1. A generalization of horseshoes. The existence of a horseshoe implies positive entropy because an exponential number of chains of intervals of a given length can be made by using the intervals forming the horseshoe. This idea can be generalized by counting the number of chains within a family of closed intervals. A convenient way to determine the possible chains of intervals is to build a directed graph. This idea is originally due to Bowen and Franks [**61**] and was improved by Block, Guckenheimer, Misiurewicz and Young [**36**].

Definition 4.35. Let f be an interval map and let $I_1, \ldots, I_p$ be non degenerate closed intervals with disjoint interiors. The *graph associated to the intervals* $I_1, \ldots, I_p$ is the directed graph G whose set of vertices is $\{I_1, \ldots, I_p\}$ and, for all

$i, j \in [\![1, p]\!]$, there are exactly k arrows from I_i to I_j if k is the maximal integer such that I_i covers k times I_j.

If $P = \{p_0 < p_1 < \cdots < p_n\}$ is a finite set containing at least two points, the *P-intervals* are $[p_0, p_1], [p_1, p_2], \ldots, [p_{n-1}, p_n]$. The graph associated to the P-intervals is denoted by $G(f|P)$ and its adjacency matrix by $M(f|P)$.

REMARK 4.36. If $P = \{x_1 < \ldots < x_n\}$ is a periodic orbit, the graph $G(f|P)$ contains the graph of the periodic orbit introduced in Definition 3.14. These two graphs coincide if f is monotone on every P-interval.

The next result follows easily from the definitions.

PROPOSITION 4.37. *Let f be an interval map and let $I_1, \ldots, I_p$ be non degenerate closed intervals with disjoint interiors. Let G be the graph associated to $I_1, \ldots, I_p$. Then, for every n-tuple $\{i_1, \ldots, i_n\} \in [\![1, p]\!]^n$, $(I_{i_1}, I_{i_2}, \ldots, I_{i_n})$ is a chain of intervals if and only if there is a path $I_{i_1} \to I_{i_2} \to \cdots \to I_{i_n}$ in G.*

We are going to show that the topological entropy of an interval map is greater than or equal to the logarithm of the spectral radius of the adjacency matrix of the graph associated to a family of intervals. We need some more definitions and results about matrices. One can refer to [**152**, Chapter 1] or [**102**, §1.3] for the proofs.

PROPOSITION 4.38. *Let M be a square matrix of size $n \times n$. A complex number λ is an eigenvalue of M if and only if λ is a root of the characteristic polynomial of M, which is $\chi_M(X) := \det(M - X\mathrm{Id})$, where Id is the identity matrix of size $n \times n$.*

DEFINITION 4.39 (spectral radius). Let M be a square matrix. The *spectral radius of M* is

$$\lambda(M) := \max\{|\lambda| \mid \lambda \text{ eigenvalue of } M\}.$$

The next results can be easily proved by using the Jordan normal form of a square matrix.

PROPOSITION 4.40 (Gelfand's formula). *Let M be a square matrix. Then $\lambda(M) = \lim_{n\to+\infty} \|M^n\|^{\frac{1}{n}}$.*

LEMMA 4.41. *Let A, B be two square matrices of the same size. Then $\lambda(A^k) = \lambda(A)^k$ for all positive integers k.*

PROOF. According to Proposition 4.40,

$$\lambda(A^k) = \lim_{n\to+\infty} \|A^{kn}\|^{\frac{1}{n}} = \lim_{n\to+\infty} \left(\|A^{kn}\|^{\frac{1}{kn}}\right)^k = \lambda(A)^k;$$

the last equality comes from the facts that $(kn)_{n\geq 1}$ is a subsequence of $\mathbb{N}$ and the map $t \mapsto t^k$ is continuous. □

DEFINITION 4.42. Let $A = (a_{ij})_{1\leq i,j\leq p}$ be a square matrix. The matrix A is *non negative*, or equivalently $A \geq 0$, if $a_{ij} \geq 0$ for all $i, j \in [\![1, p]\!]$, and *positive*, or equivalently $A > 0$, if $a_{ij} > 0$ for all $i, j \in [\![1, p]\!]$. If B is another matrix of the same size, then $A \leq B$ (resp. $A < B$) means that $B - A \geq 0$ (resp. $B - A > 0$).

For all integers $n \geq 1$, let $(a_{ij}^n)_{1\leq i,j\leq p}$ be the coefficients of A^n. The matrix A is called:

- *irreducible* if for all $i, j \in [\![1, p]\!]$, there exists $n \geq 1$ such that $a_{ij}^n > 0$,
- *primitive* if there exists $n \geq 1$ such that $A^n > 0$.

LEMMA 4.43. *Let A be a non negative square matrix. Then there exists a permutation matrix P such that $M := P^{-1}AP$ is equal to*

$$M = \begin{pmatrix} M_1 & 0 & 0 & \cdots & 0 \\ * & M_2 & 0 & \cdots & 0 \\ * & * & M_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & M_k \end{pmatrix} \tag{4.37}$$

where, for every $i \in [\![1, k]\!]$, M_i is either an irreducible square matrix or is equal to the 1×1 matrix (0), and the $*$'s represent possibly non zero submatrices.

In particular, if G is a directed graph, the vertices of G can be labeled in such a way that the adjacency matrix of G is of the form given by (4.37).

THEOREM 4.44 (Perron-Frobenius). *Let A be an irreducible non negative square matrix.*

i) *$\lambda(A)$ is a positive eigenvalue, which is a simple root of the characteristic polynomial of A. If in addition A is primitive, then $|\mu| < \lambda(A)$ for every eigenvalue $\mu \neq \lambda(A)$.*
ii) *If B is another matrix of the same size such that $0 \leq B \leq A$ and $B \neq A$, then $\lambda(B) < \lambda(A)$.*

COROLLARY 4.45. *Let A be a non negative square matrix. Then $\lambda(A)$ is a non negative eigenvalue and $|\mu| \leq \lambda(A)$ for every eigenvalue μ; $\lambda(A)$ is called the* maximal eigenvalue *of A.*

PROOF. According to Lemma 4.43, there exists a permutation matrix P such that $M := P^{-1}AP$ is of the form given by Equation (4.37). Let $i \in [\![1, k]\!]$. If M_i is the 1×1 null matrix, then 0 is its only eigenvalue, that is, $\lambda(M_i) = 0$. If M_i is irreducible, then, by Theorem 4.44, $\lambda(M_i)$ is a positive eigenvalue of M_i and every eigenvalue μ of M_i satisfies $|\mu| \leq \lambda(M_i)$. Let $\lambda := \max\{\lambda(M_1), \lambda(M_2), \ldots, \lambda(M_k)\}$. The set of eigenvalues of M is equal to the union of the sets of eigenvalues of the matrices $(M_i)_{1 \leq i \leq k}$. Moreover, the set of eigenvalues of A is equal to the set of eigenvalues of M. Consequently, $\lambda(A) = \lambda$, λ is an eigenvalue of A and, if μ is an eigenvalue of A, then $|\mu| \leq \lambda$. □

We are now ready to prove the result stated at the beginning of the section. Its proof is given by Block, Guckenheimer, Misiurewicz and Young in [**36**] with few details.

PROPOSITION 4.46. *Let f be an interval map and let $I_1, \ldots, I_p$ be non degenerate closed intervals with disjoint interiors. Let G be a subgraph of the graph associated to $I_1, \ldots, I_p$ and M the adjacency matrix of G. Then $h_{top}(f) \geq \log \lambda(M)$.*

PROOF. According to Lemma 4.43, we may re-label the intervals $I_1, \ldots, I_p$ in such a way that the adjacency matrix of G is of the form given in (4.37). We set $\lambda := \lambda(M)$. By Corollary 4.45, λ is an eigenvalue of M. We assume that $\lambda > 0$, otherwise there is nothing to prove. We keep the notation of (4.37). Since λ is the maximal eigenvalue of M, it is also the maximal eigenvalue of M_p for some

$p \in [\![1, k]\!]$. Let $\mathcal{I}$ be the finite set of indices supporting M_p. For all integers $n \geq 1$, we write $(M_p)^n = (m_{ij}^n)_{i,j\in\mathcal{I}}$. Then, by Proposition 4.40,

$$\lim_{n\to+\infty} \frac{1}{n} \log \left(\sum_{i,j\in\mathcal{I}} m_{ij}^n \right) = \log \lambda.$$

Thus, by Lemma 4.8, there exist two indices $i_0, j_0 \in \mathcal{I}$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \log m_{i_0 j_0}^n = \log \lambda. \tag{4.38}$$

According to Proposition 1.16, for all $i, j \in \mathcal{I}$ and all $n \geq 1$, m_{ij}^n is equal to the number of paths of length n from I_i to I_j in the directed graph G. Since M_p is irreducible (notice that $M_p \neq (0)$ because $\lambda(M_p) > 0$), there exists $n_0 \geq 1$ such that $m_{j_0 i_0}^{n_0} > 0$. Thus there exists a path $\mathcal{A}$ of length n_0 from I_{j_0} to I_{i_0} in G. For every path $\mathcal{B}$ of length n from I_{i_0} to I_{j_0}, the concatenated path $\mathcal{BA}$ is a path of length $n + n_0$ from I_{i_0} to itself. Therefore, the number of paths of length $n + n_0$ from I_{i_0} to itself is greater than or equal to the number of paths of length n from I_{i_0} to I_{j_0}. In other words, $m_{i_0 i_0}^{n+n_0} \geq m_{i_0 j_0}^n$ for all $n \geq 1$. Combined with (4.38), this implies:

$$\limsup_{n\to+\infty} \frac{1}{n} \log m_{i_0 i_0}^n \geq \log \lambda.$$

We fix $\varepsilon \in (0, \lambda)$ and a positive integer n such that

$$\frac{1}{n} \log m_{i_0 i_0}^n \geq \log(\lambda - \varepsilon). \tag{4.39}$$

Then $N := m_{i_0 i_0}^n$ is the number of paths of length n from I_{i_0} to itself in G and, by Proposition 4.37, there exist N distinct chains of intervals of the form $(I_{i_0}, I_{i_1}, \dots, I_{i_n})$ with i_0 fixed, $i_n = i_0$ and $i_j \in \mathcal{I}$ for all $j \in [\![1, n-1]\!]$. Then, according to Lemma 1.13(iii), there exist N closed subintervals $K_1, \dots, K_N$ with pairwise disjoint interiors such that, $\forall j \in [\![1, N]\!]$, $K_j \subset I_{i_0}$ and $f^n(K_j) = I_{i_0}$. Thus $(K_1, \dots, K_N)$ is an N-horseshoe for f^n, which implies that $h_{top}(f^n) \geq \log N$ by Proposition 4.6. Then, since $h_{top}(f) = \frac{1}{n} h_{top}(f^n)$ by Proposition 4.3 and $\frac{1}{n} \log N \geq \log(\lambda - \varepsilon)$ by (4.39), we deduce that $h_{top}(f) \geq \log(\lambda - \varepsilon)$. Finally, letting ε tend to zero, we get $h_{top}(f) \geq \log \lambda$. $\square$

REMARK 4.47. If $(I_1, \dots, I_p)$ is a p-horseshoe, then the graph associated to $(I_1, \dots, I_p)$ contains a subgraph whose transition matrix is $M = (m_{ij})_{1\leq i,j\leq p}$ with $m_{ij} = 1$ for all $i, j \in [\![1, p]\!]$. Moreover, ${}^t(1, 1, \dots, 1)$ is clearly an eigenvector of M for the eigenvalue p, so $\lambda(M) \geq p$. This shows that Proposition 4.6 is a particular case of Proposition 4.46.

4.5.2. The connect-the-dots map associated to a finite invariant set. When considering the graph associated to P-intervals, a particularly convenient case is when P is a periodic orbit, or more generally a finite invariant set. Knowing only the values of f on P is sufficient to determine a subgraph of $G(f|P)$. This subgraph is intimately related to the "connect-the-dots" map f_P associated to f on P (for all $x \in P$, plot the points $(x, f(x))$; then connect linearly the dots to get the graph $y = f_P(x)$).

DEFINITION 4.48. Let $f \colon I \to I$ be an interval map and let $P = \{p_0 < p_1 < \dots < p_k\}$ be a finite subset of I with $k \geq 1$. The map f is *P-monotone* if $f(P) \subset P$,

$I = [p_0, p_k]$ and f is monotone on every P-interval $[p_0, p_1], \dots, [p_{k-1}, p_k]$. The map f is *P-linear* if in addition f is linear on every P-interval.

If P is an invariant set, let $f_P \colon [p_0, p_k] \to [p_0, p_k]$ denote the unique P-linear map agreeing with f on P. The map f_P is called the *connect-the-dots* map associated to $f|_P \colon P \to P$. For short, we write $G(f_P)$ and $M(f_P)$ instead of $G(f_P|P)$ and $M(f_P|P)$.

Let $P = \{p_0 < p_1 < \dots < p_k\}$ be a finite invariant set with $k \geq 1$ and, for every $i \in [\![0, k]\!]$, let $\sigma(i) \in [\![0, k]\!]$ be such that $f(p_i) = p_{\sigma(i)}$. The graph $G(f_P)$ is determined by the values of f on P. Indeed, for every $i \in [\![0, k-1]\!]$, there is an arrow $[p_i, p_{i+1}] \to [p_j, p_{j+1}]$ in $G(f_P)$ if and only if $[p_j, p_{j+1}] \subset \langle p_{\sigma(i)}, p_{\sigma(i+1)} \rangle$, i.e., either $\sigma(i) \leq j < j+1 \leq \sigma(i+1)$ or $\sigma(i+1) \leq j < j+1 \leq \sigma(i)$. Notice that, if f is P-monotone, then $G(f|P) = G(f_P)$.

The next lemma follows trivially from the definition.

Lemma 4.49. *Let f be an interval map and P a finite invariant set. Suppose that f is P-monotone. Then, for every P-interval J, either f is constant on J or $f(J)$ is a nonempty union of P-intervals.*

Remark 4.50. If $P = \{p_0 < p_1 < \dots < p_k\}$, a map $f \colon I \to I$ is *Markov* with respect to the pseudo-partition $[p_0, p_1], \dots, [p_{k-1}, p_k]$ if $I = [p_0, p_k]$ and, for every P-interval J, $f|_J$ is monotone and $f(J)$ is a union of P-intervals. This notion is very close to P-monotonicity. Indeed, f is Markov with respect to the P-intervals if and only if f is P-monotone and $f(p_i) \neq f(p_{i+1})$ for all $i \in [\![0, k-1]\!]$ (i.e., f is non constant on every P-interval). The main additional property of Markov maps is that, for every point $x \in I$, there exists an infinite path $A_0 \to A_1 \to A_2 \to \cdots A_n \to \cdots$ in $G(f|P)$ such that $f^n(x) \in A_n$ for all $n \geq 0$.

Lemma 4.51. *Let f be an interval map, P a finite invariant set and n a positive integer. Then $M(f^n|P) \geq M(f_P)^n$.*

Proof. Let $I_1, \dots, I_k$ be the P-intervals. We write $M(f_P)^n := (a_{ij})_{1 \leq i,j \leq k}$. By Proposition 1.16, a_{ij} is the number of paths of length n from I_i to I_j in the graph $G(f_P)$. Each path is a chain of intervals for f_P, and also for f, which implies that I_i covers I_j a_{ij} times for f. This exactly means that the (i,j)-coefficient of $M(f^n|P)$ is greater than or equal to a_{ij}. Hence $M(f^n|P) \geq M(f_P)^n$. ☐

The next result was first stated by Coppel in [**68**]. It shows that the graph associated to a P-monotone map represents well the dynamics from the point of view of entropy.

Proposition 4.52. *Let $f \colon I \to I$ be an interval map and let P be a finite invariant set. If f is P-monotone, then*

$$h_{top}(f) = h_{top}(f_P) = \max(0, \log \lambda(M(f_P))).$$

Proof. Let $\mathcal{A}$ be the family of all P-intervals. This is a cover of I. Let $\mathcal{C} \subset \mathcal{A}$ be the family of P-intervals on which f is constant. For every $n \geq 1$, we set

$$\mathcal{B}_n^+ := \left\{ \bigcap_{i=0}^{n-1} f^{-i}(I_i) \mid I_0 \to\ I_1 \to \cdots \to I_{n-1} \text{ is a path in } G(f_P),\ I_{n-1} \notin \mathcal{C} \right\},$$

$$\mathcal{B}_n^- := \left\{ \bigcap_{i=0}^{k} f^{-i}(I_i) \mid k \in [\![0, n-1]\!],\ I_0 \to \cdots \to I_k \text{ is a path in } G(f_P),\ I_k \in \mathcal{C} \right\},$$

and $\mathcal{B}_n := \mathcal{B}_n^+ \cup \mathcal{B}_n^-$. We are going to show that $\mathcal{B}_n$ is a subcover of $\mathcal{A}^n$. It is clear that $\mathcal{B}_n^+ \subset \mathcal{A}^n$, and the elements of $\mathcal{B}_n^-$ of the form $\bigcap_{i=0}^{n-1} f^{-i}(I_i)$ are in $\mathcal{A}^n$ too. Let $J \in \mathcal{B}_n^-$ with $J = \bigcap_{i=0}^{k} f^{-i}(I_i)$ and $k < n-1$. By definition, $f(I_k)$ is reduced to one point $\{x\}$. Since $\mathcal{A}$ is a cover, there exist $I_{k+1}, \dots, I_{n-1} \in \mathcal{A}$ such that $f^i(x) \in I_{k+i}$ for all $i \in [\![1, n-1-k]\!]$. We have $J = \bigcap_{i=0}^{n-1} f^{-i}(I_i)$, so $J \in \mathcal{A}^n$. This proves that $\mathcal{B}_n \subset \mathcal{A}^n$. We now show that $\mathcal{B}_n$ is a cover of I by induction on n.

- $\mathcal{B}_1 = \mathcal{A}$ is a cover.
- Let $n \geq 2$. We have $\mathcal{B}_{n-1}^- \subset \mathcal{B}_n$. Let $J \in \mathcal{B}_{n-1}^+$ with $J = \bigcap_{i=0}^{n-2} f^{-i}(I_i)$. By definition, $I_{n-2} \notin \mathcal{C}$, so $f(I_{n-2})$ is a nonempty union of P-intervals by Lemma 4.49, say $f(I_{n-2}) = A_1 \cup \cdots \cup A_j$ with $A_1, \dots, A_j \in \mathcal{A}$. Then, for every $i \in [\![1, j]\!]$, $I_0 \to \cdots I_{n-2} \to A_i$ is a path in $G(f_P)$, $J_i := J \cap f^{-(n-1)}(A_i)$ is an element of $\mathcal{B}_n$ and $J = J_1 \cup \cdots \cup J_j$. Therefore, if $\mathcal{B}_{n-1}$ is a cover of I, then $\mathcal{B}_n$ is a cover too. This ends the induction.

Since $\mathcal{B}_n$ is a subcover of $\mathcal{A}^n$, we have $N_n(\mathcal{A}, f) \leq \#\mathcal{B}_n$. By Proposition 1.16, $\|M(f_P)^k\|$ is the number of paths of length k in $G(f_P)$. Thus

$$\#\mathcal{B}_n \leq \sum_{k=0}^{n-1} \|M(f_P)^k\|,$$

and hence

$$\frac{1}{n}\log N_n(\mathcal{A}, f) \leq \frac{1}{n}\log \#\mathcal{B}_n \leq \frac{1}{n}\log\left(n \max_{k\in[\![0,n-1]\!]} \|M(f_P)^k\|\right).$$

If the sequence $\left(\|M(f_P)^k\|\right)_{k\geq 0}$ is bounded, then

$$\lim_{n\to+\infty} \frac{1}{n}\log\left(n \max_{k\in[\![0,n-1]\!]} \|M(f_P)^k\|\right) = 0,$$

and thus $h_{top}(\mathcal{A}, f) = 0$. Otherwise, there exists an increasing sequence of integers $(n_i)_{i\geq 0}$ such that $\|M(f_P)^{n_i}\| = \max_{k\in[\![0,n_i]\!]} \|M(f_P)^k\|$ for all $i \geq 0$. This implies that

$$\begin{aligned} h_{top}(\mathcal{A}, f) &= \lim_{n\to+\infty} \frac{1}{n}\log N_n(\mathcal{A}, f) \\ &\leq \limsup \frac{1}{n}\log\left(n\|M(f_P)^n\|\right) = \limsup_{n\to+\infty} \frac{1}{n}\log(\|M(f_P)^n\|) \\ &\leq \log\lambda(M(f_P)) \quad \text{by Proposition 4.40.} \end{aligned}$$

Since $\mathcal{A}$ is a monotone cover, we have $h_{top}(f) = h_{top}(\mathcal{A}, f)$ by Proposition 4.32, so $h_{top}(f) \leq \max(0, \log\lambda(M(f_P)))$. Proposition 4.46 and the fact that $h_{top}(f) \geq 0$ imply the converse inequality $h_{top}(f) \geq \max(0, \log\lambda(M(f_P)))$. We conclude that $h_{top}(f) = \max(0, \log\lambda(M(f_P)))$. □

The converse of Proposition 4.52 does not hold in general: there exist interval maps f with a finite invariant set P such that $h_{top}(f) = h_{top}(f_P)$ although f is not P-monotone and is not constant on any subinterval. See Figure 3 for a counter-example. However, we shall see later that it does hold for transitive maps, i.e., a transitive interval map f such that $h_{top}(f) = h_{top}(f_P)$ is necessarily P-monotone (Proposition 4.74).

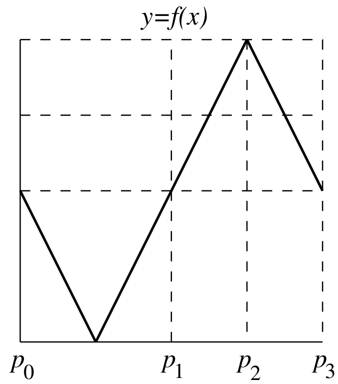


FIGURE 3. The set $P = \{p_0, p_1, p_2, p_3\}$ is invariant and it is easy to show that $h_{top}(f) = h_{top}(f_P) = \log 2$, but f is not P-monotone.

For every finite invariant set P, $G(f_P)$ is a subgraph of $G(f|P)$, and thus Proposition 4.46 implies that $h_{top}(f) \geq h_{top}(f_P)$. The next proposition shows that the entropy of f can be approached arbitrarily close in this way. This result was first stated by Takahashi [**167**], but it appears that this proof is valid only for piecewise monotone maps, as noticed by Block and Coven, who gave a complete proof in [**35**]. See also the extensive paper of Misiurewicz and Nitecki [**129**].

PROPOSITION 4.53. *Let f be an interval map. Then*

$$\begin{aligned} h_{top}(f) &= \sup\{h_{top}(f_P) \mid P \textit{ finite invariant set}\} \\ &= \sup\{h_{top}(f_P) \mid P \textit{ periodic orbit}\}. \end{aligned}$$

PROOF. The inequality $h_{top}(f) \geq \sup\{h_{top}(f_P) \mid P \text{ finite invariant set}\}$ follows from Propositions 4.46 and 4.52, and the inequality

$$\sup\{h_{top}(f_P) \mid P \text{ finite invariant set}\} \geq \sup\{h_{top}(f_P) \mid P \text{ periodic orbit}\}$$

is trivial. We are going to show that $h_{top}(f) \leq \sup\{h_{top}(f_P) \mid P \text{ periodic orbit}\}$ when $h_{top}(f) > 0$ (if $h_{top}(f) = 0$, there is nothing to prove). Let λ, λ' be such that $0 < \lambda < \lambda' < h_{top}(f)$. By Misiurewicz's Theorem 4.7, there exist an arbitrarily large integer N and a strict p-horseshoe for f^N such that $\frac{\log p}{N} \geq \lambda'$. We denote by $I_1 < I_2 < \cdots < I_p$ the intervals composing this horseshoe and we set $g := f^N$. By applying Lemma 1.13(ii) to the chain of intervals

$$\begin{gathered} I_2 \to I_1 \to I_3 \to I_1 \to \cdots \to I_{p-1} \to I_1 \to \\ I_2 \to I_p \to I_3 \to I_p \to \cdots \to I_{p-1} \to I_p \to I_2, \end{gathered}$$

we can build a periodic point x of period $4p-8$ for g such that

- for all $i = 0, \dots, p-3$, $g^{2i}(x)$ belongs successively to $I_2, I_3, \dots, I_{p-1}$ and $g^{2i+1}(x)$ belongs to I_1; we set $y_i := g^{2i}(x)$;
- for all $i = p-2, \dots, 2p-5$, $g^{2i}(x)$ belongs successively to $I_2, I_3, \dots, I_{p-1}$ and $g^{2i+1}(x)$ belongs to I_p; we set $z_i := g^{2i}(x)$.

Let $Q := \mathcal{O}_g(x)$. For every $i \in [\![2, p-1]\!]$, I_i contains only two points of Q, namely y_i, z_i, and thus $\langle y_i, z_i \rangle$ is a Q-interval. Moreover, $g(y_i) \in I_1$ and $g(z_i) \in I_p$, which implies that $\langle g(y_i), g(z_i) \rangle$ contains $I_2 \cup \cdots \cup I_{p-1}$ by connectedness. Therefore, the intervals $(\langle y_i, z_i\rangle)_{2\le i\le p-2}$ form a $(p-2)$-horseshoe for the map g_Q, which implies that $h_{top}(g_Q) \ge \log(p-2)$ by Proposition 4.6. We set $M_Q := M(g_Q)$.

Now we come back to the map f. Let $P := \mathcal{O}_f(x)$. Since x is periodic for $g = f^N$, it is also periodic for f. Moreover, Q is a periodic orbit for the map $(f_P)^N$ because f_P and f coincide on the set $P \supset Q$. Therefore,

$$h_{top}((f_P)^N) \ge \max(0, \log\lambda(M_Q)) = h_{top}(g_Q)$$

by Propositions 4.46 and 4.52, and thus

$$h_{top}(f_P) = \frac{1}{N} h_{top}((f_P)^N) \ge \frac{1}{N}\log(p-2).$$

Since $\frac{1}{N}\log p \ge \lambda'$, we have $h_{top}(f_P) \ge \lambda' - \frac{1}{N}\log\frac{p}{p-2}$. If N is large enough, then $\frac{1}{N}\log 2 < \lambda' - \lambda$ and p can be arbitrarily large. In particular, $\frac{p}{p-2} \le 2$ if $p \ge 4$, so $\frac{1}{N}\log\frac{p}{p-2} \le \frac{1}{N}\log 2 < \lambda' - \lambda$. We thus have $h_{top}(f_P) \ge \lambda$. We deduce the required result by taking λ tending to $h_{top}(f)$. □

Remarks on graph maps. The notion of graph associated to a family of intervals is meaningful for graph maps provided Definition 1.14 is used for covering, and Proposition 4.46 holds for graph maps with no change.

For graph maps, one can define P-monotone maps when the finite invariant set P contains all the branching points of the graph (which requires that the orbit of every branching point is finite), and there is no difficulty to extend Proposition 4.52 to P-monotone graph maps in this case (see, e.g., a remark in [**15**]). However, the *connect-the-dots map* associated to a finite invariant set P is not well defined in general (it is well defined when the space is a tree and P contains the branching points).

DEFINITION 4.54. Let $f\colon G \to G$ be a graph map and let P be a finite invariant set containing all the branching points and all the endpoints of G. A *P-basic interval* is any connected component of $G \setminus P$. The map f is called *P-monotone* if f is monotone in restriction to every P-basic interval. If f is P-monotone, let $M(f|P)$ denote the adjacency matrix of the graph associated to the family of all P-basic intervals.

PROPOSITION 4.55. *Let $f\colon G \to G$ be a graph map and let P be a finite invariant set containing all the branching points and all the endpoints of G. If f is P-monotone, then $h_{top}(f) = \max(0, \log(\lambda(M(f|P))))$.*

In [**5**], Alsedà, Juher and Mumbrú showed that an equality similar to Proposition 4.53 holds for graph maps; an inequality was previously proved by Alsedà, Mañosas and Mumbrú [**15**]. Since connect-the-dots maps cannot be defined, it is necessary to introduce an equivalence between actions on a pointed graph, in order to be able to tell when a P-monotone map is a “good” candidate to replace the connect-the-dots map. The next definition follows [**15**].

DEFINITION 4.56. Let G be a topological graph and let $B(G)$ denote the set of all branching points of G. Let A be a finite set of G. Let G_A denote the graph G deprived of the connected components of $G \setminus (A \cup B(G))$ containing an endpoint of G. Let $r_A\colon G \to G_A$ denote the retraction from G to G_A (that is, r_A is the identity on G_A and, if C is a connected component of $G \setminus G_A$ and $x \in C$, then $r_A(x)$ is the unique point in $\overline{C} \cap (A \cup B(G))$). Let $f\colon G \to G$ and $g\colon G \to G$

be two graph maps and assume that A is both f-invariant and g-invariant. Set $\tilde{f} := r_A \circ f|_{G_A}$ and $\tilde{g} := r_A \circ g|_{G_A}$. Then one writes $(G, A, f) \sim (G, A, g)$ if there exists a homeomorphism $\varphi\colon G_A \to G_A$ with $\varphi(A) = A$ such that $\tilde{f}$ and $\varphi^{-1} \circ \tilde{g} \circ \varphi$ are homotopic relative to A.

THEOREM 4.57. *Let $f\colon G \to G$ be a graph map, and let $B(G)$ and $E(G)$ denote respectively the set of branching points and endpoints of G. For every finite f-invariant set A, there exists a map $g_A\colon G \to G$ such that*

- $P := A \cup B(G) \cup E(G)$ *is g_A-invariant,*
- g_A *is P-monotone,*
- $(G, A, g_A) \sim (G, A, f)$,
- $h_{top}(g_A) \le h_{top}(f)$.

Furthermore, $h_{top}(f) = \sup\{h_{top}(g_A) \mid A$ is a periodic orbit of $f\}$.

4.6. Entropy and periodic points

4.6.1. Equivalent condition for positive entropy. We are going to show that an interval map has positive topological entropy if and only if it has a periodic point whose period is not a power of 2. This relation between entropy and periods is one of the most striking results in interval dynamics. This result can be expressed in term of types for Sharkovsky's order: an interval map has positive entropy if and only if it is of type n with $n \lhd 2^\infty$. This explains why Coppel calls *chaotic* an interval map having a periodic point whose period is not a power of 2 [**68, 41**], the type 2^∞ being the "frontier" between chaos and non chaos.

This result was proved in several steps. First, Sharkovsky showed in 1965 that an interval map f is of type $n \lhd 2^\infty$ if and only if f^k has a horseshoe for some k [**155**]; see [**158**] for a statement in English. The same result was re-proved by Block [**33**]. Then Bowen and Franks stated that the presence of a periodic point whose period is not a period of 2 implies positive entropy [**61**]. This result relies on the observation that horseshoes imply positive entropy (Proposition 4.6). Finally, the last step is due to Misiurewicz and Szlenk for piecewise monotone maps [**132**] and Misiurewicz for all interval maps [**125, 127**]. This is a corollary of Misiurewicz's Theorem (Theorem 4.7), proved in the same papers, stating that an interval map with positive entropy has a horseshoe for some iterate of the map.

THEOREM 4.58. *For an interval map f, the following assertions are equivalent:*

i) *the topological entropy of f is positive,*
ii) *f has a periodic point whose period is not a power of 2,*
iii) *there exists an integer $n \ge 1$ such that f^n has a strict horseshoe.*

PROOF. If $h_{top}(f) > 0$, then, according to Misiurewicz's Theorem 4.7, there exists a positive integer n such that f^n has a horseshoe. Therefore f^n has periodic points of all periods by Proposition 3.31, and thus f has a periodic point whose period is not a power of 2. This shows (i)⇒(ii).

If f has a periodic point of period $2^d q$, where q is an odd integer greater than 1 and $d \ge 0$, then f^{2^d} has a periodic point of period q and thus, by Proposition 3.32, $f^{2^{d+1}}$ has a strict horseshoe. That is, (ii)⇒(iii).

If f^n has a horseshoe, then, according to Proposition 4.6, $h_{top}(f) = \frac{1}{n} h_{top}(f^n) \ge \frac{\log 2}{n} > 0$. Hence (iii)⇒(i). ☐

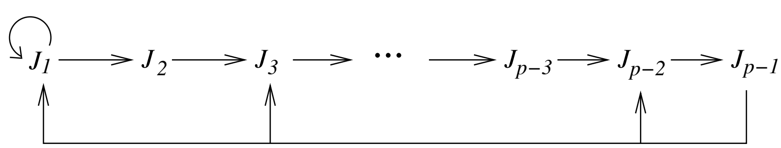
4.6.2. Lower bound for the entropy depending on Sharkovsky's type. The relation between the entropy of an interval map and its type is much more accurate than the one stated in Theorem 4.58, and one can give a lower bound for the entropy depending on the periods of the periodic points. First, Bowen and Franks proved that, if f has a periodic point of period $n = 2^d q$, where $q > 1$ is odd, then $h_{top}(f) > \frac{1}{n}\log 2$ [**61**]. Then Štefan improved this result and showed that, under the same assumption, $h_{top}(f) > \frac{\log\sqrt{2}}{2^d}$ [**166**]. Finally, Block, Guckenheimer, Misiurewicz and Young gave an optimal bound by proving that, under the same assumption, $h_{top}(f) \geq \frac{\log\lambda_q}{2^d}$, where λ_q is the maximal real root of $X^q - 2X^{q-2} - 1$ [**36**]. Actually, the value $\frac{\log\lambda_q}{2^d}$ already appears in Štefan's proof, where λ_q is proved to be greater than $\sqrt{2}$. Moreover, there exist maps of type $2^d q$ whose entropy is equal to $\frac{\log\lambda_q}{2^d}$. Examples of such maps were given without details in [**36**].

We start with a lemma, which is the key point of the proof.

Lemma 4.59. *Let f be an interval map having a periodic point of odd period greater than 1. Let p be the minimal odd period greater than 1, let G_p be the graph of a periodic orbit of period p and M_p its adjacency matrix. Then M_p is a primitive matrix and $\lambda(M_p)$ is equal to the unique positive root λ_p of $X^p - 2X^{p-2} - 1$. Moreover, for all odd $p > 1$,*

$$\sqrt{2} < \lambda_{p+2} < \lambda_p < \sqrt{2} + \frac{1}{(\sqrt{2})^{p+1}}.$$

Proof. According to Lemma 3.17, the graph G_p is of the form:



This gives the following adjacency matrix:

$$M_p = \begin{pmatrix} 1 & & & & & 1 \\ 1 & 0 & & 0 & & 0 \\ & 1 & \ddots & & & 1 \\ & & \ddots & \ddots & & \vdots \\ & 0 & & \ddots & \ddots & 1 \\ & & & & 1 & 0 \end{pmatrix}. \tag{4.40}$$

More precisely, if $M_p = (m_{ij})_{1 \leq i,j \leq p-1}$, then

- on the diagonal: $m_{11} = 1$ and $\forall i \in [\![2, p-1]\!], m_{ii} = 0$,
- below the diagonal: $\forall i \in [\![1, p-2]\!],\ m_{i+1i} = 1$,
- last column: $\forall i \in [\![1, p-1]\!],\ m_{ip-1} = \begin{cases} 1 & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even,} \end{cases}$
- for all other indices, $m_{ij} = 0$.

We write $(M_p)^n = (m_{ij}^n)_{1\le i,j\le p-1}$ for all $n \geq 1$. Then m_{ij}^n is the number of paths of length n from J_i to J_j in G_p (Proposition 1.16). For all $i, j \in [\![1, p-1]\!]$, the path

$$J_i \to J_{i+1} \to \cdots \to \underbrace{J_1 \to J_1 \to \cdots \to J_1}_{i+j-2 \text{ arrows}} \to J_2 \cdots \to J_j$$

is a path from I_i to I_j of length $2p-2$. Thus $(M_p)^{2p-2} > 0$ and M_p is primitive.

In order to find the maximal eigenvalue of M_p, we compute the characteristic polynomial $\chi_p(X) := \det(M_p - X\mathrm{Id})$ (see Proposition 4.38). We develop it with respect to the first row (the coefficients left blank are equal to zero):

$$\chi_p(X) = (1-X)\underbrace{\begin{vmatrix} -X & & & & 0 \\ 1 & \ddots & & & 1 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 1 \\ & & & 1 & -X \end{vmatrix}}_{:= Q_{p-2}(X)} - \begin{vmatrix} 1 & -X & & \\ 0 & 1 & \ddots & \\ & & \ddots & -X \\ & & & 1 \end{vmatrix}.$$

We get $\chi_p(X) = (1-X)Q_{p-2}(X) - 1$. It remains to compute $Q_k(X)$ for all odd k. If we develop twice the determinant $Q_k(X)$ with respect to the first row, we get $Q_k(X) = X^2 Q_{k-2}(X) + X$. We have $Q_1 = -X$ and an easy induction gives

$$\text{for all odd } k \geq 3,\ Q_k(X) = -X^k + \sum_{i=0}^{\frac{k-3}{2}} X^{2i+1}.$$

Therefore

$$\text{for all odd } p \geq 3,\ \chi_p(X) = X^{p-1} - X^{p-2} - \sum_{i=0}^{p-3} (-X)^i.$$

We set $P_p(X) := (X+1)\chi_p(X)$. A straightforward computation gives

$$P_p(X) = X^p - 2X^{p-2} - 1.$$

We do a short study of the polynomial function $x \mapsto P_p(x)$ on $\mathbb{R}^+$. Its differential is

$$P_p'(x) = px^{p-1} - 2(p-2)x^{p-3} = x^{p-3}(px^2 - 2(p-2)).$$

Thus, for $x \in (0, +\infty)$, $P_p'(x) > 0 \Leftrightarrow x > x_p := \sqrt{\frac{2(p-2)}{p}}$. This implies that P_p is decreasing on $[0, x_p]$ and increasing on $[x_p, +\infty)$, and also that $x_p < \sqrt{2}$. Moreover, $P_p(0) = -1$, and $\lim_{x\to+\infty} P_p(x) = +\infty$. We deduce that there exists a unique $\lambda_p > 0$ such that $P_p(\lambda_p) = 0$. Since λ_p is the maximal real root of P_p, it is also the maximal real root of $\chi_p = \frac{P_p}{X+1}$. By Corollary 4.45, this implies that $\lambda(M_p) = \lambda_p$.

Now we are going to bound λ_p. Since $P_p(x) = x^{p-2}(x^2-2) - 1$, we have $P_p(\sqrt{2}) = -1$, and thus $\lambda_p > \sqrt{2}$ (recall that P_p is increasing on $[x_p, +\infty) \supset [\sqrt{2}, +\infty)$). Moreover, since

$$\lambda_p^{p-2}(\lambda_p^2 - 2) = 1,$$

we have $\lambda_p^p(\lambda_p^2-2)=\lambda_p^2>2$, which implies $P_{p+2}(\lambda_p)>0$. Therefore, $\lambda_{p+2}<\lambda_p$ for all odd $p\geq 3$. We set $y_p:=\sqrt{2}+\frac{1}{(\sqrt{2})^{p+1}}$. We have

$$y_p^2=2+\frac{1}{2^{p+1}}+\frac{1}{(\sqrt{2})^{p-2}}$$

and $y_p>\sqrt{2}$. Therefore

$$\begin{aligned}P_p(y_p)=y_p^{p-2}(y_p^2-2)-1 &= y_p^{p-2}\left(\frac{1}{2^{p+1}}+\frac{1}{(\sqrt{2})^{p-2}}\right)-1\\ &> (\sqrt{2})^{p-2}\frac{1}{(\sqrt{2})^{p-2}}-1\\ &> 0.\end{aligned}$$

We deduce that $\lambda_p<y_p=\sqrt{2}+\frac{1}{(\sqrt{2})^{p+1}}$. This concludes the proof. □

THEOREM 4.60. *If an interval map f has a periodic point of period 2^dq with $d\geq 0$, $q>1$, q odd, then $h_{top}(f)\geq\frac{\log\lambda_q}{2^d}$, where λ_q is the unique positive root of $X^q-2X^{q-2}-1$.*

Moreover, for all integers $d\geq 0$ and all $q>1$ with q odd, there exists an interval map with a periodic point of period 2^dq and whose topological entropy is equal to $\frac{\log\lambda_q}{2^d}$.

PROOF. First we suppose that $d=0$ and that q is the minimal odd period greater than 1. Let G_q be the graph of a periodic orbit of period q. According to Lemma 4.59, the spectral radius of the adjacency matrix of G_q is equal to λ_q. Thus $h_{top}(f)\geq\log\lambda_q$ by Proposition 4.46.

Now we suppose that x is a periodic point of period 2^dq with $q>1$, q odd. The point x is periodic of period q for f^{2^d}. Let p be the minimal odd period greater than 1 for the map f^{2^d}. What precedes shows that $h_{top}(f^{2^d})\geq\log\lambda_p$. Since $p\leq q$, Lemma 4.59 implies that $\lambda_p\geq\lambda_q$, and thus

$$h_{top}(f)=\frac{1}{2^d}h_{top}(f^{2^d})\geq\frac{\log\lambda_q}{2^d}.$$

For the sharpness of the bound, see Examples 4.61 and 4.62 below. □

EXAMPLE 4.61. Let n be a positive integer and $p:=2n+1$ (i.e., p is an odd integer greater than 1). We consider the map $f_p\colon[0,2n]\to[0,2n]$ built in Example 3.21. We already proved that it is topologically mixing and that its type for Sharkovsky's order is p. We recall that the map f_p (represented in Figure 4) is linear between the points $0,n-1,n,2n-1,2n$, and

- $\forall k\in[\![1,n]\!]$, $f_p(n-k)=n+k$,
- $\forall k\in[\![0,n-1]\!]$, $f_p(n+k)=n-k-1$.

We set $P:=\{0,1,2,\dots,,2n\}$ and, for all $k\in[\![1,n]\!]$,

$$J_{2k-1}:=[n-k,n-k+1]\quad\text{and}\quad J_{2k}:=[n+k-1,n+k].$$

Then P is a periodic orbit of period p and the graph associated to P is:

$$\begin{tikzcd} J_1 \arrow[loop left] \arrow[r] & J_2 \arrow[r] & J_3 \arrow[r] & \cdots \arrow[r] & J_{p-3} \arrow[r] & J_{p-2} \arrow[r] & J_{p-1} \arrow[llllll, bend left] \arrow[llll, bend left] \arrow[l, bend left] \end{tikzcd} \tag{4.41}$$

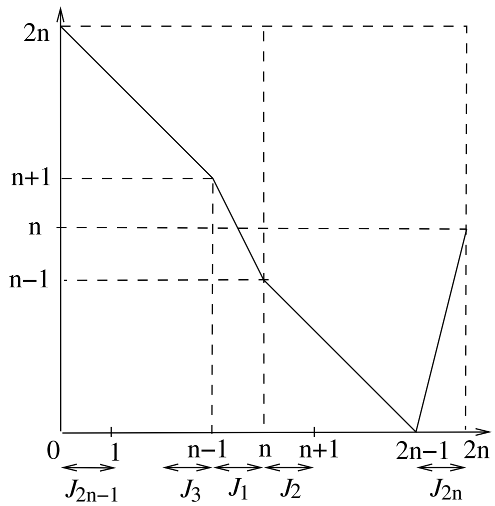


FIGURE 4. This interval map is of type $p = 2n+1$, it is topologically mixing and its topological entropy is equal to $\log \lambda_p$.

Moreover, f_p is P-linear and the matrix $M_p = M(f_p|P)$ is exactly the one given by (4.40) in the proof of Lemma 4.59. Hence $\lambda(M_p) = \lambda_p$, where λ_p is defined in Theorem 4.60. By Proposition 4.52, $h_{top}(f_p) = \log \lambda_p$. This proves that the bound of Theorem 4.60 is sharp for $d = 0$.

Finally, we remark that, for $p = 3$, we get a map of type 3 with no horseshoe because $h_{top}(f_3) < \log 2$. This shows that the converse of Proposition 3.31 does not hold, as said in Chapter 3.

EXAMPLE 4.62. Our goal is to show that, for all integers $d \geq 0$ and all $p > 1$ with p odd, there exists an interval map of type $2^d p$ such that its topological entropy is equal to $\frac{\log \lambda_p}{2^d}$, where λ_p is defined in Theorem 4.60. This will prove that the bound of Theorem 4.60 is optimal.

In Example 3.22, we defined the *square root* of an interval map and showed that the square root of a map of type n is of type $2n$. We recall this construction. If $f\colon [0,b] \to [0,b]$ is an interval map, the square root of f is the continuous map $g\colon [0,3b] \to [0,3b]$ defined by

- $\forall x \in [0,b]$, $g(x) := f(x) + 2b$,
- $\forall x \in [2b,3b]$, $g(x) := x - 2b$,
- g is linear on $[b,2b]$.

The graphs of g and g^2 are represented in Figure 5.

Suppose that f is P-monotone with $P = \{x_0 < x_1 < \cdots < x_p\}$, $x_0 = 0$ and $x_p = b$. We set

$$Q := \{x_0, \ldots, x_p, x_0 + 2p, x_1 + 2p, \ldots, x_p + 2p\}.$$

By definition of g, it is obvious that g is Q-monotone. The matrix $A := M(f|P)$ is of size $p \times p$. Let $B := M(g|Q)$, with the convention that the Q-interval $[b,2b]$ corresponds to the last column and row. The matrix B is of size $(2p+1) \times (2p+1)$

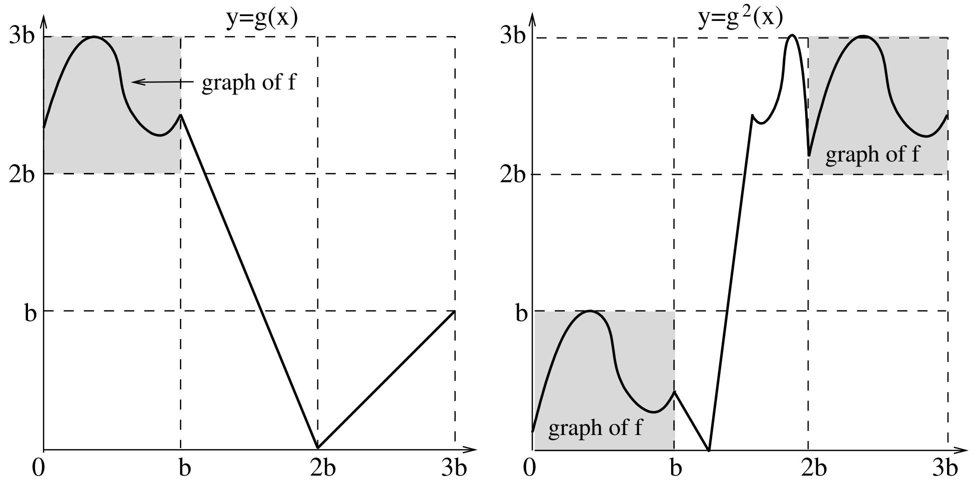


FIGURE 5. The left side represents the map g, which is the square root of f; the topological entropy of g is $\frac{h_{top}(f)}{2}$. The right side represents the map g^2.

and, looking at Figure 5, it is clear that B is of the form

$$B = \begin{pmatrix} 0_{p\times p} & A & 0_p \\ \mathrm{Id}_p & 0_{p\times p} & 0_p \\ * & {}^t1_p & 1 \end{pmatrix}$$

(where $0_{p\times p}$ denotes the $p\times p$ null matrix and x_p denotes the $1\times p$ matrix with all coefficients equal to x) and thus

$$B^2 = \begin{pmatrix} A & 0_{p\times p} & 0_p \\ 0_{p\times p} & A & 0_p \\ * & * & 1 \end{pmatrix}.$$

We deduce that $\lambda(B^2) = \lambda(A)$ provided $\lambda(A) \geq 1$. According to Proposition 4.52, $h_{top}(f) = \max(0, \log\lambda(A))$ and $h_{top}(g) = \max(0, \log\lambda(B))$. Since $\lambda(B^2) = \lambda(B)^2$, we get $h_{top}(g) = \frac{1}{2}h_{top}(f)$.

We fix an odd integer $p > 1$. Starting with the map f_p of type p and topological entropy $\log\lambda_p$ defined in Example 4.61 and applying inductively the square root construction, we can build an interval map of type $2^d p$ and topological entropy $\frac{\log\lambda_p}{2^d}$ for any integer $d \geq 0$. This completes the construction.

4.6.3. Number of periodic points. We have seen that the knowledge of the periods of the periodic points gives a lower bound on the entropy. Conversely, the entropy gives some information on the number of periodic points. The next result, due to Misiurewicz [**127**], is a straightforward consequence of Misiurewicz's Theorem 4.7. Recall that $P_n(f)$ is the set of points x such that $f^n(x) = x$.

PROPOSITION 4.63. *If f is an interval map of positive topological entropy, then*

$$\limsup_{n\to+\infty} \frac{1}{n}\log\#P_n(f) \geq h_{top}(f).$$

Proof. Let $0 < \lambda < h_{top}(f)$. According to Theorem 4.7, for all integers N, there exist integers $n \geq N$ and $p \geq 2$ such that f^n has a strict p-horseshoe $(J_1, \ldots, J_p)$ and $\frac{1}{n}\log p \geq \lambda$. In particular, $f^n(J_i) \supset J_i$ for all $i \in [\![1,p]\!]$, and thus there exists $x \in J_i$ such that $f^n(x) = x$ by Lemma 1.11. Since the intervals $J_1, \ldots, J_p$ are pairwise disjoint, this implies that $\#P_n(f) \geq p$, so

$$\limsup_{n\to+\infty} \frac{1}{n}\log \#P_n(f) \geq \lambda.$$

Since λ is arbitrarily close to $h_{top}(f)$, this gives the required result. $\square$

The next proposition follows a theorem of Štefan [**166**], which strengthens a previous result of Bowen and Franks [**61**].

Proposition 4.64. *Let $f\colon I \to I$ be an interval map. If f has a periodic point of period $2^d q$ with $d \geq 0$ and q an odd integer greater than 1, then*

$$\liminf_{n\to+\infty} \frac{1}{n}\log \#\{x \in I \mid x \textit{ periodic point of period } 2^d n\} \geq \log \lambda_q,$$

where λ_q is the unique positive root of $X^q - 2X^{q-2} - 1$.

Proof. We first assume that $d = 0$. Let p be the minimal odd period greater than 1. We fix P as a periodic orbit of period p and we denote G by the graph associated to P. For all $n \geq 1$, let N_n be the number of primitive cycles of length n in G. According to Lemma 3.16, for every primitive cycle $I_0 \to I_1 \to \cdots I_n = I_0$ in G, there exists a periodic point y of period n such that $f^i(y) \in I_i$ for all $i \in [\![0, n-1]\!]$. The periodic points y, y' corresponding to two different primitive cycles are different, except maybe if y, y' are endpoints of one of the P-intervals, which implies that they are of period p. Therefore,

$$\forall n \neq p,\ \#\{x \in I \mid x \text{ periodic point of period } n\} \geq N_n. \tag{4.42}$$

Let M be the adjacency matrix of G. We write $M^n = (m_{ij}^n)_{1\leq i,j\leq p-1}$ for every $n \geq 1$. By Proposition 1.16, the number of cycles of length n in G is equal to $\sum_{i=1}^{p-1} m_{ii}^n = \mathrm{Tr}(M^n)$. By Lemma 4.59, M is primitive and its maximal eigenvalue is λ_p. Let $(\lambda_p, \mu_2, \ldots, \mu_{p-1})$ be the set of eigenvalues (with possible repetitions corresponding to the size of the generalized eigenspaces) of M. According to the Perron-Frobenius Theorem 4.44, $|\mu_i| < \lambda_p$ for all $i \in [\![2, p-1]\!]$. By triangularization of M, the matrix M^n is equivalent to

$$\begin{pmatrix} \lambda_p^n & * & * & \cdots & * \\ 0 & \mu_2^n & * & \cdots & * \\ 0 & 0 & \mu_3^n & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{p-1}^n \end{pmatrix}.$$

We deduce that

$$\mathrm{Tr}(M^n) = \lambda_p^n + \mu_2^n + \cdots + \mu_{p-1}^n = \lambda_p^n \left(1 + \sum_{i=2}^{p-1}\left(\frac{\mu_i}{\lambda_p}\right)^n\right). \tag{4.43}$$

We fix $n \neq p$. If a cycle of length n is not primitive, then it is a multiple of a primitive cycle of length k for some k dividing n with $k < n$. Therefore

$$\operatorname{Tr}(M^n) = N_n + \sum_{\substack{k|n \\ k\,<\,n}} N_k$$

(where $k|n$ means that k divides n). Moreover, $N_k \leq \operatorname{Tr}(M^k)$ because $\operatorname{Tr}(M^k)$ is the number of cycles of length k. Thus

$$N_n \geq \operatorname{Tr}(M^n) - \sum_{\substack{k|n \\ k\,<\,n}} \operatorname{Tr}(M^k). \tag{4.44}$$

Let k be an integer dividing n such that $k < n$. Necessarily, $k \leq n/2$ and thus, by (4.43), $\operatorname{Tr}(M^k) \leq (p-1)\lambda_p^k \leq (p-1)\lambda_p^{n/2}$. Combining this with (4.44) and (4.43), we get

$$\begin{aligned} N_n &\geq \lambda_p^n \left(1 + \sum_{i=2}^{p-1} \left(\frac{\mu_i}{\lambda_p}\right)^n\right) - (p-1)\frac{n}{2}(\lambda_p)^{n/2} \\ &\geq \lambda_p^n \left(1 + \sum_{i=2}^{p-1} \left(\frac{\mu_i}{\lambda_p}\right)^n - (p-1)\frac{n}{2}(\lambda_p)^{-n/2}\right). \end{aligned}$$

Since $\lambda_p > 1$ and $|\mu_i| < \lambda_p$ for all $i \in [\![2, p-1]\!]$, we have

$$\lim_{n\to+\infty} \left(\frac{\mu_i}{\lambda_p}\right)^n = 0 \quad \text{and} \quad \lim_{n\to+\infty} \frac{n}{2}(\lambda_p)^{-n/2} = 0,$$

so

$$\liminf_{n\to+\infty} \frac{1}{n} \log N_n \geq \log \lambda_p.$$

Thus, by (4.42)

$$\liminf_{n\to+\infty} \frac{1}{n} \log \#\{x \in I \mid x \text{ periodic point of period } n\} \geq \log \lambda_p. \tag{4.45}$$

We now suppose that f has a periodic point of period $2^d q$ with $d \geq 0$ and $q > 1$, q odd. Then the map $g := f^{2^d}$ has a periodic point of period q (Lemma 3.18(i)). Let p be the minimal odd period greater than 1 for g. Since a periodic point of period $2^d n$ for f is a periodic point of period n for g, (4.45) implies that

$$\liminf_{n\to+\infty} \frac{1}{n} \log \#\{x \in I \mid x \text{ periodic point of period } 2^d n \text{ for } f\} \geq \log \lambda_p.$$

Moreover $\lambda_p \geq \lambda_q$ by Lemma 4.59. This concludes the proof. □

Remarks on graph maps. Llibre and Misiurewicz showed that Proposition 4.63 is also valid for graph maps [**114**]. The technique is similar.

For graph maps, there exist conditions equivalent to positive entropy in terms of sets of periods, but they cannot be expressed in such a simple dichotomy as the equivalence (i)⇔(ii) in Theorem 4.58.

Optimal lower bounds on entropy are known for circle maps, in the same vein as Theorem 4.60. The results for circle maps of degree different from 1 are mainly due to Block, Guckenheimer, Misiurewicz and Young [**36**]. When the degree is 0 or −1, one essentially has the same results as for interval maps. Several papers deal with entropy of circle maps of degree 1. In particular, Ito gave an optimal lower bound on entropy when there exist two periods $p, q > 1$ such that $\gcd(p, q) = 1$ [**90**]. The lower bound stated below in Theorem 4.67, which is the most precise one, depends on the rotation interval; it is due to Alsedà, Llibre, Mañosas and Misiurewicz [**12**]. The reader is advised to refer to [**14**, Section 4.7] for an extensive exposition on circle maps. Recall that the possible sets of periods of circle maps were given in Theorems 3.25 and 3.26.

Proposition 4.65. *Let $f: \mathbb{S} \to \mathbb{S}$ be a circle map of degree d with $|d| \geq 2$. Then f admits a $|d|$-horseshoe and $h_{top}(f) \geq \log |d|$.*

Proposition 4.66. *Let $f: \mathbb{S} \to \mathbb{S}$ be a circle map of degree 0 or -1. The following assertions are equivalent:*
- *$h_{top}(f) > 0$.*
- *f has a periodic point whose period is not a power of 2.*

Moreover, if a lifting of f has a periodic point of period $2^d q$ with $d \geq 0$, $q > 1$, q odd, then $h_{top}(f) \geq \frac{\lambda_q}{2^d}$, where λ_q is the unique positive root of $X^q - 2X^{q-2} - 1$.

Theorem 4.67. *Let $f: \mathbb{S} \to \mathbb{S}$ be a circle map of degree 1, and let $[a, b]$ be its rotation interval. The following conditions are equivalent:*
- *$h_{top}(f) > 0$.*
- *There exists two integers m, n with $1 < n < m$ such that f has two periodic points of periods n, m respectively and m/n is not an integer.*
- *Either $a < b$, or there exist $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $\gcd(p, q) = 1$ such that $a = b = p/q$ and f has a periodic point whose period is not of the form $2^d q$, $d \geq 0$.*

Moreover, if $a < b$, then $h_{top}(f) \geq \log \beta_{a,b}$, where $\beta_{a,b}$ is the largest root of

$$\sum_{(p,q)\in\mathbb{Z}\times\mathbb{N},\, \frac{p}{q}\in(a,b)} t^{-q} - \frac{1}{2},$$

and for all real numbers $a < b$, there exists a circle map of degree 1 and topological entropy $\log \beta_{a,b}$.

The set of periods implied by the existence of a given periodic orbit depends on its *pattern*, that is, the relative position of the points within the orbit, and not only on its period. For an interval map f, a periodic orbit P *has a division* if there exists a point $y \notin P$ such that,

$$\forall x \in P,\ x < y \Rightarrow f(x) > y \text{ and } x > y \Rightarrow f(x) < y.$$

A periodic orbit of odd period $p > 1$ has clearly no division; this fact is important (although hidden) when proving that an odd period greater than 1 implies a cofinite set of periods (a subset of $\mathbb{N}$ is cofinite if it contains all but at most finitely many integers), which is a part of Sharkovsky's Theorem 3.13. On the other hand, an interval map can have a periodic orbit with a division but a set of periods which is not cofinite (e.g., the set of all even integers and 1). The notion of division was extended to tree maps by Alsedà and Ye, and led to the following results [**19, 172**]. Since the definition of division for tree maps is more technical than for interval maps, we do not give it here and we refer the interested readers to the cited papers. See also Blokh's paper [**54**] for results about periods and entropy of tree maps.

Theorem 4.68. *Let $f: T \to T$ be a tree map. The following assertions are equivalent:*

- $h_{top}(f) > 0$.
- *There exists* $n \in \mathbb{N}$ *such that* f^n *has a periodic orbit of period greater than* 1 *with no division.*

Moreover, if f *has a periodic orbit with no division, then* $h_{top}(f) \geq \frac{1}{e(T)} \log 2$*, where* $e(T)$ *denotes the number of endpoints of* T*.*

Finally, the next theorem holds for any graph map. It was first shown by Blokh by the means of spectral decomposition [**53**], then Llibre and Misiurewicz gave a more direct proof [**114**].

THEOREM 4.69. *Let* f *be a graph map. The following assertions are equivalent:*

- $h_{top}(f) > 0$.
- *There exists* $p \in \mathbb{N}$ *such that the set of periods of* f *contains* $p\mathbb{N}$*.*

4.7. Entropy of transitive and topologically mixing maps

A transitive interval map always has a positive entropy. Moreover, this entropy can be uniformly bounded from below. The lower bound of entropy of transitive interval maps (resp. transitive interval maps with two fixed points) is classical, as well as the examples realizing the minimum. Entropy of topologically mixing interval maps can also be bounded from below, but the infimum is not reached.

In the next proposition, statement (i) was first stated by Blokh [**44**] (see [**55**] for the proof) and was also proved by Block and Coven [**34**]; statement (ii) was shown by Block and Coven [**34**]; statement (iii) follows from a result of Bobok and Kuchta [**56**] (it can also be seen as a consequence of Theorem 2.20 and Lemma 4.59). Notice that, in (ii), a transitive interval map with two fixed points is necessarily topologically mixing by Theorem 2.19.

PROPOSITION 4.70. *Let* $f\colon I \to I$ *be an interval map.*

i) *If* f *is transitive, then* $h_{top}(f) \geq \frac{\log 2}{2}$*.*
ii) *If* f *is transitive and has at least two fixed points, then* $h_{top}(f) \geq \log 2$*.*
iii) *If* f *is topologically mixing, then* $h_{top}(f) > \frac{\log 2}{2}$*.*

PROOF. First we prove (iii). Suppose that f is topologically mixing. Then f has a periodic point of odd period greater than 1 by Theorem 2.20. According to Proposition 3.32, there exist two intervals $J = [a, b]$ and $K = [c, d]$ with $b < c$, $a \neq \min I$, $d \neq \max I$ and such that (J, K) is a strict horseshoe for f^2. We set $A := [a, d]$ and $L := [b, c]$. By the intermediate value theorem, $f^2(J) \supset A$ because $f^2(J) \supset J \cup K$. Similarly, $f^2(K) \supset A$. The map f is topologically mixing and the non degenerate closed interval L does not contains the endpoints of I, thus there exists a positive integer n such that $f^{2n}(L) \supset A$ by Theorem 2.20. Applying Lemma 1.13(iii) to the family of chains of intervals

$$\{(I_0, \dots, I_{n-1}, A) \mid \forall i \in [\![0, n-1]\!], I_i \in \{J, K\}\},$$

we see that there exist 2^n closed intervals $(L_i)_{1 \leq i \leq 2^n}$ with pairwise disjoint interiors such that $L_i \subset J \cup K$ and $f^{2n}(L_i) \supset A = J \cup L \cup K$ for all $i \in [\![1, 2^n]\!]$. We deduce that $(L_1, L_2, \dots, L_{2^n}, L)$ is a $(2^n + 1)$-horseshoe for f^{2n}. Thus, by Proposition 4.6,

$$h_{top}(f) = \frac{1}{2n} h_{top}(f^{2n}) \geq \frac{\log(2^n + 1)}{2n} > \frac{\log 2}{2}.$$

This is (iii).

Now we suppose that f is transitive. If f is topologically mixing, then it follows from (iii) that $h_{top}(f) \geq \frac{\log 2}{2}$. If f is transitive but not topologically mixing, then, according to Theorem 2.19, there exists a fixed point c in the interior of I such that, if we set $J := [\min I, c]$ and $K := [c, \max I]$, then both maps $f^2|_J$ and $f^2|_K$ are topologically mixing. The point c is also fixed for the map $f^2|_J$, and c is not in the interior of J. Therefore, $f^2|_J$ has a horseshoe by Lemma 3.35, and hence $h_{top}(f^2) \geq \log 2$ by Proposition 4.6. Thus $h_{top}(f) = \frac{1}{2} h_{top}(f^2) \geq \frac{\log 2}{2}$, which gives (i).

Finally, (ii) follows straightforwardly from Lemma 3.35 and Proposition 4.6. $\square$

The bounds given in the preceding proposition are sharp. In Example 4.71 below, two maps realizing respectively the equalities in Proposition 4.70(i)-(ii) are exhibited. In Example 4.61, we saw that, for every odd integer $p > 1$, there exists a topologically mixing map f_p whose entropy is equal to $\log \lambda_p$, where λ_p is the unique positive root of $X^p - 2X^{p-2} - 1$. According to Lemma 4.59, $\lim_{p \to +\infty} \lambda_p = \sqrt{2}$. Combining this with Proposition 4.70(iii), this shows that

$$\inf\{h_{top}(f) \mid f \text{ topologically mixing interval map}\} = \frac{\log 2}{2}.$$

EXAMPLE 4.71. We are going to exhibit a transitive map S of topological entropy $\frac{\log 2}{2}$ and a transitive map T_2 with two fixed points of topological entropy $\log 2$. We define $T_2\colon [0,1] \to [0,1]$ and $S\colon [-1,1] \to [-1,1]$ by

$$\begin{cases} \forall x \in [0, \frac{1}{2}], & T_2(x) := 2x, \\ \forall x \in [\frac{1}{2}, 1], & T_2(x) := 2(1-x), \end{cases} \qquad \begin{cases} \forall x \in [-1, -\frac{1}{2}], & S(x) := 2x+2, \\ \forall x \in [-\frac{1}{2}, 0], & S(x) := -2x, \\ \forall x \in [0,1], & S(x) := -x. \end{cases}$$

These two maps are represented in Figure 6. See also Figure 2 page 20 for the graph of S^2.

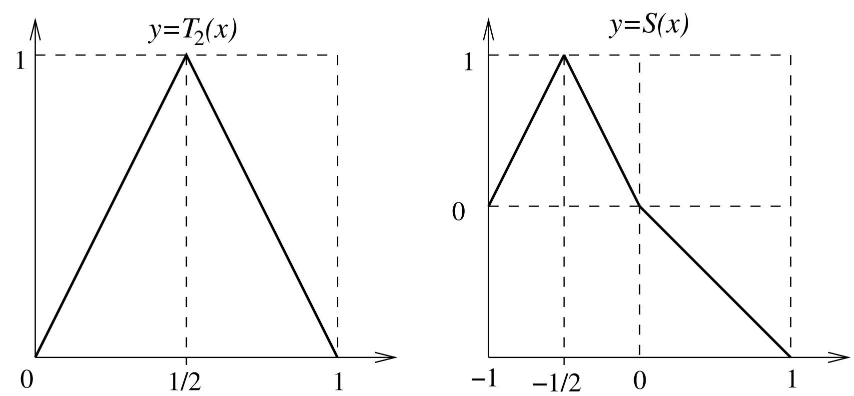


FIGURE 6. The map on the left (the *tent map*) is transitive with two fixed points and its topological entropy is $\log 2$. The map on the right is transitive with a unique fixed point and its topological entropy is $\frac{\log 2}{2}$.

It was proved in Example 2.13 that T_2 is topologically mixing. Since T_2 is 2-Lipschitz, its topological entropy is less than or equal to $\log 2$ by Proposition 4.29.

Moreover, T_2 has two fixed points (0 and $\frac{2}{3}$), and thus $h_{top}(T_2) \geq \log 2$ by Proposition 4.70(ii). Consequently, $h_{top}(T_2) = \log 2$.

The map S was proved to be transitive in Example 2.21. Thus $h_{top}(S) \geq \frac{\log 2}{2}$ by Proposition 4.70(i). Moreover, S^2 is 2-Lipschitz, and thus $h_{top}(S^2) \leq \log 2$ by Proposition 4.29. We deduce that $h_{top}(S) = \frac{\log 2}{2}$.

The two maps in the preceding example have a common property: they are P-linear for some finite invariant set P. Coven and Hidalgo proved that a transitive interval map f satisfying $h_{top}(f) = h_{top}(f_P)$ for some finite f-invariant set P is necessarily P-monotone [**69**]. This implies that there is little freedom for maps realizing the bounds in Proposition 4.70(i)-(ii). In particular, Bobok and Kuchta showed that there is a unique transitive interval map of entropy $\frac{\log 2}{2}$, up to conjugacy [**56**, Theorem 4.1].

Before proving these results, we are going to show that a transitive map f satisfying $h_{top}(f) = h_{top}(f_P)$ cannot have non accessible endpoints. We shall use the next lemma, which is an easy corollary of the Perron-Frobenius Theorem 4.44, several times.

Lemma 4.72. *Let B be a positive $n \times n$ matrix. Let E be a nonempty subset of $[\![1, n]\!]$ and let B' denote the matrix obtained from B by removing the rows and the columns with indices $i \in E$. Then $\lambda(B) > \lambda(B')$.*

Proof. It is sufficient to prove the lemma for $E = \{1\}$. We set

$$A := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{pmatrix},$$

that is, A is the matrix obtained from B by filling the first line and the first column of B with 0's. Then $0 \leq A \leq B$ and $A \neq B$ because $B > 0$. Thus $\lambda(A) < \lambda(B)$ by the Perron-Frobenius Theorem 4.44(ii). Moreover, the set of eigenvalues of A is equal to the set of eigenvalues of B' union $\{0\}$, and hence $\lambda(B') = \lambda(A) < \lambda(B)$. $\square$

Lemma 4.73. *Let $f\colon [a, b] \to [a, b]$ be a topologically mixing interval map with a non accessible endpoint.*

i) *For every finite invariant set P, $h_{top}(f) > h_{top}(f_P)$.*
ii) *If f has a horseshoe, then $h_{top}(f) > \log 2$.*

Proof. Let E denote the set of non accessible endpoints of f. By Proposition 2.30, there are four cases: either $E = \{a\}$ and $f(a) = a$, or $E = \{b\}$ and $f(b) = b$, or $E = \{a, b\}$ and both a, b are fixed, or $E = \{a, b\}$ and $f(a) = b, f(b) = a$.

Let $P = \{p_0 < p_1 < \cdots < p_k\}$ be a finite invariant set. By Proposition 4.52, $h_{top}(f_P) = \max(0, \log \lambda(M(f_P)))$. It is sufficient to consider the case $\lambda(M(f_P)) > 1$ because $h_{top}(f) > 0$ by Proposition 4.70. The set $P' := P \setminus E$ is invariant too. First we are going to show

$$\lambda(M(f_P)) = \lambda(M(f_{P'})). \tag{4.46}$$

We split the proof into four cases.

Case $\boldsymbol{P \cap E = \emptyset}$. Then $P = P'$.

Case $\boldsymbol{P \cap E = \{a\}}$. Then $f(a) = a = p_0$, and the matrix $M(f_{P'})$ is obtained from $M(f_P)$ by deleting the first row and the first column. Moreover, a does not belong to $f([p_i, p_{i+1}])$ for any $i \in [\![1, k-1]\!]$, which implies that the first column of $M(f_P)$ is ${}^t(10\cdots 0)$. Then $\det(M(f_P) - X\mathrm{Id}) = (1-X)\det(M(f_{P'}) - X\mathrm{Id})$. Since $\lambda(M(f_P)) > 1$, we have $\lambda(M(f_P)) = \lambda(M(f_{P'}))$ by Proposition 4.38.

Case $\boldsymbol{P \cap E = \{b\}}$. This case is similar to the previous one.

Case $\boldsymbol{P \cap E = \{a, b\}}$. Either both a and b are fixed or $\{a, b\}$ is a periodic orbit of period 2. In both cases, $f^2(a) = a = p_0$ and $f^2(b) = b = p_k$. We have $\{a, b\} \cap f([p_i, p_{i+1}]) = \emptyset$ for all $i \in [\![1, k-2]\!]$. This implies that, for all $i \in [\![1, k-2]\!]$, there is no path of length 2 from $[p_i, p_{i+1}]$ to $[a, p_1]$ or to $[p_{k-1}, b]$ in $G(f_P)$, and thus the first and last columns of $M(f_P)^2$ are respectively ${}^t(10\cdots 0)$ and ${}^t(0\cdots 01)$ (recall that, by Proposition 1.16, the (i, j)-coefficient of $M(f_P)^n$ is the number of paths of length n from $[p_i, p_{i+1}]$ to $[p_j, p_{j+1}]$ in $G(f_P)$). Moreover, the matrix $M(f_{P'})^2$ is obtained from $M(f_P)^2$ by deleting the first and last rows and the first and last columns. As in the second case, this implies that $\lambda(M(f_P)^2) = \lambda(M(f_{P'})^2)$, so $\lambda(M(f_P)) = \lambda(M(f_{P'}))$ by Lemma 4.41.

Now we are going to show that $h_{top}(f) > \log \lambda(M(f_P))$. According to (4.46), we can assume that $P \cap E = \emptyset$ (otherwise, we replace P by P'). Moreover, since $E \neq \emptyset$, we can assume that $a \in E$ (the case $b \in E$ is similar). If $b \in E$, then $f^2(b) = b$ by Lemma 2.32. If $b \notin E$, then there exists $x \in (a, b)$ such that $f^n(x) = b$, which implies that $f^2(b) \neq a$, otherwise a would be accessible. In both cases, $f^2(b) \neq a$. We set $c := \min f^2([p_0, b])$. Then $c > a$ because a is non accessible and $f^2(b) \neq a$, and $c < p_0$ because $[p_0, b]$ is not f^2-invariant (otherwise, it would contradict the fact that f is topologically mixing). If $b \in E$, we set $d := \max f^2([a, p_k])$, and a similar argument implies that $p_k < d < b$. If $b \notin E$, we set $d := b$. Let $Q := P \cup \{c, d\}$ (this may not be an invariant set). Then $c = \min Q$ (because $f^2([p_0, b]) \supset f^2([p_0, p_k]) \supset P$), and similarly $d = \max Q$. According to Proposition 2.30, there exists a positive integer n such that, for all Q-intervals J, $f^n(J) \supset [c, d]$. Thus the matrix $B := M(f^n|Q)$ is positive. We remove from B its first line and column (corresponding to the Q-interval $[c, p_0]$) and, if $b \in E$, its last line and column (corresponding to $[p_k, d]$); we call B' the resulting matrix. Then $B' = M(f^n|P)$. By Proposition 4.46, $h_{top}(f^n) \geq \log \lambda(B)$, and by Lemma 4.72, $\lambda(B) > \lambda(B')$. Moreover, $B' \geq M(f_P)^n$ by Lemma 4.51, which implies that $\lambda(B') \geq \lambda(M(f_P)^n)$ by the Perron-Frobenius Theorem 4.44. Combining these inequalities with the fact that $\lambda(M(f_P)^n) = (\lambda(M(f_P)))^n$ (Lemma 4.41), we get

$$h_{top}(f) = \frac{1}{n} h_{top}(f^n) \geq \frac{1}{n} \log \lambda(B) > \frac{1}{n} \log \lambda(B') \geq \log \lambda(M(f_P)).$$

This proves (i).

Now we assume that f has a horseshoe. According to Lemma 3.29, there exist three points u, v, w in $[a, b]$ such that $f(u) = f(w) = u$, $f(v) = w$ and, either $u < v < w$, or $u > v > w$. The set $P := \{u, v, w\}$ is invariant and $M(f_P)$ is the matrix of a 2-horseshoe:

$$M(f_P) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus $\lambda(M(f_P)) = 2$ and $h_{top}(f_P) = \log \lambda(M(f_P)) = \log 2$ (Proposition 4.52). Since $h_{top}(f) > h_{top}(f_P)$ by (i), we get $h_{top}(f) > \log 2$, which gives (ii). $\square$

PROPOSITION 4.74. *Let $f\colon [a,b]\to[a,b]$ be a transitive interval map and P a finite invariant set. If $h_{top}(f)=h_{top}(f_P)$, then f is P-monotone.*

PROOF. Recall that $h_{top}(f)>0$ because f is transitive (Proposition 4.70), and thus $h_{top}(f)=\log\lambda(M(f_P))$ by Proposition 4.52.

The first step of the proof consists of showing that the endpoints a,b belong to P. Suppose that f is topologically mixing. If $a\notin P$, we set $Q:=P\cup\{a\}$. According to Lemma 4.73(i), f has no non accessible endpoint and thus, by Proposition 2.30, there exists an integer $n\geq 0$ such that, for all Q-intervals J, $f^n(J)=[a,b]$. Thus the matrix $B:=M(f^n|Q)$ is positive. Let B' be the matrix obtained from B by removing the first line and the first column, that is, $B'=M(f^n|P)$. By Lemma 4.72, $\lambda(B)>\lambda(B')$. Moreover, $B'\geq M(f_P)^n$ by Lemma 4.51, and hence $\log\lambda(B')\geq\log\lambda(M(f_P)^n)=n\log\lambda(M(f_P))$ by the Perron-Frobenius Theorem 4.44(ii) and Lemma 4.41. Furthermore, $h_{top}(f^n)\geq\log\lambda(B)$ according to Proposition 4.46. These inequalities imply $h_{top}(f)>\log\lambda(M(f_P))$, a contradiction. We deduce that $a\in P$. Similarly, b belongs to P too. We have proved:

$$h_{top}(f)=h_{top}(f_P)\text{ and } f \text{ topologically mixing} \Rightarrow a,b\in P. \tag{4.47}$$

Suppose now that f is transitive but not topologically mixing. According to Theorem 2.19, there exists $c\in(a,b)$ such that

$$f(c)=c,\ f([a,c])=[c,b],\ f([c,b])=[a,c],\ \text{and } f^2|_{[a,c]},\ f^2|_{[c,b]}\text{ are mixing.} \tag{4.48}$$

If $c\notin P$, we set $P':=P\cup\{c\}$ and we consider $p:=\max(P\cap[a,c))$ and $q:=\min(P\cap(c,b])$ (these points exist by (4.48) and the f-invariance of P). According to (4.48), $f(p)\geq c>p$ and $f(q)\leq c<q$. This implies that $f_{P'}$ is decreasing on $[p,c]\cup[c,q]=[p,q]$. Thus $f_{P'}$ is P-monotone and, according to Proposition 4.52, $h_{top}(f_P)=h_{top}(f_{P'})$. If we prove the proposition for P', this will imply that the proposition holds for P too. Therefore, we may assume that $c\in P$ (otherwise we replace P by P'). We set $P_1:=P\cap[a,c]$, $P_2:=P\cap[c,b]$ and $g:=f^2$. Then the family of P-intervals splits into P_1-intervals and P_2-intervals.

One can show that $h_{top}(g_P)=h_{top}(g_{P_1})=h_{top}(g_{P_2})$ by using Bowen's formula (Theorem 4.5), the uniform continuity of f and the fact that f swaps the intervals $[a,c]$ and $[c,b]$ by (4.48). Moreover, (4.48) implies that $f_P^2=g_P$. Since $h_{top}(f)=h_{top}(f_P)$ by assumption, and $h_{top}(g)=2h_{top}(f)$, we get $h_{top}(g)=h_{top}(g_P)$ and

$$h_{top}(g)=h_{top}(g|_{[a,c]})=h_{top}(g|_{[c,b]})=h_{top}(g_{P_1})=h_{top}(g_{P_2}).$$

Moreover, according to Proposition 4.52,

$$\lambda(M(f_P)^2)=\lambda(M(g_{P_1}))=\lambda(M(g_{P_2})). \tag{4.49}$$

Applying (4.47) to $g|_{[a,c]}$ and $g|_{[c,b]}$, we see that $a,b\in P$. Moreover, f has no non accessible endpoint by Lemma 4.73 applied to $g|_{[a,c]}$ and $g|_{[c,b]}$. This concludes the proof of the first step, that is:

$$h_{top}(f)=h_{top}(f_P)\text{ and } f\text{ transitive}\Rightarrow a,b\in P.$$

In the second step, we are going to show that f is P-monotone. Suppose on the contrary that there exists a P-interval I such that $f|_I$ is not one-to-one, that is, there exist two points $u<v$ in I such that $f(u)=f(v)$. Since f is transitive, $f([u,v])$ is not degenerate, and thus

$$\begin{aligned}&\text{either }\max f([u,v])>f(u),\\ &\text{or }\min f([u,v])<f(u).\end{aligned}\tag{4.50}$$

We assume we are in case (4.50), the other case being similar. Let $w \in (u, v)$ be such that $f(w) = \max f([u, v])$, and let U be an open interval containing w such that $f(x) > f(v)$ for all $x \in U$.

By Proposition 2.15, the set of periodic points is dense, and thus we can choose a periodic point $p_0 \in U$ such that $w \notin \mathcal{O}_f(p_0)$. Let p_1, p_2 be the two points in $\mathcal{O}_f(p_0)$ such that $p_1 < w < p_2$ and there is no point of $\mathcal{O}_f(p_0)$ between w and p_i for $i \in \{1, 2\}$ and let $p \in \{p_1, p_2\}$ be such that $f(p) = \max\{f(p_1), f(p_2)\}$. It is possible that either p_1 or p_2 does not exist: if $\mathcal{O}_f(p_0) > w$ (resp. $\mathcal{O}_f(p_0) < w$), then there is no p_1 (resp. p_2); in this case p is just equal to the unique existing p_i. Since $p_0 \in U$, at least one of the points p_1, p_2 belongs to U, and hence $f(p) > f(v)$. In the sequel, we assume $p = p_1$, the case $p = p_2$ being symmetric. We define

$$z := \min\{x \in [w, v] \mid f(x) = f(p)\}.$$

This point is well defined because $f(v) < f(p) \leq f(w)$, and $z > p$. Moreover, $\mathcal{O}_f(p) \cap (w, z) = \emptyset$ according to the choice of p and z. The set $Q := P \cup \mathcal{O}_f(p) \cup \{z\}$ is a finite invariant set, and $[p, z]$ is a Q-interval. Moreover, f_Q is constant on $[p, z]$ because $f(p) = f(z)$, and thus the row of $M(f_Q)$ corresponding to $[p, z]$ is $(00 \cdots 0)$. Then, according to Proposition 4.53, $h_{top}(f_Q) \geq h_{top}(f_P)$ (because P is f_Q-invariant) and $h_{top}(f) \geq h_{top}(f_Q)$. Since $h_{top}(f) = h_{top}(f_P)$ by assumption, we have

$$h_{top}(f) = h_{top}(f_Q). \tag{4.51}$$

We split into two cases, depending on f being mixing or not.

• If f is topologically mixing, then, by Proposition 2.30, there exists an integer $n \geq 0$ such that, for all Q-intervals J, $f^n(J) = [a, b]$ (recall that we have shown that f has no non accessible endpoint). Therefore $B := M(f^n|Q)$ is a positive matrix. Moreover, $B \geq M(f_Q)^n$ (by Lemma 4.51) and $M(f_Q)^n \neq B$ because the row of $M(f_Q)^n$ corresponding to $[p, z]$ is $(0 \cdots 0)$. Thus $\lambda(B) > \lambda(M(f_Q)^n)$ by the Perron-Frobenius Theorem 4.44(ii).

• If f is transitive but not topologically mixing, then we are in the situation described in (4.48). As in the first step, we can assume that $c \in P$ and we set $g := f^2$. By Proposition 2.30, there exists an integer $k \geq 0$ such that, for all Q-intervals $J_1 \subset [a, c]$, $g^k(J_1) = [a, c]$, and for all Q-intervals $J_2 \subset [c, b]$, $g^k(J_2) = [c, b]$. Therefore the matrix $B := M(g^k|Q)$ is of the form

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \quad \text{with } B_1 > 0, B_2 > 0.$$

Moreover, $B \geq M(g_Q)^k$ (by Lemma 4.51) and

$$M(g_Q)^k = \begin{pmatrix} M(g_{P_1})^k & 0 \\ 0 & M(g_{P_2})^k \end{pmatrix}.$$

We deduce that $B_1 \geq M(g_{P_1})^k$ and $B_2 \geq M(g_{P_2})^k$. Moreover, the mixing case above implies that, if $[p, z] \subset [a, c]$ (resp. $[p, z] \subset [c, b]$), then $B_1 \neq M(g_{P_1})^k$ (resp. $B_2 \neq M(g_{P_2})^k$), so

$$\lambda(B_1) > \lambda(M(g_{P_1})^k) \quad (\text{resp. } \lambda(B_2) > \lambda(M(g_{P_2})^k))$$

by the Perron-Frobenius Theorem 4.44(ii). Combining this with (4.49), we get $\lambda(B) > \lambda(M(f_Q)^{2k})$.

In both cases (f topologically mixing or not), there exists $n \geq 0$ such that $\lambda(M(f^n|Q)) > \lambda(M(f_Q)^n)$. Recall that $\lambda(M(f_Q)^n) = \lambda(M(f_Q))^n$. This leads to

$$h_{top}(f) = \frac{1}{n} h_{top}(f^n) \geq \frac{1}{n} \log \lambda(M(f^n|Q)) > \log \lambda(M(f_Q)) = h_{top}(f_Q) = h_{top}(f)$$

(the last equality comes from (4.51)). But this is a contradiction. We conclude that f is one-to-one on every P-interval. Since a, b belong to P, we deduce that f is P-monotone. $\square$

In the next proposition, the first assertion is due to Bobok and Kuchta [**56**, Theorem 4.1].

Proposition 4.75. *Let $f \colon [a,b] \to [a,b]$ be a transitive interval map.*

- *If $h_{top}(f) = \frac{\log 2}{2}$, then f is topologically conjugate to the map S defined in Example 4.71.*
- *If f has at least two fixed points and $h_{top}(f) = \log 2$, then f is topologically conjugate to the tent map T_2 defined in Example 4.71.*

Proof. Let $f \colon [a,b] \to [a,b]$ be a transitive interval map such that $h_{top}(f) = \frac{\log 2}{2}$. By Proposition 4.70(iii), f is not topologically mixing. Thus, by Theorem 2.19. there exists $c \in (a,b)$ such that

$$f(c) = c,\ f([a,c]) = [c,b],\ f([c,b]) = [a,c], \text{ and } f^2|_{[a,c]},\ f^2|_{[c,b]} \text{ are mixing.} \tag{4.52}$$

Moreover, $f^2|_{[a,c]}$ has a horseshoe by Lemma 3.35. Thus $f^2|_{[a,c]}$ has no non accessible endpoint by Lemma 4.73(ii). This implies that there exists $d \neq c$ such that $f(d) = c$, that is,

$$\text{either } d \in [a,c) \text{ and } f(d) = c, \tag{4.53}$$

$$\text{or } d \in (c,b] \text{ and } f(d) = c. \tag{4.54}$$

Assume that (4.53) holds, the case (4.54) being symmetric. Let $m := \max f([d,c])$. Then $m \in [c,b]$ and $f([d,c]) = [c,m]$ by (4.52). Suppose that

$$\min f([c,m]) < d. \tag{4.55}$$

Then $f([c,m]) \supset [d,c]$, so $f^2([d,c]) = f([c,m]) \supset [d,c]$. Thus there exists e in $[d,c]$ such that $f^2(e) = d$. See the positions of these points in Figure 7.

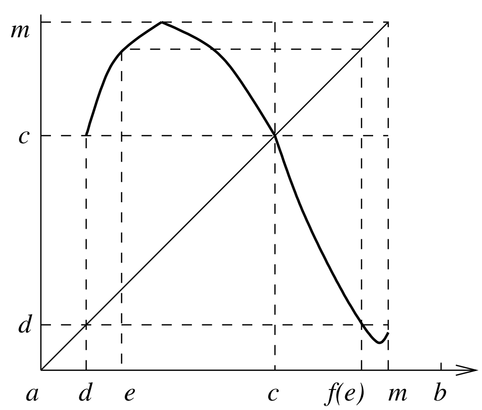


Figure 7. The positions of the various points in the case (4.55).

We set $P := \{d, e, c, f(e)\}$. Then P is invariant and $h_{top}(f_P) \geq \frac{\log 2}{2}$ because $([d, e], [e, c])$ is a horseshoe for f_P^2. Since $h_{top}(f_P) \leq h_{top}(f)$ by Proposition 4.53, we have $h_{top}(f_P) = h_{top}(f) = \frac{\log 2}{2}$. Thus f is P-monotone by Proposition 4.74, which implies that $d = a$. But this contradicts the fact that $\min f([c, m]) < d$. We deduce that (4.55) does not hold, that is, $\min f([c, m]) \geq d$. Then $f([c, m]) \subset [d, c]$ and $f([d, m]) = f([d, c]) \cup f([c, m]) \subset [c, m] \cup [d, c] = [d, m]$. Therefore the interval $[d, m]$ is invariant, which is possible only if $d = a$ and $m = b$ because f is transitive. Since f is onto, there exists $e' \in [c, b]$ such that $f(e') = a$. Let $e \in [a, c]$ be such that $f(e) = e'$. We set $Q := \{a, e, c, e'\}$. Then Q is invariant (recall that $a = d$ and $f(d) = f(c) = c$) and $h_{top}(f_Q) \geq \frac{\log 2}{2}$ because $([a, e], [e, c])$ is a horseshoe for f_Q^2. As above, Proposition 4.53 implies that $h_{top}(f_Q) = h_{top}(f) = \frac{\log 2}{2}$. Thus f is Q-monotone by Proposition 4.74. This implies that $e' = b$. The map f looks like the one represented on the left of Figure 8. Notice that the case (4.54) leads to the reverse figure (central symmetry) by exchanging the roles of a and b.

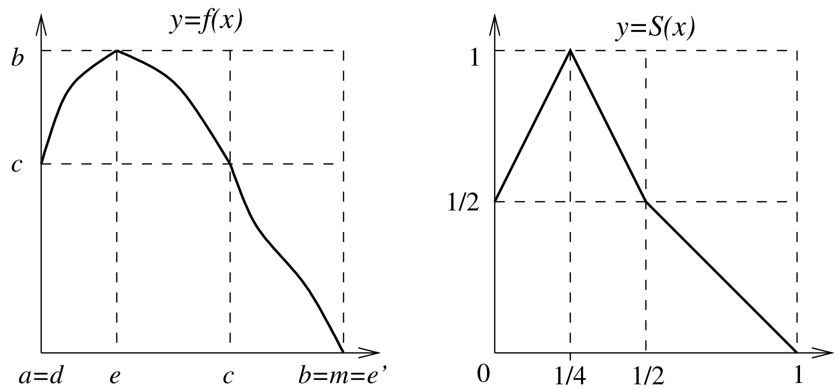


FIGURE 8. On the left, the map f is Q-monotone with $Q := \{a, e, c, b\}$. It is conjugate to the map S, on the right.

Now we consider $g \colon [a, b] \to [a, b]$ a transitive interval map with two fixed points such that $h_{top}(g) = \log 2$. Then g is topologically mixing by Theorem 2.19, and g has a horseshoe by Lemma 3.35. According to Lemma 3.29, there exist points $a' < c < b'$ such that

$$\text{either } g(a') = g(b') = a' \text{ and } g(c) = b', \tag{4.56}$$

$$\text{or } g(a') = g(b') = b' \text{ and } g(c) = a'. \tag{4.57}$$

Therefore, the set $P := \{a', c, b'\}$ is invariant, and $h_{top}(g_P) = \log 2 = h_{top}(g)$. Thus g is P-monotone by Proposition 4.74, which implies that $a' = a$ and $b' = b$. The map g looks like the one represented on the left of Figure 9 in case (4.56), and the reverse figure (central symmetry) in case (4.57).

It remains to show that f and g are topologically conjugate to S and T_2 respectively. This can be seen as a consequence of the following general result of Parry: every transitive piecewise monotone interval map is conjugate to a piecewise linear map such that the absolute value of its slope is constant [**141**]. This result is easier to prove for P-monotone maps. We are going to give a proof in the case of the map

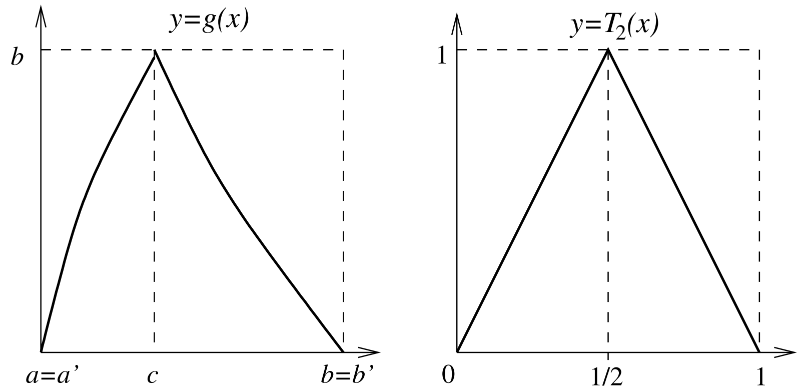


FIGURE 9. On the left, the map g is P-monotone with $P := \{a,c,b\}$. It is conjugate to the tent map T_2, on the right.

g, which is the simplest one. The conjugacy between f and S can be defined in a similar way.

We may assume that g satisfies (4.56) (otherwise, we conjugate g by $\psi\colon [a,b] \to [a,b], \psi(x) := b+a-x$ in order to get a map satisfying (4.56)). We set $J_0 := [a,c]$ and $J_1 := [c,b]$. Then, since g is P-monotone, g is increasing on J_0, decreasing on J_1, and $g(J_0) = g(J_1) = [a,b]$. For all $n \geq 1$ and all $(\alpha_0,\ldots,\alpha_{n-1}) \in \{0,1\}^n$, we define

$$J_{\alpha_0\ldots\alpha_{n-1}} := \{x \in [a,b] \mid \forall i \in [\![0,n-1]\!],\ g^i(x) \in J_{\alpha_i}\} = \bigcap_{i=0}^{n-1} g^{-i}(J_{\alpha_i}). \tag{4.58}$$

This definition implies that, for all $n \geq 2$ and all $(\alpha_0,\ldots,\alpha_{n-1}) \in \{0,1\}^n$,

$$g(J_{\alpha_0\ldots\alpha_{n-1}}) = J_{\alpha_1\ldots\alpha_{n-1}}. \tag{4.59}$$

FACT 1. *Let $n \geq 1$.*

i) $(J_{\alpha_0\ldots\alpha_{n-1}})_{(\alpha_0,\ldots,\alpha_{n-1})\in\{0,1\}^n}$ *is a cover of* $[a,b]$.
ii) $(J_{\alpha_0\ldots\alpha_{n-1}})_{(\alpha_0,\ldots,\alpha_{n-1})\in\{0,1\}^n}$ *is a family of nonempty compact intervals with pairwise disjoint interiors.*
iii) $g^n|_{J_{\alpha_0\ldots\alpha_{n-1}}}$ *is a homeomorphism from* $J_{\alpha_0\ldots\alpha_{n-1}}$ *to* $[a,b]$.

Assertion (i) follows straightforwardly from the fact that, for all $x \in [a,b]$ and all $i \in [\![0,n-1]\!]$, there is $\alpha_i \in \{0,1\}$ such that $g^i(x) \in J_{\alpha_i}$.

We prove (ii)-(iii) by induction on n.

• For $n = 1$, the sets are J_0, J_1, and (ii)-(iii) are satisfied.

• Suppose that (ii)-(iii) are satisfied for $n \geq 1$. For $i \in \{0,1\}$, the map $\widetilde{g}_i := g|_{J_i}$ is a homeomorphism from J_i to $[a,b]$. We can write $J_{\alpha_0\ldots\alpha_n}$ as

$$J_{\alpha_0\ldots\alpha_n} = \{x \in J_{\alpha_0} \mid g(x) \in J_{\alpha_1\ldots\alpha_n}\} = \widetilde{g}_{\alpha_0}^{-1}(J_{\alpha_1\ldots\alpha_n}).$$

This is a nonempty compact interval because $\widetilde{g}_{\alpha_0}^{-1}$ is continuous and $J_{\alpha_1\ldots\alpha_n}$ is a nonempty compact interval by the induction hypothesis.

Let $(\beta_0,\ldots,\beta_n) \neq (\alpha_0,\ldots,\alpha_n)$. If $\beta_0 \neq \alpha_0$, then

$$J_{\alpha_0\ldots\alpha_n} \subset J_{\alpha_0},\ J_{\beta_0\ldots\beta_n} \subset J_{\beta_0}$$

and $\mathrm{Int}\,(J_{\alpha_0}) \cap \mathrm{Int}\,(J_{\beta_0}) = \emptyset$. If $\beta_0 = \alpha_0$, then $(\beta_1, \dots, \beta_n) \neq (\alpha_1, \dots, \alpha_n)$,

$$J_{\alpha_0\dots\alpha_n} = \widetilde{g}_{\alpha_0}^{-1}(J_{\alpha_1\dots\alpha_n}),\ J_{\beta_0\dots\beta_n} = \widetilde{g}_{\alpha_0}^{-1}(J_{\beta_1\dots\beta_n})$$

and these sets have disjoint interiors because $\widetilde{g}_{\alpha_0}$ is a homeomorphism and by the induction hypothesis for $J_{\alpha_1\dots\alpha_n}, J_{\beta_1\dots\beta_n}$.

Finally, $g^n|_{J_{\alpha_0\dots\alpha_n}}$ is a homeomorphism from $J_{\alpha_0\dots\alpha_n} \subset J_{\alpha_0\dots\alpha_{n-1}}$ to its image by the induction hypothesis, and $g^n(J_{\alpha_0\dots\alpha_n}) = J_{\alpha_n}$ by (4.59). Thus

$$g^{n+1}|_{J_{\alpha_0\dots\alpha_n}} = \big(g|_{J_{\alpha_n}}\big) \circ \big(g^n|_{J_{\alpha_0\dots\alpha_n}}\big)$$

is a homeomorphism from $J_{\alpha_0\dots\alpha_n}$ to $[a,b]$. This ends the induction and the proof of Fact 1.

In a similar way, we define intervals $J'_{\alpha_0\dots\alpha_n}$ for T_2, starting with $J'_0 := [0, \frac{1}{2}]$ and $J'_1 := [\frac{1}{2}, 1]$; at level $n-1$, we get the cover $\big(\big[\frac{i}{2^n}, \frac{i+1}{2^n}\big]\big)_{0\leq i<2^n}$. The idea is to build a map $\varphi\colon [a,b] \to [0,1]$ such that the image of every interval of the form $J_{\alpha_0\dots\alpha_{n-1}}$ is $J'_{\alpha_0\dots\alpha_{n-1}}$.

We set $R_{\alpha_0\dots\alpha_{n-1}} := J_{\alpha_0\dots\alpha_{n-1}} \setminus \min J_{\alpha_0\dots\alpha_{n-1}}$ (half-open interval). For all $n \geq 1$, we define the staircase function $\varphi_n\colon [a,b] \to [0,1]$ by:

- $\varphi_n(a) = 0$, $\varphi_n(b) = 1$,
- $\forall (\alpha_0, \dots, \alpha_{n-1}) \in \{0,1\}^n$, φ_n is constant on $R_{\alpha_0\dots\alpha_{n-1}}$,
- φ_n is non decreasing and every step is of high $\frac{1}{2^n}$

(see Figure 10).

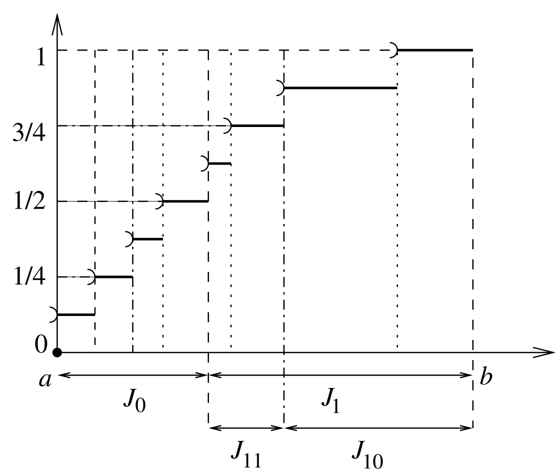


FIGURE 10. The map φ_3: the interval $[a,b]$ is divided into $2^3 = 8$ subintervals $(J_{\alpha_0\alpha_1\alpha_2})_{(\alpha_0,\alpha_1,\alpha_2)\in\{0,1\}^3}$, φ_3 is a non decreasing staircase function and takes its values in $\{\frac{i}{8} \mid i \in [\![0,8]\!]\}$.

FACT 2. *The sequence $(\varphi_n)_{n\geq 1}$ uniformly converges to a map $\varphi\colon [a,b] \to [0,1]$. Moreover, φ is an increasing homeomorphism.*

First we show that $(\varphi_n)_{n\geq 0}$ is a Cauchy sequence for the uniform distance. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \varepsilon$. Let $n > m \geq N$. Note that $\varphi_n(a) = \varphi_m(a) = 0$. Let $x \in (a,b]$. There exists $(\alpha_0, \dots \alpha_{n-1}) \in \{0,1\}^n$ such that x belongs to $R_{\alpha_0\dots\alpha_{n-1}} \subset R_{\alpha_0\dots\alpha_{m-1}}$, and there is $i \in [\![1, 2^m]\!]$ such that φ_m is equal to $\frac{i}{2^m}$

on $R_{\alpha_0\ldots\alpha_{m-1}}$. Moreover, φ_n is equal to a constant $c \in (\frac{i-1}{2^m}, \frac{i}{2^m}]$ on $R_{\alpha_0\ldots\alpha_{n-1}}$ by construction. Therefore

$$\forall x \in [a,b],\ |\varphi_n(x) - \varphi_m(x)| \leq \frac{1}{2^m} < \varepsilon. \tag{4.60}$$

This shows that $(\varphi_n)_{n\geq 0}$ is a Cauchy sequence, and thus it uniformly converges to a map $\varphi\colon [a,b] \to [0,1]$. We are going to show that φ is continuous, onto and increasing.

In (4.60), we take $n \to +\infty$. We get

$$\forall x \in [a,b],\ |\varphi(x) - \varphi_m(x)| < \varepsilon. \tag{4.61}$$

If $x \in \operatorname{Int}\left(R_{\alpha_0\ldots\alpha_{m-1}}\right)$, there exists a neighborhood U of x such that $U \subset R_{\alpha_0\ldots\alpha_{m-1}}$, and thus (4.61) implies that, for all $y \in U$,

$$\begin{array}{rcl} |\varphi(x) - \varphi(y)| & \leq & |\varphi(x) - \varphi_m(x)| + |\varphi_m(x) - \varphi_m(y)| + |\varphi_m(y) - \varphi(y)| \\ & < & \varepsilon + 0 + \varepsilon = 2\varepsilon. \end{array}$$

If $x = \max R_{\alpha_0\ldots\alpha_{m-1}}$ (resp $x = a$), there exists $(\beta_0, \ldots, \beta_{m-1}) \in \{0,1\}^m$ and a neighborhood U of x such that $U \subset R_{\alpha_0\ldots\alpha_{m-1}} \cup R_{\beta_0\ldots\beta_{m-1}}$ (resp. $U \subset \{a\} \cup R_{\beta_0\ldots\beta_{m-1}}$). There exists $i \in [\![1, 2^m]\!]$ such that φ_m takes only the values $\frac{i-1}{2^m}$ and $\frac{i}{2^m}$ on U. Thus (4.61) implies that, for all $y \in U$,

$$\begin{array}{rcl} |\varphi(x) - \varphi(y)| & \leq & |\varphi(x) - \varphi_m(x)| + |\varphi_m(x) - \varphi_m(y)| + |\varphi_m(y) - \varphi(y)| \\ & < & \varepsilon + \dfrac{1}{2^m} + \varepsilon < 3\varepsilon. \end{array}$$

This proves that φ is continuous on $[a,b]$. By definition of $(\varphi_n)_{n\geq 1}$, $\varphi(a) = 0$ and $\varphi(b) = 1$, which implies that the map φ is onto. Moreover, φ is non decreasing because, if $x \leq y$, then $\varphi_n(x) \leq \varphi_n(x)$ for all n, which implies $\varphi(x) \leq \varphi(x)$.

Now we are going to show that φ is increasing. For every $\bar\alpha = (\alpha_n)_{n\geq 0} \in \{0,1\}^{\mathbb{Z}^+}$, we set

$$J_{\bar\alpha} := \bigcap_{n=1}^{+\infty} J_{\alpha_0\ldots\alpha_{n-1}}.$$

This is a decreasing intersection of nonempty compact intervals, so $J_{\bar\alpha}$ is a nonempty compact interval. According to the definition of φ, the map φ is constant on an interval J if and only if $J \subset J_{\bar\alpha}$ for some $\bar\alpha \in \{0,1\}^{\mathbb{Z}^+}$. Moreover, (4.59) implies that $g^n(J_{\bar\alpha}) \subset J_{\alpha_n}$ for all $n \geq 1$. On the other hand, for every non degenerate interval J, there exists $n \geq 1$ such that $g^n(J) = [a,b]$ because g is topologically mixing and has no non accessible endpoint (Proposition 2.30). This implies that $J_{\bar\alpha}$ is degenerate, hence reduced to a single point for every $\bar\alpha \in \{0,1\}^{\mathbb{Z}^+}$. The map φ is non decreasing and it is non constant on any non degenerate interval. Thus φ is increasing, which implies that φ is a homeomorphism. This concludes the proof of Fact 2.

FACT 3. *For all* $(\alpha_0, \ldots, \alpha_{n-1}) \in \{0,1\}^n$ $(n \geq 1)$, $\varphi(J_{\alpha_0\ldots\alpha_n}) = J'_{\alpha_0\ldots\alpha_n}$.

We fix $n \geq 1$ and $(\alpha_0, \ldots, \alpha_{n-1}) \in \{0,1\}^n$. By construction of the sequence of maps $(\varphi_m)_{m\geq 1}$,

$$\forall m \geq n,\ \varphi_m(R_{\alpha_0\ldots\alpha_{n-1}}) = J'_{\alpha_0\ldots\alpha_n} \setminus \min J'_{\alpha_0\ldots\alpha_n}.$$

Taking the limit when m tends to infinity, we get

$$\varphi(R_{\alpha_0\ldots\alpha_{n-1}}) = J'_{\alpha_0\ldots\alpha_n} \setminus \min J'_{\alpha_0\ldots\alpha_n}.$$

Since φ is continuous and increasing by Fact 2, φ sends $\inf R_{\alpha_0\dots\alpha_{n-1}}$ to $\min J'_{\alpha_0\dots\alpha_n}$, and hence $\varphi(J_{\alpha_0\dots\alpha_n}) = J'_{\alpha_0\dots\alpha_n}$.

It remains to show that φ is a conjugacy between g and T_2, that is, $T_2 = \varphi\circ g\circ\varphi^{-1}$. First note that $\varphi\circ g\circ\varphi^{-1}(0) = \varphi\circ g\circ\varphi^{-1}(1) = 0$, $\varphi\circ g\circ\varphi^{-1}(\frac{1}{2}) = 1$ and $\varphi\circ g\circ\varphi^{-1}$ is increasing on $[0, \frac{1}{2}]$ (resp. decreasing on $[\frac{1}{2}, 1]$) because $g(a) = g(b) = a$, $g(c) = b$ and g is increasing on $J_0 = [a, c]$ (resp. decreasing on $J_1 = [c, b]$). Let $(\alpha_0, \dots, \alpha_{n-1}) \in \{0, 1\}^n$. Then $J'_{\alpha_0\dots\alpha_{n-1}}$ is an interval of length $\frac{1}{2^n}$. Moreover, Fact 3 and (4.59) imply that $J'_{\alpha_0\dots\alpha_{n-1}} = \varphi(J_{\alpha_0\dots\alpha_{n-1}})$ and

$$\varphi\circ g\circ\varphi^{-1}(J'_{\alpha_0\dots\alpha_{n-1}}) = \varphi\circ g(J_{\alpha_0\dots\alpha_{n-1}}) = \varphi(J_{\alpha_1\dots\alpha_{n-1}}) = J'_{\alpha_1\dots\alpha_{n-1}}. \tag{4.62}$$

Thus

$$|\varphi\circ g\circ\varphi^{-1}(J'_{\alpha_0\dots\alpha_{n-1}})| = \frac{1}{2^{n-1}} = 2|J'_{\alpha_0\dots\alpha_{n-1}}|. \tag{4.63}$$

Now we consider a point $x_0 \in \partial J'_{\alpha_0\dots\alpha_{n-1}}$. If $\alpha_0 = 0$ (which implies that $x_0 \in [0, \frac{1}{2}]$), then $[0, x_0]$ is the union of intervals of the form $J'_{0\beta_1\dots\beta_{n-1}}$, and the length of $[0, x_0]$ is equal to the sum of the lengths of these intervals. We have $\varphi\circ g\circ\varphi^{-1}(J'_{0\beta_1\dots\beta_{n-1}}) = J'_{\beta_1\dots\beta_{n-1}}$ by (4.62) and $|\varphi\circ g\circ\varphi^{-1}(J_{0\beta_1\dots\beta_{n-1}})| = 2|J_{0\beta_1\dots\beta_{n-1}}|$ by (4.63). Moreover, the intervals $(J'_{\beta_1\dots\beta_{n-1}})_{(\beta_1,\dots,\beta_{n-1})\in\{0,1\}^{n-1}}$ have pairwise disjoint interiors. Thus

$$|\varphi\circ g\circ\varphi^{-1}([0, x_0])| = 2|[0, x_0]| = 2x_0. \tag{4.64}$$

Since $\varphi\circ g\circ\varphi^{-1}$ fixes the point 0 and is increasing on $[0, \frac{1}{2}] \supset [0, x_0]$, we have $\varphi\circ g\circ\varphi^{-1}([0, x_0]) = [0, \varphi\circ g\circ\varphi^{-1}(x_0)]$. Combined with (4.64), this implies that

$$\forall x_0 \in \partial J'_{0\alpha_1\dots\alpha_{n-1}}, \quad \varphi\circ g\circ\varphi^{-1}(x_0) = 2x_0 = T_2(x_0). \tag{4.65}$$

If $\alpha_0 = 1$ (which implies that $x_0 \in [\frac{1}{2}, 1]$), one can show with similar arguments that $|\varphi\circ g\circ\varphi^{-1}([x_0, 1])| = 2|[x_0, 1]| = 2(1 - x_0)$, and thus

$$\forall x_0 \in \partial J'_{1\alpha_1\dots\alpha_{n-1}}, \quad \varphi\circ g\circ\varphi^{-1}(x_0) = T_2(x_0). \tag{4.66}$$

The set

$$\left\{\partial J'_{\alpha_0\dots,\alpha_{n-1}} \mid n \geq 1, (\alpha_0, \dots, \alpha_{n-1}) \in \{0, 1\}^n\right\} = \left\{\frac{i}{2^n} \mid n \geq 0, i \in [\![0, 2^n]\!]\right\}$$

is dense is $[0, 1]$. Since φ is continuous according to Fact 2, (4.65) and (4.66) imply that

$$\forall x \in [0, 1],\ \varphi\circ g\circ\varphi^{-1}(x) = T_2(x),$$

that is, g and T_2 are conjugate by φ. $\square$

Remarks on graph maps. There exist results similar to Proposition 4.70 for circle and tree maps. For transitive circle maps, the lower bound on the entropy depends on the degree, the interesting cases being the degrees $-1, 0, 1$. Indeed, we saw that, if f is a circle map of degree d with $|d| \geq 2$, then $h_{top}(f) \geq \log|d|$, regardless of whether f is transitive or not (Proposition 4.65). Moreover, for every integer $d \in \mathbb{Z}\setminus\{-1, 0, 1\}$, there exist transitive circle maps of degree d realizing the equality, for example, the map $\mathbb{S} \to \mathbb{S}$, $x \mapsto dx \bmod 1$. The cases of transitive circle maps of degree 0 or -1 were dealt with by Alsedà, Kolyada, Llibre and Snoha [**11**]. Notice that transitive circle maps of degree 0 are very similar to transitive interval maps with two fixed points. In particular, the map T_2 in Example 4.71 can be seen as a circle map by identifying the two endpoints of the interval.

PROPOSITION 4.76. *Let $f: \mathbb{S} \to \mathbb{S}$ be a transitive circle map of degree d.*

- *If $d = 0$, then $h_{top}(f) \geq \log 2$.*
- *If $d = -1$, then $h_{top}(f) \geq \frac{\log 3}{2}$.*

Moreover, there exist transitive circle maps with the prescribed degree realizing the equalities.

Irrational rotations provide examples of degree 1 circle maps that are transitive and have a null entropy. This is actually the only possibility, up to conjugacy. More generally, Blokh proved that a transitive graph map has positive entropy except if it is conjugate to an irrational rotation [**52**]. Recall that a transitive graph map with no periodic point is conjugate to an irrational rotation on the circle (Theorem 2.45).

THEOREM 4.77. *Let $f: \mathbb{S} \to \mathbb{S}$ be a transitive graph map. If f has periodic points, then $h_{top}(f) > 0$.*

For circle maps, this is the best possible lower bound: there exist transitive degree 1 circle maps with arbitrarily small positive topological entropy. This is a folklore result; see [**10**] for a proof.

For tree maps, a lower bound depending on the number of endpoints was found by Alsedà, Baldwin, Llibre and Misiurewicz [**3**].

PROPOSITION 4.78. *Let $f: T \to T$ be a transitive tree map. Let $e(T)$ denote the number of endpoints of T. Then $h_{top}(f) \geq \frac{\log 2}{e(T)}$.*

This is the best lower bound for star maps. However, this is not the case in general. The next proposition, due to Alsedà, Kolyada, Llibre and Snoha [**10**], states more specific bounds in the case of star maps.

PROPOSITION 4.79. *Let $f: S_n \to S_n$ be a transitive map, where S_n is an n-star, $n \geq 2$. Let b denote the unique branching point of S_n.*

- *If $f(b) = b$, then $h_{top}(f) \geq \frac{\log 2}{n}$. Moreover, equality is possible.*
- *If $f(b) \neq b$, then $h_{top}(f) \geq \frac{\log 2}{2}$ (it is not known whether this is the best lower bound).*

Proposition 4.74 holds for tree maps, with only obvious changes in its proof: if $f: T \to T$ is a transitive tree map such that $h_{top}(f) = h_{top}(f_P)$, where P is a finite invariant set containing all the branching points of T, then f is P-monotone. However its interest is limited since the assumption implies that every branching point has a finite orbit under f.

4.8. Uniformly positive entropy

The following notion was introduced by Blanchard [**30**], by analogy with K-systems in ergodic theory.

DEFINITION 4.80 (uniformly positive entropy). A topological dynamical system (X, f) has *uniformly positive entropy (upe)* if every open cover of X by two non dense open sets has a positive topological entropy.

A topologically mixing interval map has positive entropy by Proposition 4.70. The next theorem states that it has the stronger property of uniformly positive entropy.

THEOREM 4.81. *Let $f\colon I \to I$ be an interval map. The following assertions are equivalent:*

i) *f is topologically mixing,*
ii) *f has uniformly positive entropy.*

PROOF. We first assume that f is topologically mixing. Let $\mathcal{U} = (U_0, U_1)$ be an open cover of I such that U_0, U_1 are not dense. Since $U_0 \setminus \overline{U_1}$ is a nonempty open set, there is a non degenerate closed interval $I_0 \subset U_0 \setminus \overline{U_1}$ such that I_0 does not contains any endpoint of I. Similarly, there is a non degenerate closed interval $I_1 \subset U_1 \setminus \overline{U_0}$ containing no endpoint of I. According to Theorem 2.20, there exists an integer $k > 0$ such that $f^k(I_0) \cap f^k(I_1) \supset I_0 \cup I_1$. We set $g := f^k$. Let $n \geq 1$ and $(\varepsilon_0, \ldots, \varepsilon_{n-1}) \in \{0,1\}^n$. By Lemma 1.13(i), there exists a non degenerate closed interval J such that $g^i(J) \subset J_{\varepsilon_i}$ for all $i \in [\![0, n-1]\!]$. Consequently, for every n-tuple $(\varepsilon_0, \ldots, \varepsilon_{n-1}) \in \{0,1\}^n$, the set

$$I_{\varepsilon_0} \cap g^{-1}(J_{\varepsilon_1}) \cap \cdots \cap g^{-(n-1)}(J_{\varepsilon_{n-1}})$$

is nonempty and

$$U_{\varepsilon_0} \cap g^{-1}(U_{\varepsilon_1}) \cap \cdots \cap g^{-(n-1)}(U_{\varepsilon_{n-1}})$$

is the unique element of $\mathcal{U} \vee g^{-1}(\mathcal{U}) \vee \cdots \vee g^{-(n-1)}(\mathcal{U})$ that meets (actually contains) this set. This implies that $N_n(\mathcal{U}, g) \geq 2^n$ for all $n \geq 1$, so $h_{top}(\mathcal{U}, g) \geq \log 2$. Finally, we have

$$h_{top}(\mathcal{U}, f) = \frac{1}{k} h_{top}(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-(k-1)}(\mathcal{U}), g) \geq \frac{1}{k} h_{top}(\mathcal{U}, g) \geq \frac{\log 2}{k} > 0.$$

This proves (i)⇒(ii).

Now we are going to show that, if f is not topologically mixing, then it does not have uniformly positive entropy. This will prove (ii)⇒(i) by refutation.

Suppose that f is not transitive. This means that there exist two non degenerate closed intervals I_0, I_1 such that

$$\forall n \geq 0,\ f^{-n}(I_1) \cap I_0 = \emptyset. \tag{4.67}$$

We set $U_i := I \setminus I_i$ for $i \in \{0,1\}$. Then $\mathcal{U} := (U_0, U_1)$ is an open cover of I by two non dense sets. We see that (4.67) implies

$$\forall n \geq 0,\ I_0 \subset f^{-n}(U_1). \tag{4.68}$$

Let $x \in I$ and $n \geq 0$. If $f^i(x) \notin I_0$ for any $i \in [\![0, n-1]\!]$, then $x \in \bigcap_{i=0}^{n-1} f^{-i}(U_0)$. Otherwise, let k be the minimal non negative integer such that $f^k(x) \in I_0$. By (4.68), we have $f^k(x) \in \bigcap_{i \geq 0} f^{-i}(U_1)$, and thus

$$x \in \bigcap_{i=0}^{k-1} f^{-i}(U_0) \cap \bigcap_{i=k}^{n-1} f^{-i}(U_1).$$

This implies that $N_n(\mathcal{U}, f) \leq n + 1$ for all $n \geq 0$. We deduce that $h_{top}(\mathcal{U}, f) = 0$, so f does not have uniformly positive entropy.

Suppose now that f is transitive but not topologically mixing. Then, by Theorem 2.19, there exist two non degenerate closed intervals J, K with disjoint interiors such that $I = J \cup K$, $f(J) = K$ and $f(K) = J$. We choose two non dense open sets U_0, U_1 such that $J \subset U_0$ and $K \subset U_1$ and we set $\mathcal{U} := (U_0, U_1)$. For all $n \geq 0$,

$f^{2n}(J) \subset U_0$ and $f^{2n+1}(J) \subset U_1$. Similarly, for all $n \geq 0$, $f^{2n}(K) \subset U_1$ and $f^{2n+1}(K) \subset U_0$. This implies that I is covered by the two sets

$$\bigcap_{i=0}^{+\infty} f^{-2i}(U_0) \cap \bigcap_{i \geq 0} f^{-(2i+1)}(U_1) \quad \text{and} \quad \bigcap_{i=0}^{+\infty} f^{-2i}(U_1) \cap \bigcap_{i \geq 0} f^{-(2i+1)}(U_0).$$

This means that $N_n(\mathcal{U}, f) \leq 2$ for all $n \geq 1$. Thus, $h_{top}(\mathcal{U}, f) = 0$, and f does not have uniform positive entropy. This concludes the proof. $\square$

REMARK 4.82. Theorem 4.81 above can be seen as a consequence of results about general dynamical systems. Indeed, it is proved in [**30**] that a topological dynamical system with the specification property has uniformly positive entropy, and uniformly positive entropy implies topological weak mixing. For interval maps, topological mixing implies the specification property (Theorem 3.4), and topological weak mixing is equivalent to topological mixing (Theorem 2.20). This implies that an interval map is topologically mixing if and only if it has uniformly positive entropy.

Remarks on graph maps. Theorem 4.81 is valid for graph maps in view of Remark 4.82 and Theorem 3.5.

CHAPTER 5

Chaos in the sense of Li-Yorke, scrambled sets

5.1. Definitions

In [**113**], Li and Yorke showed that, if an interval map f has a periodic point of period 3, there exists an uncountable set S such that, for all distinct points x, y in S,

$$\limsup_{n\to+\infty} |f^n(x) - f^n(y)| > 0, \quad \liminf_{n\to+\infty} |f^n(x) - f^n(y)| = 0,$$
$$\text{and } \forall z \text{ periodic point, } \limsup_{n\to+\infty} |f^n(x) - f^n(z)| > 0.$$

They called this behavior *chaotic*, without formally defining what chaos is. This leads to the following definitions.

Definition 5.1 (Li-Yorke pair, scrambled set, Li-Yorke chaos). Let (X, f) be a topological dynamical system, $x, y \in X$ and $\delta > 0$. The pair (x, y) is a *Li-Yorke pair of modulus* δ if

$$\limsup_{n\to+\infty} d(f^n(x), f^n(y)) \geq \delta \quad \text{and} \quad \liminf_{n\to+\infty} d(f^n(x), f^n(y)) = 0, \tag{5.1}$$

and (x, y) is a *Li-Yorke pair* if it is a Li-Yorke pair of modulus δ for some $\delta > 0$. A set $S \subset X$ is a *scrambled* (resp. δ*-scrambled*) set if, for all distinct points x, y in S, (x, y) is a Li-Yorke pair (resp. a Li-Yorke pair of modulus δ).

The topological dynamical system (X, f) is *chaotic in the sense of Li-Yorke* if there exists an uncountable scrambled set.

The next proposition is straightforward (the second assertion uses the fact that f is uniformly continuous because X is compact).

Proposition 5.2. *Let (X, f) be a topological dynamical system, $S \subset X$ and $\delta > 0$.*

- *If S is a scrambled (resp. δ-scrambled) set for f^n, then it is also a scrambled (resp. δ-scrambled) set for f.*
- *If S is a scrambled (resp. δ-scrambled) set for f, then it is also a scrambled (resp. δ'-scrambled for some $\delta' > 0$) set for f^n.*

Remark 5.3. The definition of a scrambled set is not unified in the literature. In particular, in the spirit of the properties exhibited by Li and Yorke, some people say that S is a scrambled set if, for all distinct points x, y in S,

$$\limsup_{n\to+\infty} d(f^n(x), f^n(y)) > 0, \quad \liminf_{n\to+\infty} d(f^n(x), f^n(y)) = 0, \tag{5.2}$$

$$\forall z \text{ periodic point, } \limsup_{n\to+\infty} d(f^n(x), f^n(z)) > 0, \tag{5.3}$$

and the same properties with "$\geq \delta$" instead of "> 0" in (5.2) and (5.3) for δ-scrambled sets. Actually, it makes no difference for chaos in the sense of Li-Yorke, nor for existence of an uncountable δ-scrambled set for some $\delta > 0$. More precisely, if S is a scrambled set, then all but at most one point of S satisfy (5.3), and if S is an uncountable δ-scrambled set, then there exists an uncountable set S' included in S such that, for all $x \in S'$ and all periodic points z, $\limsup_{n\to+\infty} d(f^n(x), f^n(z)) \geq \delta/2$. These results are consequences of Lemmas 5.5 and 5.6 below; they were first noticed by Jiménez López [**94**, p 117–118], [**95**, Proposition 1.2.2].

DEFINITION 5.4. Let (X, f) be a topological dynamical system. A point x is *approximately periodic* if, for all $\varepsilon > 0$, there exists a periodic point z such that $\limsup_{n\to+\infty} d(f^n(x), f^n(z)) \leq \varepsilon$.

LEMMA 5.5. *Let (X, f) be a topological dynamical system and $x, x' \in X$. Suppose that x and x' are approximately periodic. Then*

$$\textit{either} \quad \lim_{n\to+\infty} d(f^n(x), f^n(x')) = 0 \quad \textit{or} \quad \liminf_{n\to+\infty} d(f^n(x), f^n(x')) > 0.$$

In particular, if S is a scrambled set, then S contains at most one approximately periodic point.

PROOF. Suppose that

$$\liminf_{n\to+\infty} d(f^n(x), f^n(x')) = 0. \tag{5.4}$$

Let $\varepsilon > 0$. By definition, there exist two periodic points z, z' and an integer N such that

$$\forall n \geq N,\ d(f^n(x), f^n(z)) \leq \varepsilon \quad \text{and} \quad d(f^n(x'), f^n(z')) \leq \varepsilon.$$

Let p be a multiple of the periods of z and z'. By continuity and (5.4), there exists $M \geq N$ such that $d(f^{M+i}(x), f^{M+i}(x')) \leq \varepsilon$ for all $i \in [\![0, p-1]\!]$. Let $n \geq N$ and let $i \in [\![0, p-1]\!]$ be such that $n - M \equiv i \bmod p$. Since $f^n(z) = f^{M+i}(z)$ and $f^n(z') = f^{m+i}(z')$, we have

$$\begin{array}{rcl} d(f^n(x), f^n(x')) & \leq & d(f^n(x), f^n(z)) + d(f^{M+i}(z), f^{M+i}(x)) \\ & & +d(f^{M+i}(x), f^{M+i}(x')) \\ & & +d(f^{M+i}(x'), f^{M+i}(z')) + d(f^n(z'), f^n(x')) \\ & \leq & 5\varepsilon. \end{array}$$

This implies that $\lim_{n\to+\infty} d(f^n(x), f^n(x')) = 0$. This proves the first statement of the lemma, which straightforwardly implies the second one. $\square$

LEMMA 5.6. *Let (X, f) be a topological dynamical system, $S \subset X$ and $\delta > 0$. Suppose that*

$$\forall x, y \in S,\ x \neq y,\ \limsup_{n\to+\infty} d(f^n(x), f^n(y)) \geq \delta.$$

Then there exists a countable set $C \subset X$ such that, for all $x \in S \setminus C$ and all periodic points $z \in X$, $\limsup_{n\to+\infty} d(f^n(x), f^n(z)) \geq \frac{\delta}{2}$.

PROOF. Let C be the set of points in S such that, for all $x \in C$, there exists a periodic point $z_x \in X$ such that $\limsup_{n\to+\infty} d(f^n(x), f^n(z_x)) < \frac{\delta}{2}$. Suppose that C is uncountable. Since C is the countable union of the sets

$$\left\{x \in S \mid \limsup_{n\to+\infty} d(f^n(x), f^n(z_x)) \leq \frac{\delta}{2} - \frac{1}{n}\right\},\ n \in \mathbb{N},$$

one of these sets is uncountable. Moreover, the set of periods of the points z_x is countable. Therefore, there exist an uncountable subset $R \subset C$, a number $\varepsilon > 0$ and an integer $p \geq 1$ such that

$$\forall x \in R,\ f^p(z_x) = z_x \text{ and } \limsup_{n \to +\infty} d(f^n(x), f^n(z_x)) \leq \frac{\delta}{2} - \varepsilon.$$

Since X is compact, f is uniformly continuous and there exists $\eta > 0$ such that

$$\forall x, y \in X,\ d(x, y) < \eta \Longrightarrow \forall i \in [\![0, p-1]\!],\ d(f^i(x), f^i(y)) < \varepsilon.$$

Since X is compact and R is infinite, the family $(z_x)_{x \in R}$ has a limit point. Thus there exist two distinct points x, x' in R such that $d(z_x, z_{x'}) < \eta$ (the case $z_x = z_{x'}$ is possible). Then $d(f^i(z_x), f^i(z_{x'})) < \varepsilon$ for all $i \in [\![0, p-1]\!]$. We have

$$\begin{aligned} \forall n \geq 0,\ d(f^n(x), f^n(x')) &\leq d(f^n(x), f^n(z_x)) + d(f^n(x'), f^n(z_{x'})) \\ &\quad + \max_{i \in [\![0, p-1]\!]} d(f^i(z_x), f^i(z_{x'})), \end{aligned}$$

so

$$\limsup_{n \to +\infty} d(f^n(x), f^n(x')) < (\delta/2 - \varepsilon) + (\delta/2 - \varepsilon) + \varepsilon < \delta.$$

This contradicts the fact that x, x' are two distinct points in the set S. We conclude that C is countable. $\square$

5.2. Weakly mixing maps are Li-Yorke chaotic

It is easy to see that every topologically weakly mixing dynamical system (X, f) has a dense G_δ-set of Li-Yorke pairs. Indeed, every $(x, y) \in X^2$ with a dense orbit is a Li-Yorke pair of modulus $\delta := \operatorname{diam}(X)$. Using results of topology (e.g., [**106**, Theorem 22.V.1]), this implies that the system has an uncountable $\operatorname{diam}(X)$-scrambled set (called an *extremally scrambled set* [**99**]). This result was first stated for interval maps by Bruckner and Hu [**62**]. More precisely, they showed that a topologically mixing interval map $f\colon [0,1] \to [0,1]$ admits a dense uncountable scrambled set S such that, for all distinct points x, y in S, the sequence $(f^n(x) - f^n(y))_{n \geq 0}$ is dense in $[-1, 1]$. Then Iwanik proved a stronger result, valid for any topologically weakly mixing dynamical system, which implies the existence of an extremally scrambled set [**91, 92**]. Iwanik's results rely on Mycielski's Theorem [**136**], that we restate under weaker hypotheses in order not to introduce irrelevant notions. We recall that a *perfect set* is a nonempty closed set with no isolated point; a perfect set is uncountable.

THEOREM 5.7 (Mycielski). *Let X be a complete metric space with no isolated point. For all integers $n \geq 1$, let r_n be a positive integer and let G_n be a dense G_δ-set of X^{r_n} such that*

$$G_n \cap \{(x_1, \ldots, x_{r_n}) \in X^{r_n} \mid \exists j, k \in [\![1, r_n]\!],\ j \neq k,\ x_j = x_k\} = \emptyset.$$

Let $(U_n)_{n \geq 1}$ be a sequence of nonempty open sets of X. Then there exists a sequence of compact perfect subsets $(K_n)_{n \geq 0}$ with $K_n \subset U_n$ such that, for all $k \geq 1$ and all distinct points $x_1, \ldots, x_{r_k}$ in $\bigcup_{n=1}^{+\infty} K_n$, $(x_1, \ldots, x_{r_k}) \in G_k$.

THEOREM 5.8. *Let (X, f) be a topological dynamical system. If (X, f) is topologically weakly mixing, then there exists a dense set $K \subset X$ which is a countable union of perfect sets and such that, for all $n \geq 1$, for all $k \geq 1$ and all distinct points $x_1, \ldots, x_n$ in K, the orbit $(f^{ik}(x_1), \ldots, f^{ik}(x_n))_{i \geq 0}$ is dense in X^n.*

PROOF. Let n, k be positive integers. By Proposition 2.6 and Theorem 2.7, the system $(X^n, f^k \times \cdots \times f^k)$ is transitive. Let G_n^k be the set of points of dense orbit in this system. According to Proposition 2.3(i), G_n^k is a dense G_δ-set and X has no isolated point (note that (X, f) cannot be weakly mixing if X is finite). Moreover, if the n-tuple $(x_1, \ldots, x_n) \in X^n$ has two equal coordinates, then its orbit is not dense. Finally, the conclusion is given by applying Mycielski's Theorem 5.7 with the countable family $(G_n^k)_{n,k \geq 1}$ and $(U_i)_{i \geq 0}$ a countable basis of nonempty open sets of X. $\square$

COROLLARY 5.9. *Let (X, f) be a topological dynamical system. If (X, f) is topologically weakly mixing, then there exists a dense set $K \subset X$ which is a countable union of perfect sets and such that, for all distinct points $x, y \in K$ and all periodic points $z \in X$,*

$$\limsup_{n \to +\infty} d(f^n(x), f^n(y)) = \mathrm{diam}(X), \quad \liminf_{n \to +\infty} d(f^n(x), f^n(y)) = 0$$

$$\textit{and } \limsup_{n \to +\infty} d(f^n(x), f^n(z)) \geq \frac{\mathrm{diam}(X)}{2}.$$

In particular, K is a δ-scrambled set for $\delta := \mathrm{diam}(X)$.

PROOF. Let K be the set given by Theorem 5.8. By compactness of X, there exist $x_0, y_0 \in X$ such that $d(x_0, y_0) = \mathrm{diam}(X)$. Let x, y be two distinct points in K. Since the orbit of (x, y) under $f \times f$ is dense in X^2, there exist two increasing sequences of positive integers $(i_n)_{n \geq 0}$ and $(j_n)_{n \geq 0}$ such that

$$\lim_{n \to +\infty} (f^{i_n}(x), f^{i_n}(y)) = (x_0, y_0) \quad \text{and} \lim_{n \to +\infty} (f^{j_n}(x), f^{j_n}(y)) = (x_0, x_0).$$

Thus

$$\limsup_{n \to +\infty} d(f^n(x), f^n(y)) = \mathrm{diam}(X) \quad \text{and} \quad \liminf_{n \to +\infty} d(f^n(x), f^n(y)) = 0.$$

Let $z \in X$ be a periodic point (if any) and let p be its period. By the triangular inequality, there exists $z' \in \{x_0, y_0\}$ such that $d(z, z') \geq \frac{\mathrm{diam}(X)}{2}$. Since x has a dense orbit under f^p by definition of K, there exists an increasing sequence of positive integers $(k_n)_{n \geq 0}$ such that $f^{pk_n}(x)$ tends to z'. Thus

$$\limsup_{n \to +\infty} d(f^n(x), f^n(z)) \geq \limsup_{n \to +\infty} d(f^{pk_n}(x), f^{pk_n}(z)) = d(z, z') \geq \frac{\mathrm{diam}(X)}{2}.$$

$\square$

REMARK 5.10. A set K satisfying the conclusion of Theorem 5.8 is called *totally independent* [**91**]. If a dynamical system (X, f) has such a set, then $(X \times X, f \times f)$ has a point of dense orbit, and thus (X, f) is topologically weakly mixing. Therefore, the existence of a totally independent set is equivalent to topological weak mixing.

The next proposition, due to Bruckner and Hu [**62**], is in some sense the converse of Corollary 5.9 for interval maps.

PROPOSITION 5.11. *Let $f\colon [0,1] \to [0,1]$ be an interval map. Assume that there exists a dense set $S \subset [0,1]$ such that*

$$\forall x, y \in S,\ x \neq y,\ \limsup_{n \to +\infty} |f^n(x) - f^n(y)| = 1.$$

Then f is topologically mixing.

Proof. Let $\varepsilon > 0$. The assumption implies that f is onto. Thus there exists $\delta \in (0, \varepsilon)$ such that $f([\delta, 1-\delta]) \supset [\varepsilon, 1-\varepsilon]$. Let J be a non degenerate subinterval of $[0, 1]$. Since S is dense, there exist two distinct points x, y in $J \cap S$. Let n be an integer such that $|f^n(x) - f^n(y)| > 1 - \delta$. Then $f^n(J) \supset [\delta, 1-\delta]$, which implies that both $f^n(J)$ and $f^{n+1}(J)$ contain $[\varepsilon, 1-\varepsilon]$ (recall that $\delta < \varepsilon$). Either n or $n+1$ is even, and thus there exists $m \geq 1$ such that $f^{2m}(J) \supset [\varepsilon, 1-\varepsilon]$. This implies that f^2 is transitive, so f is topologically mixing by Theorem 2.20. $\square$

5.3. Positive entropy maps are Li-Yorke chaotic

The original result of Li and Yorke (period 3 implies chaos in the sense of Li-Yorke [**113**]) was generalized in several steps. Nathanson stated the same result for periods which are multiple of 3, 5 or 7 [**137**]. Then, simultaneously, Butler and Pianigiani [**63**] and Oono [**138**] proved that an interval map f with a periodic point whose period is not a power of 2 (i.e., the period is $2^m q$ for some odd $q > 1$) is chaotic in the sense of Li-Yorke. Actually, this result can be derived from Li-Yorke's result using Sharkovsky's Theorem, but these authors were not aware of Sharkovsky's article. Later, Janková and Smítal proved a stronger result: an interval map with positive entropy (or equivalently with a periodic point whose period is not a power of 2, see Theorem 4.58) admits a perfect δ-scrambled set for some $\delta > 0$ [**93**]. The proof we shall give follows the spirit of [**93**], although it is slightly different.

Block [**33**] showed that, if an interval map f has a strict horseshoe, then there exists a subsystem which is semi-conjugate to a full shift on two letters, and the semi-conjugacy is "almost" a conjugacy. This semi-conjugacy with a full shift, stated in Proposition 5.15 below, is a key tool in several results.

Remark 5.12. In [**133**, Theorem 9], Moothathu stated that, if the entropy of f is positive, there exist $n \geq 0$ and an invariant set on which the action of f^{2^n} is conjugate to a full shift. Having a conjugacy rather than a semi-conjugacy would make some arguments easier. Unfortunately, there is something wrong in the proof; Li, Moothathu and Oprocha built a counter-example [**109**].

Definition 5.13. Let $\Sigma := \{0, 1\}^{\mathbb{Z}^+}$, endowed with the product topology; this is a compact metric set. The *shift* map $\sigma\colon \Sigma \to \Sigma$ is defined by $\sigma((\alpha_n)_{n\geq 0}) := (\alpha_{n+1})_{n\geq 0}$. Then (Σ, σ) is a topological dynamical system, called the *full shift* on two letters.

Lemma 5.14. *Let (X, f) and (Y, g) be two topological dynamical systems, and let $\varphi\colon X \to Y$ be a semi-conjugacy. For every $x \in X$, $\varphi(\omega(x, f)) = \omega(\varphi(x), g)$.*

Proof. Let $x \in X$ and $y := \varphi(x) \in Y$. We are going to show that $\varphi(\omega(x, f)) \subset \omega(y, g)$ and $\omega(y, g) \subset \varphi(\omega(x, f))$, which gives the equality of the two sets.

Let $x' \in \omega(x, f)$. There exists an increasing sequence of integers $(n_k)_{k\geq 0}$ such that $\lim_{k\to+\infty} f^{n_k}(x) = x'$. By continuity of φ, $\lim_{k\to+\infty} \varphi(f^{n_k}(x)) = \varphi(x')$. Since φ is a semi-conjugacy, $\varphi(f^{n_k}(x)) = g^{n_k}(\varphi(x)) = g^{n_k}(y)$. Thus $\varphi(x') \in \omega(y, g)$. This implies that $\varphi(\omega(x, f)) \subset \omega(y, g)$.

Let $y' \in \omega(y, g)$ and let $(n_k)_{k\geq 0}$ be an increasing sequence of integers such that $\lim_{k\to+\infty} g^{n_k}(y) = y'$. Since X is compact, there exist a subsequence $(n_{k_i})_{i\geq 0}$ and a point $x' \in X$ such that $\lim_{i\to+\infty} f^{n_{k_i}}(x) = x'$, and hence $x' \in \omega(x, f)$. Then $\varphi(x') = y'$ because φ is continuous. This implies that $\omega(y, g) \subset \varphi(\omega(x, f))$. $\square$

PROPOSITION 5.15. *Let $f\colon I \to I$ be an interval map and let (J_0, J_1) be a strict horseshoe for f. There exist an invariant Cantor set $X \subset I$ and a continuous map $\varphi\colon X \to \Sigma := \{0,1\}^{\mathbb{Z}^+}$ such that φ is a semi-conjugacy between $(X, f|_X)$ and (Σ, σ); the system $(X, f|_X)$ is transitive and there exists a countable set $E \subset X$ such that φ is one-to-one on $X \setminus E$ and two-to-one on E.*

Moreover, there exists a family of nonempty closed intervals

$$(J_{\alpha_0\ldots\alpha_{n-1}})_{n\geq 1, (\alpha_0,\ldots,\alpha_{n-1})\in\{0,1\}^n}$$

such that, for all $n \geq 1$ and all $(\alpha_0, \ldots, \alpha_{n-1}) \in \{0,1\}^n$,

$$J_{\alpha_0\ldots\alpha_{n-1}} \cap J_{\beta_0\ldots\beta_{n-1}} = \emptyset \text{ if } (\alpha_0,\ldots,\alpha_{n-1}) \neq (\beta_0,\ldots,\beta_{n-1}), \tag{5.5}$$

$$J_{\alpha_0\ldots\alpha_{n-1}} \subset J_{\alpha_0\ldots\alpha_{n-2}} \text{ if } n \geq 2, \tag{5.6}$$

$$f(J_{\alpha_0\ldots\alpha_{n-1}}) = J_{\alpha_1\ldots\alpha_{n-1}} \text{ and } f(\partial J_{\alpha_0\ldots\alpha_{n-1}}) = \partial J_{\alpha_1\ldots\alpha_{n-1}} \text{ if } n \geq 2, \tag{5.7}$$

$$\{x \in X \mid \varphi(x) \text{ begins with } \alpha_0\ldots\alpha_{n-1}\} = X \cap J_{\alpha_0\ldots\alpha_{n-1}}, \tag{5.8}$$

and, for all $(\alpha_n)_{n\geq 0} \in \Sigma \setminus \varphi(E)$,

$$\lim_{n\to+\infty} |J_{\alpha_0\ldots\alpha_{n-1}}| = 0. \tag{5.9}$$

PROOF. First, we show by induction on n that there exists a family of nonempty closed intervals satisfying (5.5), (5.6) and (5.7).

- For $n = 1$, the intervals J_0, J_1 satisfy (5.5) and there is nothing more to prove.
- Suppose that (5.5), (5.6), (5.7) are satisfied for some $n \geq 1$. Fix $(\alpha_0, \ldots, \alpha_n)$ in $\{0,1\}^{n+1}$. If $n = 1$, we apply Lemma 1.13(i) to the chain of intervals $(J_{\alpha_0}, J_{\alpha_1})$ and we obtain a closed interval $J_{\alpha_0\alpha_1}$ with $J_{\alpha_0\alpha_1} \subset J_{\alpha_0}$, $f(J_{\alpha_0\alpha_1}) = J_{\alpha_1}$ and $f(\partial J_{\alpha_0\alpha_1}) = \partial J_{\alpha_1}$. If $n \geq 2$, $f(J_{\alpha_0\ldots\alpha_{n-1}}) = J_{\alpha_1\ldots\alpha_{n-1}}$ and $J_{\alpha_1\ldots\alpha_n} \subset J_{\alpha_1\ldots\alpha_{n-1}}$ by the induction hypothesis. Thus we can apply Lemma 1.13(i) to the chain of intervals $(J_{\alpha_0\ldots\alpha_{n-1}}, J_{\alpha_1\ldots\alpha_n})$ and we obtain a closed interval $J_{\alpha_0\ldots\alpha_n}$ with $J_{\alpha_0\ldots\alpha_n} \subset J_{\alpha_0\ldots\alpha_{n-1}}$, $f(J_{\alpha_0\ldots\alpha_n}) = J_{\alpha_1\ldots\alpha_n}$ and $f(\partial J_{\alpha_0\ldots\alpha_n}) = \partial J_{\alpha_1\ldots\alpha_n}$. In both cases, we get (5.6) and (5.7) for $n+1$.

Let $(\alpha_0, \ldots, \alpha_n)$ and $(\beta_0, \ldots, \beta_n)$ be two distinct elements of $\{0,1\}^{n+1}$. If $(\alpha_0, \ldots, \alpha_{n-1}) \neq (\beta_0, \ldots, \beta_{n-1})$, then $J_{\alpha_0\ldots\alpha_{n-1}} \cap J_{\beta_0\ldots\beta_{n-1}} = \emptyset$ by the induction hypothesis, which implies that $J_{\alpha_0\ldots\alpha_n} \cap J_{\beta_0\ldots\beta_n} = \emptyset$ because of (5.6). If $(\alpha_0, \ldots, \alpha_{n-1}) = (\beta_0, \ldots, \beta_{n-1})$, then $\alpha_n \neq \beta_n$, and hence $J_{\alpha_1\ldots\alpha_n} \cap J_{\beta_1\ldots\beta_n} = \emptyset$ by the induction hypothesis. According to (5.7),

$$f(J_{\alpha_0\ldots\alpha_n}) \cap f(J_{\beta_0\ldots\beta_n}) = J_{\alpha_1\ldots\alpha_n} \cap J_{\beta_1\ldots\beta_n} = \emptyset,$$

which implies that $J_{\alpha_0\ldots\alpha_n} \cap J_{\beta_0\ldots\beta_n} = \emptyset$. This proves (5.5) for $n+1$ and this ends the induction.

For every $\bar\alpha = (\alpha_n)_{n\geq 0} \in \Sigma$, we set

$$J_{\bar\alpha} := \bigcap_{n=1}^{+\infty} J_{\alpha_0\ldots\alpha_{n-1}}.$$

This is a decreasing intersection of nonempty compact intervals, and thus $J_{\bar\alpha}$ is a nonempty compact interval. According to (5.5), we have

$$\forall \bar\alpha, \bar\beta \in \Sigma,\ \bar\alpha \neq \bar\beta \Longrightarrow J_{\bar\alpha} \cap J_{\bar\beta} = \emptyset. \tag{5.10}$$

We set

$$Y_0 := \bigcap_{n=1}^{+\infty} \bigcup_{\substack{\alpha_i \in \{0,1\} \\ i \in [\![0,n-1]\!]}} J_{\alpha_0\ldots\alpha_{n-1}} \quad \text{and} \quad Y := Y_0 \setminus \bigcup_{\bar\alpha\in\Sigma} \mathrm{Int}(J_{\bar\alpha}).$$

The sets Y_0 and Y are compact. One can see that $Y_0 = \bigcup_{\bar\alpha\in\Sigma} J_{\bar\alpha}$, and $(J_{\bar\alpha})_{\bar\alpha\in\Sigma}$ are the connected components of Y_0. This implies that Y is totally disconnected and $Y = \bigcup_{\bar\alpha\in\Sigma} \partial J_{\bar\alpha}$, which is a disjoint union by (5.10).

Recall that $\sigma((\alpha_n)_{n\geq 0}) = (\alpha_{n+1})_{n\geq 0}$. We are going to show that

$$\forall \bar\alpha \in \Sigma,\ f(\partial J_{\bar\alpha}) = \partial J_{\sigma(\bar\alpha)}, \tag{5.11}$$

which implies that $f(Y) = Y$.

Notice that if $x \in \partial J_{\bar\alpha}$, then

- either $x = \min J_{\bar\alpha}$ and $x = \lim_{n\to+\infty} \min J_{\alpha_0\ldots\alpha_{n-1}}$,
- or $x = \max J_{\bar\alpha}$ and $x = \lim_{n\to+\infty} \max J_{\alpha_0\ldots\alpha_{n-1}}$,

because $J_{\bar\alpha} = \bigcap_{n=1}^{+\infty} J_{\alpha_0\ldots J_{n-1}}$ is a decreasing intersection of nonempty compact intervals.

Let $\bar\alpha \in \Sigma$ and $x \in \partial J_{\bar\alpha}$. What precedes shows that there exists a sequence of points $(x_n)_{n\geq 1}$ such that $x = \lim_{n\to+\infty} x_n$ and $x_n \in \partial J_{\alpha_0\ldots\alpha_{n-1}}$ for all $n \geq 1$. Then $f(x_n) \in \partial J_{\alpha_1\ldots\alpha_{n-1}}$ by (5.7). Moreover, there is an increasing sequence $(n_k)_{k\geq 0}$ such that, either $f(x_{n_k}) = \min J_{\alpha_1\ldots\alpha_{n_k-1}}$ for all $k \geq 0$, or $f(x_{n_k}) = \max J_{\alpha_1\ldots\alpha_{n_k-1}}$ for all $k \geq 0$. Thus $\lim_{k\to+\infty} x_{n_k}$ is equal to $\min J_{\sigma(\bar\alpha)}$ or $\max J_{\sigma(\bar\alpha)}$, because $J_{\sigma(\bar\alpha)} = \bigcap_{n=1}^{+\infty} J_{\alpha_1\ldots J_n}$ is a decreasing intersection of nonempty compact intervals. Since f is continuous, $f(x) = \lim_{k\to+\infty} f(x_{n_k})$, and thus $f(x) \in \partial J_{\sigma(\bar\alpha)}$. This shows that $f(\partial J_{\bar\alpha}) \subset \partial J_{\sigma(\bar\alpha)}$.

Let $y \in \partial J_{\sigma(\bar\alpha)}$. As above, there exists a sequence of points $(y_n)_{n\geq 1}$ such that $y = \lim_{n\to+\infty} y_n$ and $y_n \in \partial J_{\alpha_1\ldots\alpha_n}$ for all $n \geq 1$. By (5.7), there exists $x_n \in \partial J_{\alpha_0\ldots\alpha_n}$ such that $f(x_n) = y_n$ for all $n \geq 1$. Then the same argument as above shows that there is an increasing sequence $(n_k)_{k\geq 0}$ such that $\lim_{k\to+\infty} x_{n_k} = x$ with $x \in \partial J_{\bar\alpha}$. Thus $f(x) = \lim_{k\to+\infty} f(x_{n_k}) = \lim_{k\to+\infty} y_{n_k} = y$. This shows that $\partial J_{\sigma(\bar\alpha)} \subset f(\partial J_{\bar\alpha})$, and this ends the proof of (5.11).

We define the map

$$\begin{array}{cccc} \varphi\colon & Y & \longrightarrow & \Sigma \\ & x & \longmapsto & \bar\alpha \text{ if } x \in J_{\bar\alpha} \end{array}$$

The map φ is trivially onto according to the definition of Y. Let $\mathcal{A}$ be the collection of $\bar\alpha \in \Sigma$ such that $J_{\bar\alpha}$ is a non degenerate interval. Then $\mathcal{A}$ is countable, and

$$F := \bigcup_{\bar\alpha\in\mathcal{A}} \partial J_{\bar\alpha} = \varphi^{-1}(\mathcal{A})$$

is a countable subset of Y. It is clear that, if $\bar\alpha \notin \mathcal{A}$, then there is a single point $x \in Y$ (with $J_{\bar\alpha} = \{x\}$) such that $\varphi(x) = \bar\alpha$; and if $\bar\alpha \in \mathcal{A}$, there are exactly two distinct points $x_1, x_2 \in Y$ (with $\partial J_{\bar\alpha} = \{x_1, x_2\}$) such that $\varphi(x_i) = \bar\alpha$. This shows that φ is one-to-one on $Y \setminus F$ and two-to-one on F.

Let δ_n be the minimal distance between two distinct intervals among

$$(J_{\alpha_0\ldots\alpha_{n-1}})_{(\alpha_0,\ldots,\alpha_{n-1})\in\{0,1\}^n}.$$

Then $\delta_n > 0$ because these intervals are compact and pairwise disjoint by (5.5). Let $x, y \in Y$. If $|x - y| < \delta_n$, then x and y are in the same interval $J_{\alpha_0\ldots\alpha_{n-1}}$

for some $(\alpha_0, \dots, \alpha_{n-1}) \in \{0,1\}^n$. This means that both $\varphi(x)$ and $\varphi(y)$ begin with $\alpha_0 \dots \alpha_{n-1}$, which implies that φ is continuous. Moreover, (5.11) implies that $\varphi \circ f(x) = \sigma \circ \varphi(x)$ for all $x \in Y$, that is, φ is a semi-conjugacy.

It is easy to see that (Σ, σ) is transitive and that Σ is uncountable. By Proposition 2.3(i), there exists a dense G_δ-set of elements $\bar\alpha \in \Sigma$ such that $\omega(\bar\alpha, \sigma) = \Sigma$. Thus there exists $\bar\alpha \in \Sigma \setminus \mathcal{A}$ such that $\omega(\bar\alpha, \sigma) = \Sigma$ because $\mathcal{A}$ is countable. Let $x_0 \in Y$ be the unique point such that $\varphi(x_0) = \bar\alpha$ and set $X := \omega(x_0, f)$. The set X is closed and invariant by Lemma 1.3(i), and $X \subset Y$. By Lemma 5.14, $\varphi(\omega(x_0, f)) = \omega(\bar\alpha, \sigma) = \Sigma$. Thus $\varphi|_X \colon X \to \Sigma$ is onto. This implies that $\varphi|_X$ is a semi-conjugacy between $(X, f|_X)$ and (Σ, σ). Moreover, there exists a countable set $E \subset F$ such that φ is two-to-one on E and one-to-one on $X \setminus E$. Since $\varphi(X) = \Sigma$ and $\varphi^{-1}(\bar\alpha) = \{x_0\}$, the point x_0 belongs to X, and the set X is infinite. According to Proposition 2.3, the fact that $X = \omega(x_0, f)$ implies that $(X, f|_X)$ is transitive and X has no isolated point. Moreover, the set X is totally disconnected because Y is totally disconnected, and thus X is a Cantor set.

By definition of φ, (5.8) is satisfied. Finally, if $\bar\alpha = (\alpha_n)_{n\geq 0}$ does not belong to $\mathcal{A}$, the fact that $J_{\bar\alpha}$ is reduced to a single point implies that

$$\lim_{n\to+\infty} |J_{\alpha_0\dots\alpha_{n-1}}| = 0,$$

which is (5.9). This concludes the proof. $\square$

For the following result of topology, one can refer, e.g., to [**106**, Theorem 37.I.3].

THEOREM 5.16 (Alexandrov-Hausdorff). *Let X be a topological space. Every uncountable Borel set contains a Cantor set.*

THEOREM 5.17. *Let f be an interval map. If $h_{top}(f) > 0$, there exists a δ-scrambled Cantor set for some $\delta > 0$. In particular, f is chaotic in the sense of Li-Yorke.*

PROOF. By Theorem 4.58, there exists an integer p such that f^p has a strict horseshoe (J_0, J_1). Let $\delta > 0$ be the distance between J_0 and J_1 and $g := f^p$. Let X, E and $\varphi \colon X \to \Sigma$ be given by Proposition 5.15 for the map g. We fix an element $\bar\omega = (\omega_n)_{n\geq 0}$ in $\Sigma \setminus \varphi(E)$. We define $\psi \colon \Sigma \to \Sigma$ by

$$\psi((\alpha_n)_{n\geq 0}) := (\omega_0\ \alpha_0\ \omega_0\omega_1\ \alpha_0\alpha_1 \dots \omega_0 \dots \omega_{n-1}\ \alpha_0\alpha_1 \dots \alpha_{n-1} \dots).$$

This map is clearly continuous and one-to-one. For every $\bar\alpha \in \Sigma$, we choose a point $x_{\bar\alpha}$ in $\varphi^{-1} \circ \psi(\bar\alpha)$ and we set $S := \{x_{\bar\alpha} \in X \mid \bar\alpha \in \Sigma\}$. According to Proposition 5.15, the set $\varphi^{-1} \circ \psi(\bar\alpha)$ contains two points if $\psi(\bar\alpha) \in \varphi(E)$ and is reduced to a single point if $\psi(\bar\alpha) \notin \varphi(E)$. Thus there exists a countable set $F \subset X$ such that $S = (\varphi^{-1} \circ \psi(\Sigma)) \setminus F$.

Let $\bar\alpha, \bar\beta$ be two distinct elements of Σ, and let $k \geq 0$ be such that $\alpha_k \neq \beta_k$. By definition of ψ, there exists an increasing sequence of integers $(n_i)_{i\geq 0}$ such that the n_i-th coordinates of $\psi(\bar\alpha)$ and $\psi(\bar\beta)$ are equal respectively to α_k and β_k, and hence are distinct. Then, by Proposition 5.15, either $g^{n_i}(x_{\bar\alpha})$ belongs to J_0 and $g^{n_i}(x_{\bar\beta})$ belongs to J_1, or the converse. In particular, $|g^{n_i}(x_{\bar\alpha}) - g^{n_i}(x_{\bar\beta})| \geq \delta$. This proves that, for all distinct points x, x' in S,

$$\limsup_{n\to+\infty} |g^n(x) - g^n(x')| \geq \delta.$$

According to Proposition 5.15 and the choice of $\bar{\omega}$,

$$\lim_{n\to+\infty} \operatorname{diam}\{x \in X \mid \varphi(x) \text{ begins with } \omega_0 \dots \omega_{n-1}\} = 0.$$

By definition of ψ, there exists an increasing sequence of integers $(m_i)_{i\geq 0}$ such that, for every $\bar{\alpha} \in \Sigma$, $\sigma^{m_i}(\psi(\bar{\alpha}))$ begins with $(\omega_0 \dots \omega_{i-1})$. Since $\sigma^{m_i}(\psi(\bar{\alpha})) = \varphi(g^{m_i}(x_{\bar{\alpha}}))$, we get

$$\forall \bar{\alpha}, \bar{\beta} \in \Sigma,\ \liminf_{n\to+\infty} |g^n(x_{\bar{\alpha}}) - g^n(x_{\bar{\beta}})| = 0.$$

Therefore, S is a δ-scrambled set for g. Moreover, by Theorem 5.16, there exists a Cantor set $K \subset S$ because $S = (\varphi^{-1} \circ \psi(\Sigma)) \setminus F$ is an uncountable Borel set. We have proved that $g = f^p$ admits a δ-scrambled Cantor set, so K is also a δ-scrambled set for f by Proposition 5.2. □

Remarks on graph maps and general dynamical systems. Proposition 5.15 can be generalized to graph maps (actually to any dynamical system having a horseshoe made of two intervals). Theorem 5.17 remains valid for graph maps, and the same proof works because, according to Theorem 4.12, if a graph map f has positive topological entropy, f^n has a strict horseshoe for some n. The proof that a graph map of positive entropy is chaotic in the sense of Li-Yorke does not appear in the literature, this result being a consequence of the next theorem, which is due to Blanchard, Glasner, Kolyada and Maass [**32**].

THEOREM 5.18. *Let (X, f) be a topological dynamical system with positive topological entropy. Then it admits a δ-scrambled Cantor set for some $\delta > 0$.*

This theorem is a remarkable result and answers a longstanding question. Its proof is much more difficult than the proof in the interval case.

REMARK 5.19. In [**32**], the main theorem states the existence of a scrambled Cantor set, but the proof actually gives a δ-scrambled Cantor set.

5.4. Zero entropy maps

The converse of Theorem 5.17 is not true: there exist zero entropy interval maps that are chaotic in the sense of Li-Yorke; we shall give an example in Section 5.7. A zero entropy map that is chaotic in the sense of Li-Yorke is sometimes called *weakly chaotic* (e.g. in [**78**]). In [**162**], Smítal proved that a zero entropy interval map that is chaotic in the sense of Li-Yorke has a δ-scrambled Cantor set for some $\delta > 0$, as it is the case for positive entropy interval maps. He also gave a necessary and sufficient condition for a zero entropy interval map to be chaotic in the sense of Li-Yorke in terms of non separable points (condition (iv) in Theorem 5.21 below). This condition looks technical, but it can be useful to show that a map is chaotic in the sense of Li-Yorke or not; in particular, it will be needed for the examples in Section 5.7.

DEFINITION 5.20 (f-non separable points). Let f be an interval map and let a_0, a_1 be two distinct points. The points a_0, a_1 are *f-separable* if there exist two disjoint intervals J_0, J_1 and two positive integers n_0, n_1 such that $a_i \in J_i$, $f^{n_i}(J_i) = J_i$ and $(f^k(J_i))_{0\leq k<n_i}$ are disjoint for $i \in \{0, 1\}$. Otherwise they are *f-non separable.*

THEOREM 5.21. *Let f be an interval map of zero topological entropy. The following properties are equivalent:*

i) *f is chaotic in the sense of Li-Yorke,*
ii) *there exists a δ-scrambled Cantor set for some $\delta > 0$,*
iii) *there exists a point x that is not approximately periodic,*
iv) *there exists an infinite ω-limit set containing two f-non separable points.*

Before proving this theorem, we need an important number of intermediate results. Some of them have an interest on their own.

The next lemma is stated in the case of zero entropy interval maps in [**162**]. We give a different proof.

LEMMA 5.22. *Let f be an interval map such that f^2 has no horseshoe. Let x_0 be a point with an infinite orbit, and $x_n := f^n(x_0)$ for all $n \geq 1$. Suppose that there exists $k_0 \geq 2$ such that either $x_{k_0} < x_0 < x_1$ or $x_{k_0} > x_0 > x_1$. Then there exist a fixed point z and an integer N such that*

$$\forall n \geq N,\ x_n > z \Longleftrightarrow x_{n+1} < z.$$

PROOF. All the points $(x_n)_{n\geq 0}$ are distinct by assumption. Let

$$U := \{x_n \mid n \geq 0, x_{n+1} > x_n\} \quad \text{and} \quad D := \{x_n \mid n \geq 0, x_{n+1} < x_n\}.$$

The map f has no horseshoe (because f^2 has no horseshoe either), and thus Lemma 3.33 applies: there exists a fixed point z such that

$$U < z < D. \tag{5.12}$$

By assumption, there exist couples of integers (p,k) with $p \geq 0$, $k \geq 2$, such that, either $x_{p+k} < x_p < x_{p+1}$, or $x_{p+k} > x_p > x_{p+1}$. We choose (p,k) satisfying this property with k minimal. We are going to show that $k = 2$. Suppose on the contrary that $k \geq 3$. We may assume that $x_{p+k} < x_p < x_{p+1}$, the case with reverse inequalities being symmetric. The minimality of k and the fact that $k \geq 3$ imply that $x_{p+2} > x_p$ and that we do not have $x_{p+k} < x_{p+1} < x_{p+2}$. Thus

$$x_{p+k} < x_p < x_{p+2} < x_{p+1}. \tag{5.13}$$

Then $x_p < z < x_{p+1}$ by (5.12). Let $q \in [\![p+1, p+k-1]\!]$ be the integer such that $x_n > z$ for all $n \in [\![p+1, q]\!]$ and $x_{q+1} < z$. By (5.12), $x_{q+2} > x_{q+1}$ and $x_{n+1} < x_n$ for all $n \in [\![p+1, q]\!]$. If $x_{q+1} > x_{p+k}$, then $x_{p+k} < x_{q+1} < x_{q+2}$, which contradicts the minimality of k. Therefore the points are ordered as follows:

$$x_{q+1} \leq x_{p+k} < x_p < z < x_q < x_{q-1} < \cdots < x_{p+2} < x_{p+1}.$$

If $q = p+1$, then $x_{p+2} \leq x_{p+k} < x_p < x_{p+1}$, which contradicts the fact that $x_p < x_{p+2}$ by (5.13). Thus $q \geq p+2$. Let $I_0 := [x_p, x_q]$ and $I_1 := [x_q, x_{p+1}]$. Then $f(I_0) \supset [x_{q+1}, x_{p+1}] \supset I_0 \cup I_1$ and $f(I_1) \supset [x_{q+1}, x_{p+2}] \supset I_0$. This implies that (I_0, I_1) is a horseshoe for f^2, which is a contradiction. We deduce that $k = 2$, that is, there exists an integer p such that

$$\text{either } x_{p+2} < x_p < x_{p+1} \quad \text{or} \quad x_{p+2} > x_p > x_{p+1}. \tag{5.14}$$

Now we are going to show that there exists an integer N such that, for all $i \geq N$, $x_i > z \Leftrightarrow x_{i+1} < z$. Assume that the contrary holds, which implies the following by (5.12):

$$\forall n \geq 0,\ \exists\, i \geq n,\ \text{either } x_i < x_{i+1} < z \quad \text{or} \quad z > x_{i+1} > x_i. \tag{5.15}$$

For every $n \geq 0$, let $i(n)$ be the minimal integer i satisfying this property. Among all integers p satisfying (5.14), we choose p such that $i(p) - p$ is minimal and we set $i := i(p)$. In (5.14), we assume that $x_{p+2} < x_p < x_{p+1}$, the case with reverse inequalities being symmetric. Then $x_p < z < x_{p+1}$ by (5.12), which implies that $i \geq p+2$. First we suppose that $i = p+2$, that is, $x_{p+2} < x_{p+3} < z$. If we set $J := [x_{p+2}, x_p]$ and $K := [x_p, z]$, then $f(J) \supset [x_{p+3}, x_{p+1}] \ni z$, so $f^2(J) \supset [x_{p+2}, z] \supset J \cup K$ and $f^2(K) \supset [x_{p+2}, z] \supset J \cup K$. Therefore J, K form a horseshoe for f^2, which is a contradiction. We deduce that $i \geq p+3$.

If n is in $[\![p+1, i]\!]$, then n does not satisfy (5.14), otherwise $i(n)-n = i-n < i-p$, which contradicts the choice of p. Thus

$$\forall n \in [\![p+1, i]\!],\ x_n \text{ is not between } x_{n+1} \text{ and } x_{n+2}. \tag{5.16}$$

We show by induction on n that, for all $n \in [\![p+2, i]\!]$ with $(n-p)$ even, we have:

$$\begin{aligned} &x_{p+2} < x_{p+4} < x_{p+6} < \cdots < x_{n-2} < x_n \\ &\qquad < z < x_{n-1} < x_{n-3} < \cdots < x_{p+3} < x_{p+1}. \end{aligned} \tag{5.17}$$

- We have seen that $x_{p+2} < z < x_{p+1}$, which is (5.17) for $n = p+2$.
- Suppose that (5.17) holds for some $n \in [\![p+2, i-2]\!]$ with $n-p$ even. Then $x_{n+1} > x_n$ by (5.12), and $x_n > x_{n+2}$ by (5.16). Moreover, $x_{n+1} > z$, otherwise we would have $x_n < x_{n+1} < z$, which would contradict the minimality of $i-p$. Furthermore, x_{n-1} cannot be between x_n and x_{n+1} by (5.16), and $x_{n-1} > z$ according to the induction hypothesis. In summary, these inequalities give $x_n < z < x_{n+1} < x_{n-1}$. Similarly, $x_{n+2} < x_{n+1}$ by (5.12) and $x_{n+2} < z$ by (5.16)+(5.12); thus $x_n < x_{n+2} < z$. This shows that (5.17) holds for $n+2$, which ends the proof of (5.17).

Now we show by induction on n that $x_n < x_p$ for all $n \in [\![p+2, i]\!]$ with $(n-p)$ even.

- We know that $x_{p+2} < x_p$.
- Suppose that the statement holds for n but not for $n+2$, for some $n \in [\![p+2, i-2]\!]$ such that $(n-p)$ is even. This means that $x_n < x_p < x_{n+2}$. Combining this with (5.17), we get $x_{p+2} \leq x_n < x_p < x_{n+2} < z$. We set $J := [x_n, x_p]$ and $K := [x_p, x_{n+2}]$. Then $f^2(J) \supset [x_{p+2}, x_{n+2}] \supset J \cup K$ and $f^2(K) \supset [x_{p+2}, x_{n+4}]$.
  - If $n+4 \leq i$, then $x_{n+4} > x_{n+2}$ by (5.17).
  - If $n+4 = i+1$, then $x_{i-1} < z$ by (5.17) and $x_i > z$ by minimality of $i-p$, so $z < x_{i+1} < x_i$ according to the definition of i. This implies that $x_{n+4} = x_{i+1} > z > x_{n+2}$.
  - If $n+4 = i+2$, then $x_i < x_{i+1} < z$ by definition of i and (5.17), and thus $x_{n+4} = x_{i+2} > x_{i+1}$ by (5.12) and we get $x_{n+4} > x_i = x_{n+2}$.

In the three cases, we have $x_{n+4} > x_{n+2}$. This implies $f^2(K) \supset J \cup K$, so (J, K) is a horseshoe for f^2, which is a contradiction. We deduce that, if $x_n < x_p$, then $x_{n+2} < x_p$ too. This completes the induction.

We end the proof by showing that (5.15) is absurd. We set $j := i$ if $i-p$ is even, and $j := i-1$ if $i-p$ is odd. Let $J' := [x_j, x_p]$ and $K' := [x_p, z]$ (recall that $x_p < z$). If $j = i$, then $x_i < x_{i+1} < z$ (by (5.15) and (5.17)) and $f(J') \supset [x_{i+1}, x_{p+1}] \ni z$, so $f^2(J') \supset [x_{p+2}, z] \supset J' \cup K'$ by (5.17). If $j = i-1$, then $z < x_{i+1} < x_i$ (by (5.15) and (5.17) again) and $f^2(J') \supset [x_{p+2}, x_{i+1}] \supset J' \cup K'$ by (5.17). Moreover, $f^2(K') \supset [x_{p+2}, z] \supset J' \cup K'$. Therefore, (J', K') is a horseshoe for f^2, which is not possible. The lemma is proved. □

The following result is due to Sharkovsky [**156**, Corollary 3]. The proof we give relies on Lemma 5.22 above.

PROPOSITION 5.23. *Let f be an interval map of zero topological entropy and let x be a point. If $\omega(x,f)$ is infinite, then $\omega(x,f)$ contains no periodic point.*

PROOF. The point x is not eventually periodic because $\omega(x,f)$ is infinite. Thus all the points $(f^n(x))_{n\geq 0}$ are distinct. Moreover,

$$\omega(x,f)\cap(\min\omega(x,f),\max\omega(x,f))\neq\emptyset$$

because $\omega(x,f)$ is infinite, and thus there exists an integer p such that $\min\omega(x,f)<f^p(x)<\max\omega(x,f)$. If $f^{p+1}(x)>f^p(x)$, then there exists $q>p$ such that $f^q(x)$ is arbitrarily close to $\min\omega(x,f)$, in such a way that $f^q(x)<f^p(x)<f^{p+1}(x)$. Similarly, if $f^{p+1}(x)<f^p(x)$, then there exists $q>p$ such that $f^q(x)>f^p(x)>f^{p+1}(x)$. Since f^2 has no horseshoe by Proposition 4.6, Lemma 5.22 applies: there exist a fixed point z and an integer N such that

$$\forall n\geq N,\ f^n(x)<z\Longleftrightarrow f^{n+1}(x)>z.$$

We define either $y:=f^N(x)$ or $y:=f^{N+1}(x)$ in order to have $y<z$, and we set $y_n:=f^n(y)$ for all $n\geq 0$. In this way, $\omega(y,f)=\omega(x,f)$ and

$$\forall i\geq 0,\ y_{2i}<z<y_{2i+1}. \tag{5.18}$$

First we prove that $\omega(x,f)$ contains no fixed point. Suppose on the contrary that there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $\lim_{i\to+\infty}y_{n_i}=a$ with $f(a)=a$. By continuity, $(y_{n_i+1})_{i\geq 0}$ tends to a too. The set $\{n_i\mid i\geq 0\}$ contains either infinitely many odd integers or infinitely many even integers, and thus there exists an increasing sequence $(k_i)_{i\geq 0}$ such that $a=\lim_{i\to+\infty}y_{2k_i}=\lim_{i\to+\infty}y_{2k_i+1}$. Then (5.18) implies that $a=z$. Hence $z\in\omega(y,f)$. Let $g:=f^2$. The map g^2 has no horseshoe (by Proposition 4.6 again), and $\omega(y,g)$ is infinite (by Lemma 1.3(vi)). Thus we can apply Lemma 5.22 to g and y: there exist a point z' and an integer N' such that $g(z')=z'$ and

$$\forall i\geq 0,\ y_{2N'+4i}<z'<y_{2N'+4i+2}<z \tag{5.19}$$

(the last inequality follows from (5.18)). Since z is in $\omega(y,g)$, there exists a sequence $(m_i)_{i\geq 0}$ such that $g^{N'+m_i}(y)=y_{2N'+2m_i}$ tends to z. By (5.19), m_i must be odd for all large enough i. By continuity, $g^{N'+m_i+1}(y)$ converges to z too, and at the same time $g^{N'+m_i+1}(y)=y_{2N'+2m_i+2}<z'<z$, which is absurd. We deduce that $\omega(x,f)$ contains no fixed point.

Let $n\geq 1$. According to Lemma 1.3, $\omega(x,f)=\bigcup_{i=0}^{2^n-1}\omega(f^i(x),f^{2^n})$ and the set $\omega(f^i(x),f^n)$ is infinite for every $i\in[\![0,n-1]\!]$. Applying the previous result to f^n, we deduce that $\omega(x,f)$ contains no periodic point of period n. $\square$

The next proposition states that an infinite ω-limit set of a zero entropy interval map is a *solenoidal set*, that is, it is included in a nested sequence of cycles of intervals of periods tending to infinity. This is a key tool when studying zero entropy interval maps. This result is implicitly contained in several papers of Sharkovsky, and stated without proof in a paper of Blokh [**43**]. A very similar result, dealing with infinite transitive sets of zero entropy interval maps, was proved by Misiurewicz [**126**]. The formulation we give follows Smítal's [**162**], except the property that the intervals can be chosen to be closed, which is due to Fedorenko, Sharkovsky and Smítal [**78**]. Although the result is mostly interesting for infinite ω-limit sets, the proposition below also deals with finite ω-limit sets because this case will be used in the sequel.

PROPOSITION 5.24. *Let f be an interval map of zero topological entropy and let x_0 be a point. If $\omega(x_0, f)$ is a periodic orbit of period 2^p for some $p \geq 0$, set $\mathcal{I} := [\![0, p]\!]$. If $\omega(x_0, f)$ is infinite, set $\mathcal{I} := \mathbb{Z}^+$. There exists a (finite or infinite) sequence of closed intervals $(L_k)_{k\in\mathcal{I}}$ such that, for all $k \in \mathcal{I}$,*

i) *$(L_k, f(L_k), \ldots, f^{2^k-1}(L_k))$ is a cycle of intervals, that is, these intervals are pairwise disjoint and $f^{2^k}(L_k) = L_k$,*
ii) *$\forall i, j \in [\![0, 2^k - 1]\!]$, $i \neq j$, there is a point z between $f^i(L_k)$ and $f^j(L_k)$ such that $f^{2^{k-1}}(z) = z$,*
iii) *$L_{k+1} \cup f^{2^k}(L_{k+1}) \subset L_k$ provided $k + 1 \in \mathcal{I}$,*
iv) *$\omega(x_0, f) \subset \displaystyle\bigcup_{i=0}^{2^k-1} f^i(L_k)$,*
v) *$f^i(L_k)$ is the smallest f^{2^k}-invariant interval containing $\omega(f^i(x_0), f^{2^k})$,*
vi) *if $k + 2 \in \mathcal{I}$, $\exists N \geq 0$, $\forall n \geq N$, $f^n(x_0) \in f^n(L_k)$.*

Moreover, if $\omega(x_0, f)$ is infinite, then f is of type 2^∞ for Sharkovsky's order.

PROOF. According to Theorem 4.58, the period of any periodic point is a power of 2, so the map f is of type $\trianglerighteq 2^\infty$. If $\omega(x_0, f)$ is infinite, the fact that f is of type 2^∞ follows from the existence of the infinite sequence of intervals $(L_k)_{k\geq 0}$. Indeed, if L_k satisfies (i), then there exists $x \in L_k$ such that $f^{2^k}(x) = x$ (by Lemma 1.11), and x is of period 2^k because $L_k, f(L_k), \ldots, f^{2^k-1}(L_k)$ are pairwise disjoint; thus the set of periods of f contains $\{2^k \mid k \geq 0\}$. The rest of the proof is devoted to the definition and the properties of $(L_k)_{k\in\mathcal{I}}$.

Let $k \in \mathcal{I}$. We set $g_k := f^{2^k}$,

$$I_k := [\min \omega(x_0, g_k), \max \omega(x_0, g)]$$
$$\text{and} \quad L_k := \overline{\bigcup_{n\geq 0} (g_k)^n(I_k)}.$$

Trivially, $g_k(L_k) \subset L_k$. For all $n \geq 0$, the set $(g_k)^n(I_k)$ is an interval containing $\omega(x_0, g_k)$ because $g_k(\omega(x_0, g_k)) = \omega(x_0, g_k)$ by Lemma 1.3(iii), so $(g_k)^n(I_k) \supset I_k$. Thus L_k is an interval and $g_k(L_k) = L_k$. Therefore

(5.20) $\qquad L_k$ is the smallest g_k-invariant interval containing $\omega(x_0, g_k)$.

Let $i \in [\![0, 2^k - 1]\!]$. It is clear that $f^i(L_k)$ is a g_k-invariant interval containing $f^i(\omega(x_0, g_k)) = \omega(f^i(x_0), g_k)$ (by Lemma 1.3(iii)). Let J be a g_k-invariant interval containing $\omega(f^i(x_0), g_k)$. Then $f^{2^k-i}(J) \supset \omega(f^{2^k}(x_0), g_k) = \omega(x_0, g_k)$ (by Lemma 1.3(iv)+(ii)). Thus $f^{2^k-i}(J) \supset L_k$ by (5.20), so $g_k(J) \supset f^i(L_k)$. This implies that, for all $i \in [\![0, 2^k - 1]\!]$,

(5.21) $\qquad f^i(L_k)$ is the smallest g_k-invariant interval containing $\omega(f^i(x_0), g_k)$

which is (v). Moreover, by Lemma 1.3(v),

$$\omega(x_0, f) = \bigcup_{i=0}^{2^k-1} (\omega(f^i(x_0), g_k)) \subset \bigcup_{i=0}^{2^k-1} f^i(L_k),$$

which gives (iv). If $k+1$ belongs to $\mathcal{I}$, the interval I_{k+1} is included in I_k because $\omega(x_0,(g_k)^2)\subset\omega(x_0,g_k)$. Thus

$$L_{k+1}\cup g_k(L_{k+1})=\overline{\bigcup_{n\geq 0}(g_k)^{2n}(I_{k+1})}\cup\overline{\bigcup_{n\geq 0}(g_k)^{2n+1}(I_{k+1})}\subset\overline{\bigcup_{n\geq 0}(g_k)^{n}(I_{k})}=L_k,$$

which is (iii).

We are going to prove (ii) by induction on k. This will show at the same time that the intervals $(f^i(L_k))_{0\leq i<2^k}$ are pairwise disjoint, which in turn implies (i) in view of the fact that L_k is strongly invariant under g_k.

- There is nothing to prove for $k=0$.
- Suppose that $k:=1$ belongs to $\mathcal{I}$. If $\#\omega(x_0,f)=2$, then $\omega(x_0,f)$ is a periodic orbit of period 2 and the interval I_1 is reduced to a single point $\{y\}$ satisfying $f^2(y)=y$ and $f(y)\neq y$; moreover, $L_1=\{y\}$ and $f(L_1)=\{f(y)\}$. Then $f(\langle y,f(y)\rangle)\supset\langle y,f(y)\rangle$, which implies that there exists a point $z\in\langle y,f(y)\rangle$ such that $f(z)=z$, and z is different from $y,f(y)$. This proves (ii) for $k=1$ in this case. From now on, we suppose that $\#\omega(x_0,f)\geq 3$ and we set $g:=g_1=f^2$. We write $I_1=[a,b]$ and $L_1=[c,d]$. We have $I_1\subset L_1$, that is, $c\leq a\leq b\leq d$. Since $\omega(x_0,f)$ contains at least 3 points,

$$\omega(x_0,f)\cap(\min\omega(x_0,f),\max\omega(x_0,f))\neq\emptyset,$$

and thus there exists $n\geq 0$ such that $\min\omega(x_0,f)<f^n(x_0)<\max\omega(x_0,f)$. If $f^{n+1}(x_0)>f^n(x_0)$, there exists $j\geq 1$ such that $f^{n+j}(x_0)$ is arbitrarily close to $\min\omega(x_0,f)$, in such a way that $f^{n+j}(x_0)<f^n(x_0)<f^{n+1}(x_0)$. Similarly, if $f^{n+1}(x_0)<f^n(x_0)$, there exists $j\geq 1$ such that $f^{n+j}(x_0)>f^n(x_0)>f^{n+1}(x_0)$. Since f has zero topological entropy, f has no horseshoe by Proposition 4.6. According to Lemma 5.22, there exist a point z and an integer N such that $f(z)=z$ and

$$\text{either}\quad\forall n\geq N,\ f^{2n}(x_0)<z<f^{2n+1}(x_0),\qquad\qquad (5.22)$$
$$\text{or}\quad\forall n\geq N,\ f^{2n}(x_0)>z>f^{2n+1}(x_0).$$

We assume that we are in case (5.22), the other case being symmetric. This implies that $b\leq z$. We are going to show that $z>d$. Suppose on the contrary that

$$z\in[b,d].\qquad\qquad (5.23)$$

Since z is a fixed point, we can define $z':=\min\{x\in[b,d]\mid g(x)=x\}$. Since $b\in\omega(x_0,g)$, we have $g(b)\in\omega(x_0,g)$; moreover b is not a fixed point for g (by Lemma 1.4 when $\omega(x_0,g)$ is finite, and by Proposition 5.23 when $\omega(x_0,g)$ is infinite). Hence

$$g(b)<b<z'$$

(recall that $\omega(x_0,f)\subset[a,b]$). Since z' is the minimal fixed point for g greater than b, this implies that

$$\forall x\in[b,z'),\ g(x)<x.\qquad\qquad (5.24)$$

See Figure 1.

Let $v:=\max g([c,b])$. Then $v\geq b$ because $g(\omega(x_0,g))=\omega(x_0,g)$, and $v\leq d$ because $g(L_1)\subset L_1$. Suppose that $v<z'$. Then $g([b,v])\subset[c,v]$ by (5.24), and $g([c,v])=g([c,b])\cup g([b,v])\subset[c,v]\cup[c,v]=[c,v]$. Thus $[c,v]$ is a g-invariant interval containing I_1. But this is a contraction to (5.20) because $[c,v]\neq L_1$. We deduce that $\max g([c,b])\geq z'$, and thus there exists $y\in[c,b]$ such that $g(y)=z'$.

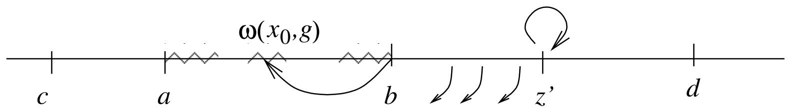


FIGURE 1. The points a, b, c, d, z' and the set $\omega(x_0, g)$ (represented by a zigzag).

We choose y maximal with this property; $y < b$ by (5.24). Let $x \in (y, z')$. The maximality of y and (5.24) imply that $g(x) < z'$. If $g(x) \leq y$, then $g(x) \leq y < x < z' = g(y)$, and thus $[y, x], [x, z']$ form a horseshoe for g, which contradicts the fact that $h_{top}(f) = 0$ by Proposition 4.6. Therefore,

$$\forall x \in (y, z'),\ y < g(x) < z'. \tag{5.25}$$

Using (5.25) and the fact that $g(y) = g(z') = z' > y$, we get

$$w := \inf g((y, z')) = \min g([y, z']) > y.$$

Thus $[y, w]$ is mapped into $[w, z']$, and $[w, z']$ is g-invariant (see Figure 2). We are

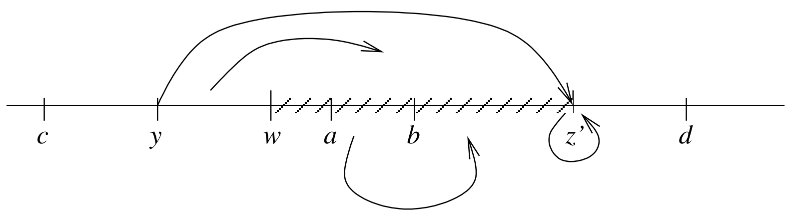


FIGURE 2. The interval $[y, w]$ is mapped into $[w, z']$, and $[w, z']$ (hatched) is g-invariant.

going to show that $w \leq a$. Recall that $y < b < z'$, so $[y, z']$ is a neighborhood of b. Since $b \in \omega(x_0, g)$, there exists i such that $g^i(x_0) \in [y, z'] = [y, w] \cup [w, z']$. Thus $g^j(x_0) \in [w, z']$ for all $j \geq i + 1$, which implies that $\omega(x_0, g) \subset [w, z']$. Since $a \in \omega(x_0, g)$, this implies

$$w \leq a.$$

Therefore, $[w, z']$ is a g-invariant interval containing $I_1 = [a, b]$, but this is a contradiction to (5.20) because $[w, z']$ does not contain L_1. We deduce that (5.23) is false, that is, $z > d = \max L_1$. According to (5.22), we have $z \leq \omega(f(x_0), f^2)$. Moreover, z does not belong to $f(L_1)$ because $f^2(L_1) = L_1$ and $z \notin L_1$. Since $f(L_1)$ is an interval containing $\omega(f(x_0), f^2)$, we conclude that $L_1 < z < f(L_1)$. This ends the step $k = 1$.

• Let $k \geq 1$ such that $k + 1 \in \mathcal{I}$ and suppose that (ii) is satisfied for k. Let i, j be such that $0 \leq i < j < 2^{k+1}$. If $j - i \neq 2^k$, then, according to the induction hypothesis, there exists a point z between $f^i(L_k)$ and $f^j(L_k)$, satisfying $f^{2^{k-1}}(z) = z$. Thus z is also between $f^i(L_{k+1})$ and $f^j(L_{k+1})$ because $L_{k+1} \subset L_k$. If $j = i + 2^k$, then, using (5.21), we can apply the case $k = 1$ to the map g_k and the point $f^i(x_0)$: there exists a point z strictly between $f^i(L_{k+1})$ and $f^j(L_{k+1}) = g_k(f^i(L_{k+1}))$ such that $f^{2^k}(z) = z$. This completes the induction.

It remains to show (vi). Let $k \geq 0$ be an integer such that $k+2 \in \mathcal{I}$. Then L_k contains the four disjoint intervals $L_{k+2}, f^{2^k}(L_{k+2}), f^{2^{k+1}}(L_{k+2}), f^{2^{k+2}}(L_{k+2})$ by (iii). One of these intervals is included in $\mathrm{Int}\,(L_k)$. Moreover, this interval contains $\omega(f^{2^k i}(x_0), f^{2k+2})$ for some $i \in [\![0,3]\!]$; thus there exists an integer $N \geq 0$ such that $f^N(x) \in L_k$. Since L_k is a 2^k-periodic interval containing $\omega(x_0, f^{2^k})$, this implies that N is a multiple of 2^k, so $f^N(L_k) = L_k$. Therefore $f^n(x) \in f^n(L_k)$ for all $n \geq N$. This concludes the proof of the proposition. □

LEMMA 5.25. *Let f be an interval map of zero topological entropy. If J is a nonempty (non necessarily closed) interval such that $f^p(J) = J$ and $(f^i(J))_{0\leq i<p}$ are pairwise disjoint, then p is a power of* 2.

PROOF. If J is reduced to one point, then it is a periodic point and thus p is a power of 2 by Theorem 4.58. From now on, we assume that J is non degenerate, which implies that $f^n(J)$ is a non degenerate interval for all $n \geq 0$. Since $f^p(\overline{J}) = \overline{J}$, there exists $x \in \overline{J}$ such that $f^p(x) = x$ by Lemma 1.11. By Theorem 4.58, the period of x is equal to 2^k for some $k \geq 0$, and thus there exists $m \geq 1$ such that $p = m2^k$. If $x \in J$, then $(f^i(x))_{0\leq i<p}$ are pairwise distinct, so $p = 2^k$.

Suppose that $x \in \partial J$ and $m \geq 3$. The point $x = f^{2^k}(x)$ belongs to $f^{2^k}(\overline{J})$. Since $f^{2^k}(J) \cap J = \emptyset$, this implies that x is an endpoint of $f^{2^k}(J)$. We also have $x = f^{2^{k+1}}(x) \in f^{2^{k+1}}(\overline{J})$, which implies that $x \in \partial f^{2^{k+1}}(J)$. But this contradicts the fact that $J, f^{2^k}(J), f^{2^{k+1}}(J)$ are pairwise disjoint non degenerate intervals. Therefore, if $x \in \partial J$, then $m = 1$ or 2 and p is a power of 2. □

The next lemma states that two points in the same infinite ω-limit set are f-separable if and only if they are separable by intervals in the family $(f^i(L_k))$ given by Proposition 5.24.

LEMMA 5.26. *Let f be an interval map of zero topological entropy and let a_0, a_1 be two distinct points in the same infinite ω-limit set $\omega(x_0, f)$. Let $(L_n)_{n\geq 0}$ be the intervals given by Proposition 5.24. Then the following assertions are equivalent:*

i) *a_0, a_1 are f-separable,*
ii) *there exist $n \geq 1$ and $i, j \in [\![0, 2^n - 1]\!]$ such that $i \neq j$, $a_0 \in f^i(L_n)$ and $a_1 \in f^j(L_n)$.*

PROOF. The implication (ii) ⇒ (i) is trivial. Suppose that the points a_0 and a_1 are f-separable. By definition, there exist an interval J and an integer $p \geq 1$ such that $a_0 \in J$, $a_1 \notin J$, $f^p(J) = J$ and $(f^i(J))_{0\leq i<p}$ are pairwise disjoint. By Lemma 5.25, p is a power of 2, that is, $p = 2^n$ for some $n \geq 0$. Since a_0 is in $\omega(x_0, f) \subset \bigcup_{i=0}^{2^n-1} f^i(L_n)$, there exists $i \in [\![0, 2^n - 1]\!]$ such that $a_0 \in f^i(L_n)$. We set $K := f^i(L_n) \cap J$. Then $f^{2^n}(K) \subset f^i(L_n) \cap J = K$ (recall that $f^{2^n}(L_n) = L_n$). Therefore, the interval K contains the three points $a_0, f^{2^n}(a_0), f^{2^{n+1}}(a_0)$, which belong to $\omega(x_0, f)$ (by Lemma 1.3(iii)) and are distinct by Proposition 5.23. This implies that $\mathrm{Int}\,(K) \cap \omega(x_0, f) \neq \emptyset$, and thus there exists an integer $m \geq 0$ such that $f^m(x_0) \in K$. Hence

$$\omega(x_0, f) \subset \bigcup_{k=0}^{2^n-1} f^k(\overline{K}) \quad \text{and} \quad \omega(f^m(x_0), f^{2^n}) \subset \overline{K}. \tag{5.26}$$

According to Proposition 5.24, $f^i(L_n)$ is the smallest f^{2^n}-invariant interval containing $\omega(f^i(x_0), f^{2^n})$. Therefore, (5.26) and the fact that $\overline{K} \subset f^i(L_n)$ imply that $\overline{K} = f^i(L_n)$. Since $a_1 \in \omega(x_0, f)$, there exists $j \in [\![0, 2^n - 1]\!]$ such that $a_1 \in f^j(L_n)$. We are going to show that $j \neq i$. Suppose on the contrary that $j = i$, that is, $a_1 \in f^i(L_n) = \overline{K}$. This implies that a_1 is an endpoint of $f^i(L_n)$ because $a_1 \notin J$ and $J \supset K$. Let a_2 denote the other endpoint of $f^i(L_n)$. There are two cases:

- Case 1: $K = f^i(L_n) \setminus \{a_1\}$.
- Case 2: $K = f^i(L_n) \setminus \{a_1, a_2\}$.

Recall that $f^{2^n}(K) \subset K$ and $f^{2^n}(f^i(L_n)) = f^i(L_n)$. In Case 1, this implies that $f^{2^n}(a_1) = a_1$. In Case 2, this implies that $f^{2^n}(a_1) \in \{a_1, a_2\}$ and $f^{2^n}(a_2) \in \{a_1, a_2\}$. In both cases, $f^{2^n}(a_1)$ is a periodic point, which is impossible by Proposition 5.23 because $f^{2^n}(a_1) \in \omega(x_0, f)$. We conclude that $i \neq j$, which proves the implication (i) $\Rightarrow$ (ii). ☐

Remark 5.27. In the previous proof, we saw that $\overline{J \cap f^i(L_n)} = f^i(L_n)$. Therefore, if J is any periodic closed interval containing a_0 with $a_0 \in \omega(x_0, f)$, then J contains $f^i(L_n)$ for some integers n, i such that $a_0 \in f^i(L_n)$.

Lemma 5.28. *Let f be an interval map of zero topological entropy and let x_0 be a point such that $\omega(x_0, f)$ is infinite.*

i) *If J is an interval containing three distinct points of $\omega(x_0, f)$, then J contains a periodic point.*
ii) *If U is an open interval such that $U \cap \omega(x_0, f) \neq \emptyset$, then there exists an integer $n \geq 0$ such that $f^n(U)$ contains a periodic point.*

Proof. i) Let J be an interval and let $z_1 < z_2 < z_3$ be three points in $J \cap \omega(x_0, f)$. Let $(L_n)_{n \geq 0}$ be the intervals given by Proposition 5.24 for $\omega(x_0, f)$. Suppose that, for every integer $n \geq 0$, there is $i_n \in [\![0, 2^n - 1]\!]$ such that $z_1, z_3 \in f^{i_n}(L_n)$. Since $z_2 \in (z_1, z_3) \cap \omega(x_0, f)$, there exist two positive integers $m > k$ such that the two points $f^{m-k}(x_0)$, $f^m(x_0)$ belong to (z_1, z_3). Since $[z_1, z_3] \subset f^{i_n}(L_n)$, this implies that $f^m(x_0) \in f^{k+i_n}(L_n) \cap f^{i_n}(L_n)$ for all $n \geq 0$. On the other hand, $f^{k+i_n}(L_n) \cap f^{i_n}(L_n) = \emptyset$ if $2^n > k$ because the intervals $(f^k(L_n))_{0 \leq k < 2^n}$ are pairwise disjoint; we get a contradiction. We deduce that there exist $n \geq 0$ and $i, j \in [\![0, 2^n - 1]\!]$ such that $i \neq j$, $z_1 \in f^i(L_n)$ and $z_3 \in f^j(L_n)$. We know by Proposition 5.24 that there exists a periodic point z between $f^i(L_n)$ and $f^j(L_n)$, so $z \in [z_1, z_3] \subset J$.

ii) Now we consider an open interval U such that $U \cap \omega(x_0, f) \neq \emptyset$ and let $y \in U \cap \omega(x_0, f)$. If U contains a periodic point, the proof is over. From now on, we suppose that U contains no periodic point. Let $L \supset U$ be the maximal interval containing no periodic point. Since U is open and contains $y \in \omega(x_0, f)$, there exist integers $k \geq 0$ and $n_2 > n_1 > 0$ such that the points $f^k(x_0), f^{k+n_1}(x_0)$ and $f^{k+n_2}(x_0)$ belong to U. The points $y, f^{n_1}(y)$ and $f^{n_2}(y)$ belong to $\omega(x_0, f)$ (by Lemma 1.3(iii)), and they are distinct by Proposition 5.23. If $y, f^{n_1}(y), f^{n_2}(y)$ belong to L, then L contains a periodic point by (i), which contradicts the definition of L. Thus there exists $i \in \{1, 2\}$ such that $f^{n_i}(y) \notin L$. The interval $f^{n_i}(U)$ contains both $f^{k+n_i}(x_0)$ and $f^{n_i}(y)$, with $f^{k+n_i}(x_0) \in L$ and $f^{n_i}(y) \notin L$, and thus the maximality of the interval L implies that $f^{n_i}(U)$ contains a periodic point. ☐

Proposition 5.30 below was shown by Smítal [**162**] in the case $x_0 = x_1$. It states that, if a_0, a_1 belong to the same infinite ω-limit set and are f-non separable, where f is a zero entropy interval map, then f admits a δ-scrambled Cantor set with $\delta = |a_1 - a_0|$. The next lemma is the first step of the proof of this result.

LEMMA 5.29. *Let f be an interval map of zero topological entropy and let x_0, x_1, a_0, a_1 be four points such that $\omega(x_0, f)$ and $\omega(x_1, f)$ are infinite, $a_0 \in \omega(x_0, f)$ and $a_1 \in \omega(x_1, f)$. Let $(L_n)_{n\geq 0}$ denote the intervals given by Proposition 5.24 for $\omega(x_0, f)$. Suppose that, for all $n \geq 0$, $\omega(x_1, f) \subset \bigcup_{i=0}^{2^n-1} f^i(L_n)$ and there exists $i_n \in [\![0, 2^n - 1]\!]$ such that both points a_0, a_1 belong to $J_n := f^{i_n}(L_n)$. Let A_0, A_1 be two intervals such that $a_0 \in \operatorname{Int}(A_0)$ and $a_1 \in \operatorname{Int}(A_1)$. Then there exists $m \geq 0$ such that $f^{2^m}(A_0) \cap f^{2^m}(A_1) \supset J_m$.*

PROOF. By Lemma 5.28(ii), there exist n_0 and n_1 such that $f^{n_0}(A_0)$ contains a periodic point y_0 and $f^{n_1}(A_1)$ contains a periodic point y_1. According to Theorem 4.58, the periods of y_0, y_1 are some powers of 2. Let 2^p be a common multiple of their periods and let q be such that $q > p$ and $2^q \geq \max\{n_0, n_1\}$. We fix $j \in \{0, 1\}$ and we set $y'_j := f^{2^q - n_j}(y_j)$. Then $f^{2^p}(y'_j) = y'_j$ and $y'_j \in f^{2^q}(A_j)$. Moreover, $y'_j \notin J_q$ because $J_q = f^{i_q}(L_q)$ is a periodic interval of period $2^q > 2^p$. Suppose that

$$y'_j < J_q, \tag{5.27}$$

the case with the reverse inequality being symmetric.

Let $g := f^{2^q}$. Then $g(y'_j) = y'_j$. The intervals J_{q+1} and $g(J_{q+1})$ are disjoint, $g^2(J_{q+1}) = J_{q+1}$ and $J_{q+1} \cup g(J_{q+1}) \subset J_q$. Moreover,

$$\{y'_j, g(a_j)\} \subset g(A_j) \quad \text{and} \quad g(a_j) \in g(J_{q+1}). \tag{5.28}$$

We consider two cases.

Case 1. If $g(J_{q+1}) > J_{q+1}$ (Figure 3), then, by connectedness, $g(A_j) \supset J_{q+1}$ by (5.28) and (5.27). Thus $g^2(A_j) \supset g(J_{q+1}) \cup \{y'_j\}$ and, by connectedness, $g^2(A_j)$ contains J_{q+1}. This implies that $g^4(A_j) \supset J_{q+1}$.

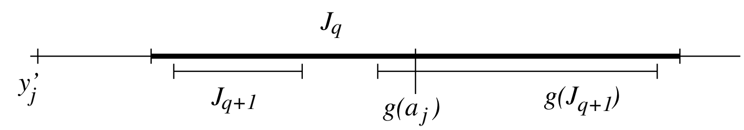


FIGURE 3. Relative positions in Case 1 of the proof of Lemma 5.29. The interval $g(A_j)$ contains y'_j and $g(a_j)$, so $g(A_j) \supset J_{q+1}$.

Case 2. If $g(J_{q+1}) < J_{q+1}$ (Figure 4), then $g^2(A_j)$ contains the points $g^2(a_j)$ and y'_j by (5.28). Since $a_j \in J_{q+1}$, the point $g^2(a_j)$ belongs to J_{q+1} too. Thus, by connectedness, $g^2(A_j)$ contains $g(J_{q+1})$ by (5.27). Then $g^3(A_j) \supset J_{q+1} \cup \{y'_j\}$, so

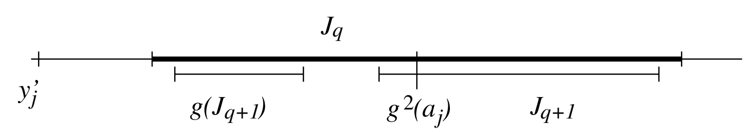


FIGURE 4. Relative positions in Case 2 of the proof of Lemma 5.29. The interval $g^2(A_j)$ contains y'_j and $g^2(a_j)$, so $g^2(A_j) \supset g(J_{q+1})$.

$g^3(A_j) \supset g(J_{q+1})$ by connectedness, and finally $g^4(A_i) \supset J_{q+1}$.

We conclude that $g^4(A_j) \supset J_{q+1} \supset J_{q+2}$ for $j \in \{0,1\}$. This is the required result with $m := q+2$. $\square$

PROPOSITION 5.30. *Let $f\colon I \to I$ be an interval map of zero topological entropy and let x_0, x_1, a_0, a_1 be four points such that $\omega(x_0, f)$ and $\omega(x_1, f)$ are infinite, $a_0 \in \omega(x_0, f)$, $a_1 \in \omega(x_1, f)$ and $a_0 \neq a_1$. Let $(L_n)_{n\geq 0}$ denote the intervals given by Proposition 5.24 for $\omega(x_0, f)$. Suppose that, for all $n \geq 0$, $\omega(x_1, f) \subset \bigcup_{i=0}^{2^n-1} f^i(L_n)$ and there exists $i_n \in [\![0, 2^n-1]\!]$ such that both points a_0, a_1 belong to $f^{i_n}(L_n)$. Then f has a δ-scrambled Cantor set with $\delta := |a_1 - a_0|$. Moreover, if K_0, K_1 are disjoint closed intervals such that $a_i \in \mathrm{Int}\,(K_i)$ for $i \in \{0,1\}$, then there exists an increasing sequence of integers $(n_k)_{k\geq 0}$ such that*

$$\forall (\alpha_k)_{k\geq 0} \in \{0,1\}^{\mathbb{Z}^+},\ \exists x \in I,\ \forall k \geq 0,\ f^{n_k}(x) \in K_{\alpha_k}. \tag{5.29}$$

PROOF. For every $n \geq 0$, we set $J_n := f^{i_n}(L_n)$. According to Proposition 5.24, the intervals $(f^i(J_n))_{0\leq i<2^n}$ are disjoint, $f^{2^n}(J_n) = J_n$ and $J_{n+1} \cup f^{2^n}(J_{n+1}) \subset J_n$.

First we build by induction two sequences of integers $(n(k))_{k\geq 0}$ and $(m(k))_{k\geq 0}$ and a family of closed subintervals $\{I_{\alpha_0\ldots\alpha_k} \mid k \geq 0, \alpha_i \in \{0,1\}\}$ satisfying the following properties for all $k \geq 0$ and all $(\alpha_0, \ldots, \alpha_{k+1}) \in \{0,1\}^{k+2}$:

i) $I_{\alpha_0\ldots\alpha_k\alpha_{k+1}} \subset I_{\alpha_0\ldots\alpha_k}$,
ii) $I_{\alpha_0\ldots\alpha_k} \cap I_{\beta_0\ldots\beta_k} = \emptyset$ if $(\alpha_0, \ldots, \alpha_k) \neq (\beta_0, \ldots, \beta_k)$, where $(\beta_0, \ldots, \beta_k) \in \{0,1\}^{k+1}$,
iii) $f^{n(k)}(I_{\alpha_0\ldots\alpha_k}) = J_{m(k)}$,
iv) $m(k) \geq k$ and $n(k+1) - n(k) = 2^{m(k+1)}$,
v) for $i \in \{0,1\}$, $f^{n(k)}(I_{\alpha_0\ldots\alpha_k i}) \subset [a_i - \frac{1}{k}, a_i + \frac{1}{k}]$.

Step $k = 0$. Let $\varepsilon \in (0, \frac{\delta}{2})$. We set $A_i := [a_i - \varepsilon, a_i + \varepsilon] \cap I$ for $i \in \{0,1\}$. According to the choice of ε, the intervals A_0, A_1 are disjoint. By Lemma 5.29, there exists an integer m such that $f^{2^m}(A_0) \cap f^{2^m}(A_1) \supset J_m$. Thus there exist closed subintervals $I_0 \subset A_0$ and $I_1 \subset A_1$ such that $f^{2^m}(I_i) = J_m$ for $i \in \{0,1\}$ (Lemma 1.13(i)). Letting $m(0) = m$ and $n(0) = 2^m$, this ends the construction at step $k = 0$.

Step $k+1$. Suppose that $n(k), m(k)$ and $(I_{\alpha_0\ldots\alpha_k})_{(\alpha_0,\ldots,\alpha_k)\in\{0,1\}^{k+1}}$ are already defined. Let $\varepsilon \in (0, \min\{\frac{1}{k+1}, \frac{\delta}{2}\})$. We set $B_i := [a_i - \varepsilon, a_i + \varepsilon] \cap I$ for $i \in \{0,1\}$. According to the choice of ε, the intervals B_0, B_1 are disjoint.

We set $g := f^{2^{m(k)}}$. The interval $J_{m(k)}$ contains the four disjoint intervals $(g^i(J_{m(k)+2}))_{0\leq i\leq 3}$. We order these intervals from left to right, and we call $J'_{m(k)+2}$ the second one. Let $j \in [\![0, 3]\!]$ be such that $g^j(J'_{m(k)+2}) = J_{m(k)+2}$. For $i \in \{0,1\}$, let a'_i be a point in $J'_{m(k)+2}$ such that $g^j(a'_i) = a_i$. It is clear that, for all $n \geq 1$, the points a'_0, a'_1 are in the same interval among $(f^k(L_n))_{0\leq k\leq 2^n-1}$, otherwise it would be false for a_0, a_1. Moreover, the points a'_0, a'_1 belong to $\mathrm{Int}\left(J_{m(k)}\right)$ according to the choice of $J'_{m(k)+2}$.

Since B_i is a neighborhood of a_i, the set $g^{-j}(B_i)$ is a neighborhood of a'_i for $i \in \{0,1\}$. Let U_i be the connected component of $g^{-j}(B_i)$ containing a'_i. Then $U_0 \cap J_{m(k)}$ and $U_1 \cap J_{m(k)}$ are a connected neighborhood of a'_0 and a'_1 respectively. Thus, according to Lemma 5.29, there exists an integer $q \geq 0$ such that, for $i \in \{0,1\}$, $f^{2^q}(U_i \cap J_{m(k)}) \supset f^p(L_q)$, where $p \in [\![0, 2^q-1]\!]$ is the integer such that

$a'_0, a'_1 \in f^p(L_q)$. Since $g^j(U_i \cap J_{m(k)}) \subset B_i \cap J_{m(k)}$, we have

$$f^{2^q}(B_i \cap J_{m(k)}) \supset f^{2^q}(g^j(U_i \cap J_{m(k)})) \supset g^j(f^p(L_q)). \tag{5.30}$$

Moreover, $g^j(f^p(L_q))$ contains $a_0 = g^j(a'_0)$ and $a_1 = g^j(a'_1)$, which implies that

$$g^j(f^p(L_q)) = J_q \tag{5.31}$$

because $(f^k(J_q))_{0\le k\le 2^q-1}$ is a cycle of intervals and J_q is the unique interval of this cycle containing a_0, a_1. We choose $m(k+1) \geq \max\{q, k+1\}$. Then $f^{2^{m(k+1)}-2^q}(J_q) = J_q$, and thus, by (5.30) and (5.31),

$$f^{2^{m(k+1)}}(B_i \cap J_{m(k)}) \supset f^{2^{m(k+1)}-2^q}(g^j(f^p(L_q))) = f^{2^{m(k+1)}-2^q}(J_q) = J_q \supset J_{m(k+1)}.$$

Then, for $i \in \{0,1\}$, there exists a closed subinterval $F_i \subset B_i \cap J_{m(k)}$ such that $f^{2^{m(k+1)}}(F_i) = J_{m(k+1)}$ (by Lemma 1.13(i)). Let $(\alpha_0, \ldots, \alpha_k) \in \{0,1\}^{k+1}$ and $i \in \{0,1\}$. Since $f^{n(k)}(I_{\alpha_0\ldots\alpha_k}) = J_{m(k)}$ by the induction hypothesis, there exists a closed subinterval $I_{\alpha_0\ldots\alpha_k i} \subset I_{\alpha_0\ldots\alpha_k}$ such that $f^{n(k)}(I_{\alpha_0\ldots\alpha_k i}) = F_i$. By choice of F_i, this implies that $f^{n(k)+2^{m(k+1)}}(I_{\alpha_0\ldots\alpha_k i}) = J_{m(k+1)}$. We define $n(k+1) := n(k) + 2^{m(k+1)}$. It is clear that properties (i), (iii), (iv), (v) are satisfied. The intervals $I_{\alpha_0\ldots\alpha_k 0}$ and $I_{\alpha_0\ldots\alpha_k 1}$ are disjoint because their images under $f^{n(k+1)}$ are included respectively in B_0 and B_1. Moreover, $I_{\alpha_0\ldots\alpha_k\alpha_{k+1}} \cap I_{\beta_0\ldots\beta_k\beta_{k+1}} = \emptyset$ if $(\alpha_0, \ldots, \alpha_k) \neq (\beta_0, \ldots, \beta_k)$ because $I_{\alpha_0\ldots\alpha_k\alpha_{k+1}} \subset I_{\alpha_0\ldots\alpha_k}$, $I_{\beta_0\ldots\beta_k\beta_{k+1}} \subset I_{\beta_0\ldots\beta_k}$ and these sets are disjoint by the induction hypothesis. This gives (ii) and the induction is over.

Now we prove the proposition. Let $\Sigma := \{0,1\}^{\mathbb{Z}^+}$, endowed with the product topology. For every $\bar\alpha = (\alpha_n)_{n\ge 0} \in \Sigma$, we set

$$I_{\bar\alpha} := \bigcap_{n=0}^{+\infty} I_{\alpha_0\ldots\alpha_n}.$$

By (i), this is a decreasing intersection of nonempty compact intervals, and thus $I_{\bar\alpha}$ is a nonempty compact interval. Moreover, $I_{\bar\alpha} \cap I_{\bar\beta} = \emptyset$ if $\bar\alpha \neq \bar\beta$, $\bar\alpha, \bar\beta \in \Sigma$. We define

$$E := \{\bar\alpha \in \Sigma \mid I_{\bar\alpha} \text{ is not reduced to a single point}\}.$$

The set E is at most countable because the sets $(I_{\bar\alpha})_{\bar\alpha\in E}$ are disjoint intervals and they are non degenerate by definition. We set

$$X := \left(\bigcap_{n=0}^{+\infty} \bigcup_{\substack{\alpha_i \in \{0,1\} \\ i \in [\![0,n]\!]}} I_{\alpha_0\ldots\alpha_n} \right) \setminus \bigcup_{\bar\alpha\in E} \operatorname{Int}(I_{\bar\alpha}).$$

This is a totally disconnected compact set. We define

$$\begin{array}{rccll} \varphi\colon & X & \longrightarrow & \Sigma & \\ & x & \longmapsto & \bar\alpha & \text{if } x \in I_{\bar\alpha}. \end{array}$$

The map φ is well defined and is clearly onto. Let δ_n be the minimal distance between two distinct intervals of the form $I_{\alpha_0\ldots\alpha_n}$. Clearly $\delta_n > 0$ because these intervals are closed and pairwise disjoint. Let $x, y \in X$ and $(\alpha_n)_{n\ge0} := \varphi(x)$, $(\beta_n)_{n\ge0} := \varphi(y)$. If $|x-y| < \delta_n$, then necessarily $\alpha_0\ldots\alpha_n = \beta_0\ldots\beta_n$, and thus φ is continuous.

Let K_0, K_1 be two disjoint closed intervals such that $a_i \in \mathrm{Int}(K_i)$ for $i \in \{0,1\}$. Then there exists a positive integer N such that $[a_i - \frac{1}{N}, a_i + \frac{1}{N}] \subset K_i$ for $i \in \{0,1\}$. Let $\bar{\alpha} = (\alpha_n)_{n\geq 0} \in \Sigma$ and $x \in \varphi^{-1}(\alpha)$. Then, according to (v), for every $k \geq N$, $f^{n(k)}(x) \in f^{n(k)}(I_{\alpha_0\ldots\alpha_{k+1}}) \subset K_{\alpha_{k+1}}$. This proves statement (5.29) in the proposition (with the the sequence $(n_{N+k})_{k\geq 0}$).

We define $\psi\colon \Sigma \to \Sigma$ by

$$\psi((\alpha_n)_{n\geq 0}) := (0\alpha_0 00\alpha_0\alpha_1 \ldots 0^n\alpha_0\alpha_1\ldots\alpha_{n-1}\ldots) \quad \text{where } 0^n = \underbrace{0\cdots 0}_{n \text{ times}}.$$

The map ψ is clearly continuous and one-to-one, and thus $\psi(\Sigma)$ is compact and uncountable. For every $\bar{\alpha} \in \Sigma$, we choose $x_{\bar{\alpha}} \in X$ such that $\varphi(x_{\bar{\alpha}}) = \psi(\bar{\alpha})$ and we set $S := \{x_{\bar{\alpha}} \in X \mid \bar{\alpha} \in \Sigma\}$. If $\psi(\bar{\alpha}) \notin E$, there is a unique choice for x_α, and if $\psi(\bar{\alpha}) \in E$, there are two possible choices. Therefore, S is equal to $\varphi^{-1}(\psi(\Sigma))$ deprived of a countable set, and thus it is an uncountable Borel set. By Theorem 5.16, there exists a Cantor set $C \subset S$.

Let $\bar{\alpha}, \bar{\beta}$ be two distinct elements of Σ, and let $i \geq 0$ be such that $\alpha_i \neq \beta_i$. According to the definition of ψ, for every $N \geq 0$, there exists $k \geq N$ such that the $(k+1)$-th coordinate of $\psi(\bar{\alpha})$ and $\psi(\bar{\beta})$ are equal respectively to α_i and β_i, and thus they are distinct. Using (v), this implies that either $f^{n(k)}(x_{\bar{\alpha}})$ belongs to $[a_0 - \frac{1}{k}, a_0 + \frac{1}{k}]$ and $f^{n(k)}(x_{\bar{\beta}})$ belongs to $[a_1 - \frac{1}{k}, a_1 + \frac{1}{k}]$, or the converse. In particular, $|f^{n(k)}(x_{\bar{\alpha}}) - f^{n(k)}(x_{\bar{\beta}})| \geq \delta - \frac{2}{k}$, which implies that

$$\text{for all } x, x' \in S,\ x \neq x',\ \limsup_{n\to+\infty} |f^n(x) - f^n(x')| \geq \delta.$$

By definition of ψ, for every $N \geq 0$, there exists $k \geq N$ such that, for all $\bar{\alpha} \in \Sigma$, the $(k+1)$-th coordinate of $\psi(\bar{\alpha})$ is equal to 0. Using (v), we have that, for all $\bar{\alpha} \in \Sigma$, $f^{n(k)}(x_{\bar{\alpha}}) \in [a_0 - \frac{1}{k}, a_0 + \frac{1}{k}]$, and hence

$$\text{for all } x, x' \in S,\ \liminf_{n\to+\infty} |f^n(x) - f^n(x')| = 0.$$

We deduce that S is a δ-scrambled set, and thus C is a δ-scrambled Cantor set. $\square$

In the next proposition, assertion (ii) is stated in [**162**, Theorem 2.4]. In view of Lemma 5.5, it implies that, if f is a zero entropy map admitting no pair of f-non separable points in the same infinite ω-limit set, then f is not chaotic in the sense of Li-Yorke.

Remark 5.31. In [**162**], it is claimed without explanation that, for every $\varepsilon > 0$, there exists $n \geq 0$ such that $\max_{i\in[\![0,2^n-1]\!]} |f^i(L_n)| < \varepsilon$, where $(L_n)_{n\geq 0}$ are the intervals given by Proposition 5.24. It is not clear to us whether this property does hold. The weaker assertion (i) below is sufficient to prove Theorem 5.21. See also Lemma 5.48 for a refinement.

Proposition 5.32. *Let f be an interval map of zero topological entropy.*

i) *If $\omega(x,f)$ is infinite and contains no f-non separable points, then*

$$\lim_{n\to+\infty} \max_{i\in[\![0,2^n-1]\!]} \mathrm{diam}(\omega(f^i(x), f^{2^n})) = 0.$$

ii) *Suppose that all pairs of distinct points in an infinite ω-limit set are f-separable. Then every point x is approximately periodic.*

Proof. i) Suppose that $\omega(x,f)$ is infinite and contains no f-non separable points. We set $a_n^i := \min \omega(f^i(x), f^{2^n})$, $b_n^i := \max \omega(f^i(x), f^{2^n})$ and $I_n^i := [a_n^i, b_n^i]$ for all $n \geq 0$ and all $i \in [\![0, 2^n - 1]\!]$. Then by Lemma 1.3, $a_n^i, b_n^i \in \omega(x,f)$ and

$$\forall i \in [\![0, 2^n - 1]\!],\ I_{n+1}^i \cup I_{n+1}^{i+2^n} \subset I_n^i. \tag{5.32}$$

Suppose that there exists $\varepsilon > 0$ such that, for all $n \geq 0$, there is $i \in [\![0, 2^n - 1]\!]$ with $|I_n^i| \geq \varepsilon$. Using (5.32), we can build a sequence $(i_n)_{n \geq 0}$ such that

$$\forall n \geq 0,\ i_n \in [\![0, 2^n - 1]\!],\ I_{n+1}^{i_{n+1}} \subset I_n^{i_n} \text{ and } |I_n^{i_n}| \geq \varepsilon.$$

We set $J := \bigcap_{n \geq 1} I_n^{i_n}$. Since it is a decreasing intersection of compact intervals, J is a compact interval and $|J| \geq \varepsilon$. We write $J = [a, b]$. Then

$$a = \lim_{n \to +\infty} a_n^{i_n} \quad \text{and} \quad b = \lim_{n \to +\infty} b_n^{i_n}.$$

Since $\omega(x,f)$ is a closed set (by Lemma 1.3(i)), the points a, b belong to $\omega(x,f)$. The intervals $(L_n)_{n \geq 0}$ given by Proposition 5.24 are defined in such a way that $I_n^i \subset f^i(L_n)$ for all $n \geq 0$ and all $i \in [\![0, 2^n - 1]\!]$. Therefore $a, b \in f^{i_n}(L_n)$ for all $n \geq 0$. Then Lemma 5.26 implies that a, b are f-non separable, which is a contradiction (notice that $a \neq b$ because $b - a \geq \varepsilon$). We deduce that, for all $\varepsilon > 0$, there exists $m \geq 0$ such that $|I_m^i| < \varepsilon$ for all $i \in [\![0, 2^m - 1]\!]$. Combined with the fact that these intervals are nested, this gives (i).

ii) Let x be a point and $\varepsilon > 0$. We split the proof into two cases depending on $\omega(x,f)$ being finite or not.

First we suppose that $\omega(x,f)$ is infinite. The intervals I_n^i are defined as above. It was shown in the proof of (i) that there exists an integer m such that $|I_m^i| < \varepsilon$ for all $i \in [\![0, 2^m - 1]\!]$. Moreover, there exists a point $z \in I_m^0$ such that $f^{2^m}(z) = z$ and $f^i(z) \in I_m^i$ for all $i \in [\![0, 2^m - 1]\!]$ (Lemma 1.13(ii)). Since f is uniformly continuous, there exists $\delta > 0$ such that

$$\forall y, y',\ |y - y'| \leq \delta \Rightarrow \forall i \in [\![0, 2^m - 1]\!],\ |f^i(y) - f^i(y')| \leq \varepsilon.$$

Let N be a positive integer such that, for all $k \geq N$, there exists a point a_k in $\omega(x, f^{2^m})$ with $|f^{k2^m}(x) - a_k| \leq \delta$. For all $i \in [\![0, 2^m - 1]\!]$, the two points $f^i(z), f^i(a_k)$ belong to I_m^i, so

$$|f^{k2^m+i}(x) - f^{k2^m+i}(z)| \leq |f^{k2^m+i}(x) - f^i(a_k)| + |f^i(a_k) - f^i(z)| \leq 2\varepsilon.$$

We get: $\forall n \geq N2^m$, $|f^n(x) - f^n(z)| \leq 2\varepsilon$.

Now we suppose that $\omega(x,f)$ is finite. By Lemma 1.4, the set $\omega(x,f)$ is a periodic orbit. Let p be the period of this orbit and $z := \lim_{n \to +\infty} f^{np}(x)$; the point z is periodic and $f^p(z) = z$. Since f is continuous, there exists $\delta > 0$ such that

$$\forall y,\ |y - z| \leq \delta \Rightarrow \forall i \in [\![0, p - 1]\!],\ |f^i(y) - f^i(z)| \leq \varepsilon.$$

Let N be an integer such that $|f^{np}(x) - z| \leq \delta$ for all $n \geq N$. Then

$$\forall m \geq Np,\ |f^m(x) - f^m(z)| \leq \varepsilon.$$

This completes the proof of (ii). □

Now we are ready to give the proof of Theorem 5.21, which follows from Propositions 5.30 and 5.32. For clarity, we recall the statement of the theorem.

Theorem 5.21. Let f be an interval map of zero topological entropy. The following properties are equivalent:

i) f is chaotic in the sense of Li-Yorke,
ii) there exists a δ-scrambled Cantor set for some $\delta > 0$,
iii) there exists a point x that is not approximately periodic,
iv) there exists an infinite ω-limit set containing two f-non separable points.

PROOF. If (iv) does not hold, then, according to Proposition 5.32(ii), all points x are approximately periodic. By refutation, we get (iii)$\Rightarrow$(iv).

Suppose that (iv) holds, that is, there exists an infinite ω-limit set $\omega(x_0, f)$ containing two f-non separable points a_0, a_1. Then, according to Lemma 5.26 and Proposition 5.30 applied with $x_1 = x_0$, there exists a δ-scrambled Cantor set with $\delta := |a_1 - a_0|$, which is (ii). Obviously, (ii)$\Rightarrow$(i).

Suppose that (iii) does not hold, that is, every point is approximately periodic. Then Lemma 5.5 implies that there is no Li-Yorke pair. By refutation, we get (i)$\Rightarrow$(iii). $\square$

5.5. One Li-Yorke pair implies chaos in the sense of Li-Yorke

Kuchta and Smítal showed that, for interval maps, the existence of one Li-Yorke pair of points is enough to imply the existence of a δ-scrambled Cantor set [**104**]. We give a different proof, suggested by Jiménez López, which follows easily from Theorem 5.21.

PROPOSITION 5.33. *Let f be an interval map. If there exists one Li-Yorke pair, then there exists a δ-scrambled Cantor for some $\delta > 0$.*

PROOF. If $h_{top}(f) > 0$, the result follows from Theorem 5.17. We assume that $h_{top}(f) = 0$. Let (x, y) be a Li-Yorke pair. By Lemma 5.5, either x or y is not approximately periodic. Therefore, the result is given by the implication (iv)$\Rightarrow$(ii) in Theorem 5.21. $\square$

As a corollary, we get the following summary theorem. We shall see another condition equivalent to chaos in the sense of Li-Yorke in the next section.

THEOREM 5.34. *Let f be an interval map. The following properties are equivalent:*

i) *there exists one Li-Yorke pair,*
ii) *f is chaotic in the sense of Li-Yorke,*
iii) *f admits a δ-scrambled Cantor set for some $\delta > 0$,*
iv) *there exists a point x that is not approximately periodic.*

PROOF. The first three assertions are equivalent by Theorem 5.33. According to Lemma 5.5, we have (ii)$\Rightarrow$(iv). If $h_{top}(f) = 0$, then (iv)$\Rightarrow$(ii) by Theorem 5.21. If $h_{top}(f) > 0$, then the equivalence follows from Theorem 5.18. $\square$

Remarks on graph maps. A key tool to generalize the results of the last two sections to graphs is the topological characterization of ω-limit sets of graph maps, which was given by Blokh [**46, 51**]; see also the more recent paper of Hric and Málek [**89**] (the classification of ω-limit sets in [**89**] is equivalent to the one in [**46, 51**], although the equivalence is not straightforward and does not seem to be explicitly proved in the literature). We rather follow Blokh's works.

THEOREM 5.35. *Let $f: G \to G$ be a graph map of zero topological entropy and $x \in G$. If $\omega(x, f)$ is infinite, it is of one of the following kinds:*

• Solenoidal*: there exist a sequence of subgraphs $(G_n)_{n\geq 1}$ and an increasing sequence of positive integers $(k_n)_{n\geq 1}$ such that $(f^i(G_n))_{0\leq i<k_n}$ is a cycle of graphs and, for all $n \geq 1$, $G_{n+1} \subset G_n$, k_{n+1} is a multiple of k_n and $\omega(f^i(x), f^{k_n}) \subset f^i(G_n)$ for all $i \in [\![0, k_n - 1]\!]$ (which implies that $\omega(x, f) \subset \bigcup_{i=0}^{k_n-1} f^i(G_n)$).*
• Circumferential*: $\omega(x, f)$ contains no periodic point and there exists a minimal cycle of graphs $(f^i(G'))_{0\leq i<k}$ such that $\omega(x, f^k) \subset G'$ (which implies that $\omega(x, f) \subset \bigcup_{i=0}^{k-1} f^i(G')$).*

Notice that a solenoidal set cannot contain periodic points, and thus, for a zero entropy graph map, any infinite ω-limit set contains no periodic point. That is, Proposition 5.23 is valid for graph maps too.

Blokh [**48, 49**] showed that, in the case of a circumferential ω-limit set, $f^k|_{G'}$ is almost conjugate to an irrational rotation, that is, semi-conjugate by a map that collapses any connected component of $G' \setminus \omega(x, f^k)$ to a single point. In particular, this implies that a tree map has no circumferential set.

THEOREM 5.36. *Let $f\colon G \to G$ be a graph map and $x \in G$. Suppose that $\omega(x, f)$ is circumferential, and let $(f^i(G'))_{0\leq i<k}$ denote the minimal cycle of graphs containing $\omega(x, f)$, with $G' \supset \omega(x, f^k)$. Then there exists an irrational rotation $R\colon \mathbb{S} \to \mathbb{S}$, and a semi-conjugacy $\varphi\colon G' \to \mathbb{S}$ between $f^k|_{G'}$ and R such that*

- $\varphi(\omega(x, f^k)) = \mathbb{S}$,
- $\forall y \in \mathbb{S}$, *$\varphi^{-1}(y)$ is connected,*
- $\forall y \in \mathbb{S}$, $\varphi^{-1}(y) \cap \omega(x, f^k) = \partial\varphi^{-1}(y)$.

In [**150**], the author and Snoha studied chaos in the sense of Li-Yorke for graph maps. We present the main ideas. Suppose that (x, y) is a Li-Yorke pair for the graph map f. We showed that neither $\omega(x, f)$ nor $\omega(y, f)$ can be circumferential [**150**, proof of Theorem 3]. Moreover, it is easy to see that either $\omega(x, f)$ or $\omega(y, f)$ is infinite. Therefore, if $h_{top}(f) = 0$, one of these ω-limit sets is solenoidal. If $\omega(x, f)$ is solenoidal, with the notation of Theorem 5.35, then for all large enough n, there exists $i \geq 0$ such that $J := f^i(G_n)$ is an interval (because the graph has finitely many branching points, and thus one of the graphs $(f^i(G_n))_{0\leq i<k_n}$ contains no branching point if k_n is large enough). We have $\omega(f^i(x), f^{k_n}) \subset J$; in addition, it is possible to show that one can choose n, i such that $f^i(x), f^i(y) \in J$. Thus $(f^i(x), f^i(y))$ is a Li-Yorke pair for $f^{k_n}|_J$, and we come down to the interval case. On the other hand, Theorem 5.18 applies when $h_{top}(f) > 0$. These ideas make it possible to show the following result [**150**, Theorem 3].

PROPOSITION 5.37. *Let $f\colon G \to G$ be a graph map. The following properties are equivalent:*

i) *there exists one Li-Yorke pair,*
ii) *f is chaotic in the sense of Li-Yorke,*
iii) *there exists a δ-scrambled Cantor set for some $\delta > 0$.*

REMARK 5.38. Contrary to what happens for graph maps, there exist topological dynamical systems admitting a finite (resp. countable) scrambled set but no infinite (resp. uncountable) scrambled set [**31**].

5.6. Topological sequence entropy

Any positive entropy interval map is chaotic in the sense of Li-Yorke (Theorem 5.17), but the converse is not true (see Example 5.59 below). We are going to see that an interval map is chaotic in the sense of Li-Yorke if and only if it has positive topological sequence entropy.

5.6.1. Definition of sequence entropy. The notion of topological sequence entropy was introduced by Goodman [**84**]. Its definition is analogous to the one of topological entropy, the difference is that one considers a subsequence of the family of all iterates of the map. The definition we give is analogous to Bowen's formula (Theorem 4.5), but topological sequence entropy can also be defined using open covers in a similar way as topological entropy in Section 4.1.1.

DEFINITION 5.39. Let (X, f) be a topological dynamical system and let $A = (a_n)_{n\geq 0}$ be an increasing sequence of non negative integers. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. A set $E \subset X$ is (A, n, ε)-*separated* if for all distinct points x, y in E, there exists $k \in [\![0, n-1]\!]$ such that $d(f^{a_k}(x), f^{a_k}(y)) > \varepsilon$. Let $s_n(A, f, \varepsilon)$ denote the maximal cardinality of an (A, n, ε)-separated set. The set E is an (A, n, ε)-*spanning set* if for all $x \in X$, there exists $y \in E$ such that $d(f^{a_k}(x), f^{a_k}(y)) \leq \varepsilon$ for all $k \in [\![0, n-1]\!]$. Let $r_n(A, f, \varepsilon)$ denote the minimal cardinality of an (A, n, ε)-spanning set.

The *topological sequence entropy* of f with respect to the sequence A is

$$h_A(f) := \lim_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{n}\log s_n(A, f, \varepsilon) = \lim_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{n}\log r_n(A, f, \varepsilon).$$

REMARK 5.40. As in Lemma 4.4, we have

- if $0 < \varepsilon' < \varepsilon$, then $s_n(A, f, \varepsilon') \geq s_n(A, f, \varepsilon)$ and $r_n(A, f, \varepsilon') \geq r_n(A, f, \varepsilon)$,
- $r_n(A, f, \varepsilon) \leq s_n(A, f, \varepsilon) \leq r_n(A, f, \frac{\varepsilon}{2})$.

This implies that the two limits in the definition above exist and are equal. Therefore, $h_A(f)$ is well defined.

According to the definition, $h_{top}(f) = h_A(f)$ with $A := (n)_{n\geq 0}$.

5.6.2. Li-Yorke chaos and sequence entropy. The rest of this section will be devoted to proving the following theorem, due to Franzová and Smítal [**80**].

THEOREM 5.41. *Let f be an interval map. Then f is chaotic in the sense of Li-Yorke if and only if there exists an increasing sequence A such that $h_A(f) > 0$.*

The "only if" part can be easily shown by using previous results in this chapter; this is done in Proposition 5.42. Before proving the reverse implication, we shall need to show several preliminary results; this will be done in Subsections 5.6.4 and 5.6.5.

5.6.3. Li-Yorke chaos implies positive sequence entropy.

PROPOSITION 5.42. *If an interval map $f: I \to I$ is chaotic in the sense of Li-Yorke, there exists an increasing sequence A such that $h_A(f) > 0$.*

PROOF. If f has positive topological entropy, then $h_A(f) = h_{top}(f) > 0$ with $A := (n)_{n\geq 0}$. From now on, we assume that $h_{top}(f) = 0$. According to Theorem 5.21, there exist two f-non separable points a_0, a_1 belonging to the same infinite ω-limit set. Let J_0, J_1 be two disjoint closed intervals such that $a_i \in \operatorname{Int}(J_i)$ for $i \in \{0, 1\}$. By Lemma 5.26 and Proposition 5.30, there exists an increasing sequence of positive integers $A = (n_k)_{k\geq 0}$ such that

$$\forall \bar{\alpha} = (\alpha_k)_{k\geq 0} \in \{0,1\}^{\mathbb{Z}^+},\ \exists x_{\bar{\alpha}} \in I,\ \forall k \geq 0,\ f^{n_k}(x_{\bar{\alpha}}) \in J_{\alpha_k}.$$

For all $n \geq 1$, we set

$$E_n := \{x_{(\alpha_k)_{k\geq 0}} \mid \forall k \geq n, \alpha_k = 0 \text{ and } \alpha_0, \dots, \alpha_{n-1} \in \{0,1\}\}.$$

Let $\delta > 0$ be the distance between J_0 and J_1. Then E_n is an (A, n, ε)-separated set for all $\varepsilon \in (0, \delta)$, and $\#E = 2^n$. Thus $s_n(A, f, \varepsilon) \geq 2^n$ for all $\varepsilon \in (0, \delta)$, and so $h_A(f) \geq \log 2 > 0$. $\square$

5.6.4. Preliminary results on ω-limit set. In this subsection, we are going to show several results concerning the ω-limit set of an interval map. These results are due to Sharkovsky [**157**]; see also [**41**, Chapter IV] (in English). In [**119**], Mai and Sun generalized these results to graph maps. For Lemma 5.44, we follow the ideas of Mai and Sun [**119**, Proposition 2], whose proof is simpler. Recall that the ω-limit set of a map $f\colon I \to I$ is

$$\omega(f) := \bigcup_{x \in I} \omega(x, f).$$

REMARK 5.43. In the previous version of this book (v4 on arxiv) as well as in [**157**] and [**41**, Proposition IV.6], Lemma 5.44 was stated without the part (i)-(ii), and its proof was split into two cases, the second one being more difficult to deal with. In the previous version of this book, the main part of the second case was isolated in Lemma 5.43, whose assumptions were the following ones.

Let $f\colon I \to I$ be an interval map and $c \in I \setminus \{\max I\}$. Let $J_0 := [c, c']$ for some $c' > c$. Suppose that:

$$\forall n \geq 1,\ \forall d > c,\ [c, d] \not\subset f^n(J_0), \tag{5.33}$$

$$\forall n \geq 1,\ f^n(c) \notin J_0. \tag{5.34}$$

Let $\mathcal{V}_c := \{U$ nonempty open subinterval $\mid U \subset J_0, \inf U = c\}$. Suppose that

$$\forall U \in \mathcal{V}_c,\ \exists k \geq 1,\ U \cap f^k(U) \neq \emptyset. \tag{5.35}$$

Actually, one can show, by using the proof of Lemma 5.44, that these assumptions are never satisfied, and thus the second case of the previous proofs was void.

LEMMA 5.44. *Let $f\colon I \to I$ be an interval map and $c \in I \setminus \{\max I\}$ (resp. $c \in I \setminus \{\min I\}$). Suppose that for every nonempty open interval U such that $\inf U = c$ (resp. $\sup U = c$),*

$$\exists k \geq 1, U \cap f^k(U) \neq \emptyset. \tag{5.36}$$

Then $c \in \omega(f)$. Moreover, one of the following statements holds:

i) *There exists $x > c$ (resp. $x < c$) and an increasing sequence of positive integers $(m_k)_{k\geq 0}$ such that for all $k \geq 0$, $f^{m_k}(x) > c$ (resp. $f^{m_k}(x) < c$) and $\lim_{k\to+\infty} f^{m_k}(x) = c$.*

ii) *c is a periodic point.*

PROOF. We deal only with the case $c \in I \setminus \max\{I\}$ and $\inf U = c$, the other case being symmetric. First we suppose

$$\forall \varepsilon > 0, \exists a, b \in (c, c+\varepsilon] \text{ with } a < b,\ \exists d > c, \exists n \geq 1, f^n([a, b]) = [c, d]. \tag{5.37}$$

A straightforward induction shows that there exist decreasing sequences of points $(a_k)_{k\geq 0}$, $(b_k)_{k\geq 0}$ and a sequence of positive integers $(n_k)_{k\geq 0}$ such that, for all $k \geq 0$,

$c < a_k < b_k$, $\lim_{k\to+\infty} b_k = c$ and $f^{n_k}([a_k, b_k]) \supset [a_{k+1}, b_{k+1}]$. Thus we have the following coverings:

$$[a_0, b_0] \xrightarrow[f^{n_0}]{} [a_1, b_1] \xrightarrow[f^{n_1}]{} [a_2, b_2] \xrightarrow[f^{n_2}]{} \cdots [a_k, b_k] \xrightarrow[f^{n_k}]{} [a_{k+1}, b_{k+1}] \cdots . \tag{5.38}$$

Using inductively Lemma 1.13, we can build a sequence of closed intervals $(I_k)_{k\geq 0}$ such that

$$I_0 := [a_0, b_0],\ I_{k+1} \subset I_k \text{ and } \forall k \geq 0,\ f^{n_0+\cdots+n_k}(I_k) = [a_{k+1}, b_{k+1}]. \tag{5.39}$$

Let x be in $\bigcap_{k\geq 0} I_k$ (this set is nonempty because it is a decreasing intersection of nonempty compact sets). For all $k \geq 0$, we set $m_k := n_0 + \cdots + n_k$. Then $f^{m_k}(x) \in [a_{k+1}, b_{k+1}]$ for all $k \geq 0$. Thus $\lim_{k\to+\infty} f^{m_k}(x) = c$, which implies that $c \in \omega(x, f)$. Moreover, $f^{m_k}(x) > c$ for all $k \geq 0$ and statement (i) holds.

From now on, we assume that (5.37) does not hold. We are going to show that c is a periodic point, that is, statement (ii) holds, which implies that $c \in \omega(c, f)$. Assume on the contrary that c is not a periodic point. The negation of (5.37) means that there exists $\varepsilon > 0$ such that:

$$\forall a, b \in (c, c+\varepsilon] \text{ with } a < b, \forall d > c, \forall k \geq 1, f^k([a, b]) \neq [c, d]. \tag{5.40}$$

We set $J := [c, c+\varepsilon]$. We first prove the following fact:

$$\forall m \geq 1, \exists \delta_m > 0,\ (c, c+\delta_m) \cap f^m(J) = \emptyset. \tag{5.41}$$

Suppose that the contrary holds, that is, there exists $m \geq 1$ such that, for all $\delta > 0$, $(c, c+\delta) \cap f^m(J) \neq \emptyset$. This implies that $c \in \overline{f^m(J)}$, and thus $c \in f^m(J)$ because $f^m(J)$ is compact. Let $c' \in (c, c+\varepsilon) \cap f^m(J)$. Then $[c, c'] \subset f^m(J)$ because $f^m(J)$ is connected. We choose a sequence of points $y_n \in (c, c')$ such that $\lim_{n\to+\infty} y_n = c$. For all $n \geq 0$, there exists $x_n \in J$ such that $f^m(x_n) = y_n$. By taking a subsequence, we may assume that the sequence $(x_n)_{n\geq 0}$ converges to a point x, and $x \in J$ by compactness. Then $f^m(x) = c$ by continuity, $x \neq c$ because c is not periodic by assumption, and for all $n \geq 0$, $x \neq x_n$ because $f^m(x_n) \neq c$. Since $\lim_{n\to+\infty} x_n = x > c$, there exists $n \geq 0$ such that $x_n > c$. We set

$$\begin{aligned} x' &:= \max\{t \in [x, x_n] \mid f^m(t) = c\} \quad \text{if } x < x_n,\\ x' &:= \min\{t \in [x_n, x] \mid f^m(t) = c\} \quad \text{if } x > x_n. \end{aligned}$$

Then $f^m(\langle x, x'\rangle) \geq c$ by definition of x' and continuity of f. Moreover, $f^m(\langle x, x'\rangle)$ contains both $f^m(x') = c$ and $f^m(x_n) = y_n > c$. Thus there exists $d > c$ such that $f^m(\langle x, x'\rangle) = [c, d]$. We set $\{a, b\} := \{x, x'\}$ with $a < b$. Note that $a > c$ because $x > c$ and $x_n > c$. Then $f^m([a, b]) = [c, d]$ with $a, b \in J \setminus \{c\} = (c, c+\varepsilon]$, which contradicts (5.40). This proves that the fact (5.41) holds.

We set

$$Y := \bigcup_{n=1}^{\infty} f^n(J). \tag{5.42}$$

According to the assumption (5.36), for all $\varepsilon' \in (0, \varepsilon]$, there exists an integer $k \geq 1$ such that $(c, c+\varepsilon') \cap f^k((c, c+\varepsilon')) \neq \emptyset$. Thus for all $\varepsilon' \in (0, \varepsilon]$, $(c, c+\varepsilon') \cap Y \neq \emptyset$, which implies that $c \in \overline{Y}$. On the other hand, the fact (5.41) implies that $c \notin Y$. Thus $c \in \overline{Y} \setminus Y$. Moreover, according to the assumption (5.36), there exists $k \geq 1$ such that $(c, c+\varepsilon) \cap f^k((c, c+\varepsilon)) \neq \emptyset$, and thus $J \cap f^k(J) \neq \emptyset$. This implies that Y

has at most k connected components and that $\overline{Y} \setminus Y$ is a finite set. By definition, $Y = f(Y) \cup f(J)$. Thus, since J is compact, $\overline{Y} = f(\overline{Y}) \cup f(J)$. Moreover,

$$\begin{aligned}
\overline{Y} \setminus Y &= \big(f(\overline{Y}) \cup f(J)\big) \setminus Y, && (5.43)\\
&= f(\overline{Y}) \setminus Y \quad \text{because } f(J) \subset Y, && (5.44)\\
&\subset f(\overline{Y}) \setminus f(Y) \quad \text{because } f(Y) \subset Y, && (5.45)\\
&\subset f(\overline{Y} \setminus Y). && (5.46)
\end{aligned}$$

Since $\overline{Y}\setminus Y$ is finite, (5.46) implies that $\overline{Y}\setminus Y = f(\overline{Y}\setminus Y)$ and all points in $\overline{Y}\setminus Y$ are periodic. Since $c \in \overline{Y}\setminus Y$, this contradicts the fact that c is not periodic. Conclusion: if (5.37) does not hold, then c is periodic. This concludes the proof. ☐

The next result gives a characterization of the points in the ω-limit set. Note that its statement is not optimal since one can replace the bound 4 by 3 in (5.47). Since the value of this bound has no consequence on the other results of the book, we have chosen to give a simple proof with a non optimal bound. To prove this result with the bound 3, one can either use additional lemmas about interval maps (which gives a longer proof) as in [**41**, Proposition V.11], or use Sierpiński's Theorem[1] (which gives a short but non elementary proof) as in [**119**, Theorem 2].

PROPOSITION 5.45. *Let $f\colon I \to I$ be an interval map and $c \in I$. Then $c \in \omega(f)$ if and only if*

$$\text{for every neighborhood } U \text{ of } c,\ \exists x \in I,\ \#\{n \geq 0, \mid f^n(x) \in U\} \geq 4. \tag{5.47}$$

PROOF. If $c \in \omega(f)$, there exists x such that $c \in \omega(x, f)$, and we trivially have $\#\{n \geq 0, \mid f^n(x) \in U\} \geq 4$ for every neighborhood U of c.

Assume that (5.47) holds. For every set $U \subset I$, we define

$$U^- := \{x \in U \mid x < c\} \quad \text{and} \quad U^+ := \{x \in U \mid x > c\}.$$

We assume

$$\exists\, U \text{ neighborhood of } c,\ \forall k \geq 1,\ U^- \cap f^k(U^-) = \emptyset \text{ and } U^+ \cap f^k(U^+) = \emptyset. \tag{5.48}$$

Let U be such a neighborhood. We also assume that $c \notin \omega(c, f)$ (otherwise there is nothing to prove). In this way, we may replace U by a smaller neighborhood in order to have

$$\forall k \geq 1,\ f^k(c) \notin U. \tag{5.49}$$

By assumption (5.47), there exist a point x and positive integers $p < q < r$ such that $x, f^p(x), f^q(x), f^r(x) \in U$. By (5.49), the point x is not equal to c because $f^p(x) \in U$. Thus $x \in U^- \cup U^+$. We suppose $x \in U^+$, the case $x \in U^-$ being symmetric. Similarly, (5.49) implies that $f^p(x) \neq c$ because $f^{q-p}(f^p(x)) \in U$; and $f^q(x) \neq c$ because $f^{r-q}(f^q(c)) \in U$. Since $f^p(x) \in f^p(U^+)$ and $f^p(x) \in U \setminus \{c\}$, we have $f^p(x) \in U^-$ by (5.48). The same argument with the points $x, f^q(x)$ (resp. $x' := f^p(x), f^{q-p}(x') = f^q(x)$) leads to $f^q(x) \in U^-$ (resp. $f^q(x) \in U^+$), which is impossible. Thus (5.48) does not hold. It is easy to see, by considering the neighborhoods $(c - \varepsilon_k, c + \varepsilon_k) \cap I$, where $(\varepsilon_k)_{k\geq 0}$ is a decreasing sequence of positive numbers tending to 0, that there exists $s \in \{+, -\}$ such that, for every neighborhood U of c, there exists $k \geq 1$ such that $U^s \cap f^k(U^s) \neq \emptyset$. Then $c \in \omega(f)$ according to Lemma 5.44. ☐

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[1] Sierpiński's Theorem: If $(F_n)_{n\geq 0}$ is a pairwise disjoint closed cover of the compact connected Hausdorff set S, then there exists $n \geq 0$ such that $F_n = S$. See e.g. [**77**, Theorem 6.1.27].

COROLLARY 5.46. *Let $f\colon I \to I$ be an interval map. The set $\omega(f)$ is compact.*

PROOF. Let $(c_n)_{n\geq 0}$ be a sequence of points in $\omega(f)$ that converges to some point c. Let U be a neighborhood of c. There exists $n \geq 0$ such that U is a neighborhood of c_n. Thus, according to Proposition 5.45, there exists a point x such that $\#\{n \geq 0, \mid f^n(x) \in U\} \geq 4$. Then, by Proposition 5.45, $c \in \omega(f)$. This shows that $\omega(f)$ is closed, and hence compact because I is compact. □

By definition, for every open set U containing $\omega(f)$ and every point x, all but finitely many points of the trajectory of x lie in U. The next result states that the number of points of the trajectory of x falling outside U is bounded independently of x.

COROLLARY 5.47. *Let $f\colon I \to I$ be an interval map. For every open set U containing $\omega(f)$, there exists a positive integer N such that, for all points $x \in I$, $\#\{n \geq 0 \mid f^n(x) \notin U\} \leq N$.*

PROOF. Let U be an open set containing $\omega(f)$. Let $y \in I \setminus U$. According to Proposition 5.45, there exists an open set V_y containing y such that V_y contains at most three points of any trajectory. Since $I \setminus U$ is compact, there exist finitely many points $y_1, \ldots, y_p \in I \setminus U$ such that $I \setminus U \subset V_{y_1} \cup \cdots \cup V_{y_p}$. Therefore, the open set $V := V_{y_1} \cup \cdots \cup V_{y_p}$ contains at most $3p$ points of any trajectory. This gives the conclusion with $N = 3p$. □

5.6.5. Positive sequence entropy implies Li-Yorke chaos. We are going to show several preliminary results about interval maps that are not chaotic in the sense of Li-Yorke. Then we shall be able to show that such a map has zero sequence entropy for any sequence.

The next lemma is a refinement of Proposition 5.32(i).

LEMMA 5.48. *Let f be an interval map that is not chaotic in the sense of Li-Yorke and let x_0 be a point. Suppose that $\omega(x_0, f)$ is infinite and let $(L_n)_{n\geq 0}$ be the intervals given by Proposition 5.24. Then*

$$\lim_{n\to+\infty} \max_{i\in[\![0,2^n-1]\!]} \operatorname{diam}(f^i(L_n) \cap \omega(f)) = 0.$$

PROOF. Recall that the intervals $(L_n)_{n\geq 0}$ satisfy: for all $n, i \geq 0$, $f^i(L_{n+1})$ and $f^{i+2^n}(L_{n+1})$ are included in $f^i(L_n)$, and $(f^i(L_n))_{0\leq i<2^n}$ is the smallest cycle of intervals of period 2^n containing $\omega(x_0, f)$. Suppose that the lemma does not hold; this implies

$$\exists \delta > 0,\ \forall n \geq 0,\ \exists i \in [\![0, 2^n - 1]\!],\ \operatorname{diam}(f^i(L_n) \cap \omega(f)) \geq \delta. \tag{5.50}$$

Using (5.50), one can build a sequence $(i_n)_{n\geq 0}$ such that

$$\forall n \geq 0,\ f^{i_{n+1}}(L_{n+1}) \subset f^{i_n}(L_n) \text{ and } \operatorname{diam}(f^{i_n}(L_n) \cap \omega(f)) \geq \delta.$$

We set $J_n := f^{i_n}(L_n)$. For every $n \geq 0$, let b_n, c_n be two points in $J_n \cap \omega(f)$ such that $|b_n - c_n| \geq \delta$. By compactness, there exist two points $b, c \in I$ and an increasing sequence of integers $(n_k)_{k\geq 0}$ such that $\lim_{k\to+\infty} b_{n_k} = b$ and $\lim_{k\to+\infty} c_{n_k} = c$. Since $\omega(f)$ is closed by Corollary 5.46, the points b, c belong to $\omega(f)$. Moreover, $[b - c| \geq \delta$ and b, c belong to J_n for all $n \geq 0$ (because $(J_n)_{n\geq 0}$ is a decreasing sequence of closed intervals).

According to Proposition 5.32(i), $\mathrm{diam}(J_n \cap \omega(x_0, f))$ tends to 0 when n goes to infinity. Thus there exists a unique point $a \in \omega(x_0, f)$ such that

$$\bigcap_{n \geq 0} J_n \cap \omega(x_0, f) = \{a\}$$

because this is a decreasing intersection of nonempty compact sets. By the triangular inequality, either $|a-b| \geq \frac{\delta}{2}$ or $|a-c| \geq \frac{\delta}{2}$. With no loss of generality, we suppose $|a-b| \geq \frac{\delta}{2}$. For every $n \geq 0$, the point b is in the interval J_n, which belongs to a periodic cycle of intervals of period 2^n. This implies that the points $(f^k(b))_{k \geq 0}$ are all distinct. Therefore, since $b \in \omega(f)$, there exists a point x_1 such that $b \in \omega(x_1, f)$ and $\omega(x_1, f)$ is infinite. Moreover, since $b, f^{2^n}(b), f^{2^{n+1}}(b)$ are three distinct points in the interval J_n, one of them is in $\mathrm{Int}\,(J_n)$, and thus there exists $k \geq 0$ such that $f^k(x_1) \in J_n$. The periodicity of J_n implies that $\omega(f^k(x_1), f^{2^n}) \subset J_n$. Then, by Lemma 1.3, we get

$$\forall n \geq 0,\ \omega(x_1, f) \subset \bigcup_{i=0}^{2^n-1} f^i(L_n).$$

Since f is not chaotic in the sense of Li-Yorke by assumption, we have $h_{top}(f) = 0$ by Theorem 5.17. Then the assumptions of Proposition 5.30 are fulfilled (with $a_0 := a, a_1 := b$), and this proposition implies that f is chaotic in the sense of Li-Yorke, a contradiction. This ends the proof of the lemma. $\square$

The next result is due to Fedorenko, Sharkovsky and Smítal [**78**, Theorem 2.1]. Recall that the notion of an unstable point is defined in Definition 2.37.

Proposition 5.49. *Let f be an interval map that is not chaotic in the sense of Li-Yorke. Then $f|_{\omega(f)}$ has no unstable point, that is,*

$$\forall a \in \omega(f),\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall b \in \omega(f),\ |a-b| \leq \delta \Rightarrow \forall n \geq 0,\ |f^n(a) - f^n(b)| \leq \varepsilon.$$

Proof. According to Theorem 5.17, $h_{top}(f) = 0$. We fix $\varepsilon > 0$ and $a \in \omega(f)$. Let x_0 be a point such that $a \in \omega(x_0, f)$. We split the proof depending on $\omega(x_0, f)$ being finite or infinite.

First we suppose that $\omega(x_0, f)$ is infinite. Let $(L_n)_{n \geq 0}$ be the closed intervals given by Proposition 5.24: for all $n, i \geq 0$, $f^i(L_{n+1})$ and $f^{i+2^n}(L_{n+1})$ are included in $f^i(L_n)$, and $(f^i(L_n))_{0 \leq i < 2^n}$ is the smallest cycle of intervals of period 2^n containing $\omega(x_0, f)$. By Lemma 5.48, there exists $n \geq 0$ such that

$$\forall i \in [\![0, 2^n - 1]\!],\ \mathrm{diam}(f^i(L_n) \cap \omega(f)) < \varepsilon. \tag{5.51}$$

We set $J := f^j(L_n)$, where $j \in [\![0, 2^n - 1]\!]$ is such that $a \in f^j(L_n)$. Since the four intervals $(f^{j+i2^n}(L_{n+2}))_{0 \leq i \leq 3}$ are pairwise disjoint and included in J, one of them is included in $\mathrm{Int}\,(J)$. Since a is in one of these four intervals, there exists $i \in [\![0, 3]\!]$ such that $a' := f^{i2^n}(a)$ belongs to $\mathrm{Int}\,(J)$. Let $\delta > 0$ be such that $(a' - \delta, a' + \delta) \subset J$. Let $b \in \omega(f)$ be such that $|a' - b| < \delta$. Then b belongs to $J \cap \omega(f)$, so $f^k(b) \in f^k(J) \cap \omega(f)$ for all $k \geq 0$ (the set $\omega(f)$ is invariant by Lemma 1.3(vi)). Using (5.51) and the fact that $f^{2^n}(J) = J$, we get

$$\forall k \geq 0,\ |f^k(a') - f^k(b)| < \varepsilon.$$

We deduce that a' is not ε-unstable for the map $f|_{\omega(f)}$. Consequently, a is not ε-unstable for the map $f|_{\omega(f)}$ by Lemma 2.38(iii), and this holds for any $\varepsilon > 0$.

Now, we suppose that $\omega(x_0, f)$ is finite, that is, a is a periodic point (by Lemma 1.4). Let p denote the period of a and $g := f^p$. In this way, a is a fixed point for g. We are going to prove that a is not unstable for $g|_{\omega(g)}$, which is enough to ensure that a is not unstable for $f|_{\omega(f)}$ according to Lemma 2.38(ii) and the fact that $\omega(f) = \omega(g)$ (Lemma 1.3(vii)). Since g is uniformly continuous, there exist $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$ and

$$\forall x, y,\ |x - y| < \varepsilon_1 \quad\Rightarrow\quad \forall i \in [\![0, 3]\!],\ |g^i(x) - g^i(y)| < \varepsilon, \tag{5.52}$$

$$\forall x, y,\ |x - y| < \varepsilon_2 \quad\Rightarrow\quad \forall i \in [\![0, 3]\!],\ |g^i(x) - g^i(y)| < \varepsilon_1. \tag{5.53}$$

Let b be in $\omega(g) \cap (a - \varepsilon_2, a + \varepsilon_2)$. If $g^2(b) = b$, then (5.53) implies

$$\forall n \geq 0,\ \forall i \in \{0, 1\},\ |g^{2n+i}(a) - g^{2n+i}(b)| = |g^i(a) - g^i(b)| < \varepsilon_1 < \varepsilon.$$

From now on, we suppose that $g^2(b) \neq b$. Let x_1 be a point such that $b \in \omega(x_1, g)$; by assumption, $\omega(x_1, g)$ contains more than 2 points. Then, according to Proposition 5.24, there exists a closed interval L_2 such that $(g^i(L_2))_{0 \leq i \leq 3}$ is a cycle of intervals of period 4 for g and $\omega(x_1, g) \subset \bigcup_{i=0}^{3} g^i(L_2)$. We can choose L_2 such that $b \in L_2$. By (5.53), $|a - g^i(b)| < \varepsilon_1$ for every $i \in [\![0, 3]\!]$, so $g^i(L_2) \cap (a - \varepsilon_1, a + \varepsilon_1) \neq \emptyset$. Since the four intervals $(g^i(L_2))_{0 \leq i \leq 3}$ are pairwise disjoint, there exists $i_0 \in [\![0, 3]\!]$ such that $g^{i_0}(L_2) \subset (a - \varepsilon_1, a + \varepsilon_1)$. Then (5.52) implies that $g^{i_0+j}(L_2) \subset (a - \varepsilon, a + \varepsilon)$ for all $j \in [\![0, 3]\!]$. Since L_2 is a periodic interval of period 4 for g, we get

$$\forall n \geq 0,\ g^n(L_2) \subset (a - \varepsilon, a + \varepsilon).$$

Moreover, $g^n(b)$ is in $g^n(L_2)$, so $|a - g^n(b)| < \varepsilon$ for all $n \geq 0$. We deduce that a is not ε-unstable for $g|_{\omega(g)}$, for any $\varepsilon > 0$. This concludes the proof. $\square$

The next lemma is due to Fedorenko, Sharkovsky and Smítal [**79**]; see also [**159**, Theorem 3.13] for a statement in English.

Lemma 5.50. *Let f be an interval map that is not chaotic in the sense of Li-Yorke. Then every point in $\omega(f)$ is almost periodic, that is,*

$$\forall y \in \omega(f),\ \forall U \text{ neighborhood of } y,\ \exists p \geq 1,\ \forall n \geq 0,\ f^{np}(y) \in U.$$

Proof. Let y belong to $\omega(x_0, f)$ for some point x_0. If $\omega(x_0, f)$ is finite, then y is periodic (Lemma 1.4) and the conclusion is trivial with p the period of y. Suppose that $\omega(x_0, f)$ is infinite and let U be a neighborhood of y. For every $k \geq 0$, we set

$$I_k := \left[\min \omega(f^{i_k}(x_0), f^{2^k}), \max \omega(f^{i_k}(x_0), f^{2^k})\right],$$

where i_k is an integer such that $y \in \omega(f^{i_k}(x_0), f^{2^k})$ (such an integer exists by Lemma 1.3(iv)). According to Theorem 5.21 and Proposition 5.32(i),

$$\lim_{k \to +\infty} |I_k| = 0.$$

This implies that there exists $k \geq 0$ such that $I_k \subset U$ because y belongs to I_k for all k. Moreover, $f^{2^k}(\omega(f^{i_k}(x_0), f^{2^k})) = \omega(f^{i_k}(x_0), f^{2^k})$ (Lemma 1.3(i)), so $f^{n2^k}(y) \in I_k \subset U$ for all $n \geq 0$. This is the expected result with $p := 2^k$. $\square$

The next lemma will be a key tool in the proof of Theorem 5.53; it is due to Franzová and Smítal [**80**].

Remark 5.51. In the proof in [**80**], the fact that the open sets must satisfy (5.55) is omitted, although the proof does not work without this condition.

LEMMA 5.52. *Let f be an interval map that is not chaotic in the sense of Li-Yorke. Let $\varepsilon > 0$. Then there exist finitely many points $y_1, \ldots, y_r$ in $\omega(f)$ and an open set U containing $\omega(f)$ such that, for every point x satisfying*

$$\exists N_0, N_1 \in \mathbb{Z}^+, N_0 \le N_1,\ \forall n \in [\![N_0, N_1]\!],\ f^n(x) \in U,$$

then there exists $i \in [\![1, r]\!]$ such that

$$\forall n \in [\![N_0, N_1]\!],\ |f^n(x) - f^n(y_i)| \le \varepsilon.$$

PROOF. According to Proposition 5.49, for every $x \in \omega(f)$ there exists a connected neighborhood $W(x)$ of x such that

$$\forall z \in W(x) \cap \omega(f),\ \forall n \ge 0,\ |f^n(x) - f^n(z)| \le \frac{\varepsilon}{2}. \tag{5.54}$$

Since $\omega(f)$ is compact by Corollary 5.46, there exist finitely many distinct points $x_1, \ldots, x_s$ in $\omega(f)$ such that $\omega(f) \subset W(x_1) \cup \cdots \cup W(x_s)$. We would like these sets not to overlap too much, so we replace them by smaller but more numerous sets. We define inductively on $k \in [\![1, s]\!]$ a family of connected open sets $(W_k^j)_{1 \le j \le \alpha_k}$ that are subsets of $W(x_k)$, and points $(x_k^j)_{1 \le j \le \alpha_k}$ such that $x_k^j \in W_k^j \cap \omega(f)$.

Construction at step $k \in [\![1, s]\!]$. Suppose that $(W_i^j)_{1 \le j \le \alpha_i}$ and $(x_i^j)_{1 \le j \le \alpha_i}$ have been defined for all $i \le k-1$ (for $k = 1$, these two families are empty). We consider all the connected components C of

$$W(x_k) \setminus \{x_i^j \mid i \in [\![1, k-1]\!], j \in [\![1, \alpha_i]\!]\}$$

such that

$$\left(C \setminus \left(\bigcup_{i \in [\![1, k-1]\!], j \in [\![1, \alpha_i]\!]} W_i^j \right) \right) \cap \omega(f) \ne \emptyset.$$

We call them $W_k^1, \ldots, W_k^{\alpha_k}$ (notice that $W(x_k) \setminus \{x_i^j \mid i \in [\![1, k-1]\!], j \in [\![1, \alpha_i]\!]\}$ has finitely many connected components because $W(x_k)$ is connected and the set $\{x_i^j \mid i \in [\![1, k-1]\!], j \in [\![1, \alpha_i]\!]\}$ is finite). For every $j \in [\![1, \alpha_k]\!]$, we choose a point x_k^j in

$$\left(W_k^j \setminus \left(\bigcup_{i \in [\![1, k-1]\!], j \in [\![1, \alpha_i]\!]} W_i^j \right) \right) \cap \omega(f).$$

This ends the construction at step k. Note that

$$\bigcup_{i \in [\![1, k]\!], j \in [\![1, \alpha_i]\!]} W_i^j \cap \omega(f) = \bigcup_{i=1}^{k} W(x_i) \cap \omega(f).$$

To simplify the notation, we call $V_1, \ldots, V_r$ and $y_1, \ldots, y_r$ the family of sets $(W_i^j)_{i \in [\![1, s]\!], j \in [\![1, \alpha_i]\!]}$ and the associated points $(x_i^j)_{i \in [\![1, s]\!], j \in [\![1, \alpha_i]\!]}$, and we order them in order to have $y_1 < y_2 < \cdots < y_r$. Then V_i is a connected open set containing $y_i \in \omega(f)$, V_i is included in $W(x_j)$ for some $j \in [\![1, s]\!]$, and $\omega(f) \subset V_1 \cup \cdots \cup V_r$. Moreover, the construction above ensures that:

$$\forall i, j \in [\![1, r]\!], i \ne j, V_i \cap V_j \subset \langle y_i, y_j \rangle$$

because V_i (resp. V_j) is an interval and does not contain y_j (resp. y_i). This implies that $V_i \cap V_j = \emptyset$ if $|i - j| \ge 2$ (that is, only intervals corresponding to consecutive points may intersect).

We modify once more these sets by an inductive construction for $i = 1, \dots, r-1$:
• if $V_i \cap V_{i+1}$ is not included in $\omega(f)$, we choose a point $x \in (V_i \cap V_{i+1}) \setminus \omega(f)$ and we replace V_i and V_{i+1} by $V_i \cap (-\infty, x)$ and $V_{i+1} \cap (x, +\infty)$ respectively; we still call these sets V_i and V_{i+1};
• if $V_i \cap V_{i+1} \subset \omega(f)$, we do not change the sets at step i.

At the end of this construction, we get intervals $V_1, \dots, V_r$ that are open set and satisfy:

$$\omega(f) \subset V_1 \cup \cdots V_r,$$
$$\forall i \in [\![1, r]\!],\ y_i \in V_i \cap \omega(f),$$
$$\forall i, j \in [\![1, r]\!],\ i \neq j,\ V_i \cap V_j \subset \omega(f).$$

This last condition implies:

$$\forall x \in \bigcup_{i=1}^{r} V_i,\ x \notin \omega(f) \Longrightarrow \exists! i \in [\![1, r]\!],\ x \in V_i. \tag{5.55}$$

Moreover, since $V_i \subset W(x_j)$ for some $j \in [\![1, s]\!]$, the triangular inequality and (5.54) imply:

$$\forall i \in [\![1, r]\!],\ \forall y, z \in V_i \cap \omega(f),\ \forall n \geq 0,\ |f^n(y) - f^n(z)| \leq \varepsilon. \tag{5.56}$$

Let $i \in [\![1, r]\!]$ and $z \in V_i \cap \omega(f)$. According to Lemma 5.50, there exists a positive integer $p_i(z)$ such that

$$\forall n \geq 0,\ f^{np_i(z)}(z) \in V_i. \tag{5.57}$$

We can assume that

$$p_i(z) \text{ is a multiple of } p_i(y_i). \tag{5.58}$$

Since f is continuous, there exists an open neighborhood $U_i(z)$ of z such that:

$$U_i(z) \subset V_i$$
$$f^{p_i(z)}(U_i(z)) \subset V_i \tag{5.59}$$
$$\forall n \in [\![0, p_i(z)]\!],\ \operatorname{diam}\left(f^n(U_i(z))\right) \leq \varepsilon. \tag{5.60}$$

We set

$$U_i := \bigcup_{z \in \omega(f) \cap V_i} U_i(z) \quad \text{and} \quad U := \bigcup_{i=1}^{r} U_i.$$

The sets U_i are open and satisfy:

$$\forall i \in [\![1, r]\!],\ U_i \cap \omega(f) = V_i \cap \omega(f). \tag{5.61}$$

Indeed, the inclusion $U_i \cap \omega(f) \subset V_i \cap \omega(f)$ is trivial because $U_i \subset V_i$. Conversely, if $z \in V_i \cap \omega(f)$, then $z \in U_i(z) \subset U_i$, so $V_i \cap \omega(f) \subset U_i \cap \omega(f)$.

By definition, the set U is open and contains $\omega(f)$. Let $x_0 \in U$ and $N \geq 0$ be such that

$$\forall n \in [\![0, N]\!],\ f^n(x_0) \in U. \tag{5.62}$$

We are going to show by induction the following:

FACT 1. *There exist integers $k \geq 0$ and $i_0 \in [\![1, r]\!]$ and finite sequences of points $(z_n)_{0 \leq n \leq k}$ and $(x_n)_{0 \leq n \leq k}$ such that, for all $n \in [\![0, k]\!]$,*

$$z_n \in \omega(f) \cap V_{i_0}, \quad x_n \in U_{i_0}(z_n), \quad x_{n+1} = f^{p_{i_0}(z_n)}(x_n).$$

If we set $q_0 := 0$ *and* $q_n := p_{i_0}(z_0) + \cdots + p_{i_0}(z_{n-1})$ *for all* $n \in [\![1, k+1]\!]$, *the integer* k *is such that* $q_k \leq N < q_{k+1}$.

• According to the definition of U, there exists $i_0 \in [\![1, r]\!]$ and $z_0 \in \omega(f) \cap V_{i_0}$ such that $x_0 \in U_{i_0}(z_0)$. If $q_1 := p_{i_0}(z_0) > N$, then the construction is over with $k := 0$.

• Suppose that the points $(z_n)_{0 \leq n \leq j}$ and $(x_n)_{0 \leq n \leq j}$ are already defined up to some integer j with $q_{j+1} \leq N$. We set $x_{j+1} := f^{p_{i_0}(z_j)}(x_j)$. Thus $x_{j+1} = f^{q_{j+1}}(x_0)$. Then $x_{j+1} \in U$ by (5.62) because $q_{j+1} \leq N$. Since $x_j \in U_{i_0}(z_j)$, we have $x_{j+1} \in V_{i_0}$ by (5.59). If $x_{j+1} \in \omega(f)$, then $x_{j+1} \in U_{i_0}$ by (5.61), and we set $z_{j+1} := x_{j+1}$; trivially $x_{j+1} \in U_{i_0}(z_{j+1})$. If $x_{j+1} \notin \omega(f)$, the fact that $x_{j+1} \in U$ implies that there exists $i \in [\![1, r]\!]$ such that $x_{j+1} \in U_i \subset V_i$. Necessarily, $i = i_0$ because of (5.55). Thus there exists $z_{j+1} \in \omega(f) \cap U_{i_0}$ such that $x_{j+1} \in U_{i_0}(z_{j+1})$. If $q_{j+2} := q_{j+1} + p_{i_0}(z_{j+1}) > N$, then the construction is over with $k := j + 1$.

Since all the integers $p_{i_0}(z)$ are positive, the sequence (q_n) is increasing, and thus the construction finishes. This ends the proof of Fact 1.

Let x_0 satisfy (5.62) and $n \in [\![0, N]\!]$. We keep the notation of Fact 1. Let $j \in [\![0, k+1]\!]$ be such that $q_j \leq n < q_{j+1}$. We have

$$
\begin{aligned}
(5.63)\quad & |f^n(x_0) - f^n(y_{i_0})| \\
& \leq |f^{n-q_j}(f^{q_j}(x_0)) - f^{n-q_j}(z_j)| + |f^{n-q_j}(z_j) - f^{n-q_j}(f^{q_j}(y_{i_0}))|.
\end{aligned}
$$

Since $n - q_j < q_{j+1} - q_j = p_{i_0}(z_j)$, the fact that the points $x_j = f^{q_j}(x_0)$ and z_j belong to $U_{i_0}(z_j)$, combined with (5.60), implies that

$$|f^{n-q_j}(f^{q_j}(x_0)) - f^{n-q_j}(z_j)| \leq \varepsilon.$$

By (5.57)+(5.58) and the f-invariance of $\omega(f)$, the point $f^{q_j}(y_{i_0})$ is in $V_{i_0} \cap \omega(f)$. Moreover, z_j belongs to $V_{i_0} \cap \omega(f)$. Therefore, (5.56) implies that

$$|f^{n-q_j}(z_j) - f^{n-q_j}(f^{q_j}(y_{i_0}))| \leq \varepsilon.$$

Inserting these inequalities in (5.63), we get

$$(5.64) \qquad \text{if } x_0 \text{ satisfies (5.62)}, \exists i_0 \in [\![1, r]\!], \forall n \in [\![0, N]\!],\ |f^n(x_0) - f^n(y_{i_0})| \leq 2\varepsilon.$$

Now, let x be a point and let $N_0 \leq N_1$ be integers such that

$$\forall n \in [\![N_0, N_1]\!],\ f^n(x) \in U.$$

We apply (5.64) to $x_0 := f^{N_0}(x)$ and $N := N_1 - N_0$:

$$(5.65) \qquad \exists i_0 \in [\![1, r]\!],\ \forall n \in [\![0, N]\!],\ |f^n(x_0) - f^n(y_{i_0})| \leq 2\varepsilon.$$

Since $f(\omega(f)) = \omega(f)$, there exists $y \in \omega(f)$ such that $f^{N_0}(y) = y_{i_0}$, and this point satisfies: $\forall n \in [\![0, N_1]\!], f^n(y) \in \omega(f) \subset U$. Thus we can apply (5.64) to $x_0 := y$ and $N := N_1$:

$$(5.66) \qquad \exists i \in [\![1, r]\!],\ \forall n \in [\![0, N_1]\!],\ |f^n(y) - f^n(y_i)| \leq 2\varepsilon.$$

Combining (5.65) and (5.66), we get, for all $n \in [\![N_0, N_1]\!]$:

$$
\begin{aligned}
|f^n(x) - f^n(y_i)| &\leq |f^n(x) - f^{n-N_0}(y_{i_0})| + |f^{n-N_0}(y_{i_0}) - f^n(y_i)| \\
&= |f^{n-N_0}(x_0) - f^{n-N_0}(y_{i_0})| + |f^n(y) - f^n(y_i)| \\
&\leq 2\varepsilon + 2\varepsilon = 4\varepsilon,
\end{aligned}
$$

which gives the expected result. □

At last, we are able to show the following result, which is the "if" part of Theorem 5.41. Together with Proposition 5.42, this finally proves Theorem 5.41 and concludes this section.

THEOREM 5.53. *Let $f\colon I \to I$ be an interval map. If there exists a sequence A such that $h_A(f) > 0$, then f is chaotic in the sense of Li-Yorke.*

PROOF. Suppose that f is not chaotic in the sense of Li-Yorke. We are going to show that $h_A(f) = 0$ for every increasing sequence of non negative integers A, which proves the theorem by refutation.

Let $\varepsilon > 0$. Let U and $y_1, \dots, y_r$ be given by Lemma 5.52. Let $(X_i)_{r+1 \le i \le s}$ be pairwise disjoint nonempty sets such that $\mathrm{diam} X_i \le \varepsilon$ for all $i \in [\![r+1, s]\!]$ and $X_{r+1} \cup \dots \cup X_s \supset I \setminus U$. We choose a point $y_i \in X_i$ for every $i \in [\![r+1, s]\!]$. According to Corollary 5.47, there exists an integer $N \ge 0$ such that

$$\forall x \in I,\ \#\{n \ge 0 \mid f^n(x) \notin U\} \le N.$$

Let x be a point in I. Let $n_1 < n_2 < \dots < n_M$ be the integers such that $f^{n_i}(x) \notin U$; we have $M \le N$. We set

$$\begin{aligned} \mathcal{I}_{2i-1} &:= \{n_i\} \quad \text{for all } i \in [\![1, M]\!],\\ \mathcal{I}_{2i} &:= [\![n_i + 1, n_{i-1} - 1]\!] \quad \text{for all } i \in [\![1, M-1]\!],\\ \mathcal{I}_0 &:= [\![0, n_1 - 1]\!], \quad \mathcal{I}_{2M} := \{n \mid n \ge M+1\}. \end{aligned}$$

The sets $(\mathcal{I}_i)_{0 \le i \le 2M}$ form a partition of $\mathbb{Z}^+$ into intervals of integers; the point $f^n(x)$ belongs to U if and only if $n \in \mathcal{I}_i$ for some i even. According to Lemma 5.52, for every $i \in [\![0, M]\!]$, there exists $j_{2i} \in [\![1, r]\!]$ such that

$$\forall n \in \mathcal{I}_{2i},\ |f^n(x) - f^n(y_{j_{2i}})| \le \varepsilon.$$

For every $i \in [\![1, M]\!]$, let j_{2i-1} be the integer in $[\![r+1, s]\!]$ such that $f^{n_i}(x) \in X_{j_{2i-1}}$. We then have

$$\forall i \in [\![0, 2M]\!],\ \forall n \in \mathcal{I}_i,\ |f^n(x) - f^n(y_{j_i})| \le \varepsilon.$$

We associate to x a sequence $\overline{C}(x) = (C_n(x))_{n \ge 0} \in [\![1, s]\!]^{\mathbb{Z}^+}$ coding the trajectory of x and defined by

$$\forall i \in [\![0, 2M]\!],\ \forall n \in \mathcal{I}_i,\ C_n(x) = j_i.$$

Let $A = (a_n)_{n \ge 0}$ be an increasing sequence of non negative integers. If x, y satisfy $C_k(x) = C_k(y)$, then $|f^k(x) - f^k(y)| \le 2\varepsilon$. Thus it is easy to see that $r_n(A, f, 2\varepsilon)$ is bounded by the number of different sequences $(C_{a_i}(x))_{0 \le i < n}$ when x varies in I. We are going to bound this number.

For a given point x, the sets $(\{k \in [\![0, n-1]\!] \mid a_k \in \mathcal{I}_i\})_{0 \le i \le 2M}$ form a partition of $[\![0, n-1]\!]$ into $2M+1$ intervals of integers; some may be empty. We call $\mathcal{J}_1, \dots, \mathcal{J}_m$ the nonempty sets among them, with $m \le 2M+1 \le 2N+1$. To determine the sets $(\mathcal{J}_i)_{1 \le i \le m}$, it is sufficient to give the positions of the first integer of each $\mathcal{J}_i$; the number of such choices is bounded by $\binom{n}{m}$. Then for every $i \in [\![1, m]\!]$, there exists $j_i \in [\![1, s]\!]$ such that $C_{a_k}(x) = j_i$ for all $k \in \mathcal{J}_i$; the number of choices of $(j_i)_{1 \le i \le m}$ is bounded by s^m. Therefore we have

$$\begin{aligned} r_n(A, f, 2\varepsilon) &\le \sum_{m=0}^{2N+1} \binom{n}{m} s^m = \sum_{m=0}^{2N+1} \frac{n(n-1)\cdots(n-m+1)}{m!} s^m \\ &\le (2N+2) n^{2N+1} s^{2N+1}. \end{aligned}$$

This implies that $\lim_{n\to+\infty}\frac{1}{n}\log r_n(A,f,2\varepsilon)=0$. We deduce that $h_A(f)=0$ for any sequence A. $\square$

Remarks on graph maps. Theorem 5.41 was generalized to circle maps by Hric [**88**] and to some star maps by Cánovas [**65**].

THEOREM 5.54. *Let $f\colon \mathbb{S}\to\mathbb{S}$ be a circle map. Then f is chaotic in the sense of Li-Yorke if and only if there exists an increasing sequence A such that $h_A(f)>0$.*

THEOREM 5.55. *Let $f\colon S_n\to S_n$ be a continuous map, where $n\geq 3$ and $S_n:=\{z\in\mathbb{C}\mid z^n\in[0,1]\}$. Suppose that $f(0)=0$. Then f is chaotic in the sense of Li-Yorke if and only if there exists an increasing sequence A such that $h_A(f)>0$.*

5.7. Examples of maps of type 2^∞, Li-Yorke chaotic or not

According to Theorem 5.17, all interval maps of positive entropy are chaotic in the sense of Li-Yorke; positive entropy interval maps are exactly the maps of type 2^nq for some odd $q>1$ by Theorem 4.58. On the other hand, an interval map of type 2^n for some finite $n\geq 0$ has no infinite ω-limit set by Proposition 5.24, and thus it is not chaotic in the sense of Li-Yorke according to Theorem 5.21. What about maps of type 2^∞? There exist maps of type 2^∞ that are not chaotic in the sense of Li-Yorke; some have no infinite ω-limit set, such as Example 3.23, whereas some have an infinite ω-limit set as in the example built by Smítal [**162**]. On the other hand, there exist zero entropy interval maps that are chaotic in the sense of Li-Yorke, as it was shown simultaneously by Smítal [**162**] and Xiong [**170**]. We are going to give two examples of maps of type 2^∞ with an infinite ω-limit set, the first one (Example 5.56) is not chaotic in the sense of Li-Yorke and the second one (Example 5.59) is chaotic in the sense of Li-Yorke.

EXAMPLE 5.56. We define the map $f\colon[0,1]\to[0,1]$ by

$$f(0):=2/3,$$
$$\forall n\geq 1,\ f\left(1-\frac{2}{3^n}\right):=\frac{1}{3^{n-1}}\quad\text{and}\quad f\left(1-\frac{1}{3^n}\right):=\frac{2}{3^{n+1}},$$

and f is linear between two consecutive points among the values we have just defined; see Figure 5. Finally, $f(1):=0$. It is clear that f is continuous at 1, and thus f is continuous on $[0,1]$.

Let us give the idea of the construction. The map f swaps the intervals $[0,1/3]$ and $[2/3,1]$, and we "fill the gap" linearly on $[1/3,2/3]$ to get a continuous map (we shall see that the core of the dynamics is in $[0,1/3]\cup[2/3,1]$). More precisely, f sends $[0,1/3]$ linearly onto $[2/3,1]$ and it sends $[2/3,1]$ to $[0,1/3]$ in such a way that $f^2|_{[2/3,1]}$ is the same map as f up to a rescaling. On the graph of f, it means that, if one magnifies the square $[2/3,1]\times[0,1/3]$ (bottom right square among the 9 big squares in Figure 5), then one sees the same picture as the graph of the whole map.

This map appears in [**72**], where Delahaye proved that it is of type 2^∞ and has an infinite ω-limit set. We are going to show in addition that f is not chaotic in the sense of Li-Yorke. We follow Smítal's ideas [**162**], although the construction is slightly different from the one in [**162**].

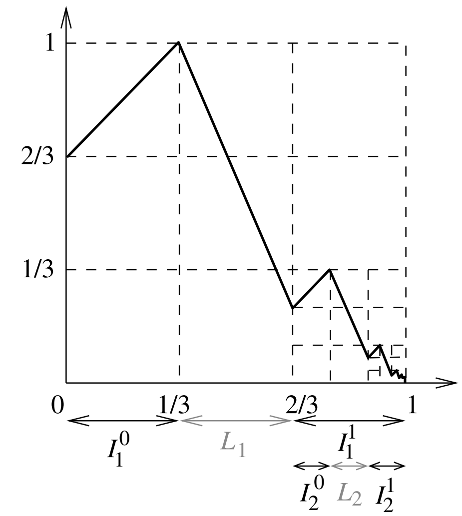

FIGURE 5. This map f is of type 2^∞, the set $\omega(0,f)$ is infinite but f is not chaotic in the sense of Li-Yorke.

We set $I_0^1 := [0,1]$ and, for all $n \geq 1$,

$$I_n^0 := \left[1 - \frac{1}{3^{n-1}}, 1 - \frac{2}{3^n}\right],\ L_n := \left[1 - \frac{2}{3^n}, 1 - \frac{1}{3^n}\right],\ I_n^1 := \left[1 - \frac{1}{3^n}, 1\right].$$

It is clear that

$$\forall n \geq 1,\ I_n^0 \cup L_n \cup I_n^1 = I_{n-1}^1 \quad \text{and} \quad |I_n^0| = |L_n| = |I_n^1| = \frac{1}{3^n}.$$

Moreover, one can check from the definition of f that, for all $n \geq 1$,

$$f^{2^{n-1}}|_{I_n^0} \text{ is linear increasing and } f^{2^{n-1}}(I_n^0) = I_n^1, \tag{5.67}$$

$$f^{2^{n-1}}|_{L_n} \text{ is linear decreasing and } f^{2^{n-1}}(L_n) \supset L_n \cup I_n^1, \tag{5.68}$$

$$f^{2^{n-1}}(I_n^1) = I_n^0. \tag{5.69}$$

Then $f^{2^n}(I_n^1) = I_n^1$ by (5.67) and (5.69). Moreover, the intervals $(f^i(I_n^1))_{0 \leq i < 2^n}$ are pairwise disjoint, they have the same length $\frac{1}{3^n}$ and, if we set

$$\forall n \geq 0,\ K_n := \bigcup_{i=0}^{2^n-1} f^i(I_n^1) \quad \text{and} \quad K := \bigcap_{n \geq 0} K_n,$$

then K is the triadic Cantor set (see Example 8.59 in the Appendix).

First we are going to show that f is of type 2^∞ and that all but finitely many trajectories eventually fall in $\mathcal{O}_f(I_n^1)$. We fix $n \geq 1$. Since $f^{2^{n-1}}(L_n) \supset L_n$ by (5.68), there exists a point $z_n \in L_n$ such that $f^{2^{n-1}}(z_n) = z_n$ (Lemma 1.11). The period of z_n is exactly 2^{n-1} because $L_n \subset I_{n-1}^1$ and the intervals $(f^i(I_{n-1}^1))_{0 \leq i < 2^{n-1}}$ are pairwise disjoint. Moreover, using (5.68) and the fact that $|L_n| = |I_n^1|$, we see that $\mathsf{slope}(f^{2^{n-1}}|_{L_n}) \leq -2$. This implies that, if $x, f^{2^{n-1}}(x), \ldots, f^{k2^{n-1}}(x)$ are in L_n, then $|f^{(k+1)2^{n-1}}(x) - z_n| \geq 2^k |x - z_n|$. Therefore, for all $x \in L_n \setminus \{z_n\}$, there exists $k \geq 1$ such that $f^{k2^{n-1}}(x) \notin L_n$. Thus the point $f^{k2^{n-1}}(x)$ belongs to

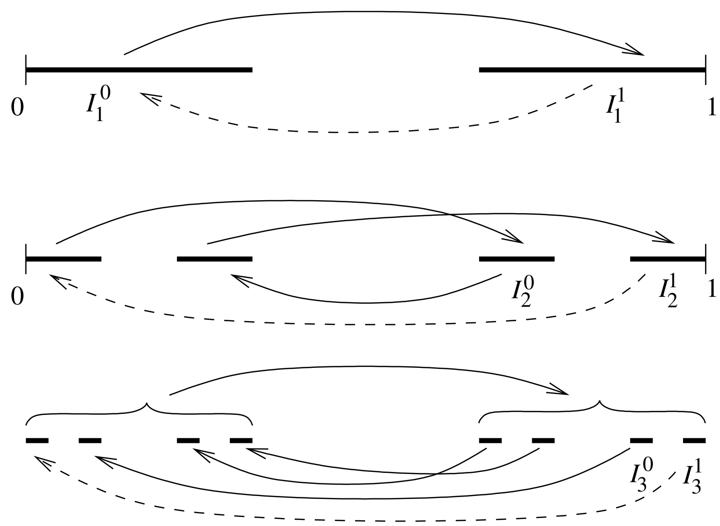

FIGURE 6. Orbits of I_n^1 for $n = 1, 2, 3$. An arrow means that an interval is sent onto another. The arrows are solid when the action of the map is linear increasing, dotted otherwise.

$I_n^0 \cup I_n^1$ because L_n is included in I_{n-1}^1, which is $f^{2^{n-1}}$-invariant. Using the fact that $f^{2^{n-1}}(I_n^0) = I_n^1$, we see that for all $n \geq 1$ and all $x \in I_{n-1}^1 \setminus \{z_n\}$, there exists $k \geq 0$ such that $f^k(x) \in I_n^1$. Starting with $I_0^1 = [0, 1]$, a straightforward induction shows that

$$\forall x \in [0, 1],\ \mathcal{O}_f(x) \cap \{z_n \mid n \geq 1\} = \emptyset \Longrightarrow \forall n \geq 1,\ \exists k \geq 0,\ f^k(x) \in I_n^1.$$

This implies that, for all $x \in [0, 1]$, either $\omega(x, f) = \mathcal{O}_f(z_n)$ for some $n \geq 1$, or $\omega(x, f) \subset K$. Thus

$$\text{all infinite } \omega\text{-limit sets are included in } K. \tag{5.70}$$

Moreover, K contains no periodic point because, for all $n \geq 1$, K is included in $\mathcal{O}_f(I_n^1)$, which is a cycle of intervals of period 2^n (and thus $\mathcal{O}_f(I_n^1)$ contains no periodic point of period less than 2^n). Therefore, the only periodic orbits are $(\mathcal{O}_f(z_n))_{n \geq 1}$, and f is of type 2^∞.

Now we are going to show that $\omega(0, f) = K$. For all $n \geq 1$, $f^{2^{n-1}}(\min I_n^0) = \min I_n^1$ by (5.67). Since $\min I_n^0 = \min I_{n-1}^1$ and $\min I_0^1 = 0$, we get

$$\forall n \geq 1,\ f^{2^n - 1}(0) = \min I_n^1 = 1 - \frac{1}{3^n}. \tag{5.71}$$

Let $x \in K$ and $\varepsilon > 0$. We fix $n \geq 1$ such that $\frac{1}{3^n} < \varepsilon$. By definition of K, there exists $i \in [\![0, 2^n - 1]\!]$ such that $x \in f^i(I_n^1)$. The point $f^{2^n - 1 + i}(0)$ belongs to $f^i(I_n^1)$ by (5.71), so $|f^{2^n - 1 + i}(0) - x| \leq |f^i(I_n^1)| = \frac{1}{3^n} < \varepsilon$. Since ε is arbitrarily small and n is arbitrarily large, this implies that $x \in \omega(0, f)$, that is, $K \subset \omega(0, f)$. We deduce that $\omega(0, f)$ is infinite, and $\omega(0, f) = K$ by (5.70).

Finally, we are going to show that f is not chaotic in the sense of Li-Yorke. Let x, y be two distinct points in K and let n be a positive integer such that $\frac{1}{3^n} < |x - y|$. There exist $i, j \in [\![0, 2^n - 1]\!]$ such that $x \in f^i(I_n^1)$ and $y \in f^j(I_n^1)$. Because of the choice of n, the integers i and j are not equal. Thus x, y are f-separable. Since K contains all infinite ω-limit sets, Theorem 5.21 implies that f is not chaotic in the sense of Li-Yorke.

REMARK 5.57. Let $\Sigma := \{0,1\}^{\mathbb{Z}^+}$ and let $A\colon \Sigma \to \Sigma$ be the map consisting in adding $(1, 0, 0, \dots)$ mod 2 with carrying. More formally,

$$A((\alpha_n)_{n\geq 0}) = (\beta_n)_{n\geq 0} \quad \text{with} \quad \beta_n = \begin{cases} 1-\alpha_n & \text{if } \forall i \in [\![0, n-1]\!], \alpha_i = 1, \\ \alpha_n & \text{otherwise.} \end{cases}$$

For instance $A(0,0,1,0,0,...) = (1,0,1,0,0,...)$ and $A(1,1,0,0,...) = (0,0,1,0,...)$. The map A is called the *dyadic adding machine*.

Let f, K_n and K be as defined in Example 5.56. Let $h\colon K \to \Sigma$ be the map defined by $h(x) = (\alpha_n)_{n\geq 0}$ such that: if C_n is the connected component of K_n containing x, then $\alpha_n = 0$ if x is in the left connected component of K_{n+1} contained in C_n, and $\alpha_n = 1$ otherwise (recall that each connected component of K_n contains two connected components of K_{n+1}). One can show that h is a homeomorphism and that it is a topological conjugacy between $(K, f|_K)$ and (Σ, A).

The readers interested in the dynamics of adding machines and their relations with interval maps can look at [**37, 38, 39**]. The adding machine belongs to the wider family of dynamical systems called *odometers*, see e.g., the survey [**75**].

REMARK 5.58. The map in Example 5.56 is made of countably many linear pieces. The Feigenbaum map is another example of completely different nature, since it is C^∞ and unimodal. Recall that the Feigenbaum map $f_{\lambda_{2^\infty}}$ is a map of the logistic family $f_\lambda\colon [0,1] \to [0,1], x \mapsto \lambda x(1-x)$ (see Remark 3.24). The map $f_{\lambda_{2^\infty}}$ is of type 2^∞, it has an infinite ω-limit set S (which is a Cantor set) and it is not Li-Yorke chaotic. Moreover, the restriction of $f_{\lambda_{2^\infty}}$ to S is topologically conjugate to the dyadic adding machine. See e.g. [**87**, Theorem 11.3.11].

EXAMPLE 5.59. We are going to build a zero topological entropy interval map g that is chaotic in the sense of Li-Yorke. This map will look like the map f of the previous example, except that the intervals I_n^0, I_n^1 are rescaled in such a way that the set $K = \bigcap_{n\geq 0} \mathcal{O}_g(I_n^1)$ is not a Cantor set any more, its interior being nonempty. This example is inspired by Smítal's [**162**]. We shall first define g, then prove several lemmas in order to show the expected properties.

Let $(a_n)_{n\geq 0}$ be an increasing sequence of numbers less than 1 such that $a_0 = 0$. We define $I_0^1 := [a_0, 1]$ and, for all $n \geq 1$,

$$I_n^0 := [a_{2n-2}, a_{2n-1}],\ L_n := [a_{2n-1}, a_{2n}],\ I_n^1 := [a_{2n}, 1].$$

It is clear that $I_n^0 \cup L_n \cup I_n^1 = I_{n-1}^1$. We choose $(a_n)_{n\geq 0}$ such that the lengths of the intervals I_n^0, I_n^1 satisfy:

$$|I_n^0| = \frac{1}{3^n}|I_{n-1}^1|, \quad |I_n^1| = \left(1 - \frac{2}{3^n}\right)|I_{n-1}^1| \quad \text{if } n \text{ is odd,}$$
$$|I_n^0| = \left(1 - \frac{2}{3^n}\right)|I_{n-1}^1|, \quad |I_n^1| = \frac{1}{3^n}|I_{n-1}^1| \quad \text{if } n \text{ is even.}$$

This implies that $|L_n| = \frac{1}{3^n}|I_{n-1}^1|$ for all $n \geq 1$. Note that $|I_n^1| \to 0$ when n goes to infinity, that is, $\lim_{n\to+\infty} a_n = 1$; hence $\bigcup_{n\geq 1}(I_n^0 \cup L_n) = [0,1)$.

For all $n \geq 1$, let $\varphi_n\colon I_n^0 \to I_n^1$ be the increasing linear homeomorphism mapping I_n^0 onto I_n^1; its slope is $\mathsf{slope}(\varphi_n) = \frac{|I_n^1|}{|I_n^0|}$. We define the map $g\colon [0,1] \to [0,1]$ such that g is continuous on $[0, 1)$ and

$$\forall n \geq 1,\ \forall x \in I_n^0, \qquad g(x) := \varphi_1^{-1} \circ \varphi_2^{-1} \circ \cdots \circ \varphi_{n-1}^{-1} \circ \varphi_n(x), \tag{5.72}$$

$$\forall n \geq 1, \qquad g|_{L_n} \text{ is linear,} \tag{5.73}$$

$$g(1) := 0.$$

Note that $g|_{I_n^0}$ is linear increasing. We shall show below that g is continuous at 1.

Let us explain the underlying construction. At step $n = 1$, the interval I_1^0 is sent linearly onto I_1^1 (hence $g|_{I_1^0} = \varphi_1$) and we require that $g(I_1^1) \subset I_1^0$ (i.e., the graph of $g|_{I_1^1}$ is included in the gray area in Figure 7). Then we do the same kind of construction in the gray area with respect to $I_2^0, I_2^1 \subset I_1^1$: we rescale I_2^0, I_2^1 as $\varphi_1^{-1}(I_2^0), \varphi_1^{-1}(I_2^1) \subset I_1^0$ (on the vertical axis), then we send linearly I_2^0 onto $\varphi_1^{-1}(I_2^1)$ and we decide that $g(I_2^1) \subset \varphi^{-1}(I_2^0)$; in this way, $g|_{I_2^0} = \varphi_1^{-1} \circ \varphi_2$ (which is (5.72) for $n = 2$) and the graph of $g|_{I_2^1}$ is included in the black area in Figure 7. We repeat this construction on I_2^1 (black area), and so on. Finally, we fill the gaps in a linear way (which is (5.73)) to get the whole map, which is represented on the right side of Figure 7.

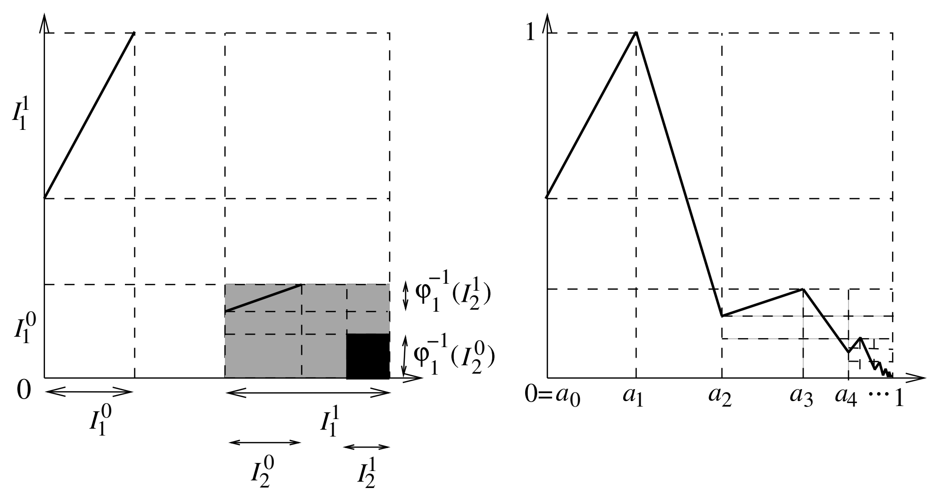

FIGURE 7. On the left: the first steps of the construction of g. On the right: the graph of g.

We introduce an auxiliary family of intervals. We set $J_0^0 := [0, 1]$ and, for all $n \geq 1$, the subintervals $J_n^0, J_n^1 \subset J_{n-1}^0$ are defined by

$$\min J_n^0 = 0,\ \max J_n^1 = \max J_{n-1}^0 \quad \text{and} \quad \frac{|J_n^i|}{|J_{n-1}^0|} = \frac{|I_n^i|}{|I_{n-1}^1|} \quad \text{for } i \in \{0, 1\}.$$

We have

$$|J_n^0| = \prod_{i=1}^{n} \frac{|I_i^0|}{|I_{i-1}^1|} = \prod_{\substack{k \,\in\, [\![1,\, n]\!] \\ k \text{ even}}} \left(1 - \frac{2}{3^k}\right) \prod_{\substack{k \,\in\, [\![1,\, n]\!] \\ k \text{ odd}}} \frac{1}{3^k}.$$

In this product, we bound by 1 all the factors except the last one, and we get

$$|J_n^0| \leq \frac{1}{3^{n-1}}. \tag{5.74}$$

To show that g is continuous at 1, it is enough to prove that $\max(g|_{I_n^1}) \to 0$ when n goes to infinity. For all $n \geq 1$, we have

$$\begin{aligned}
\varphi_n(\max I_n^0) &= \max I_n^1 = 1 = \min I_{n-1}^1 + |I_{n-1}^1| \\
\varphi_{n-1}^{-1} \circ \varphi_n(\max I_n^0) &= \min I_{n-1}^0 + |I_{n-1}^1| \mathsf{slope}(\varphi_{n-1}^{-1}) \\
&= \min I_{n-2}^1 + |I_{n-1}^1| \mathsf{slope}(\varphi_{n-1}^{-1}) \\
\varphi_{n-2}^{-1} \circ \varphi_{n-1}^{-1} \circ \varphi_n(\max I_n^0) &= \min I_{n-2}^0 + |I_{n-1}^1| \mathsf{slope}(\varphi_{n-2}^{-1}) \mathsf{slope}(\varphi_{n-1}^{-1}) \\
&\vdots \\
\varphi_1^{-1} \circ \varphi_2^{-1} \circ \cdots \circ \varphi_{n-1}^{-1} \circ \varphi_n(\max I_n^0) &= \min I_1^0 + |I_{n-1}^1| \prod_{i=1}^{n-1} \mathsf{slope}(\varphi_i^{-1}).
\end{aligned}$$

We have $\max I_1^0 = 1$, $\mathsf{slope}(\varphi_i^{-1}) = \frac{|I_i^0|}{|I_i^1|}$ and $|I_0^1| = 1$. Thus

$$\varphi_1^{-1} \circ \varphi_2^{-1} \circ \cdots \circ \varphi_{n-1}^{-1} \circ \varphi_n(\max I_n^0) = \prod_{i=1}^{n-1} \frac{|I_i^0|}{|I_{i-1}^1|} = |J_{n-1}^0|.$$

Consequently

$$g(\max I_n^0) = |J_{n-1}^0| = \max J_{n-1}^0. \tag{5.75}$$

According to the definition of g, $\max(g|_{I_{n-1}^1}) = g(\max I_n^0)$, so $\max(g|_{I_{n-1}^1}) = |J_{n-1}^0|$. By (5.74), $\lim_{n\to+\infty} \max(g|_{I_{n-1}^1}) = 0$; therefore g is continuous at 1.

The next lemma describes the action of g on the intervals (J_n^i), (I_n^i), and collects the properties that we shall use later. As in Example 5.56, the interval I_n^1 is periodic of period 2^n and the map $g^{2^{n-1}}$ swaps I_n^0 and I_n^1; Figure 6 is still valid, except that the intervals have different lengths. However, we prefer to deal with $J_n^0 = g(I_n^1)$ instead of I_n^1; this will simplify the proofs because $g|_{I_n^1}$ is not monotone, whereas $g^i|_{J_n^0}$ is linear for all $i \in [\![1, 2^n - 1]\!]$ (assertion (iii) in the lemma below).

LEMMA 5.60. *Let g be the map defined above. Then for all $n \geq 1$:*

i) $g(I_n^1) = J_n^0$,
ii) $g(I_n^0) = J_n^1$,
iii) $g^i|_{J_n^0}$ *is linear increasing for all* $i \in [\![1, 2^n - 1]\!]$,
iv) $g^{2^{n-1}-1}(J_n^0) = I_n^0$ *and* $g^{2^n-1}(J_n^0) = I_n^1$,
v) $g^i(J_n^0) \subset \bigcup_{k=1}^n I_k^0$ *for all* $i \in [\![0, 2^n - 2]\!]$,
vi) $\left(g^i(J_n^0)\right)_{0\leq i<2^n}$ *are pairwise disjoint.*

Moreover, the previous assertions imply:

vii) $g^{2^{n-1}}(J_n^0) = J_n^1$,
viii) $g^{2^n}(J_n^0) = J_n^0$,
ix) $g^{2^{n-1}}|_{I_n^0}$ *is linear increasing and* $g^{2^{n-1}}(I_n^0) = I_n^1$,
x) $g^{2^{n-1}}(I_n^1) = I_n^0$,
xi) $(g^i(I_n^0))_{0\leq i<2^n}$ *are pairwise disjoint and* $g^{2^n}(I_n^1) = I_n^1$.

PROOF. According to (5.75),

$$\max(g|_{I_n^1}) = g(\max I_{n+1}^0) = |J_n^0| = \max J_n^0. \tag{5.76}$$

Moreover, $\min(g|_{I_n^1}) = g(1) = 0 = \min J_n^0$. Thus $g(I_n^1) = J_n^0$ by the intermediate value theorem; this is (i).

According to the definition of g,

$$|g(I_n^0)| = |I_n^0|\mathsf{slope}(\varphi_n)\prod_{i=1}^{n-1}\mathsf{slope}(\varphi_i^{-1}) = |I_n^1|\prod_{i=1}^{n-1}\frac{|I_i^0|}{|I_i^1|} = \frac{|I_n^1|}{|I_{n-1}^1|}\prod_{i=1}^{n-1}\frac{|I_i^0|}{|I_{i-1}^1|} = |J_n^1|.$$

Moreover, $g(\max I_n^0) = \max J_{n-1}^0 = \max J_n^1$ by (5.76), so $g(I_n^0) = J_n^1$. This is (ii).

We show by induction on n that assertions (iii) and (iv) are satisfied.
• This is true for $n = 1$ because $J_1^0 = I_1^0$, $g(I_1^0) = I_1^1$ and $g|_{I_1^0} = \varphi_1$ is linear increasing.
• Suppose that (iii) and (iv) are true for n. Since $J_{n+1}^0 \subset J_n^0$, the map $g^i|_{J_{n+1}^0}$ is linear increasing for all $i \in [\![1, 2^n - 1]\!]$, and $g^{2^n-1}(J_{n+1}^0) \subset I_n^1$. Moreover, the facts that $g^{2^n-1}|_{J_n^0}$ is linear increasing and $g^{2^n-1}(J_n^0) = I_n^1$ imply

$$\min g^{2^n-1}(J_{n+1}^0) = \min g^{2^n-1}(J_n^0) = \min I_n^1 = \min I_{n+1}^0, \tag{5.77}$$

and

$$\frac{|g^{2^n-1}(J_{n+1}^0)|}{|I_n^1|} = \frac{|g^{2^n-1}(J_{n+1}^0)|}{|g^{2^n-1}(J_n^0)|} = \frac{|J_{n+1}^0|}{|J_n^0|}. \tag{5.78}$$

Then (5.78) implies that $|g^{2^n-1}(J_{n+1}^0)| = |I_{n+1}^0|$ because $\frac{|J_{n+1}^0|}{|J_n^0|} = \frac{|I_{n+1}^0|}{|I_n^1|}$. Combined with (5.77), we get $g^{2^n-1}(J_{n+1}^0) = I_{n+1}^0$. Then $g^{2^n}(J_{n+1}^0) = J_{n+1}^1$ by (ii). Since $J_{n+1}^1 \subset J_n^0$, the induction hypothesis implies that $g^i|_{J_{n+1}^1}$ is linear increasing for all $i \in [\![1, 2^n - 1]\!]$ and $g^{2^n-1}(J_{n+1}^1) \subset I_n^1$. Moreover,

$$\max g^{2^n-1}(J_{n+1}^1) = \max g^{2^n-1}(J_n^0) = 1 = \max I_{n+1}^1,$$

and by linearity

$$\frac{|g^{2^n-1}(J_{n+1}^1)|}{|I_n^1|} = \frac{|g^{2^n-1}(J_{n+1}^1)|}{|g^{2^n-1}(J_n^0)|} = \frac{|J_{n+1}^1|}{|J_n^0|} = \frac{|I_{n+1}^1|}{|I_n^1|}.$$

By the same argument as above, we get $g^{2^n-1}(J_{n+1}^1) = I_{n+1}^1$. Since $g^{2^{n+1}-1}(J_{n+1}^0) = g^{2^n-1}(J_{n+1}^1)$, this shows that (iii) and (iv) hold for $n+1$. This ends the induction and proves (iii) and (iv).

Now we prove (v) by induction on n.
• This is true for $n = 1$ because $J_1^0 = I_1^0$.
• Suppose that (v) is true for n. Since $J_{n+1}^0 \subset J_n^0$, it follows that $g^i(J_{n+1}^0) \subset \bigcup_{k=1}^n I_k^0$ for all $i \in [\![0, 2^n - 2]\!]$. Moreover, $g^{2^n-1}(J_{n+1}^0) = I_{n+1}^0$ by (iv) and $g^{2^n}(J_{n+1}^0) = g(I_{n+1}^0) = J_{n+1}^1$ by (ii). Since $J_{n+1}^1 \subset J_n^0$, we can use the induction hypothesis again, so $g^{2^n+i}(J_{n+1}^0) \subset \bigcup_{k=1}^n I_k^0$ for all $i \in [\![0, 2^n - 2]\!]$. This gives (v) for $n+1$. This ends the induction and proves (v).

Next we prove (vi). Suppose that $g^i(J_n^0) \cap g^j(J_n^0) \neq \emptyset$ for some $i, j \in [\![0, 2^n - 1]\!]$ with $i < j$. Then $g^{2^n-1-j}(g^i(J_n^0)) \cap g^{2^n-1-j}(g^j(J_n^0)) \neq \emptyset$. But $g^{2^n-1}(J_n^0) = I_n^1$ by (iv) and $g^{2^n-1-(j-i)}(J_n^0) \subset [0, \max I_n^0]$ by (v), so these two sets are disjoint, which is a contradiction. We deduce that $(g^i(J_n^0))_{0 \le i < 2^n}$ are pairwise disjoint.

Finally we indicate how to obtain the other assertions. Assertions (vii) and (viii) follow respectively from (iv)+(ii) and (iv)+(i). Assertion (ix) follows from (iii)+(iv). Assertion (x) follows from (i)+(iv). Assertion (xi) follows from the combination of (i), (iv) and (vi). ☐

We define

$$\forall n \geq 0,\ K_n := \mathcal{O}_g(I_n^1) = \bigcup_{i=0}^{2^n-1} g^i(J_n^0), \quad K := \bigcap_{n=0}^{+\infty} K_n \quad \text{and} \quad B_K := \mathrm{Bd}_{\mathbb{R}}(K),$$

that is, B_K is the boundary of K for the topology of $\mathbb{R}$ (i.e. the points $0,1$ are not excluded). According to Lemma 5.60, K_n is the disjoint union of the intervals $(g^i(J_n^0))_{0\leq i\leq 2^n-1}$. The set K has a Cantor-like construction: at each step, a middle part of every connected component of K_n is removed to get K_{n+1}. However we shall see that K is not a Cantor set because its interior is not empty (see Lemma 5.63), contrary to the situation in Example 5.56. The next lemma states that g is of type 2^∞. Next we shall show that the set $\omega(0,f)$ contains B_K. Then we shall prove that g is chaotic in the sense of Li-Yorke.

LEMMA 5.61. *Let g be the map defined above. Then g is of type 2^∞.*

PROOF. The same arguments as in Example 5.56 can be used to show that g is of type 2^∞. We do not repeat the proof, we just check that the required conditions are satisfied:
- By definition of g, the map $g|_{L_n}$ is linear decreasing, so $g(L_n) \subset [0, g(\max I_n^0)]$. Moreover, $g(\max I_n^0) = \max J_{n-1}^0$ by (5.76), so $g(L_n) \subset J_{n-1}^0$. Thus $g^{2^{n-1}}|_{L_n}$ is linear decreasing by Lemma 5.60(iii).
- The map $g^{2^{n-1}}|_{I_n^0}$ is linear increasing and $g^{2^{n-1}}(I_n^0) = I_n^1$ by Lemma 5.60(ix), so $g^{2^{n-1}}(\min L_n) = \max I_n^1 = 1$. Moreover, $g^{2^{n-1}}(I_n^1) = I_n^0$ by Lemma 5.60(x), so $g^{2^{n-1}}(\max L_n) \in I_n^0$. This implies that $g^{2^{n-1}}(L_n) \supset L_n \cup I_n^1$. Since $|L_n| \leq |I_n^1|$, we have $\mathsf{slope}(g^{2^{n-1}}|_{L_n}) \leq -2$.
- $g^{2^{n-1}}(I_n^0) = I_n^1$ by Lemma 5.60(ix).
- $g^{2^n}(I_n^1) = I_n^1$ and $(g^i(I_n^1))_{0\leq i<2^n}$ are pairwise disjoint by Lemma 5.60(xi). ☐

Since $\min J_n^0 = 0$, the orbit of 0 obviously enters $g^i(J_n^0)$ for all $n \geq 0$ and all $i \in [\![0, 2^n-1]\!]$, which implies that $\omega(0,g)$ meets all connected components of K. The next lemma states that $\omega(0,g)$ contains B_K; the proof relies on the idea that the smaller interval among J_{n+1}^0 and J_{n+1}^1 contains alternatively either $\min J_n^0$ or $\max J_n^0$, when n varies, so that both endpoints of a connected component of K can be approximated by small intervals of the form $g^i(J_n^0)$.

LEMMA 5.62. *Let g and K be as defined above. Then $B_K \subset \omega(0,g)$. In particular, $\omega(0,g)$ is infinite.*

PROOF. According to the definition of K, the connected components of K are of the form $C := \bigcap_{n\geq 0} C_n$, where C_n is a connected component of K_n and $C_{n+1} \subset C_n$ for all $n \geq 0$. That is, the connected components of K are exactly the nonempty sets of the form $C := \bigcap_{n=0}^{+\infty} g^{j_n}(J_n^0)$ with $j_n \in [\![0, 2^n-1]\!]$, and

$$\min C = \lim_{n\to+\infty} \min g^{j_n}(J_n^0) \quad \text{and} \quad \max C = \lim_{n\to+\infty} \max g^{j_n}(J_n^0) \tag{5.79}$$

because C is a decreasing intersection of compact intervals. Moreover, B_K is equal to the union of the endpoints of all connected components of K. Let $y \in B_K$. By (5.79), there exists a sequence of points $(y_n)_{n\geq 0}$ such that $y = \lim_{n\to+\infty} y_n$ and $y_n \in \partial g^{j_n}(J_n^0) = \{\min g^{j_n}(J_n^0), \max g^{j_n}(J_n^0)\}$ for all $n \geq 0$, where $j_n \in [\![0, 2^n-1]\!]$ is such that $y \in g^{j_n}(J_n^0)$. Let $\varepsilon > 0$ and $N \geq 0$. Let n be an even integer such that $\frac{1}{3^{n+1}} < \varepsilon$ and $|y_n - y| < \varepsilon$, and let $k \geq 0$ be such that $k2^{n+1} \geq N$.

First we suppose that $y_n = \min g^{j_n}(J_n^0)$. The point 0 belongs to J_{n+1}^0 and, by Lemma 5.60(viii), $g^{2^{n+1}}(J_{n+1}^0) = J_{n+1}^0$. Thus $g^{k2^{n+1}+j_n}(0)$ belongs to $g^{j_n}(J_{n+1}^0)$. According to Lemma 5.60(iii), $y_n = \min g^{j_n}(J_{n+1}^0)$ and

$$\frac{|g^{j_n}(J_{n+1}^0)|}{|g^{j_n}(J_n^0)|} = \frac{|J_{n+1}^0|}{|J_n^0|} = \frac{|I_{n+1}^0|}{|I_n^0|} = \frac{1}{3^{n+1}} < \varepsilon.$$

Therefore $|g^{k2^{n+1}+j_n}(0) - y_n| < \varepsilon|g^{j_n}(J_n^0)| \leq \varepsilon$.

Secondly we suppose that $y_n = \max g^{j_n}(J_n^0)$. The point $g^{k2^{n+2}}(0)$ belongs to J_{n+2}^0 and $g^{2^{n+1}}(J_{n+2}^0) = J_{n+2}^1$ by Lemma 5.60(viii) and (vii) respectively, so

$$g^{k2^{n+2}+2^{n+1}+2^n+j_n}(0) \in g^{2^n+j_n}(J_{n+2}^1).$$

According to Lemma 5.60(iii)+(vii),

$$\max g^{2^n+j_n}(J_{n+2}^1) = \max g^{2^n+j_n}(J_{n+1}^0) = \max g^{j_n}(J_{n+1}^1) = \max g^{j_n}(J_n^0) = y_n.$$

Moreover, by Lemma 5.60(vii),

$$g^{2^n}(J_{n+2}^1) \subset g^{2^n}(J_{n+1}^0) = J_{n+1}^1 \subset J_n^0.$$

Thus, using the linearity given by Lemma 5.60(iii) and the definitions of (J_n^i) and (I_n^i),

$$\frac{|g^{j_n+2^n}(J_{n+2}^1)|}{|g^{j_n}(J_n^0)|} = \frac{|g^{2^n}(J_{n+2}^1)|}{|J_n^0|} = \frac{|g^{2^n}(J_{n+2}^1)|}{|g^{2^n}(J_{n+1}^0)|} \cdot \frac{|J_{n+1}^1|}{|J_n^0|} = \frac{1}{3^{n+2}} \cdot \left(1 - \frac{2}{3^{n+1}}\right).$$

Consequently, $y_n \in g^{2^n+j_n}(J_{n+2}^1)$ and

$$|g^{k2^{n+2}+2^{n+1}+2^n+j_n}(0) - y_n| \leq |g^{j_n+2^n}(J_{n+2}^1)| < \varepsilon.$$

In both cases, there exists $p \geq N$ such that $|g^p(0) - y_n| < \varepsilon$, so $|g^p(0) - y| < 2\varepsilon$. This means that $y \in \omega(0, g)$, that is, $B_K \subset \omega(0, g)$. Finally, for every $n \geq 0$, K_n has 2^n connected components, each of which containing two connected components of K_{n+1}; thus K has an infinite number of connected components, which implies that B_K is infinite. $\square$

In the proof of the next lemma, we shall first show that K contains a non degenerate connected component C; then we shall see that the two endpoints of C are g-non separable.

Lemma 5.63. *Let g and K be as defined above. Then B_K contains two g-non separable points and g is chaotic in the sense of Li-Yorke.*

Proof. First we define by induction a sequence of intervals $C_n := g^{i_n}(J_n^0)$ for some $i_n \in [\![0, 2^n - 1]\!]$ such that

$$\forall n \geq 1,\ C_n \subset C_{n-1} \quad \text{and} \quad |C_n| = \left(1 - \frac{2}{3^n}\right)|C_{n-1}|.$$

- We set $i_0 := 0$ and $C_0 := J_0 = [0, 1]$.
- Suppose that $C_{n-1} = g^{i_{n-1}}(J_{n-1}^0)$ is already built. If n is even, we set $i_n := i_{n-1}$ and $C_n := g^{i_n}(J_n^0)$. The map $g^{i_{n-1}}|_{J_{n-1}^0}$ is linear increasing by Lemma 5.60(iii) and $J_n^0 \subset J_{n-1}^0$, so

$$\frac{|C_n|}{|C_{n-1}|} = \frac{|J_n^0|}{|J_{n-1}^0|} = 1 - \frac{2}{3^n}.$$

If n is odd, we set $i_n := i_{n-1} + 2^{n-1}$ and $C_n := g^{i_n}(J_n^0)$. By Lemma 5.60(iii)+(vii), the map $g^{i_{n-1}}|_{J^0_{n-1}}$ is linear increasing and $C_n = g^{i_{n-1}}(J_n^1)$, so

$$\frac{|C_n|}{|C_{n-1}|} = \frac{|J_n^1|}{|J_{n-1}^0|} = 1 - \frac{2}{3^n}.$$

We set $C := \bigcap_{n\geq 0} C_n$. It is a compact interval, and it is non degenerate because

$$\log|C| = \log|C_0| + \sum_{n\geq 1} \log\left(1 - \frac{2}{3^n}\right) > -\infty;$$

the last inequality follows from the facts that $\log(1+x) \sim x$ when $x \to 0$ and $\sum \frac{1}{3^n} < +\infty$. Moreover, C is a connected component of K, so $\partial C \subset B_K$. Let $c_0 := \min C$ and $c_1 := \max C$. By Lemma 5.62, the points c_0, c_1 belong to $\omega(0, g)$, which is an infinite ω-limit set. Suppose that c_0, c_1 are g-separable and let A_0, A_1 be two disjoint periodic intervals containing respectively c_0, c_1. Let k be a common multiple of their periods. We choose an integer n such that $2^n > k$. Then there exists $i \in [\![0, 2^n - 1]\!]$ such that $C \subset g^i(J_n^0)$. Since $g^i(J_n^0) \cap g^{i+k}(J_n^0) = \emptyset$ by Lemma 5.60(vi)+(viii), we have $g^k(C) \cap C = \emptyset$. Suppose for instance that $g^k(C) < C$. Then $A_1 = g^k(A_1)$ contains both c_1 and $g^k(c_1)$, and we have $g^k(c_1) \in g^k(C) < c_0 < c_1$. Thus c_0 belongs to A_1 by connectedness, which is a contradiction. The same conclusion holds if $g^k(C) > C$. We deduce that c_0, c_1 are two g-non separable points in $\omega(0, g)$. By Theorem 5.21, we conclude that g is chaotic in the sense of Li-Yorke. $\Box$

According to Lemmas 5.61 and 5.63, the map g is of type 2^∞ (and thus has zero topological entropy by Theorem 4.58) and is chaotic in the sense of Li-Yorke. These are the required properties for the map g. We show one more property for further reference.

LEMMA 5.64. *Let g be the map defined above. Then $g|_{\omega(0,g)}$ is transitive and sensitive to initial conditions.*

PROOF. The point $\{0\} = \bigcap_{n=0}^{+\infty} J_n^0$ is in B_K, so $0 \in \omega(0, g)$. This implies that 0 has a dense orbit in $\omega(0, g)$, that is, $g|_{\omega(0,g)}$ is transitive.

We consider $(i_n)_{n\geq 0}$ and c_0, c_1 as in the proof of Lemma 5.63. Let $\varepsilon > 0$. Let $n \geq 0$ be such that $|J_n^0| < \varepsilon$. Since $B_K \subset \omega(0, g)$ by Lemma 5.62, the points c_0, c_1 belong to $g^{i_n}(J_n^0) \cap \omega(0, g)$. Since J_n^0 is a periodic compact interval (Lemma 5.60(vi)+(viii)) and $\omega(0, g)$ is strongly invariant (Lemma 1.3(i)), there exist $x_0, x_1 \in J_n^0 \cap \omega(0, g)$ such that $g^{i_n}(x_0) = c_0$ and $g^{i_n}(x_1) = c_1$. Moreover, $g^{i_n}(0) \in J_n^0$. Then the triangular inequality implies that, either $|g^{i_n}(0) - g^{i_n}(x_0)| \geq \frac{|c_1 - c_0|}{2}$, or $|g^{i_n}(0) - g^{i_n}(x_1)| \geq \frac{|c_1 - c_0|}{2}$. This implies that the point 0 is δ-unstable with $\delta := \frac{|c_1 - c_0|}{2}$ for the map $g|_{\omega(0,g)}$. Since $g|_{\omega(0,g)}$ is transitive, the map $g|_{\omega(0,g)}$ is $\frac{\delta}{2}$-sensitive by Lemma 2.38(iv). $\Box$

REMARK 5.65. As in Example 5.56, the map in Example 5.59 is made of countably many linear pieces. There exist completely different examples. Indeed, there exist C^∞ weakly unimodal maps of zero topological entropy and chaotic in the sense of Li-Yorke; an interval map $f\colon [0, 1] \to [0, 1]$ is *weakly unimodal* if $f(0) = f(1) = 0$ and there is $c \in (0, 1)$ such that $f|_{[0,c]}$ is non decreasing and $f|_{[c,1]}$ is non increasing. See the articles of Jiménez López [**96**] or Misiurewicz-Smítal [**130**]+[**97**, p674] ([**97**] contains a correction concerning [**130**]).

CHAPTER 6

Other notions related to Li-Yorke pairs: generic and dense chaos, distributional chaos

6.1. Generic and dense chaos

6.1.1. Definitions and general results. Let (X, f) be a topological dynamical system. We recall that $(x, y) \in X^2$ is a *Li-Yorke pair of modulus* $\delta > 0$ if

$$\limsup_{n\to+\infty} d(f^n(x), f^n(y)) \geq \delta \quad \text{and} \quad \liminf_{n\to+\infty} d(f^n(x), f^n(y)) = 0,$$

and (x, y) is a *Li-Yorke pair* if it is a Li-Yorke pair of modulus δ for some $\delta > 0$. Let $\mathrm{LY}(f, \delta)$ and $\mathrm{LY}(f)$ denote respectively the set of Li-Yorke pairs of modulus δ and the set of Li-Yorke pairs for f.

The definition of *generic chaos* is due to Lasota (see [**143**]). It is somehow a two-dimensional notion since the Li-Yorke pairs live in X^2. Being inspired by this definition, Snoha defined *generic δ-chaos*, *dense chaos* and *dense δ-chaos* [**163**].

Definition 6.1 (generic chaos, dense chaos). Let (X, f) be a topological dynamical system and $\delta > 0$.

- f is *generically chaotic* if $\mathrm{LY}(f)$ contains a dense G_δ-set of X^2.
- f is *generically δ-chaotic* if $\mathrm{LY}(f, \delta)$ contains a dense G_δ-set of X^2.
- f is *densely chaotic* if $\mathrm{LY}(f)$ is dense in X^2.
- f is *densely δ-chaotic* if $\mathrm{LY}(f, \delta)$ is dense in X^2.

Some results hold for any dynamical system. Trivially, generic δ-chaos implies both generic chaos and dense δ-chaos; and generic chaos (resp. dense δ-chaos) implies dense chaos. In [**135**] Murinová showed that generic δ-chaos and dense δ-chaos are equivalent (Proposition 6.3). Moreover, topological weak mixing implies generic δ-chaos for some $\delta > 0$ (Proposition 6.4); and dense δ-chaos implies sensitivity to initial conditions (Proposition 6.5). We start with a lemma; then we prove these three results.

Lemma 6.2. *Let (X, f) be a topological dynamical system and $\delta \geq 0$. Let*

$$A(\delta) := \{(x, y) \in X^2 \mid \limsup_{n\to+\infty} d(f^n(x), f^n(y)) \geq \delta\},$$
$$B(\delta) := \{(x, y) \in X^2 \mid \liminf_{n\to+\infty} d(f^n(x), f^n(y)) \leq \delta\}.$$

Then $A(\delta)$ and $B(\delta)$ are G_δ-sets.

Proof. Let

$$\Delta_\varepsilon := \{(x, y) \in X^2 \mid d(x, y) < \varepsilon\} \quad \text{and} \quad \overline{\Delta}_\varepsilon = \{(x, y) \in X^2 \mid d(x, y) \leq \varepsilon\}.$$

Then Δ_ε is open and $\overline{\Delta}_\varepsilon$ is closed. For every integer $n \geq 0$ and every $\varepsilon \geq 0$, we set

$$A_n(\varepsilon) := \{(x, y) \in X^2 \mid \exists i \geq n,\ d(f^i(x), f^i(y)) > \varepsilon\} = \bigcup_{i \geq n} (f \times f)^{-i}(X^2 \setminus \overline{\Delta}_\varepsilon).$$

Since $f \times f$ is continuous, the set $A_n(\varepsilon)$ is open. Moreover,

$$A(\delta) = \bigcap_{k \geq 1} \bigcap_{n \geq 0} A_n(\delta - 1/k).$$

Thus $A(\delta)$ is a G_δ-set. Similarly, we set

$$B_n(\varepsilon) := \{(x, y) \in X^2 \mid \exists i \geq n,\ d(f^i(x), f^i(y)) < \varepsilon\} = \bigcup_{i \geq n} (f \times f)^{-i}(\Delta_\varepsilon).$$

The set $B_n(\varepsilon)$ is open, and thus $B(\delta)$ is a G_δ-set because

$$B(\delta) = \bigcap_{k \geq 1} \bigcap_{n \geq 0} B_n(\delta + 1/k).$$

□

PROPOSITION 6.3. *Let (X, f) be a topological dynamical system and $\delta > 0$. Then f is generically δ-chaotic if and only if it is densely δ-chaotic.*

PROOF. It is sufficient to notice that $\mathrm{LY}(f, \delta) = A(\delta) \cap B(0)$, and thus $\mathrm{LY}(f, \delta)$ is a G_δ-set by Lemma 6.2. □

PROPOSITION 6.4. *Let (X, f) be a topological dynamical system. If f is topologically weakly mixing, then it is generically δ-chaotic with $\delta := \mathrm{diam}(X)$.*

PROOF. By assumption, the system $(X \times X, f \times f)$ is transitive. Let G be the set of pairs of dense orbits in X^2; it is a dense G_δ-set by Proposition 2.3. Since X is compact, there exist $x_1, x_2 \in X$ such that $d(x_1, x_2) = \delta$, where $\delta = \mathrm{diam}(X)$. Then, for every $(x, y) \in G$, there exist two increasing sequences of integers $(n_i)_{i \geq 0}$ and $(m_i)_{i \geq 0}$ such that

$$\lim_{i \to +\infty} (f^{n_i}(x), f^{n_i}(y)) = (x_1, x_2) \quad \text{and} \quad \lim_{i \to +\infty} (f^{m_i}(x), f^{m_i}(y)) = (x_1, x_1).$$

Therefore

$$\limsup_{n \to +\infty} d(f^n(x), f^n(y)) \geq \delta \quad \text{and} \quad \liminf_{n \to +\infty} d(f^n(x), f^n(y)) = 0,$$

that is, $G \subset \mathrm{LY}(f, \delta)$. □

PROPOSITION 6.5. *Let (X, f) be a topological dynamical system and $\delta > 0$. If f is densely δ-chaotic, then f is ε-sensitive for all $\varepsilon \in (0, \frac{\delta}{2})$.*

PROOF. We fix $\varepsilon \in (0, \frac{\delta}{2})$. Let $x \in X$ and let U be a neighborhood of x. By density of $\mathrm{LY}(f, \delta)$ in X^2, there exists (y_1, y_2) in $U \times U$ such that

$$\limsup_{n \to +\infty} d(f^n(y_1), f^n(y_2)) \geq \delta > 2\varepsilon.$$

Using the triangular inequality, we see that there exist $n \geq 0$ and $i \in \{1, 2\}$ such that $d(f^n(x), f^n(y_i)) \geq \varepsilon$. Thus x is ε-unstable. □

REMARK 6.6. The same argument as in the proof of Proposition 6.5 leads to the following result: if f is densely chaotic, then every point x is ε-unstable, for some $\varepsilon > 0$ depending on x.

6.1.2. Preliminary results. In this section, we state several lemmas for densely chaotic interval maps. They will be used to study both generic chaos and dense chaos.

LEMMA 6.7. *Let f be a densely chaotic interval map.*

i) *If J is a non degenerate interval, then $f^n(J)$ is non degenerate for all $n \geq 0$.*
ii) *Let $J_1, \ldots, J_p$ be disjoint non degenerate intervals such that $f(J_i) \subset J_{i+1}$ for all $i \in [\![1, p-1]\!]$ and $f(J_p) \subset J_1$. Then either $p = 2$ and J_1, J_2 have a common endpoint, or $p = 1$. If the intervals $(J_i)_{1 \leq i \leq p}$ are closed, then $p = 1$.*
iii) *If J, J' are non degenerate invariant intervals, then $J \cap J' \neq \emptyset$.*

PROOF. i) Let J be a non degenerate interval. By density of $\mathrm{LY}(f)$, there exists $(x, y) \in J \times J$ such that

$$\limsup_{n \to +\infty} |f^n(x) - f^n(y)| > 0.$$

Thus $f^n(J)$ is non degenerate for infinitely many n. Since the image of a degenerate interval is degenerate, $f^n(J)$ is non degenerate for all $n \geq 0$.

ii) Let $J_1, \ldots, J_p$ be disjoint non degenerate intervals such that $f(J_i) \subset J_{i+1}$ for all $i \in [\![1, p-1]\!]$ and $f(J_p) \subset J_1$. Suppose that there exist two integers $i, j \in [\![1, p]\!]$ such that the distance δ between J_i and J_j is positive. Since f is uniformly continuous, there exists $\eta > 0$ such that

$$\forall x, y,\ |x - y| < \eta \Rightarrow \forall k \in [\![0, p]\!],\ |f^k(x) - f^k(y)| < \delta.$$

Let $(x, y) \in J_i \times J_j$. Then $(f^{kp}(x), f^{kp}(y)) \in J_i \times J_j$ for all $k \geq 0$, and thus $|f^{kp}(x) - f^{kp}(y)| \geq \delta$. Thus $|f^n(x) - f^n(y)| \geq \eta > 0$ for all $n \geq 0$, which contradicts the fact that $I_i \times J_j$ contains Li-Yorke pairs. Therefore, the distance between any two intervals J_i, J_j is null. If the intervals $J_1, \ldots, J_p$ are closed, this implies that $p = 1$. Otherwise, this implies that $p = 1$ or $p = 2$; and, if $p = 2$, then J_1 and J_2 have a common endpoint.

iii) Let J, J' be two non degenerate invariant intervals. Since $\mathrm{LY}(f)$ is dense, there exist (x, x') in $J \times J'$ such that $\liminf_{n \to +\infty} |f^n(x) - f^n(x')| = 0$. By compactness, there exist an increasing sequence of integers $(n_i)_{i \geq 0}$ and a point z such that

$$\lim_{i \to +\infty} f^{n_i}(x) = \lim_{i \to +\infty} f^{n_i}(x') = z.$$

Since J and J' are invariant (and hence closed), the point z belongs to $J \cap J'$. $\square$

LEMMA 6.8. *Let f be a densely chaotic interval map. Suppose that there exists a sequence of non degenerate invariant intervals $(J_n)_{n \geq 0}$ such that $\lim_{n \to +\infty} |J_n| = 0$. Then there exists a fixed point z in $\bigcap_{n \geq 0} J_n$. Moreover, there exists a sequence of non degenerate invariant intervals $(J'_n)_{n \geq 0}$ such that $\lim_{n \to +\infty} |J'_n| = 0$ and, for all $n \geq 0$, J'_{n+1} is included in the interior of J'_n with respect to the induced topology on J'_0.*

PROOF. First we are going to show that

$$\bigcap_{n=0}^{+\infty} J_n \neq \emptyset. \tag{6.1}$$

If the interval $\bigcap_{n=0}^N J_n$ is nonempty for all $N \geq 0$, then $\bigcap_{n=0}^{+\infty} J_n \neq \emptyset$ by compactness. Otherwise, let N be the greatest integer such that $\bigcap_{n=0}^N J_n$ is non degenerate. Then the interval $K := \bigcap_{n=0}^N J_n$ is closed, non degenerate and invariant. By Lemma 6.7(iii), $J_{N+1} \cap K \neq \emptyset$, and thus the set $J_{N+1} \cap K$ is reduced to one point z according to the definition of N. Then, again by Lemma 6.7(iii), $J_n \cap K \neq \emptyset$ and $J_n \cap J_{N+1} \neq \emptyset$ for all $n \geq 0$. Thus z belongs to J_n by connectedness. We deduce that $z \in \bigcap_{n=0}^{+\infty} J_n$. This proves (6.1). The set $\bigcap_{n=0}^{+\infty} J_n$ is reduced to $\{z\}$ because $\lim_{n\to+\infty} |J_n| = 0$. Moreover, $f(z) = z$ because, $f(J_n) \subset J_n$ for all $n \geq 0$.

Either there exist infinitely many integers n such that $J_n \cap (z, +\infty) \neq \emptyset$, or there exist infinitely many n such that $J_n \cap (-\infty, z) \neq \emptyset$. Without loss of generality, we may suppose that the first case holds, that is, there exists an increasing sequence $(n_i)_{i\geq 0}$ such that, $\forall i \geq 0$, $J_{n_i} \supset [z, z + \varepsilon_i]$ for some $\varepsilon_i > 0$. We set $K_m := \bigcap_{i=0}^m J_{n_i}$ for all $m \geq 0$. Then K_m is a non degenerate invariant interval and $K_{m+1} \subset K_m$. We split the end of the proof into two cases.

Case 1. There exists an increasing sequence $(m_i)_{i\geq 0}$ such that $K_{m_{i+1}} \subset \mathrm{Int}\,(K_{m_i})$ for all $i \geq 0$. Then we define $J'_i := K_{m_i}$ and we get a suitable sequence of intervals.

Case 2. Suppose that the assumption of Case 1 is not satisfied. Then there exists $M \geq 0$ such that $K_m \not\subset \mathrm{Int}\,(K_M)$ for all $m \geq M$. Since $(K_m)_{m\geq 0}$ is a sequence of nested intervals, this implies that

$$\text{(6.2)}\quad \text{either } \forall m \geq M,\ \min K_m = \min K_M, \quad \text{or } \forall m \geq M,\ \max K_m = \max K_M.$$

Since $\lim_{m\to+\infty} |K_m| = 0$, there exists an increasing sequence of integers $(m_i)_{i\geq 0}$ with $m_0 = M$ such that $|K_{m_{i+1}}| < |K_{m_i}|$ for all $i \geq 0$. Together with (6.2), this implies that $K_{m_{i+1}}$ is included in the interior of K_{m_i} for the induced topology on K_M. Then $J'_i := K_{m_i}$ gives a suitable sequence of intervals. □

The next lemma will be an important tool. It gives a sufficient condition for a densely chaotic map to be generically δ-chaotic for some $\delta > 0$.

LEMMA 6.9. *Let $f: I \to I$ be a densely chaotic interval map. Suppose that there exists $\varepsilon > 0$ such that every non degenerate invariant interval has a length greater than or equal to ε. Then there exists $\delta > 0$ such that f is generically δ-chaotic.*

PROOF. Suppose that

$$\text{(6.3)}\quad \forall \delta > 0,\ \exists J \text{ non degenerate closed interval such that } \forall n \geq 0,\ |f^n(J)| \leq \delta.$$

We are going to show that this is impossible. We fix $\delta \in (0, \frac{\varepsilon}{4})$. Let J be a non degenerate closed interval such that $|f^n(J)| \leq \delta$ for all $n \geq 0$. There exists a Li-Yorke pair in $J \times J$, which implies that

$$\text{(6.4)}\qquad \limsup_{n\to+\infty} |f^n(J)| > 0.$$

Thus there exist positive integers N, p such that $f^N(J) \cap f^{N+p}(J) \neq \emptyset$ (otherwise, all $(f^n(J))_{n\geq 0}$ would be disjoint, and (6.4) could not hold because I has a finite length). Let $X := \bigcup_{n\geq N} f^n(J)$. The set X has at most p connected components, which are cyclically mapped under f. Moreover, the connected components of X are non degenerate by Lemma 6.7(i). Thus, according to Lemma 6.7(ii), X has either one connected component or two connected components with a common endpoint. In both cases, $\overline{X}$ is an interval.

Suppose that there exist a point z and an integer $n_0 \geq N$ such that $f^2(z) = z$ and $z \in f^{n_0}(J)$. Then $z \in f^{n_0+2k}(J)$ for all $k \geq 0$. By assumption on J, $|f^n(J)| \leq \delta$ for all integers $n \geq 0$, which implies that $\left|\bigcup_{k\geq 0} f^{n_0+2k}(J)\right| \leq 2\delta$ and $\left|\bigcup_{k\geq 0} f^{n_0+2k+1}(J)\right| \leq 2\delta$ (these two sets are intervals containing respectively z and $f(z)$). Let $Y := \overline{\bigcup_{n\geq n_0} f^n(J)} = f^{n_0-N}(\overline{X})$. Then Y is a non degenerate closed interval, $f(Y) \subset Y$ and $|Y| \leq 4\delta$. Moreover, $|Y| \geq \varepsilon$ according to the hypothesis of the lemma, which is a contradiction because $\delta < \varepsilon/4$. We deduce that

$$X \text{ contains no point } z \text{ such that } f^2(z) = z. \tag{6.5}$$

Let X_0 be the connected component of X containing $f^N(J)$. We set $g := f^2$. Then $g(X_0) \subset X_0$ (because X has at most two connected components) and $g|_{X_0}$ has no fixed point by (6.5). By continuity, either

$$\forall x \in X_0,\ g(x) < x, \tag{6.6}$$

or

$$\forall x \in X_0,\ g(x) > x.$$

We assume that (6.6) holds, the other case being symmetric. Let $a := \inf X_0$. The fact that $g(X_0) \subset X_0$ combined with (6.6) implies that $g(a) = a$ (and $a \notin X_0$). Let $b := \max f^N(J)$. We set $b_n := \max g^n([a,b])$ for every $n \geq 0$. Then, for every $n \geq 0$, there exists $x_n \in [a,b]$ such that $g^{n+1}(x_n) = b_{n+1}$. Thus, by (6.6), $b_{n+1} = g(g^n(x_n)) \leq g^n(x_n) \leq b_n$. Therefore, the sequence $(b_n)_{n\geq 0}$ is non increasing, and thus has a limit in $\overline{X_0}$. We set $b_\infty := \lim_{n\to+\infty} b_n$. Note that $b_\infty \in X_0 \cup \{a\}$ because $a \leq b_\infty \leq b$ and $b \in X_0$. We have

$$\bigcap_{n\geq 0} g^n([a,b]) = \bigcap_{n\geq 0} [a,b_n] = [a,b_\infty]$$

and

$$\begin{aligned} \bigcap_{n\geq 0} g^n([a,b]) &= g(\bigcap_{n\geq 0} g^n([a,b])) \text{ because the intersection is decreasing} \\ &= g([a,b_\infty]) \supset [a, g(b_\infty)]. \end{aligned}$$

Hence $g([a,b_\infty]) = [a,b_\infty]$. Thus there exists $x \in [a,b_\infty]$ such that $g(x) = b_\infty$, which implies that $g(x) \geq x$. According to (6.6), this is possible only if $x = a$, and hence $b_\infty = g(a) = a$. Since $|g^{n+N}(J)| \leq |b_n - a|$, we have $\lim_{n\to+\infty} |g^{n+N}(J)| = 0$. By continuity of f, this implies that $\lim_{n\to+\infty} |f^n(J)| = 0$, which contradicts (6.4). We conclude that (6.3) does not hold, that is, there exists $\delta > 0$ such that

$$\text{for every non degenerate closed interval } J,\ \exists n \geq 0,\ |f^n(J)| > \delta. \tag{6.7}$$

Let J be a non degenerate closed interval. Then the closed interval $f^n(J)$ is also non degenerate by Lemma 6.7(i). Thus, according to (6.7),

$$\limsup_{n\to+\infty} |f^n(J)| \geq \delta. \tag{6.8}$$

We define

$$A_k(\eta) := \{(x,y) \in I \times I \mid \ \exists i \geq k,\ |f^i(x) - f^i(y)| > \eta\}$$

and

$$A(\delta) := \bigcap_{n\geq 1} \bigcap_{k\geq 0} A_k(\delta - 1/n) = \{(x,y) \in I \times I \mid \limsup_{k\to+\infty} |f^k(x) - f^k(y)| \geq \delta\}.$$

We are going to show that $A_k(\eta)$ is dense for all $\eta < \delta$ and all $k \geq 0$. Let J_1, J_2 be two non degenerate closed intervals. We consider two cases.
• For some $m \geq 0$, $f^m(J_1) \subset f^m(J_2)$. By (6.8), there exists $n \geq \max\{k, m\}$ such that $|f^n(J_1)| \geq \delta > \eta$, thus there exist $x, x' \in J_1$ such that $|f^n(x) - f^n(x')| > \eta$ and there exists $y \in J_2$ such that $f^n(y) = f^n(x')$. Consequently $A_k(\eta) \cap (J_1 \times J_2) \neq \emptyset$.
• For all $m \geq 0$, $f^m(J_1) \setminus f^m(J_2) \neq \emptyset$. By (6.8), there exists $n \geq k$ such that $|f^n(J_2)| \geq \delta > \eta$, thus there exist $x, x' \in J_2$ such that $|f^n(x) - f^n(x')| > \eta$. By assumption there exists $y \in J_1$ such that $f^n(y) \notin f^n(J_2)$. Since $f^n(J_2)$ is an interval containing x, x' but not y, we have either $|f^n(y) - f^n(x)| > \eta$ or $|f^n(y) - f^n(x')| > \eta$. Consequently $A_k(\eta) \cap (J_1 \times J_2) \neq \emptyset$.

The sets $A_k(\eta)$ are open and dense in $I \times I$. Thus, according to the Baire category theorem, $A(\delta)$ is a dense G_δ-set. Moreover, the set

$$B(0) := \{(x, y) \in I \times I \mid \liminf_{n \to +\infty} |f^n(x) - f^n(y)| = 0\}$$

is a G_δ-set by Lemma 6.2 and it is dense because f is densely chaotic. Therefore the set $\mathrm{LY}(f, \delta) = B(0) \cap A(\delta)$ is a dense G_δ-set and f is generically δ-chaotic. □

6.1.3. Generic chaos and transitivity. Using the structure of transitive non mixing interval maps (Theorem 2.19), one can show that Proposition 6.4 implies that any transitive interval map is generically δ-chaotic for some $\delta > 0$. The converse is not true (see Example 6.14 below), yet it is partially true since a generically δ-chaotic interval map has exactly one or two transitive intervals, as it was shown by Snoha; he also proved that for an interval map, the notions of generic δ-chaos, generic chaos and dense δ-chaos are equivalent (Theorem 6.11 below).

We shall need the Kuratowski-Ulam Theorem [**140, 105**].

THEOREM 6.10 (Kuratowski-Ulam). *Let X, Y be complete metric spaces. If G is a dense G_δ-set in $X \times Y$, then there exists a dense G_δ-set $A \subset X$ such that, for all $x \in A$, the set $\{y \in Y \mid (x, y) \in G\}$ is a dense G_δ-set.*

The next theorem is due to Snoha [**163**].

THEOREM 6.11. *Let f be an interval map. The following properties are equivalent:*

i) *f is generically chaotic,*
ii) *f is generically δ-chaotic for some $\delta > 0$,*
iii) *f is densely δ-chaotic for some $\delta > 0$,*
iv) *either there exists a unique non degenerate transitive interval, or there exist exactly two non degenerate transitive intervals having a common endpoint; moreover, if J is a non degenerate interval, then $f(J)$ is non degenerate, and there exist a transitive interval I_0 and an integer $n \geq 0$ such that $f^n(J) \cap \mathrm{Int}(I_0) \neq \emptyset$.*

Moreover, (ii) and (iii) hold with the same δ.

PROOF. The implication (ii)⇒(i) is trivial and the equivalence (ii)⇔(iii) with the same δ is given by Proposition 6.3. We are going to show the implications (i)⇒(ii) and (iii)⇒(iv)⇒(ii).

(i)⇒(ii).

Assume that f is generically chaotic. Suppose that there exists a sequence of non degenerate invariant intervals $(J_n)_{n \geq 0}$ such that $|J_n| \to 0$ when n goes to infinity. We are going to show that this situation is impossible. According to Lemma 6.8, we may assume that $J_{n+1} \subset \mathrm{Int}\,(J_n)$ for the induced topology of J_0. From now on, we work in J_0; notice that $f|_{J_0}\colon J_0 \to J_0$ is a generically chaotic interval map and $\mathrm{LY}(f|_{J_0}) = \mathrm{LY}(f) \cap (J_0 \times J_0)$. By compactness, $\bigcap_{n\geq 0} J_n$ is nonempty, and hence is reduced to a single point $\{z\}$.

The set $\mathrm{LY}(f) \cap (J_0 \times J_n)$ is a dense G_δ-set in $J_0 \times J_n$. Thus, by Theorem 6.10, there exists a dense G_δ-set A_n in J_0 such that for all $x \in A_n$, there exists $y \in J_n$ with $(x, y) \in \mathrm{LY}(f)$. According to the Baire category theorem (see Corollary 8.53), $A := \bigcap_{n\geq 0} A_n$ is a dense G_δ-set in J_0. Let $x \in A$ and $n \geq 0$. There exists $y \in J_{n+1}$ such that $(x, y) \in \mathrm{LY}(f)$; in particular $\liminf_{k\to+\infty} |f^k(x) - f^k(y)| = 0$. Since J_{n+1} is included in $\mathrm{Int}\,(J_n)$ and the intervals $(J_n)_{n\geq 0}$ are invariant, this implies that there exists $p \geq 0$ such that $f^p(x) \in \mathrm{Int}\,(J_n)$, and hence $f^k(x) \in J_n$ for all $k \geq p$. Since this is true for all $n \geq 0$, we have $\lim_{k\to+\infty} f^k(x) = z$ (recall that $\bigcap_{i\geq 0} J_n = \{z\}$). On the other hand, $A \times A$ is a dense G_δ-set in $J_0 \times J_0$, and thus $(A \times A) \cap \mathrm{LY}(f) \neq \emptyset$. This leads to a contradiction because

$$\forall (x, x') \in A \times A, \lim_{k\to+\infty} f^k(x) = \lim_{n\to+\infty} f^k(x') = z,$$

and thus (x, x') is not a Li-Yorke pair. This shows that there exists $\varepsilon > 0$ such that

$$\text{if } J \text{ is a non degenerate invariant interval, then } |J| \geq \varepsilon.$$

Then Lemma 6.9 applies: the map f is generically δ-chaotic for some $\delta > 0$.

(iii)$\Rightarrow$(iv).

Suppose that f is densely δ-chaotic. According to Proposition 6.5, the map f is sensitive to initial conditions. By Proposition 2.40, there exist some non degenerate closed intervals $I_1, \ldots, I_p$ such that $f(I_i) = I_{i+1 \bmod p}$ for all $i \in [\![1, p]\!]$ and $f|_{I_1\cup\cdots\cup I_p}$ is transitive; by Lemma 6.7(ii), we have $p = 1$, that is, the interval I_1 is transitive.

Suppose that I_2 is another non degenerate transitive interval. Then $I_1 \cap I_2 \neq \emptyset$ by Lemma 6.7. If $\mathrm{Int}\,(I_1 \cap I_2) \neq \emptyset$, then $I_1 = I_2 = \overline{\mathcal{O}_f(I_1 \cap I_2)}$; otherwise $I_1 \cap I_2$ is reduced to a single point. Since the ambient space is an interval, we conclude that either there is a unique non degenerate transitive interval or there are exactly two non degenerate transitive intervals which have a common endpoint.

Finally consider a non degenerate interval J. According to Lemma 6.7(i), $f(J)$ is non degenerate. Since $(J \times J) \cap \mathrm{LY}(f, \delta) \neq \emptyset$, we have $\limsup_{n\to+\infty} |f^n(J)| \geq \delta$, which implies that there exist some integers $i, p > 0$ such that $f^i(J) \cap f^{i+p}(J) \neq \emptyset$. Let $X := \overline{\bigcup_{n\geq 0} f^{n+i}(J)}$. The set X has at most p connected components, which are non degenerate closed intervals and are mapped cyclically under f. Thus, by Lemma 6.7(ii), X is an interval. Moreover, $f(X) \subset X$ and $f|_X$ is sensitive. Thus, by Proposition 2.40 and Lemma 6.7(ii), there exists a non degenerate invariant interval $K \subset X$ such that $f|_K$ is transitive. According to the definition of X, there is some integer $n \geq 0$ such that $f^n(J) \cap \mathrm{Int}\,(K) \neq \emptyset$.

(iv)$\Rightarrow$(ii).

First we show the following fact.

Fact 1. Suppose that the image of a non degenerate interval is non degenerate, that there exist two non degenerate invariant intervals I_1, I_2 such that $f|_{I_i}$ is topologically mixing for $i \in \{1, 2\}$ ($I_1 = I_2$ is allowed) and that, for every non degenerate interval J, there exist $n \geq 0$ and $i \in \{1, 2\}$ such that $f^n(J) \cap \mathrm{Int}\,(I_i) \neq \emptyset$. Then f is generically δ-chaotic with $\delta := \min\{|I_1|, |I_2|\}$.

Let $i, j \in \{1, 2\}$. Both $f|_{I_i}$ and $f|_{I_j}$ are topologically mixing, thus $(f \times f)|_{I_i \times I_j}$ is transitive by Proposition 2.6. Let G_{ij} be the set of points $(x, y) \in I_i \times I_j$ whose orbit is dense in $I_i \times I_j$. According to Proposition 2.3, G_{ij} is a dense G_δ-set in $I_i \times I_j$. By Lemma 6.7(iii), there exists a point z in $I_1 \cap I_2$. We choose $(x_1, x_2) \in I_1 \times I_2$ such that $|x_1 - x_2| = \delta$. For every $(x, y) \in G_{ij}$, there exists a subsequence of $(f^n(x), f^n(y))_{n\geq 0}$ that converges to (z, z) and another one that converges to (x_1, x_2); thus (x, y) is a Li-Yorke pair of modulus δ. It is clear that $(f \times f)^{-1}\mathrm{LY}(f, \delta) \subset \mathrm{LY}(f, \delta)$, and thus we get

$$G := \bigcup_{i,j\in\{1,2\}} \bigcup_{n\geq 0} (f \times f)^{-n}(G_{ij}) \subset \mathrm{LY}(f, \delta).$$

Then G is a G_δ-set (see Propositions 8.24 and 8.72). We are going to show that G is dense.

Let U, V be two nonempty open intervals. By assumption, there exist integers $N, M \geq 0$ and $i, j \in \{1, 2\}$ such that $f^N(U) \cap \mathrm{Int}\,(I_i) \neq \emptyset$ and $f^M(V) \cap \mathrm{Int}\,(I_j) \neq \emptyset$. Let $U_0 \subset U$ and $V_0 \subset V$ be nonempty open subintervals such that $f^N(U_0) \subset I_i$ and $f^M(V_0) \subset I_j$. If $n := \max\{N, M\}$, then $f^n(U_0) \subset I_i$ and $f^n(V_0) \subset I_j$ because I_1, I_2 are invariant. Since the intervals $f^n(U_0), f^n(V_0)$ are non degenerate by assumption, there exists $(x, y) \in (f^n(U_0) \times f^n(V_0)) \cap G_{ij}$; in other words,

$$(U_0 \times V_0) \cap (f \times f)^{-n}(G_{ij}) \neq \emptyset.$$

Therefore the set G is dense and f is generically δ-chaotic. This proves Fact 1.

Now we assume that (iv) holds with no additional hypothesis. If there is a unique non degenerate transitive interval I_1, then, by Theorem 2.19, either $f|_{I_1}$ is topologically mixing or there exist two non degenerate closed intervals J, K such that $I_1 = J \cup K$ and $f^2|_J, f^2|_K$ are topologically mixing. In the first case, Fact 1 gives the conclusion (taking $I_2 := I_1$). In the second case, Fact 1 applied to f^2 (with $I_1 := J$ and $I_2 := K$) shows that f^2 is generically δ-chaotic, and thus f is generically δ-chaotic too. If there are two different non degenerate transitive intervals I_1, I_2 then, by Lemma 6.7(iii), $I_1 \cap I_2 \neq \emptyset$, and this intersection must be reduced to a single point because $I_1 \neq I_2$; we call z this common endpoint. Since $f(z) \in I_1 \cap I_2 = \{z\}$, we have $f(z) = z$. Thus, by Theorem 2.19, both maps $f|_{I_1}$ and $f|_{I_2}$ are topologically mixing. Consequently, Fact 1 applies and f is generically δ-chaotic. $\square$

Remark 6.12. As indicated by Snoha (see [**135**]), there is a misprint in the statement of Theorem 1.2 in [**163**], where the condition "if J is a non degenerate interval, then $f(J)$ is non degenerate" in point (h) (equivalent to our Theorem 6.11(iv)) is omitted.

Remark 6.13. In [**163**], Snoha gave several other properties equivalent to generic chaos for an interval map f, in particular f is generically δ-chaotic if and only if, for every two non degenerate intervals J, J', one has

$$\liminf_{n\to+\infty} d(f^n(J), f^n(J')) = 0 \text{ and } \limsup_{n\to+\infty} |f^n(J)| \geq \delta.$$

Murinová proved that this result is still true in more general spaces except that the equivalence does not hold with the same δ [**135**]. She also built a continuous map on a compact subset of $\mathbb{R}^2$ which is generically chaotic but not generically δ-chaotic for any $\delta > 0$.

EXAMPLE 6.14. Figure 1 represents an interval map $f\colon [0,a] \to [0,a]$ (for some fixed $a > 1$) which is generically chaotic but not transitive. The restriction of f to $[0,1]$ is the tent map T_2 (Example 2.13). Thus $f|_{[0,1]}$ is transitive and f is not transitive. The interval $[1,a]$ is mapped linearly onto $[0,1]$ (thus, for every non degenerate interval $J \subset [1,a]$, $f(J)$ is non degenerate and $f(J) \subset [0,1]$). It is then clear that condition (iv) of Theorem 6.11 is satisfied, and thus f is generically chaotic.

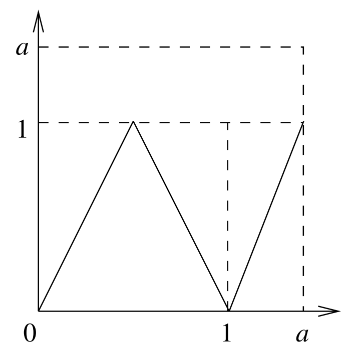

FIGURE 1. This map is generically chaotic but not transitive.

6.1.4. Dense chaos. For interval maps, generic δ-chaos, generic chaos and dense δ-chaos are equivalent and imply dense chaos; but the reverse implication does not hold, as shown in Example 6.15. Next we shall give a result on the structure of densely, non generically chaotic interval maps: in this situation, there exists a decreasing sequence of invariant intervals, and each of them contains a horseshoe for the second iterate of the map (Theorem 6.16).

EXAMPLE 6.15. We are going to exhibit an interval map that is densely chaotic but has no non degenerate transitive interval, and hence is not generically chaotic according to Theorem 6.11. By Proposition 2.40, this map is not sensitive either. This example is originally due to Mizera (see [**163**]). For all $n \geq 0$, we set

$$a_n := 1 - \frac{1}{3^n}, \quad b_n := 1 - \frac{1}{4 \cdot 3^{n-1}}, \quad c_n := 1 - \frac{1}{2 \cdot 3^n} \quad \text{and} \quad J_n := [a_n, 1].$$

These points are ordered as follows:

$$a_0 := 0 < b_0 < c_0 < a_1 < b_1 < c_1 < a_2 < \cdots < a_n < b_n < c_n < a_{n+1} < \cdots < 1.$$

Then we define the continuous map $f\colon [0,1] \to [0,1]$ by

$$\forall n \geq 0,\ f(a_n) := a_n,\ f(b_n) := 1,\ f(c_n) := a_n,$$
$$f(1) := 1,$$

and f is linear on the intervals $[a_n, b_n]$, $[b_n, c_n]$ and $[c_n, a_{n+1}]$ for all $n \geq 0$; see Figure 2.

It is clear from the definition that J_n is invariant and $([a_n, b_n], [b_n, c_n])$ is a horseshoe for all $n \geq 0$. Thus, $f|_{J_n}$ is chaotic in the sense of Li-Yorke by Theorems 4.7 and 5.17. In particular,

$$\forall n \geq 0,\ \mathrm{LY}(f) \cap (J_n \times J_n) \neq \emptyset. \tag{6.9}$$

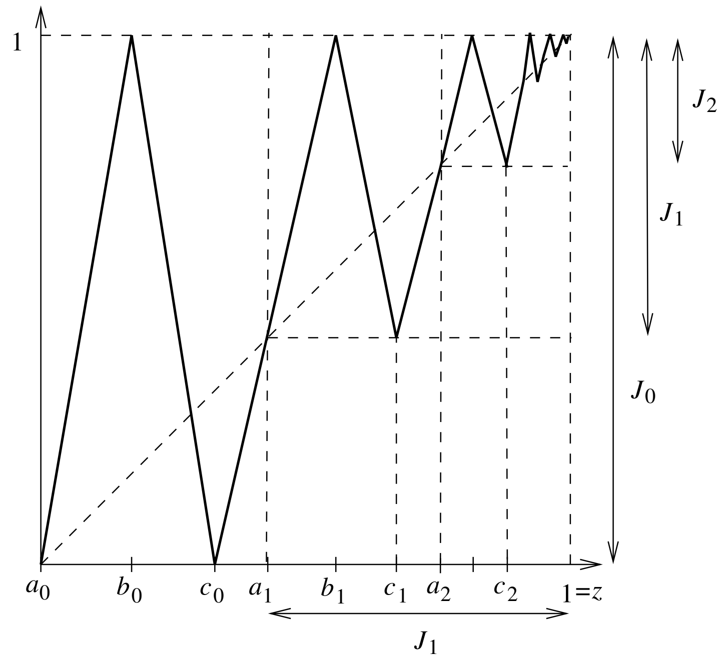

FIGURE 2. This map is densely chaotic but not generically chaotic.

A straightforward computation shows that the absolute value of the slope of f is equal to 4 on each linear piece. Let J, J' be two non degenerate intervals. By Lemma 2.10, there exists $n \geq 0$ such that $f^n(J)$ contains three distinct points in $\{a_k, b_k, c_k \mid k \geq 0\}$, which implies that $f^{n+1}(J) \supset J_p$ for some $p \geq 0$. Similarly, there exist $m, q \geq 0$ such that $f^{m+1}(J') \supset J_q$. Let $N := \max\{n+1, m+1\}$ and $k := \max\{p, q\}$. Then $f^N(J) \cap f^N(J') \supset J_k$ because $J_k \subset J_p \cap J_q$ and $f(J_k) = J_k$. It is obvious that $(f \times f)^{-N}(\mathrm{LY}(f)) \subset \mathrm{LY}(f)$. Thus (6.9) implies that $(J \times J') \cap \mathrm{LY}(f) \neq \emptyset$. In other words, f is densely chaotic.

Let J be a non degenerate invariant interval. Then, as shown above, there exist $n, k \geq 0$ such that $f^n(J) \supset J_k$. Moreover, J_k strictly contains the invariant interval J_{k+1}. Thus $f|_J$ is not transitive. We conclude that f has no non degenerate transitive interval.

The next result is due to the author [**148**].

THEOREM 6.16. *Let f be a densely chaotic interval map that is not generically chaotic. Then there exists a sequence of non degenerate invariant intervals $(J_n)_{n\geq 0}$ such that* $\lim_{n\to+\infty} |J_n| = 0$*,* $J_{n+1} \subset J_n$ *and* $f^2|_{J_n}$ *has a horseshoe for all* $n \geq 0$*.*

PROOF. According to Lemma 6.9, for every $\varepsilon > 0$, there exists a non degenerate invariant interval J such that $|J| < \varepsilon$. Thus there exists a sequence of invariant non degenerate closed subintervals $(I_n)_{n\geq 0}$ such that $\lim_{n\to+\infty} |I_n| = 0$. Then, by Lemma 6.8, there exists a sequence of invariant non degenerate intervals $(J_n)_{n\geq 0}$ such that $\lim_{n\to+\infty} |J_n| = 0$ and $J_{n+1} \subset \mathrm{Int}\,(J_n)$ with respect to the induced topology on J_0 for all $n \geq 0$; moreover, there is a fixed point z such that $\bigcap_{n\geq 0} J_n = \{z\}$. From now on, we consider J_0 as the ambient space; in particular, when speaking about the interior of a set, it is with respect to J_0. We fix an integer $n_0 \geq 0$. We are going to show that $f^2|_{J_{n_0}}$ has a horseshoe. Assume on the contrary that $f^2|_{J_{n_0}}$ has no horseshoe. We set

$$\mathcal{P} := \{x \in J_{n_0} \mid \exists p \geq 1, \lim_{n \to +\infty} f^{np}(x) \text{ exists}\}.$$

If $x, y \in \mathcal{P}$, then (x, y) is not a Li-Yorke pair. Thus the set $J_{n_0} \setminus \mathcal{P}$ is not empty because the map $f|_{J_{n_0}}$ is densely chaotic. Let $x_0 \in J_{n_0} \setminus \mathcal{P}$; we set $x_n := f^n(x_0)$ for all $n \geq 1$. Since $x_0 \notin \mathcal{P}$, the sequence $(x_n)_{n \geq 0}$ is not eventually monotone. Thus, according to Lemma 5.22, there exist a fixed point c and an integer N such that $x_{N+2n} < c < x_{N+2n+1}$ for all $n \geq 0$. We assume $c \leq z$, the case $c \geq z$ being symmetric. Since $x_0 \notin \mathcal{P}$, the sequence $(x_{N+2n})_{n \geq 0}$ is not eventually monotone, so there exists $i \geq 0$ such that

$$x_{N+2i+2} < x_{N+2i} < c \leq z.$$

By continuity, there exists a non degenerate closed interval K containing x_{N+2i} such that $z \notin K$ and

$$\forall y \in K,\ f^2(y) < y. \tag{6.10}$$

Since $z \in \bigcap_{k \geq 0} J_k$ and $\lim_{k \to +\infty} |J_k| = 0$, there exists $k_0 \geq n_0$ such that

$$K < J_{k_0}. \tag{6.11}$$

The set $K \times K$ contains a Li-Yorke pair because f is densely chaotic. Thus $\limsup_{n \to +\infty} |f^n(K)| > 0$ and there exist positive integers p, q such that $f^{q+p}(K) \cap f^q(K) \neq \emptyset$. Let $L := \overline{\bigcup_{n \geq q} f^n(K)}$. The set L is invariant, and the same argument as for $\overline{X}$ in the proof of Lemma 6.9 shows that L is a non degenerate interval. Moreover, $L \cap J_k \neq \emptyset$ for all $k \geq n_0$ by Lemma 6.7(iii). Since $J_{k_0+1} \subset \operatorname{Int}(J_{k_0})$, this implies that there exists $n \geq 0$ such that $f^n(K) \cap \operatorname{Int}(J_{k_0}) \neq \emptyset$. Thus there exists a non degenerate closed subinterval $K' \subset K$ such that $f^n(K') \subset J_{k_0}$. We set $g := f^2|_{J_{n_0}}$ and we fix $m_0 \geq n/2$. For all $y \in K'$ and all $m \geq m_0$, we have $g^m(y) \in J_{k_0}$ because J_{k_0} is invariant. Hence, by (6.10) and (6.11),

$$\forall m \geq m_0,\ g(y) < y < g^m(y). \tag{6.12}$$

This implies that there exists $j \in [\![1, m_0 - 1]\!]$ such that $g^j(y) < g^{j+1}(y)$. Let

$$U(y) := \{y' \in \mathcal{O}_g(y) \mid g(y') > y'\},$$
$$D(y) := \{y' \in \mathcal{O}_g(y) \mid g(y') < y'\}.$$

We have $y \in D(y)$ by (6.12) and $g^j(y) \in U(y)$ according to the choice of j. By assumption, the map g has no horseshoe. Thus, according to Lemma 3.33,

$$U(y) \leq D(y). \tag{6.13}$$

Moreover, for all $m \geq m_0$, $y \leq g^m(y)$, so $g^m(y) \in D(y)$ by (6.13) and because $y \in D(y)$. This implies that $g^{m+1}(y) \leq g^m(y)$. Therefore, the sequence $(g^m(y))_{m \geq m_0}$ is non increasing, and hence convergent. But this implies that $K' \times K'$ contains no Li-Yorke pair, which contradicts the fact that f is densely chaotic. We conclude that $f^2|_{J_{n_0}}$ has a horseshoe for every integer $n_0 \geq 0$. □

Consider a densely, non generically chaotic interval map f, and let $(J_n)_{n \geq 0}$ be the decreasing sequence of invariant intervals given by Theorem 6.16. By Lemma 6.8, the intersection $\bigcap_{n \geq 0} J_n$ is reduced to a fixed point z. Figure 2 is an example of a such a map when z is an endpoint of all intervals J_n. Figure 3 illustrates what the graph of f may look like when z is in $\mathrm{Int}\,(J_n)$ for all n: in one case, the left and right parts of J_n are exchanged under f; in the other case, the left and right parts of J_n are invariant (left and right parts are with respect to z).

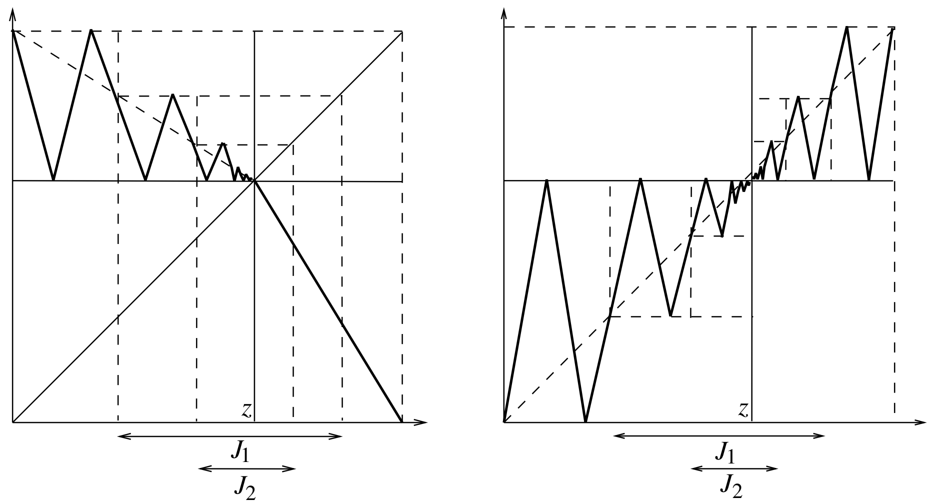

FIGURE 3. The map f on the left is the square root of the map represented in Figure 2. The map on the right is f^2. Both are densely, non generically chaotic.

Using the structure of generically chaotic interval maps and densely non generically chaotic interval maps, it is possible to have information on the entropy and the type of a densely chaotic interval map. Notice also that a densely chaotic interval map is chaotic in the sense of Li-Yorke by Theorem 5.34.

COROLLARY 6.17. *If f is a densely chaotic interval map, then f^2 has a horseshoe. Moreover, $h_{top}(f) \geq \frac{\log 2}{2}$ and f is of type n for some $n \trianglelefteq 6$ for Sharkovsky's order (i.e., f has a periodic point of period* 6*).*

PROOF. If f is generically chaotic, then f^2 has a horseshoe by Theorem 6.11 and Proposition 3.36. Otherwise, f^2 has a horseshoe by Theorem 6.16. In both cases, according to Propositions 4.6 and 3.31, $h_{top}(f) \geq \frac{\log 2}{2}$ and f^2 has a periodic point of period 3. Thus f has a periodic point of period 3 or 6, and hence the type of f is $\trianglelefteq 6$ by Sharkovsky's Theorem 3.13. □

Equalities are possible in Corollary 6.17: the map S in Example 4.71 is transitive, and hence densely chaotic by Theorem 6.11; its entropy is equal to $\frac{\log 2}{2}$ and, since S is not topologically mixing, it is of type 6 by Proposition 3.36.

Example 6.15 shows that there exists densely chaotic maps that are not generically chaotic. The next result states that such a map cannot be piecewise monotone nor C^1. The fact that a densely chaotic piecewise monotone map is generically chaotic is due to Snoha [**164**].

PROPOSITION 6.18. *Let f be a densely chaotic interval map. If f is piecewise monotone or C^1, then f is generically chaotic.*

PROOF. Suppose that f is not generically chaotic. According to Theorem 6.16, there exists a sequence of non degenerate invariant intervals $(J_n)_{n\geq 0}$ such that $\lim_{n\to+\infty}|J_n| = 0$ and $J_{n+1} \subset J_n$ for all $n \geq 0$. Moreover, $\bigcap_{n\geq 0} J_n = \{z\}$, where z is a fixed point, by Lemma 6.8. We write $J_n = [a_n, b_n]$ for all $n \geq 0$. Thus

$$(a_n)_{n\geq 0} \text{ is non decreasing, } (b_n)_{n\geq 0} \text{ is non increasing,}$$
$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n = z.$$

First we assume that f is piecewise monotone in a neighborhood of z. Then there exists $k \geq 0$ such that both $f|_{[a_k,z]}$ and $f|_{[z,b_k]}$ are monotone. If $z = a_k$ or $z = b_k$, we set $J := J_k$, and thus the map $f^2|_J$ is non decreasing because the endpoint z is fixed. If $a_k < z < b_k$, the fact that J_k is invariant implies that either $f|_{[a_k,z]}$ is non decreasing and $f([a_k, z]) \subset [a_k, z]$, or $f|_{[a_k,z]}$ is non increasing and $f([a_k, z]) \subset [z, b_k]$; the symmetric statement holds for $[z, b_k]$. Therefore, there exists an interval J among $[a_k, z]$ and $[z, b_k]$ such that $f^2(J) \subset J$ and $f^2|_J$ is non decreasing. In all cases, we get a non degenerate f^2-invariant interval J such that $f^2|_J$ is non decreasing. We are going to show that

$$\forall x \in J, \text{ the sequence } (f^{2n}(x))_{n\geq 0} \text{ converges.} \tag{6.14}$$

Let P_2 be the set of fixed points of f^2; this is a closed set. For all $x \in P_2$, the sequence $(f^{2n}(x))_{n\geq 0}$ is stationary, and hence convergent. Suppose that $J \setminus P_2 \neq \emptyset$ and let U be a connected component of $J \setminus P_2$. Either $\inf U \in P_2$ and hence $f^2(\inf U) = \inf(U)$, or $\inf U = \min J$ and hence $f^2(\inf U) \geq \inf U$ because J is f^2-invariant. Similarly, $\sup U$ belongs to $P_2 \cup \{\max J\}$, and $f^2(\sup U) \leq \sup(U)$. This implies that $f(\overline{U}) \subset \overline{U}$ because $f^2|_J$ is non decreasing. Moreover, the fact that $U \cap P_2 = \emptyset$ implies, by continuity:

$$\text{either } \forall x \in U,\ f^2(x) > x,$$
$$\text{or } \forall x \in U,\ f^2(x) < x.$$

Therefore, for every $x \in \overline{U}$, the orbit of x is included in $\overline{U}$ and $(f^n(x))_{n\geq 0}$ is either non decreasing or non increasing, and thus it converges. This proves (6.14). But this implies that $J \times J$ contains no Li-Yorke pair, which is a contradiction. We conclude that f is not piecewise monotone in a neighborhood of z.

Secondly we assume that f is C^1. If $f'(z) \neq 0$, then f is monotone in a neighborhood of z and the previous case leads to a contradiction. Thus $f'(z) = 0$. Then there exists $k \geq 0$ such that

$$\max_{x\in J_k} |f'(x)| \leq \frac{1}{2}.$$

We recall the mean value inequality: Let $\varphi\colon I \to \mathbb{R}$ be a differentiable map (where I is an interval) and let $M \in \mathbb{R}$ be such that $|f'(x)| \leq M$ for all $x \in I$. Then for all $x, y \in I$, $|f(y) - f(x)| \leq M|y - x|$.

Since $f(J_k) \subset J_k$ and $f(z) = z$, the mean value inequality implies that

$$\forall x \in J_k,\ \forall n \geq 0,\ |f^n(x) - z| \leq \frac{1}{2^n}|x - z|.$$

Therefore, for all $x \in J_k$ the sequence $(f^n(x))_{n\geq 0}$ converges, and thus $J_k \times J_k$ contains no Li-Yorke pair, which is a contradiction.

Conclusion: if f is piecewise monotone or C^1, then it is generically chaotic. $\square$

REMARK 6.19. In Proposition 6.18, we get the same result if we only assume that f is piecewise monotone or C^1 in the neighborhood of every fixed point.

6.2. Distributional chaos

In [**151**], Schweizer and Smítal defined lower and upper distribution functions of two points in a dynamical system, and studied them for interval maps.

DEFINITION 6.20. Let (X, f) be a topological dynamical system and $x, y \in X$. For all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$, set

$$\xi(f, x, y, n, t) := \#\{i \in [\![0, n-1]\!] \mid d(f^i(x), f^i(y)) < t\}.$$

The lower and upper distribution functions $F_{xy}, F^*_{xy} \colon \mathbb{R} \to [0,1]$ are defined respectively by:

$$\forall t \in \mathbb{R}, \quad F_{xy}(t) = \liminf_{n\to+\infty} \frac{1}{n}\xi(f, x, y, n, t),$$
$$F^*_{xy}(t) = \limsup_{n\to+\infty} \frac{1}{n}\xi(f, x, y, n, t).$$

The next properties are straightforward from the definition.

PROPOSITION 6.21. *Let (X, f) be a topological dynamical system and $x, y \in X$.*

- $\forall t \leq 0$, $F_{xy}(t) = F^*_{xy}(t) = 0$ *and* $\forall t > \operatorname{diam}(X)$, $F_{xy}(t) = F^*_{xy}(t) = 1$.
- *The maps F_{xy} and F^*_{xy} are non decreasing and $F_{xy} \leq F^*_{xy}$.*

The notion of distributional chaos was introduced in [**151**] (although the name "distributional chaos" was given later). Three variants of distributional chaos are now known in the literature (see, e.g., [**23**]). Distributional chaos of type 1 is considered as the original definition of distributional chaos.

DEFINITION 6.22. Let (X, f) be a topological dynamical system. Then (X, f) is called *distributionally chaotic of type 1, 2, 3* respectively (for short, DC1, DC2, DC3) if the condition (DC1), (DC2), (DC3) respectively is satisfied:

(DC1) $\exists x, y \in X,\ \exists \delta > 0,\ \forall t \in (0, \delta),\ F_{xy}(t) = 0$ and $\forall t > 0,\ F^*_{xy}(t) = 1$,

(DC2) $\exists x, y \in X,\ \exists \delta > 0,\ \forall t \in (0, \delta),\ F_{xy}(t) < 1$ and $\forall t > 0,\ F^*_{xy}(t) = 1$,

(DC3) $\exists x, y \in X,\ \exists\, 0 < a < b,\ \forall t \in (a, b),\ F_{xy}(t) < F^*_{xy}(t)$.

Notice that, if condition (DC2) holds for some x, y, then (x, y) is a Li-Yorke pair of modulus δ. Therefore, DC2 is a refinement of the definition of Li-Yorke pair.

It is clear that (DC1)⇒(DC2)⇒(DC3). In [**151**], Schweizer and Smítal showed that, for interval maps, DC1, DC2 and DC3 coincide and are equivalent to positive entropy (Corollary 6.27 below).

We start with the case of zero entropy interval maps. The proofs of Lemma 6.23 and Theorem 6.25 follow the ideas from [**151**].

LEMMA 6.23. *Let $f \colon I \to I$ be an interval map such that $h_{top}(f) = 0$. For all $x \in I$ and all $\varepsilon > 0$, there exist a periodic point z and a positive integer K such that*

$$\forall k \geq K,\ \forall t \geq \varepsilon,\ \frac{1}{k}\xi(f, x, z, k, t) \geq 1 - \varepsilon. \tag{6.15}$$

PROOF. Let $x \in I$ and $\varepsilon > 0$. We split the proof depending on $\omega(x, f)$ being finite or infinite.

First we suppose that $\omega(x, f)$ is finite, that is, there exists a periodic point z of period p such that $\omega(x, f) = \mathcal{O}_f(z)$ (Lemma 1.4); we choose z such that $\lim_{n \to +\infty} f^{np}(x) = z$. Thus, by continuity, there exists an integer N such that $|f^n(x) - f^n(z)| < \varepsilon$ for all $n \geq N$. Then

$$\forall n \geq N,\ \forall t \geq \varepsilon,\ \xi(f, x, z, n, t) = \xi(f, x, z, N, t) + (n - N).$$

This implies that $\lim_{n \to +\infty} \frac{1}{n}\xi(f, x, z, n, t) = 1$. Therefore (6.15) holds for some integer K.

Now we suppose that $\omega(x, f)$ is infinite. Let $(L_n)_{n \geq 0}$ be the sequence of intervals given by Proposition 5.24. We fix a positive integer n that will be chosen later. Since $f^{2^n}(L_n) = L_n$, there exists a point $z \in L_n$ such that $f^{2^n}(z) = z$ (Lemma 1.11). Let N be an integer such that $f^i(x) \in f^i(L_n)$ for all $i \geq N$ (such an integer N exists by Proposition 5.24(vi)). Thus, if $i \geq N$, both points $f^i(x), f^i(z)$ belong to $f^i(L_n)$, which is an interval of the family $(f^j(L_n))_{0 \leq j < 2^n}$. Let $k \geq N$; we write $k = N + k'2^n + r$ with $k' \geq 0$ and $r \in [\![0, 2^n - 1]\!]$. Since L_n is a periodic interval of period 2^n, we have

$$\begin{aligned}
&\#\{i \in [\![N, k-1]\!] \mid |f^i(x) - f^i(z)| \geq \varepsilon\} \leq \#\{i \in [\![N, k-1]\!] \mid |f^i(L_n)| \geq \varepsilon\} \\
&\quad \leq \quad \#\{i \in [\![0, r-1]\!] \mid |f^{N+i}(L_n)| \geq \varepsilon\} + k'\#\{j \in [\![0, 2^n - 1]\!] \mid |f^j(L_n)| \geq \varepsilon\} \\
&\quad \leq \quad (k'+1)\#\{j \in [\![0, 2^n - 1]\!] \mid |f^j(L_n)| \geq \varepsilon\}.
\end{aligned}$$

Among the intervals $(f^i(L_n))_{0 \leq i < 2^n}$, at most $\frac{|I|}{\varepsilon}$ have a length greater than or equal to ε because these intervals are pairwise disjoint. Thus

$$\#\{i \in [\![N, k-1]\!] \mid |f^i(x) - f^i(z)| \geq \varepsilon\} \leq \frac{(k'+1)|I|}{\varepsilon},$$

and hence

$$\begin{aligned}
\xi(f, x, z, k, \varepsilon) \quad &\geq \quad \#\{i \in [\![N, k-1]\!] \mid |f^i(x) - f^i(z)| < \varepsilon\} \\
&\geq \quad (k - N) - \frac{(k'+1)|I|}{\varepsilon} = k - N - \frac{|I|}{\varepsilon} - k'\frac{|I|}{\varepsilon}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{1}{k}\xi(f, x, z, k, \varepsilon) \quad &\geq \quad 1 - \frac{N + \frac{|I|}{\varepsilon}}{k} - \frac{k'}{N + k'2^n + r} \cdot \frac{|I|}{\varepsilon} \\
&\geq \quad 1 - \frac{N + \frac{|I|}{\varepsilon}}{k} - \frac{|I|}{2^n\varepsilon}.
\end{aligned}$$

We choose n such that $\frac{|I|}{2^n\varepsilon} < \frac{\varepsilon}{2} \Leftrightarrow 2^n > \frac{2|I|}{\varepsilon^2}$, and we choose $K \geq N$ such that $\frac{N + \frac{|I|}{\varepsilon}}{K} < \frac{\varepsilon}{2}$. Then

$$\forall k \geq K,\ \forall t \geq \varepsilon,\ \frac{1}{k}\xi(f, x, z, k, t) \geq \frac{1}{k}\xi(f, x, z, k, \varepsilon) \geq 1 - \varepsilon,$$

which concludes the proof. ☐

LEMMA 6.24. *Let (X, f) be a topological dynamical system and let z, z' be periodic points. Then $F_{zz'} = F^*_{zz'}$.*

PROOF. Let $p \in \mathbb{N}$ be a common multiple of the periods of z and z'. For all integers $k, i \geq 0$, we have $f^{kp+i}(z) = f^i(z)$ and $f^{kp+i}(z') = f^i(z')$. Let n be a positive integer and $t \in \mathbb{R}$. We write $n = kp + r$ with $k \geq 0$ and $r \in [\![0, p-1]\!]$. Then

$$\xi(f, z, z', n, t) = k\xi(f, z, z', p, t) + \xi(f, z, z', r, t).$$

This implies that $\lim_{n\to+\infty} \frac{1}{n}\xi(f, z, z', n, t)$ exists and is equal to $\frac{1}{p}\xi(f, z, z', p, t)$. Hence $F_{zz'}(t) = F^*_{zz'}(t) = \frac{1}{p}\xi(f, z, z', p, t)$ for all $t \in \mathbb{R}$. $\square$

THEOREM 6.25. *Let $f: I \to I$ be an interval map of zero topological entropy. Then, for all points x, y in I, $\|F_{xy} - F^*_{xy}\|_1 = 0$, where $\|\cdot\|_1$ is the L^1 norm, that is, $\|\varphi\|_1 := \int_{-\infty}^{+\infty} |\varphi(t)|\, dt$.*

PROOF. We fix two points x, y in I and a positive number ε. According to Lemma 6.23, there exist periodic points z, z' and an integer K such that

$$\forall k \geq K,\ \forall t \geq \varepsilon,\ \xi(f, x, z, k, t) \geq k(1-\varepsilon) \text{ and } \xi(f, y, z', k, t) \geq k(1-\varepsilon). \tag{6.16}$$

We set

$$\mathcal{I}_k := \{i \in [\![0, k-1]\!] \mid |f^i(x) - f^i(z)| < \varepsilon \text{ and } |f^i(y) - f^i(z')| < \varepsilon\}.$$

Then (6.16) implies that $\#\mathcal{I}_k \geq k(1-2\varepsilon)$ if $k \geq K$. For all integers i, we have

$$|f^i(x) - f^i(y)| \leq |f^i(x) - f^i(z)| + |f^i(z) - f^i(z')| + |f^i(y) - f^i(z')|. \tag{6.17}$$

If $i \in \mathcal{I}_k$ and $|f^i(z) - f^i(z')| < t - 2\varepsilon$, then $|f^i(x) - f^i(y)| < t$ by (6.17). Thus, for all $k \geq K$ and all $t \geq \varepsilon$,

$$\begin{array}{rcl} \xi(f, x, y, k, t) & \geq & \#\{i \in \mathcal{I}_k \mid |f^i(z) - f^i(z')| < t - 2\varepsilon\} \\ & \geq & k(1-2\varepsilon)\xi(f, z, z', k, t-2\varepsilon). \end{array}$$

Dividing by k and taking the limit inf, we get

$$\forall t \geq \varepsilon,\ F_{xy}(t) \geq (1-2\varepsilon)F_{zz'}(t-2\varepsilon) \geq F_{zz'}(t-2\varepsilon) - 2\varepsilon.$$

As in (6.17), we have:

$$|f^i(z) - f^i(z)| \leq |f^i(x) - f^i(z)| + |f^i(x) - f^i(y)| + |f^i(y) - f^i(z')|.$$

Similar arguments as above (with $t + 2\varepsilon$ and lim sup instead of t and lim inf) give

$$\forall t \geq \varepsilon,\ F^*_{zz'}(t+2\varepsilon) \geq F^*_{xy}(t) - 2\varepsilon.$$

According to Lemma 6.24, $F_{zz'} = F^*_{zz'}$. Thus

$$\forall t \geq \varepsilon,\ F_{zz'}(t-2\varepsilon) - 2\varepsilon \leq F_{xy}(t) \leq F^*_{xy}(t) \leq F_{zz'}(t+2\varepsilon) + 2\varepsilon.$$

By Proposition 6.21, $\|F^*_{xy} - F_{xy}\|_1 = \int_0^{|I|} \big(F^*_{xy}(t) - F_{xy}(t)\big)\, dt$ and $F^*_{xy}(t) - F_{xy}(t) \leq 1$ for all $t \in [0, \varepsilon]$. This implies that

$$\|F^*_{xy} - F_{xy}\|_1 \leq \varepsilon + \int_\varepsilon^{|I|} (F_{zz'}(t+2\varepsilon) - F_{zz'}(t-2\varepsilon))\, dt + 4\varepsilon|I|. \tag{6.18}$$

We set

$$A := \int_\varepsilon^{|I|} (F_{zz'}(t+2\varepsilon) - F_{zz'}(t-2\varepsilon))\, dt.$$

By Proposition 6.21, $F_{zz'}(t) = 0$ if $t \leq 0$ and $0 \leq F_{zz'}(t) \leq 1$ for all t. Thus

$$\begin{aligned} A &= \int_{3\varepsilon}^{|I|+2\varepsilon} F_{zz'}(u)\,du - \int_{-\varepsilon}^{|I|-2\varepsilon} F_{zz'}(u)\,du \\ &\leq \int_0^{|I|} F_{zz'}(u)\,du + 2\varepsilon - \int_0^{|I|} F_{zz'}(u)\,du + 2\varepsilon = 4\varepsilon. \end{aligned}$$

Including this result in (6.18), we have $\|F_{xy}^* - F_{xy}\|_1 \leq \varepsilon(5+4|I|)$. Taking the limit when $\varepsilon \to 0$, we get $\|F_{xy}^* - F_{xy}\|_1 = 0$. $\square$

The next theorem deals with positive entropy interval maps. The proof is different from the one in [**151**]; it uses the semi-conjugacy of a subsystem with a full shift, and the arguments are similar to the ones in the proof of Theorem 5.17.

THEOREM 6.26. *Let $f\colon I \to I$ be an interval map of positive topological entropy. Then there exist a Cantor set $K \subset I$ and a positive number δ such that, for all distinct points x, y in K,*

$$\forall t \in [0,\delta),\ F_{xy}(t) = 0 \quad \text{and} \quad \forall t > 0,\ F_{xy}^*(t) = 1.$$

PROOF. By Theorem 4.58, there exists an integer r such that f^r has a strict horseshoe (J_0, J_1). Let X, E, $\varphi\colon X \to \Sigma$ and $(J_{\alpha_0\ldots\alpha_{n-1}})_{n\geq 1,(\alpha_0,\ldots,\alpha_{n-1})\in\{0,1\}^n}$ be given by Proposition 5.15 for the map $g := f^r$.

We set $\gamma^n := \underbrace{\gamma\cdots\gamma}_{n \text{ times}}$ if $\gamma \in \{0,1\}$. We first prove the following fact:

$$\lim_{n\to+\infty} |J_{0^n1}| = 0. \tag{6.19}$$

By Proposition 5.15, $\bigcap_{n=1}^{+\infty} J_{0^n}$ is a decreasing intersection of nonempty compact intervals (this intersection may be a non degenerate interval because $(0000\ldots)$ may be in $\varphi(E)$). Thus $(\min J_{0^n})_{n\geq 0}$ is a non decreasing sequence that converges to some point x_1, and $(\max J_{0^n})_{n\geq 0}$ is a non increasing sequence that converges to some point x_2. One has $x_1, x_2 \in \bigcap_{n=1}^{+\infty} J_{0^n}$ and $x_1 \leq x_2$. Let $\varepsilon > 0$ and let $N \geq 1$ be such that

$$\forall n \geq N,\ |x_1 - \min J_{0^n}| < \varepsilon \quad \text{and} \quad |x_2 - \max J_{0^n}| < \varepsilon. \tag{6.20}$$

Let $n \geq N$. The intervals J_{0^n1} and $J_{0^{n+1}}$ are disjoint and included in J_{0^n}, and x_1, x_2 belong to $J_{0^{n+1}}$. This implies that

- either $J_{0^n1} < J_{0^{n+1}}$ and $J_{0^n1} \subset [\min J_{0^n}, x_1]$ (see Figure 4 on the left),
- or $J_{0^n1} > J_{0^{n+1}}$ and $J_{0^n1} \subset [x_2, \max J_{0^n}]$ (see Figure 4 on the right).

In both cases, $|J_{0^n1}| < \varepsilon$ according to (6.20). This proves (6.19).

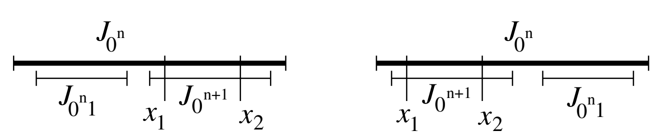

FIGURE 4. The two cases $J_{0^n1} < J_{0^{n+1}}$ and $J_{0^n1} > J_{0^{n+1}}$.

Let $(n_k)_{k\geq 0}$ be a sequence of positive integers increasing fast enough to have

$$\lim_{k\to+\infty}\frac{1}{n_k}\sum_{i=0}^{k-1}n_i=0. \tag{6.21}$$

In particular, (6.21) implies that

$$\lim_{k\to+\infty}\frac{k}{n_k}=0 \tag{6.22}$$

$$\text{and}\quad \lim_{k\to+\infty}\frac{n_k}{\sum_{i=0}^{k}n_i}=1. \tag{6.23}$$

For all $i\geq 1$ and all $\bar\alpha=(\alpha_n)_{n\geq 0}\in\Sigma$, we set $W_i:=0^{i-1}1$ and

$$B_i(\bar\alpha):=W_{n_{k_i}}(\alpha_0)^{n_{k_i+1}}(\alpha_1)^{n_{k_i+2}}\dots(\alpha_{i-1})^{n_{k_i+i}},$$

where $(k_i)_{i\geq 1}$ is the sequence defined by $k_1=0$ and $k_{i+1}=k_i+i+1$ (in this way, $B_i(\bar\alpha)$ ends with $(\alpha_{i-1})^{n_{k_i+i}}$ and $B_{i+1}(\bar\alpha)$ begins with $W_{n_{k_i+i+1}}$).

We define $\psi\colon\Sigma\to\Sigma$ by

$$\begin{array}{rcl}\psi(\bar\alpha) & := & (B_1(\bar\alpha)B_2(\bar\alpha)B_3(\bar\alpha)\dots)\\ & = & (W_{n_0}(\alpha_0)^{n_1}W_{n_2}(\alpha_0)^{n_3}(\alpha_1)^{n_4}W_{n_5}(\alpha_0)^{n_6}(\alpha_1)^{n_7}(\alpha_2)^{n_8}\dots)\end{array}$$

The map ψ is clearly continuous. For every $\bar\alpha\in\Sigma$, we chose a point $x_{\bar\alpha}$ in $\varphi^{-1}\circ\psi(\bar\alpha)$ and we set $S:=\{x_{\bar\alpha}\in X\mid\bar\alpha\in\Sigma\}$. According to Proposition 5.15, the set $\varphi^{-1}\circ\psi(\bar\alpha)$ contains two points if $\psi(\bar\alpha)\in\varphi(E)$ and one point if $\psi(\bar\alpha)\notin\varphi(E)$. Thus there exists a countable set $F\subset X$ such that $S=\varphi^{-1}\circ\psi(\Sigma)\setminus F$.

We fix $\bar\alpha=(\alpha_n)_{n\geq 0}$ and $\bar\beta=(\beta_n)_{n\geq 0}$ two distinct elements of Σ.

Let $t>0$. Since f is uniformly continuous, there exists $\varepsilon>0$ such that

$$\forall x,y\in I,\ |x-y|<\varepsilon\Rightarrow\forall i\in[\![0,r-1]\!],\ |f^i(x)-f^i(y)|<t.$$

According to (6.19), there exists a positive integer N such that $|J_{0^{n-1}1}|<\varepsilon$ for all $n\geq N$. If $j\geq 0$ is such that both $\sigma^j(\psi(\bar\alpha))$ and $\sigma^j(\psi(\bar\beta))$ begin with W_n with $n\geq N$, then $|g^j(x_{\bar\alpha})-g^j(x_{\bar\beta})|<\varepsilon$ because both points $g^j(x_{\bar\alpha}),g^j(x_{\bar\beta})$ belong to $J_{0^{n-1}1}$. We set

$$m_i:=\sum_{k=0}^{k_i}n_k.$$

The integer m_i is the length of the sequence $B_1(\bar\alpha)\dots B_{i-1}(\bar\alpha)W_{n_{k_i}}$. Then, by the definition of ψ, for all i such that $n_{k_i}>N$,

$$\forall j\in[\![N,n_{k_i}]\!],\ |g^{m_i-j}(x_{\bar\alpha})-g^{m_i-j}(x_{\bar\beta})|<\varepsilon.$$

This implies that $\xi(g,x_{\bar\alpha},x_{\bar\beta},m_i,\varepsilon)\geq n_{k_i}-N$ and $\xi(f,x_{\bar\alpha},x_{\bar\beta},r.m_i,t)\geq r(n_{k_i}-N)$. According to (6.23), $\lim_{i\to+\infty}\frac{n_{k_i}-N}{m_i}=1$, and hence $F^*_{x_{\bar\alpha}x_{\bar\beta}}(t)=1$.

Since $\bar\alpha\neq\bar\beta$, there is an integer q such that $\alpha_q\neq\beta_q$. Let $D>0$ be the distance between J_0 and J_1, and let $\delta>0$ be such that

$$\forall x,y\in I,\ |x-y|<\delta\Rightarrow\forall i\in[\![0,r-1]\!],\ |f^i(x)-f^i(y)|<D.$$

We set $p_i:=m_i+(k_i+1)+\dots+(k_i+q)$. If $k_{i+1}-k_i>q+1$, then p_i is the length of the sequence

$$B_1(\bar\alpha)\dots B_{i-1}(\bar\alpha)W_{k_i}(\alpha_0)^{n_{k_i+1}}\dots(\alpha_{q-1})^{n_{k_i+q}},$$

and $\sigma^{p_i}(\psi(\bar{\alpha}))$, $\sigma^{p_i}(\psi(\bar{\beta}))$ begin respectively with $(\alpha_q)^{n_{k_i+q+1}}$ and $(\beta_q)^{n_{k_i+q+1}}$. Then

$$\forall j \in [\![0, n_{k_i+q+1} - 1]\!],\ |g^{p_i+j}(x_{\bar{\alpha}}) - g^{p_i+j}(x_{\bar{\beta}})| \geq D$$

because either $g^{p_i+j}(x_{\bar{\alpha}}) \in J_0$ and $g^{p_i+j}(x_{\bar{\beta}}) \in J_1$, or the converse. This implies that $\xi(g, x_{\bar{\alpha}}, x_{\bar{\beta}}, p_i + n_{k_i+q+1}, D) \leq p_i$ and $\xi(f, x_{\bar{\alpha}}, x_{\bar{\beta}}, r(p_i + n_{k_i+q+1}), \delta) \leq r.p_i$. One can compute that $p_i = m_i + qk_i + \frac{q(q+1)}{2}$. Thus

$$\frac{p_i}{p_i + n_{k_i+q+1}} \leq \frac{p_i}{n_{k_i+1}} \leq \frac{m_i + qk_i + q(q+1)/2}{n_{k_i+1}}.$$

This last quantity tends to 0 according to (6.21) and (6.22), and hence

$$\lim_{i \to +\infty} \frac{p_i}{p_i + n_{k_i+q+1}} = 0.$$

We deduce that $F_{x_{\bar{\alpha}} x_{\bar{\beta}}}(t) = 0$ for all $t \in [0, \delta)$. Finally, by Theorem 5.16, there exists a Cantor set $K \subset S$ because $S = \varphi^{-1} \circ \psi(\Sigma) \setminus F$ is a Borel set. $\square$

COROLLARY 6.27. *Let f be an interval map. The following properties are equivalent:*

- *f is DC1,*
- *f is DC2,*
- *f is DC3,*
- *$h_{top}(f) > 0$.*

PROOF. It is clear than DC1⇒DC2⇒DC3. Theorem 6.25 implies that, if $h_{top}(f) = 0$, then f is not DC3. By refutation, we get DC3 $\Rightarrow h_{top}(f) > 0$. Finally, if $h_{top}(f) > 0$, then f is DC1 by Theorem 6.26. $\square$

Remarks on graph maps and general dynamical systems. The results of Schweizer and Smítal on distributional chaos was generalized to graph maps by steps, first to circle maps [**121, 122**], then to tree maps [**64, 66**] and finally to general graph maps. The next result is due to Hric and Málek [**89**].

THEOREM 6.28. *Let $f\colon G \to G$ be a graph map. The following properties are equivalent:*

- *f is DC2,*
- *$h_{top}(f) > 0$.*

For general dynamical systems, Downarowicz showed the following implication [**76**].

THEOREM 6.29. *Let (X, f) be a topological dynamical system. If f has positive topological entropy, then f is DC2.*

The equivalence of the three types of distributional chaos is not true for general dynamical systems. On the one hand, Pikuła showed that positive topological entropy does not imply DC1 [**142**]. On the other hand, Balibrea, Smítal and Štefánková exhibited a dynamical system which is DC3 and distal (i.e., for all $x \neq y$, $\liminf_{n\to+\infty} d(f^n(x), f^n(y)) > 0$) [**24**], and thus DC3 does not even imply the existence of Li-Yorke pairs (recall that, on the contrary, positive entropy implies Li-Yorke chaos according to Theorem 5.18). Therefore, DC1, DC2 and DC3 are distinct notions in general. Moreover, DC3 is not invariant by conjugacy [**24**], whereas DC1 and DC2 are.

CHAPTER 7

Chaotic subsystems

7.1. Subsystems chaotic in the sense of Devaney

In [**74**], Devaney mainly studied maps on the interval or on the real line. Observing some chaotic behavior, he introduced a definition of *chaos*. For Devaney, chaos is seen as a combination of unpredictability (sensitivity) and regular behaviors (periodic points), transitivity ensuring that the system is undecomposable.

DEFINITION 7.1 (chaos in the sense of Devaney). A topological dynamical system (X, f) is *chaotic in the sense of Devaney* if

- f is transitive,
- the set of periodic points is dense in X,
- f is sensitive to initial conditions.

For interval maps, transitivity is enough to imply the other two conditions, as it was pointed out by Silverman [**160**] and Vellekoop and Berglund [**168**]. It is a straightforward corollary of Propositions 2.15 and 2.39.

PROPOSITION 7.2. *An interval map is chaotic in the sense of Devaney if and only if it is transitive.*

Devaney was actually interested in systems having a chaotic subsystem. Shihai Li showed that, for interval maps, this is equivalent to positive entropy [**110**].

THEOREM 7.3. *Let f be an interval map. The following are equivalent:*

i) $h_{top}(f) > 0$,
ii) *there exists an invariant set X such that $(X, f|_X)$ is chaotic in the sense of Devaney,*
iii) *there exists an infinite invariant set X such that $(X, f|_X)$ is transitive and X contains a periodic point.*

PROOF. First we suppose that $h_{top}(f) > 0$. By Theorem 4.58, there exist two closed intervals J_0, J_1 and an integer $n \geq 1$ such that (J_0, J_1) is a strict horseshoe for f^n. Let X, E and $\varphi\colon X \to \Sigma$ be given by Proposition 5.15 for the map $g := f^n$. Then $(X, g|_X)$ is transitive and X is a g-invariant Cantor set. We are going to show that $(X, g|_X)$ is sensitive to initial conditions and has a dense set of periodic points.

We define the following distance on Σ: for all $\bar{\alpha} = (\alpha_n)_{n\geq 0}, \bar{\beta} = (\beta_n)_{n\geq 0}$ in Σ,

$$d(\bar{\alpha}, \bar{\beta}) := \sum_{n=0}^{+\infty} \frac{|\beta_n - \alpha_n|}{2^n}$$

(see also Definition 8.28 in the Appendix). Since X is compact, the map φ is uniformly continuous and there exists $\delta > 0$ such that

$$\forall x, y \in X,\ |x - y| < \delta \Rightarrow d(\varphi(x), \varphi(y)) < 1. \tag{7.1}$$

Let $x_0 \in X$ and $\varepsilon > 0$. The Cantor set X has no isolated point and φ is at most two-to-one, thus there exists $y \in X$ such that $|x_0 - y| < \varepsilon$ and $\varphi(y) \neq \varphi(x_0)$. Let $\bar{\alpha} = (\alpha_n)_{n\geq 0} := \varphi(x_0)$ and $\bar{\beta} = (\beta_n)_{n\geq 0} := \varphi(y)$, and let $k \geq 0$ be an integer such that $\alpha_k \neq \beta_k$. Then $d(\sigma^k(\bar{\alpha}), \sigma^k(\bar{\beta})) \geq 1$. Since φ is a semi-conjugacy, $\sigma^k(\bar{\alpha}) = \varphi(g^k(x_0))$ and $\sigma^k(\bar{\beta}) = \varphi(g^k(y))$. According to (7.1), this implies that $|g^k(x_0) - g^k(y)| \geq \delta$. This proves that $(X, g|_X)$ is δ-sensitive.

Let $x_0 \in X \setminus E$ and $\varepsilon > 0$. Let $(\alpha_n)_{n\geq 0} := \varphi(x_0)$. Since $x_0 \notin E$, there exists an integer k such that

$$\operatorname{diam}\{x \in X \mid \varphi(x) \text{ begins with } \alpha_0 \ldots \alpha_{k-1}\} < \varepsilon. \tag{7.2}$$

Let $\bar{\beta} = (\beta_n)_{n\geq 0} \in \Sigma$ be the periodic point such that $\beta_0 \ldots \beta_{k-1} = \alpha_0 \ldots \alpha_{k-1}$ and $\sigma^k(\bar{\beta}) = \bar{\beta}$ (i.e., $\bar{\beta}$ is the infinite repetition of $\alpha_0 \ldots \alpha_{k-1}$). Since φ is onto and at most two-to-one, there exist two (possibly equal) points y_1, y_2 in X such that $\varphi^{-1}(\bar{\beta}) = \{y_1, y_2\}$ (one has $y_1 = y_2$ if $\bar{\beta} \notin \varphi(E)$). Then, for $i \in \{1, 2\}$, $\varphi(g^k(y_i)) = \sigma^k(\varphi(y_i)) = \sigma^k(\bar{\beta}) = \bar{\beta}$, and $g^k(y_i) \in \varphi^{-1}(\bar{\beta}) = \{y_1, y_2\}$. This implies that either $g^{2k}(y_1) = y_1$ or $g^{2k}(y_2) = y_2$. Thus there is a periodic point among y_1, y_2; we call it y. By (7.2), $|x_0 - y| < \varepsilon$ because $\varphi(y) = \bar{\beta}$. Thus the set of periodic points is dense in $X \setminus E$. This implies that the set of periodic points is dense in X because X is an uncountable set with no isolated point and E is countable.

We set $X' := X \cup f(X) \cup \cdots f^{n-1}(X)$. Then X' is closed, f-invariant, and $(X', f|_{X'})$ is chaotic in the sense of Devaney. Thus (i)⇒(ii).

The implication (ii)⇒(iii) is trivial (notice that a sensitive system is necessarily infinite).

Now we suppose that there exists an infinite f-invariant set X such that $f|_X$ is transitive and X contains a periodic point. By Proposition 2.3, X has no isolated point and there exists $x \in X$ such that $\omega(x, f) = X$. If $h_{top}(f) = 0$, then, by Proposition 5.23, the set $\omega(x, f)$ contains no periodic point, which contradicts the fact that X contains a periodic point. We conclude that $h_{top}(f) > 0$, that is, (iii)⇒(i). □

Remarks on graph maps and general dynamical systems. The results of this section are still valid for graph maps. The generalization of Proposition 7.2 is given by Theorem 2.45, Corollary 2.46 and the fact that a rotation is not sensitive to initial conditions. The proof of Theorem 7.3 for graph maps is the same since Propositions 5.15 and 5.23 remain valid for graph maps (see "Remarks on graph maps" at the end of Sections 5.3 and 5.5).

It was shown simultaneously in several papers that there is a redundancy in the definition of chaos in the sense of Devaney, sensitivity being implied by the other two conditions [**25, 160, 83**].

THEOREM 7.4. *Let (X, f) be a topological dynamical system where X is an infinite compact space. Suppose that f is transitive and that the set of periodic points is dense. Then f is sensitive to initial conditions, and thus f is chaotic in the sense of Devaney.*

7.2. Topologically mixing subsystems

Xiong showed that an interval map f has an infinite mixing subsystem in which the set of periodic points is dense if and only if f has a periodic point of odd period greater than 1 [**171**]. The "if" part, which is a variant of Proposition 5.15, relies on the fact that f has a subsystem "almost" conjugate to the subshift associated to the graph of a periodic orbit of odd period $p > 1$, and this graph is known when p is minimal. The "only if" part can be strengthened: the existence of an infinite subsystem on which f^2 is transitive is sufficient to imply that f has a periodic point of odd period greater than 1.

REMARK 7.5. According to Xiong's terminology [**171**], an interval map f is called *strongly chaotic* if there exists an invariant subset X such that $(X, f|_X)$ is topologically mixing, the set of periodic points is dense in X and the periods of periodic points in X form an infinite set.

Much can be said about subshifts associated to a directed graph, which belong to the class of subshifts of finite type (see, e.g., [**102**]). We just give the definition; we shall not explicitly use the properties of such systems.

DEFINITION 7.6. Let G be a directed graph and V its set of vertices (recall that directed graphs are defined in Section 1.5). Let $\Gamma(G)$ denote the set of infinite paths in G, that is,

$$\Gamma(G) := \{(\alpha_n)_{n\geq 0} \in V^{\mathbb{Z}^+} \mid \forall n \geq 0, \alpha_n \to \alpha_{n+1} \text{ is an arrow in } G\}.$$

The set $V^{\mathbb{Z}^+}$ is endowed with the product topology (where V has the discrete topology) and $\Gamma(G) \subset V^{\mathbb{Z}^+}$ is endowed with the induced topology. The shift map $\sigma\colon \Gamma(G) \to \Gamma(G)$ is defined by $\sigma((\alpha_n)_{n\geq 0}) := (\alpha_{n+1})_{n\geq 0}$. Then $(\Gamma(G), \sigma)$ is a topological dynamical system, called the *subshift (or topological Markov shift) associated to the graph G.*

THEOREM 7.7. *Let $f\colon I \to I$ be an interval map. Assume that f has a periodic point of odd period greater than* 1. *Then there exists an uncountable invariant set X such that $f|_X\colon X \to X$ is topologically mixing and the set of periodic points is dense in X. Moreover, the set of periods of periodic points in X is infinite.*

PROOF. Let p be the smallest odd period greater than 1, and let G_p be the graph of a periodic orbit of period p given by Lemma 3.17. According to Proposition 4.37, for every n-tuple of vertices $(\alpha_0, \dots, \alpha_{n-1})$, if $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_{n-1}$ is a path in G_p, then $(\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ is a chain of intervals for f. For every $n \geq 0$, let Γ_n denote the set of paths of lengths n in G_p. We apply Lemma 1.13(iii) to the family of chains of intervals $(\alpha_0, \alpha_1)_{(\alpha_0,\alpha_1)\in\Gamma_1}$, and we obtain non degenerate closed subintervals with disjoint interiors $(J_{\alpha_0\alpha_1})_{(\alpha_0,\alpha_1)\in\Gamma_1}$ such that $J_{\alpha_0\alpha_1} \subset \alpha_0$ and $f(J_{\alpha_0\alpha_1}) = \alpha_1$. Using Lemma 1.13 inductively, we define non degenerate closed subintervals $(J_{\alpha_0\dots\alpha_n})_{(\alpha_0,\dots,\alpha_n)\in\Gamma_n}$ such that, for all $(\alpha_0, \dots, \alpha_n), (\beta_0, \dots, \beta_n)$ in Γ_n:

$$J_{\alpha_0\dots\alpha_n} \subset J_{\alpha_0\dots\alpha_{n-1}}, \tag{7.3}$$

$$f(J_{\alpha_0\dots\alpha_n}) = J_{\alpha_1\dots\alpha_n}, \tag{7.4}$$

$$(\alpha_0, \dots, \alpha_n) \neq (\beta_0, \dots, \beta_n) \Longrightarrow \operatorname{Int}(J_{\alpha_0\dots\alpha_n}) \cap \operatorname{Int}(J_{\beta_0\dots,\beta_n}) = \emptyset. \tag{7.5}$$

For every $\bar{\alpha} = (\alpha_n)_{n\geq 0} \in \Gamma(G_p)$, we set

$$J_{\bar{\alpha}} := \bigcap_{n=0}^{+\infty} J_{\alpha_0\ldots\alpha_n}.$$

This is a decreasing intersection of nonempty compact intervals, so $J_{\bar{\alpha}}$ is a nonempty compact interval too. Moreover, (7.5) implies that

$$\forall \bar{\alpha}, \bar{\beta} \in \Gamma(G_p),\ \bar{\alpha} \neq \bar{\beta} \Longrightarrow \mathrm{Int}\,(J_{\bar{\alpha}}) \cap \mathrm{Int}\left(J_{\bar{\beta}}\right) = \emptyset.$$

Now we are going to show that

$$\forall \bar{\alpha} \in \Gamma(G_p), \quad f(J_{\bar{\alpha}}) = \bigcap_{n=0}^{+\infty} f(J_{\alpha_0\ldots\alpha_n}). \tag{7.6}$$

The inclusion $\subset$ is obvious according to the definition of $J_{\bar{\alpha}}$. Let y be a point in $\bigcap_{n\geq 0} f(J_{\alpha_0\ldots\alpha_n})$ and, for every $n \geq 0$, let $x_n \in J_{\alpha_0\ldots\alpha_n}$ be such that $f(x_n) = y$. By compactness, there exist an increasing sequence of positive integers $(n_i)_{i\geq 0}$ and a point x such that $\lim_{i\to+\infty} x_{n_i} = x$. Moreover, $x \in \bigcap_{n=0}^{+\infty} J_{\alpha_0\ldots\alpha_n}$ because this is a decreasing intersection of compact sets; and $f(x) = y$ by continuity of f. This proves that $\bigcap_{n=0}^{\infty} f(J_{\alpha_0\ldots\alpha_n}) \subset f(J_{\bar{\alpha}})$, and thus (7.6) holds. Then (7.4) and (7.6) imply that

$$\forall \bar{\alpha} \in \Gamma(G_p), \quad f(J_{\bar{\alpha}}) = J_{\sigma(\bar{\alpha})}. \tag{7.7}$$

Let

$$E := \{\bar{\alpha} \in \Gamma(G_p) \mid J_{\bar{\alpha}} \text{ is not reduced to one point}\}.$$

By definition, we have

$$\forall (\alpha_n)_{n\geq 0} \in \Gamma(G_p) \setminus E,\ \lim_{n\to+\infty} |J_{\alpha_0\ldots\alpha_n}| = 0. \tag{7.8}$$

The set E is countable because the intervals $(J_{\bar{\alpha}})_{\bar{\alpha}\in E}$ are non degenerate and have disjoint interiors (see Lemma 1.5). Moreover, (7.7) implies that $\sigma(\Gamma(G_p) \setminus E) \subset \Gamma(G_p) \setminus E$. We define the map

$$\begin{array}{rccl} \varphi\colon & \Gamma(G_p) \setminus E & \longrightarrow & I \\ & \bar{\alpha} & \longmapsto & x \text{ such that } J_{\bar{\alpha}} = \{x\}. \end{array}$$

It is easy to show that this map is continuous using (7.3) and (7.8), and $\varphi\circ\sigma = f\circ\varphi$ by (7.7). Moreover, φ is at most two-to-one. Indeed, if $(\alpha_n)_{n\geq 0}, (\beta_n)_{n\geq 0}, (\gamma_n)_{n\geq 0}$ are three distinct elements of $\Gamma(G_p) \setminus E$, there exists $n \geq 0$ such that $(\alpha_0, \ldots, \alpha_n)$, $(\beta_0, \ldots, \beta_n)$, $(\gamma_0, \ldots, \gamma_n)$ are not all three equal to the same $(n+1)$-tuple, and thus (7.5) implies that $J_{\alpha_0\ldots\alpha_n} \cap J_{\beta_0\ldots\beta_n} \cap J_{\gamma_0\ldots\gamma_n}$ is empty.

We set

$$X_0 := \varphi(\Gamma(G_p) \setminus E) \quad \text{and} \quad X := \overline{X_0}.$$

These sets satisfy $f(X_0) \subset X_0$ and $f(X) \subset X$ because $\sigma(\Gamma(G_p) \setminus E) \subset \Gamma(G_p) \setminus E$. Moreover, X_0 and X are uncountable because φ is at most two-to-one and $\Gamma(G_p)\setminus E$ is uncountable. Looking at the graph G_p described in Lemma 3.17, we see that there exists $k \geq 0$ ($k := 2p - 3$ is suitable) such that, for all vertices α, β of G_p,

(7.9) there exists a path $(\omega_0^{\alpha\beta}, \ldots, \omega_k^{\alpha\beta}) \in \Gamma_k$ such that $\omega_0^{\alpha\beta} = \alpha$ and $\omega_k^{\alpha\beta} = \beta$.

We are going to show that

$$\forall i \geq 0,\ \forall (\alpha_0, \ldots, \alpha_i) \in \Gamma_i,\ f^{i+k}(J_{\alpha_0\ldots\alpha_i} \cap X) = X. \tag{7.10}$$

We fix $(\alpha_0, \dots, \alpha_i)$ in Γ_i. Let $\varepsilon > 0$. Let $y \in X_0$ and $(\beta_n)_{n\geq 0} \in \Gamma(G_p) \setminus E$ be such that $\varphi((\beta_n)_{n\geq 0}) = y$. By (7.8), there exists $q \geq 0$ such that $|J_{\beta_0\dots\beta_q}| < \varepsilon$. We define the map $\psi\colon \Gamma(G_p) \to \Gamma(G_p)$ by

$$\psi((\gamma_n)_{n\geq 0}) := (\alpha_0 \dots \alpha_i \omega_1^{\alpha_i\beta_0} \dots \omega_{k-1}^{\alpha_i\beta_0} \beta_0 \dots \beta_q \omega_1^{\beta_q\gamma_0} \dots \omega_{k-1}^{\beta_q\gamma_0} \gamma_0\gamma_1 \dots \gamma_n \dots),$$

where $\omega_0^{\alpha_i\beta_0} \dots \omega_k^{\alpha_i\beta_0}$ (resp. $\omega_0^{\beta_q\gamma_0} \dots \omega_k^{\beta_q\gamma_0}$) is the path from α_i to β_0 (resp. from β_q to γ_0) defined in (7.9). The map ψ is one-to-one. Since $\Gamma(G_p)$ is uncountable and E is countable, there exists $\bar\gamma \in \Gamma(G_p)$ such that $\psi(\bar\gamma) \notin E$. Let $x := \varphi \circ \psi(\bar\gamma) \in X_0$. Then $x \in J_{\alpha_0\dots\alpha_i}$ and $f^{i+k}(x) \in J_{\beta_0\dots\beta_q}$, so $|f^{i+k}(x) - y| < \varepsilon$. Since this is true for any $\varepsilon > 0$, this implies that the set $f^{i+k}(J_{\alpha_0\dots\alpha_i} \cap X_0)$ is dense in X. By compactness, we get $f^{i+k}(J_{\alpha_0\dots\alpha_i} \cap X) = X$; this is (7.10).

Finally we are going to show that $f|_X\colon X \to X$ is topologically mixing and that the set of periodic points is dense in X. Let U be an open set of I such that $U \cap X \neq \emptyset$. By denseness of X_0 in X, there exists y in $U \cap X_0$. Let $(\alpha_n)_{n\geq 0} \in \Gamma(G_p) \setminus E$ be such that $\varphi((\alpha_n)_{n\geq 0}) = y$. Since $\lim_{n\to+\infty} |J_{\alpha_0\dots\alpha_n}| = 0$ by (7.8), there exists an integer q such that $J_{\alpha_0\dots\alpha_{q-1}} \subset U$. Then $f^{q+k}(U \cap X) = X$ by (7.10). Therefore, $f|_X$ is topologically mixing. Let $\bar\gamma = (\gamma_n)_{n\geq 0}$ be the periodic sequence of period q beginning with $(\alpha_0 \dots \alpha_{q-1})$, that is, $\gamma_n = \alpha_r$ if $n = pq + r$ with $r \in [\![0, q-1]\!]$. The difficulty to find a periodic point in $U \cap X$ is that $\bar\gamma$ may belong to E (if $\bar\gamma \notin E$, then we have $z := \varphi(\bar\gamma) \in X_0 \cap U$ and $f^q(z) = z$). For every $n \geq 0$, there exists $z_n \in (J_{\gamma_0\dots\gamma_n} \cap X_0) \setminus J_{\bar\gamma}$. Let $(n_i)_{i\geq 0}$ be an increasing sequence of integers such that $(z_{n_i})_{i\geq 0}$ converges, and let z denote the limit. The point z necessarily belongs to $\partial J_{\bar\gamma}$ because $\bigcap_{n\geq 0} J_{\gamma_0\dots\gamma_n} = J_{\bar\gamma}$ is a decreasing intersection of intervals and $z_n \notin J_{\bar\gamma}$. Moreover, $z \in X$ because X is closed. For every $n \geq q$, $f^q(z_n) \in J_{\gamma_0\dots\gamma_{n-q}} \setminus J_{\bar\gamma}$ because $\sigma^q(\bar\gamma) = \bar\gamma$. Therefore the sequence $(f^q(z_{n_i}))_{i\geq 0}$ converges to the point $f^q(z)$ by continuity, $f^q(z) \in X$ because X is invariant and $f^q(z) \in \partial J_{\bar\gamma}$ for the same reason as above. Similarly, $f^{2q}(z) \in \partial J_{\bar\gamma} \cap X$. The three points $\{z, f^q(z), f^{2q}(z)\}$ belong to $\partial J_{\bar\gamma}$, and thus two of these points are equal. Therefore, either z or $f^q(z)$ is a periodic point and belongs to $U \cap X$. This shows that the set of periodic points is dense in X. Finally, the facts that $f|_X$ is topologically mixing and has a dense set of periodic points ensure that the set of periods of periodic points in X is infinite (if the set of periods is finite and if N is a common multiple of all the periods, then $f^N|_X$ is the identity map by denseness of the set of periodic points, and thus $f|_X$ is not mixing). ☐

THEOREM 7.8. *Let f be an interval map. The following are equivalent:*

i) *f has a periodic point of odd period greater than 1,*
ii) *there exists an infinite f-invariant set X such that $(X, f^2|_X)$ is transitive.*

PROOF. The implication (i)⇒(ii) is given by Theorem 7.7.

We suppose that there exists an infinite f-invariant set X such that $f^2|_X$ is transitive. Let $y \in X$ be a point whose orbit under f^2 is dense in X. Since X is infinite, the points $(f^n(y))_{n\geq 0}$ are pairwise distinct. We may assume that $\min X < f(y) < \max X$ (otherwise, we can replace y by some iterate). We also assume that $f(y) < f^2(y)$, the case with reverse inequality being symmetric. Since $\mathcal{O}_{f^2}(y)$ is dense in X, there exists $n \geq 2$ such that $f^{2n}(y) \in [\min X, f(y))$. Thus we have $f^{2n}(y) < f(y) < f^2(y)$. According to Proposition 3.34 applied to the point $x := f(y)$, there exists a periodic point of odd period greater than 1. That is, (ii)⇒(i). ☐

According to Theorem 4.58, an interval map has positive entropy if and only if it has a periodic point of period $2^n p$ for some $n \geq 0$ and some odd $p > 1$. Therefore the next corollary follows straightforwardly from Theorems 7.7 and 7.8.

COROLLARY 7.9. *Let f be an interval map. The following are equivalent:*

i) $h_{top}(f) > 0$,
ii) *there exist a positive integer n and an uncountable f^n-invariant set X such that $(X, f^n|_X)$ is topologically mixing and the set of periodic points is dense in X,*
iii) *there exist a positive integer n and an f^n-invariant set X such that $(X, f^n|_X)$ is topologically mixing,*
iv) *there exist a positive integer n and an infinite f^n-invariant set X such that $(X, f^{2n}|_X)$ is transitive.*

REMARK 7.10. A result similar to, but weaker than, the equivalence (i)$\Leftrightarrow$(iii) in Corollary 7.9 was stated by Osikawa and Oono in [**139**]: they proved that an interval map f has a periodic point whose period is not a power of 2 if and only if there exists a mixing f^n-invariant measure for some positive integer n. The proof relies on the construction of a set X such that $f^n(X) \subset X$ and $(X, f^n|_X)$ is Borel conjugate to the full shift on two symbols (i.e., the conjugacy map is only Borel and may not be continuous); in particular, X may not be closed in this construction.

Remarks on graph maps. For graph maps, there is no simple relation between positive entropy and the periods of periodic points. Moreover, an irrational rotation is totally transitive but not topologically mixing and has zero entropy; thus there is no way to get a result similar to Theorem 7.8. However, Corollary 7.9 can be partially generalized to graph maps. Indeed, a graph map f has positive entropy if and only if f^n has a horseshoe for some n (Theorem 4.12), which implies that there exists an f^n-invariant set X such that (X, f^n) is "almost" conjugate to the shift (Σ, σ) (Proposition 5.15). Moreover, the properties of ω-limit sets for zero entropy graph maps (Theorems 5.35 and 5.36) imply that a zero entropy graph map admits no topologically mixing subsystem. This leads to the following result.

THEOREM 7.11. *Let f be a graph map. The following are equivalent:*

i) $h_{top}(f) > 0$,
ii) *there exist a positive integer n and an uncountable f^n-invariant set X such that $(X, f^n|_X)$ is topologically mixing and the set of periodic points is dense in X,*
iii) *there exist a positive integer n and an f^n-invariant set X such that $(X, f^n|_X)$ is topologically mixing.*

7.3. Transitive sensitive subsystems

One may consider a variant of Devaney's definition of chaos by omitting the assumption on periodic points. What can be said about interval maps having transitive sensitive subsystems? By Theorem 7.3, a positive entropy interval map has a transitive sensitive subsystem. The converse is not true: the map built in Example 5.59 has zero entropy but has a transitive sensitive subsystem by Lemma 5.64. We are going to show that the existence of a transitive sensitive subsystem implies chaos in the sense of Li-Yorke. The converse is not true either: a (rather complicated) counter-example is given in Subsection 7.3.2. It follows that, for interval maps, the existence of a transitive sensitive subsystem is a strictly intermediate notion between positive entropy and chaos in the sense of Li-Yorke. These results were shown by the author in [**149**].

REMARK 7.12. A topological dynamical system (X, f) is sometimes called *chaotic in the sense of Auslander-Yorke* or *chaotic in the sense of Ruelle and Takens* if it is transitive and sensitive to initial conditions [**20, 147**], and *chaotic in the sense of Wiggins* if there exists an invariant set $Y \subset X$ such that $(Y, f|_Y)$ is transitive and sensitive to initial conditions [**81**].

7.3.1. Transitive sensitive subsystem implies Li-Yorke chaos. The next result is [**149**, Theorem 1.7].

THEOREM 7.13. *Let f be an interval map. If Y is an invariant set such that $f|_Y$ is transitive and sensitive to initial conditions, then f is chaotic in the sense of Li-Yorke.*

PROOF. We show the result by refutation. Suppose that f is not chaotic in the sense of Li-Yorke. By Theorem 5.17, the topological entropy of f is zero. Consider an invariant set Y such that $f|_Y$ is transitive. If Y is finite, then $f|_Y$ is not sensitive. If Y is infinite, there exists $y_0 \in Y$ such that $\omega(y_0, f) = Y$ (Proposition 2.3(i)). By Theorem 5.21, Y does not contain two f-non separable points. Let $\varepsilon > 0$. According to Proposition 5.32(i), there exists an integer $n \geq 1$ such that

$$\max_{i \in [\![0, 2^n - 1]\!]} \operatorname{diam}(\omega(f^i(y_0), f^{2^n})) < \varepsilon.$$

We set $I_i := [\min \omega(f^i(y_0), f^{2^n}), \max \omega(f^i(y_0), f^{2^n})]$ for all $i \in [\![0, 2^n - 1]\!]$. Then $f(Y \cap I_i) = Y \cap I_{i+1 \bmod 2^n}$ because $f(\omega(f^i(y_0), f^{2^n})) = \omega(f^{i+1 \bmod 2^n}(y_0), f^{2^n})$ and $Y = \bigcup_{i \in [\![0,2^n-1]\!]} \omega(f^i(y_0), f^{2^n})$ by Lemma 1.3. Moreover the intervals $(I_i)_{0 \leq i \leq 2^n - 1}$ are pairwise disjoint by Proposition 5.24. Let $\delta > 0$ be such that the distance between two different intervals among $(I_i)_{0 \leq i \leq 2^n-1}$ is greater than δ. Let $x, y \in Y$ be such that $|x - y| < \delta$. Then there exists $i \in [\![0, 2^n - 1]\!]$ such that $x, y \in I_i$ and, for all $k \geq 0$, $f^k(x), f^k(y) \in I_{i+k \bmod 2^n}$, so $|f^k(x) - f^k(y)| < \varepsilon$. We conclude that $f|_Y$ is not sensitive to initial conditions. $\square$

7.3.2. Li-Yorke chaos does not imply a transitive sensitive subsystem. The aim of this subsection is to exhibit an interval map $h\colon [0, 3/2] \to [0, 3/2]$ that is chaotic in the sense of Li-Yorke but has no transitive sensitive subsystem. This example is taken from [**149**]. Let us first explain the main underlying ideas of the construction of h. This map is obtained by modifying the construction of the map g of Example 5.59. The maps h and g have the same construction on the set $\bigcup_{n \geq 1} I_n^0$ – which is the core of the dynamics of g – but the the lengths of the intervals $(I_n^0)_{n \geq 1}$ are not the same and the definition of h on the intervals $(L_n)_{n \geq 1}$ is different. For g, we showed that $K := \bigcap_{n \geq 0} \bigcup_{i=0}^{2^n-1} g^i(J_n^0)$ has a non degenerate connected component C and that the endpoints of C are g-non separable. The same remains true for h with $C := \bigcap_{n \geq 0} I_n^1 = [a, 1]$ (the fact that $a, 1$ are h-non separable will be proved in Proposition 7.17). For g, we proved that $\mathrm{Bd}_{\mathbb{R}} K \subset \omega(0, g)$, hence $\partial C \subset \omega(0, g)$. For h, it is not true that $\partial C \subset \omega(0, h)$ because the orbit of 0 stays in $[0, a]$. The construction of h on the intervals L_n allows one to approach 1 from outside $[0, 1]$: we shall see in Proposition 7.17 that $\omega(3/2, h)$ contains both a and 1, which implies chaos in the sense of Li-Yorke because a and 1 are h-non separable.

On the other hand, the proof showing that $g|_{\omega(0,g)}$ is transitive and sensitive fails for h because $\omega(0,h)$ does not contain $\{a,1\}$, and $\omega(3/2,h)$ is not transitive. We shall see in Proposition 7.20 that h has no transitive sensitive subsystem at all.

Let $(a_n)_{n\geq 0}$ be an increasing sequence of numbers less than 1 such that $a_0=0$. We set $I_0^1:=[a_0,1]$ and

$$\forall n\geq 1,\ I_n^0:=[a_{2n-2},a_{2n-1}],\ L_n:=[a_{2n-1},a_{2n}],\ I_n^1:=[a_{2n},1].$$

It is clear that $I_n^0\cup L_n\cup I_n^1=I_{n-1}^1$. We fix $(a_n)_{n\geq 0}$ such that the lengths of the intervals satisfy

$$\forall n\geq 1,\ |I_n^0|=|L_n|=\frac{1}{3^n}|I_{n-1}^1|\quad\text{and}\quad |I_n^1|=\left(1-\frac{2}{3^n}\right)|I_{n-1}^1|.$$

Let $a:=\lim_{n\to+\infty}a_n$. Then $\bigcup_{n\geq 1}(I_n^0\cup L_n)=[0,a)$. Moreover, $a<1$ because

$$\log(1-a)=\sum_{n=1}^{+\infty}\log\left(1-\frac{2}{3^n}\right)>-\infty,$$

the last inequality follows from the facts that $\log(1+x)\sim x$ when $x\to 0$ and $\sum\frac{1}{3^n}<+\infty$.

Notation. If I is an interval, let $\mathsf{mid}(I)$ denote its middle point (that is, $\mathsf{mid}([b,c])=\frac{b+c}{2}$). For short, we write $\uparrow$ (resp. $\downarrow$) for "increasing" (resp. "decreasing").

For all $n\geq 1$, let $\varphi_n\colon I_n^0\to I_n^1$ be the increasing linear homeomorphism mapping I_n^0 onto I_n^1. We define the map $h\colon[0,3/2]\to[0,3/2]$ such that h is continuous on $[0,3/2]\setminus\{a\}$ and

$$\begin{aligned}
&h(x)=\varphi_1^{-1}\circ\varphi_2^{-2}\circ\cdots\circ\varphi_{n-1}^{-1}\circ\varphi_n(x)\text{ for all }x\in I_n^0,\ n\geq 1,\\
&h\text{ is linear}\uparrow\text{ of slope }\lambda_n\text{ on }[\min L_n,\mathsf{mid}(L_n)]\quad\text{for all }n\geq 1,\\
&h\text{ is linear}\downarrow\text{ on }[\mathsf{mid}(L_n),\max L_n]\quad\text{for all }n\geq 1,\\
&h(x)=0\quad\text{for all }x\in[a,1],\\
&h(x)=x-1\quad\text{for all }x\in[1,3/2],
\end{aligned}$$

where the slopes (λ_n) will be defined below. We shall also show below that h is continuous at a. The map h is represented on Figure 1.

We set $J_0^0:=[0,1]$ and, for all $n\geq 1$, we define J_n^0,J_n^1 as subinterval of J_{n-1}^0 such that $\min J_n^0=0$, $\max J_n^1=\max J_{n-1}^0$ and $\frac{|J_n^i|}{|J_{n-1}^0|}=\frac{|I_n^i|}{|I_{n-1}^1|}$ for $i\in\{0,1\}$. We also set $M_n:=[\max J_n^0,\min J_n^1]$.

Notice that, on the set $\bigcup_{n\geq 1}I_n^0$, the map h is defined similarly to the map g of Example 5.59 (the reader can refer to Figure 7 page 148 and the explanations of the underlying construction of g on this set). Therefore, the assertions of Lemma 5.60 remain valid for h, except the point (i) and its derived results (viii), (x), (xi).

Lemma 7.14. *Let h be the map defined above. Then, for all $n\geq 1$,*

i) $h(I_n^0)=J_n^1$,
ii) $h^i|_{J_n^0}$ *is linear* $\uparrow$ *for all* $i\in[\![0,2^n-1]\!]$,
iii) $h^{2^{n-1}-1}(J_n^0)=I_n^0$ *and* $h^{2^n-1}(J_n^0)=I_n^1$,
iv) $h^i(J_n^0)\subset\bigcup_{k=1}^n I_k^0$ *for all* $i\in[\![0,2^n-2]\!]$,
v) $(h^i(J_n^0))_{0\leq i<2^n}$ *are pairwise disjoint,*

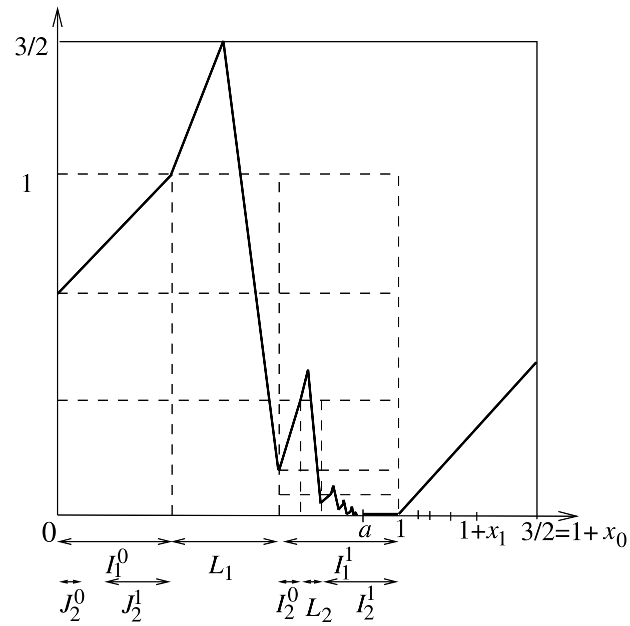

FIGURE 1. The graph of h; this map is chaotic in the sense of Li-Yorke but has no transitive sensitive subsystem.

vi) $h^{2^{n-1}}|_{I_n^0}$ *is linear* $\uparrow$ *and* $h^{2^{n-1}}(I_n^0) = I_n^1$,
vii) $h^{2^{n-1}-1}|_{M_n}$ *is linear* $\uparrow$ *and* $h^{2^{n-1}-1}(M_n) = L_n$,
viii) $h(\min L_n) = \min M_{n-1}$,
ix) $h^{2^{n-2}}(\min L_n) = \min L_{n-1}$.

PROOF. For the assertions (i) to (vi), see the proof of Lemma 5.60(ii)-(vi)+(ix).

According to (ii), the map $h^{2^{n-1}-1}|_{M_n}$ is linear $\uparrow$ because M_n is included in J_{n-1}^0. Since $M_n = [\max J_n^0, \min J_n^1]$ and $L_n = [\max I_n^0, \min I_n^1]$, the combination of (i), (ii) and (iii) implies that $h^{2^{n-1}-1}(M_n) = L_n$; this is (vii).

The map $h|_{I_n^0}$ is increasing and $\min L_n = \max I_n^0$. Hence, according to (i), we have $h(\min L_n) = \max J_n^1 = \max J_{n-1}^0 = \min M_{n-1}$; this is (viii).

Finally, (ix) follows from (vii) and (viii). □

For all $n \geq 0$, we set $x_n := \mathsf{mid}(M_{n+1})$, that is, $x_n = \frac{3}{2}\prod_{i=1}^{n+1}\frac{1}{3^i}$. It is a decreasing sequence and $x_0 = 1/2$. Therefore $h(1 + x_n)$ is well defined and equal to x_n for all $n \geq 0$.

For all $n \geq 0$, let $t_n := \mathsf{slope}\left(h^{2^n-1}|_{J_n^0}\right)$; by convention, h^0 is the identity map, so $t_0 = 1$. We fix $\lambda_1 := \frac{2x_1}{|L_1|}$ and we define inductively $(\lambda_n)_{n\geq 2}$ such that

$$\frac{|L_n|}{2}\prod_{i=1}^{n}\lambda_i\prod_{i=0}^{n-2}t_i = x_n. \tag{7.11}$$

By convention, an empty product is equal to 1, so (7.11) is satisfied for $n = 1$ too.

The slopes $(\lambda_n)_{n\geq 1}$ are such that $h^{2^{n-1}}([\min L_n, \mathsf{mid}(L_n)]) = [1, 1 + x_n]$, as proved in the next lemma. This means that, under the action of $h^{2^{n-1}}$, the image of L_n falls outside of $[0,1)$ but remains close to 1. We also list some properties of h on the intervals L_n, I_n^1 and $[1,1+x_n]$.

LEMMA 7.15. *Let h be the map defined above. Then*

i) $h^{2^n}|_{[1,1+x_n]}$ *is linear* $\uparrow$ *and* $h^{2^n}([1,1+x_n]) = [\min I_{n+1}^0, \mathsf{mid}(L_{n+1})]$ *for all* $n \geq 0$,
ii) $h^{2^{n-1}}|_{[\min L_n, \mathsf{mid}(L_n)]}$ *is linear* $\uparrow$ *and* $h^{2^{n-1}}([\min L_n, \mathsf{mid}(L_n)]) = [1, 1+x_n]$ *for all* $n \geq 1$,
iii) $h^{2^{n+1}}|_{[1,1+x_n]}$ *is* $\uparrow$ *and* $h^{2^{n+1}}([1,1+x_n]) = I_{n+1}^1 \cup [1, 1+x_{n+1}]$ *for all* $n \geq 1$,
iv) $h(I_n^1) \subset [0, \mathsf{mid}(M_n)]$ *for all* $n \geq 1$,
v) $h^{2^n}([\min I_n^1, 1+x_n]) \subset [\min I_n^1, 1+x_n]$ *and* $h^i([\min I_n^1, 1+x_n]) \subset [0,1]$ *for all* $n \geq 1$ *and all* $i \in [\![1, 2^n-1]\!]$.

PROOF. The map $h|_{[1,1+x_n]}$ is linear $\uparrow$ and $h([1,1+x_n]) = [0, \mathsf{mid}(M_{n+1})] \subset J_n^0$, thus $h^{2^n}|_{[1,1+x_n]}$ is linear $\uparrow$ by Lemma 7.14(ii). Moreover $h^{2^n-1}(0) = \min I_{n+1}^0$ and $h^{2^n-1}(\mathsf{mid}(M_{n+1})) = \mathsf{mid}(L_{n+1})$ by Lemma 7.14(ii)+(iii)+(vii); this implies (i).

Before proving (ii), we show some intermediate results. Let n, k be integers with $n \geq 2$ and $k \in [\![2, n]\!]$. Then

$$\begin{aligned}
\lambda_n \dots \lambda_k \cdot t_{n-2} \dots t_{k-2} &= \frac{\displaystyle\prod_{i=1}^{n} \lambda_i \prod_{i=0}^{n-2} t_i}{\displaystyle\prod_{i=1}^{k-1} \lambda_i \prod_{i=0}^{k-3} t_i} \\
&= \frac{x_n}{x_{k-1}} \cdot \frac{|L_{k-1}|}{|L_n|} \quad \text{by (7.11)} \\
&= \prod_{i=k+1}^{n+1} \frac{1}{3^i} \prod_{i=k-1}^{n-1} \frac{1}{1-\frac{2}{3^i}} \cdot \frac{3^n}{3^{k-1}} \\
&= \frac{1}{3^{n-k+1}} \prod_{i=k-1}^{n-1} \frac{1}{3^i - 2}
\end{aligned}$$

and so

$$\lambda_n \dots \lambda_k \cdot t_{n-2} \dots t_{k-2} < 1. \tag{7.12}$$

By definition, $h|_{[\min L_n, \mathsf{mid}(L_n)]}$ is linear $\uparrow$ and $h(\mathsf{mid}(L_n)) = h(\min L_n) + \lambda_n \frac{|L_n|}{2}$. By (7.11),

$$\begin{aligned}
\lambda_n \frac{|L_n|}{2} &= \frac{x_n}{t_{n-2} \displaystyle\prod_{i=1}^{n-1} \lambda_i \prod_{i=0}^{n-3} t_i} \\
&= \frac{x_n |L_{n-1}|}{2 x_{n-1} t_{n-2}} \\
&= \frac{1}{3^{n+1}} \frac{|M_{n-1}|}{2};
\end{aligned}$$

the last equality is because $J^0_{n-2} \supset M_{n-1}$, so $t_{n-2} = \frac{|L_{n-1}|}{|M_{n-1}|}$ by Lemma 7.14(vii). Therefore

$$\lambda_n \frac{|L_n|}{2} < \frac{|M_{n-1}|}{2}.$$

Moreover, $h(\min L_n) = \min M_{n-1}$ by Lemma 7.14(viii), hence

$$h([\min L_n, \mathsf{mid}(L_n)]) \subset [\min M_{n-1}, \mathsf{mid}(M_{n-1})] \quad \text{for all } n \geq 2. \tag{7.13}$$

Fact 1. *For all $k \in [\![2, n]\!]$,*

- *the map $h^{2^{n-2}+2^{n-3}+\cdots+2^{k-2}}$ is linear $\uparrow$ of slope $\lambda_n \ldots \lambda_k t_{n-2} \ldots t_{k-2}$ on $[\min L_n, \mathsf{mid}(L_n)]$ and sends $\min L_n$ to $\min L_{k-1}$,*
- *$h^i([\min L_n, \mathsf{mid}(L_n)]) \subset [0, 1]$ for all $i \in [\![0, 2^{n-2} + 2^{n-3} + \cdots + 2^{k-2}]\!]$.*

We show this fact by induction on k, where k decreases from n to 2.

• By (7.13) we have $h([\min L_n, \mathsf{mid}(L_n)]) \subset M_{n-1} \subset J^0_{n-2}$, so $h^{2^{n-2}}|_{[\min L_n, \mathsf{mid}(L_n)]}$ is linear $\uparrow$ of slope $\lambda_n t_{n-2}$. By Lemma 7.14(ix), $h^{2^{n-2}}(\min L_n) = \min L_{n-1}$. Then (7.13) and Lemma 7.14(iii)+(iv) imply that $h^i([\min L_n, \mathsf{mid}(L_n)]) \subset [0, 1]$ for all $i \in [\![1, 2^{n-2}]\!]$. This is Fact 1 for $k = n$.

• Suppose that Fact 1 holds for some $k \in [\![3, n]\!]$. By (7.12), we have

$$\lambda_n \ldots \lambda_k \cdot t_{n-2} \ldots t_{k-2} \frac{|L_n|}{2} \leq \frac{|L_{k-1}|}{2}$$

so that

$$h^{2^{n-2}+2^{n-3}+\cdots+2^{k-2}}([\min L_n, \mathsf{mid}(L_n)]) \subset [\min L_{k-1}, \mathsf{mid}(L_{k-1})].$$

The map h is of slope λ_{k-1} on this interval, $h(\min L_{k-1}) = \min M_{k-2}$ according to Lemma 7.14(viii) and $h([\min L_{k-1}, \mathsf{mid}(L_{k-1})]) \subset M_{k-2}$ by (7.13). Since $M_{k-2} \subset J^0_{n-1}$, the map $h^{2^{n-2}+2^{n-3}+\cdots+2^{k-2}+2^{k-3}}$ is linear $\uparrow$ of slope $\lambda_n \ldots \lambda_{k-1} \cdot t_{n-2} \ldots t_{k-3}$ on $[\min L_n, \mathsf{mid}(L_n)]$, and it sends $\min L_n$ to $\min L_{k-2}$ by Lemma 7.14(ix). Moreover, $h^i([\min L_n, \mathsf{mid}(L_n)]) \subset [0, 1]$ for all $i \in [\![0, 2^{n-2} + 2^{n-3} + \cdots + 2^{k-2} + 2^{k-3}]\!]$ by Lemma 7.14(iv) and the induction hypothesis. This is Fact 1 for $k - 1$. This ends the induction and proves Fact 1.

For $k = 2$, Fact 1 implies that $h^{2^{n-2}+\cdots+2^0} = h^{2^{n-1}-1}$ is linear increasing of slope $\prod_{i=2}^n \lambda_i \prod_{i=0}^{n-2} t_i$ on $[\min L_n, \mathsf{mid}(L_n)]$, with

$$h^{2^{n-1}-1}(\min L_n) = \min L_1$$
$$\text{and } h^{2^{n-1}-1}([\min L_n, \mathsf{mid}(L_n)]) \subset [\min L_1, \mathsf{mid}(L_1)].$$

The map h is of slope λ_1 on this interval, hence, according to the definition of (λ_n), (ii) holds for all $n \geq 2$; it also trivially holds for $n = 1$. Fact 1 for $k = 2$ also shows that

$$h^i([\min L_n, \mathsf{mid}(L_n)]) \subset [0, 1] \quad \text{for all } i \in [\![0, 2^{n-1} - 1]\!] \text{ and all } n \geq 1. \tag{7.14}$$

Then (iii) follows from (i), (ii) and Lemma 7.14(vi).

We have $I^1_n = \bigcup_{k \geq n+1} (I^0_k \cup L_k) \cup [a, 1]$. From the definition of h, we can see that

$$\max\{h(x) \mid x \in I^0_k \cup L_k\} = h(\mathsf{mid}(L_k)),$$

so $h(I^0_k \cup L_k) \subset [0, \mathsf{mid}(M_{k-1})]$ by (7.13). Hence

$$h(I^1_n) \subset [0, \mathsf{mid}(M_n)] = J^0_n \cup [\min M_n, \mathsf{mid}(M_n)]; \tag{7.15}$$

this is (iv).

According to Lemma 7.14(iii)+(vii),

$$h^{2^n-1}(J_n^0) = I_n^1 \text{ and } h^{2^{n-1}-1}([\min M_n, \mathsf{mid}(M_n)]) = [\min L_n, \mathsf{mid}(L_n)],$$

and by (ii), $h^{2^{n-1}}([\min L_n, \mathsf{mid}(L_n)]) = [1, 1+x_n]$. Combined with (7.15), we get

$$h^{2^n}(I_n^1) \subset I_n^1 \cup [1, 1+x_n]. \tag{7.16}$$

Moreover, $h^i(J_n^0) \subset [0,1]$ for all $i \in [\![0, 2^n-2]\!]$ and $h^i([\min M_n, \mathsf{mid}(M_n)]) \subset [0,1]$ for all $i \in [\![0, 2^{n-1}-2]\!]$ according to Lemma 7.14(iv). In addition,

$$h^{2^{n-1}+i-1}([\min M_n, \mathsf{mid}(M_n)]) = h^i([\min L_n, \mathsf{mid}(L_n)]) \subset [0,1]$$

for all $i \in [\![0, 2^{n-1}-1]\!]$ by Lemma 7.14(vii) and (7.14). Therefore,

$$h^i(I_n^1) \subset [0,1] \text{ for all } i \in [\![0, 2^n-1]\!]. \tag{7.17}$$

Finally, since $h([1,1+x_n]) = [0, \mathsf{mid}(M_{n+1})] \subset J_n^0$, statement (i) implies that $h^{2^n}([1,1+x_n]) \subset I_n^1$. Combined with (7.17), (7.16) and Lemma 7.14(iv), this implies (v). $\square$

Now we show that h is continuous at the point a as claimed at the beginning of the section.

LEMMA 7.16. *The map h defined above is continuous.*

PROOF. We just have to show the continuity at a. It is clear from the definition that h is continuous at a^+. According to Lemma 7.15(iv), we have $h(I_n^1) \subset J_{n-1}^0$. By definition, $h(a) = 0$ and $a = \max I_n^1$ for all n. Moreover, by definition of the intervals $(J_n^0)_{n\geq 0}$,

$$\lim_{n\to+\infty} \max J_n^0 = \lim_{n\to+\infty} |J_n^0| = 0,$$

and thus $\lim_{n\to+\infty} \max h(I_n^1) = 0$. This implies that h is continuous at a^-. $\square$

PROPOSITION 7.17. *Let h be the map defined above. Then the set $\omega(1+x_0, h)$ is infinite and contains the points a and 1, which are h-non separable. Consequently, the map h is chaotic in the sense of Li-Yorke.*

PROOF. Lemma 7.15(iii) implies that $h^{2^{n+1}}(1+x_n) = 1 + x_{n+1}$ for all $n \geq 0$. Since $x_n \to 0$ when n goes to infinity, this implies that 1 belongs to $\omega(1+x_0, h)$ (recall that $\omega(1+x_0, h)$ is closed by Lemma 1.3(i)). Moreover, Lemma 7.15(i) implies that $h^{2^n}(1) = \min I_{n+1}^0 = a_{2n}$ for all $n \geq 1$, so a belongs to $\omega(1,h) \subset \omega(1+x_0, h)$.

Suppose that A_1, A_2 are two periodic intervals such that $a \in A_1$ and $1 \in A_2$, and let p be a common multiple of their periods. Since $h(a) = h(1) = 0$, it follows that $h^p(a) = h^p(1) \in A_1 \cap A_2$, so A_1, A_2 are not disjoint. This means that a and 1 are h-non separable.

A finite ω-limit set is a periodic orbit (Lemma 1.4). Therefore, if y_0, y_1 are two distinct points in a finite ω-set, the degenerate intervals $\{y_0\}$, $\{y_1\}$ are periodic and y_0, y_1 are h-separable. This implies that $\omega(1+x_0, h)$ is infinite. We deduce that the map h is chaotic in the sense of Li-Yorke by Theorem 5.21. $\square$

The next lemma is about the location of transitive subsystems.

LEMMA 7.18. *Let h be the map defined above and let Y be an invariant set with no isolated point such that $h|_Y$ is transitive. Then*

i) $Y \subset [0, a]$,

ii) $Y \subset \bigcup_{i=0}^{2^n-1} h^i(J_n^0)$ *for all* $n \geq 1$,

iii) $h^i(J_n^0 \cap Y) = h^i(J_n^0) \cap Y = h^{i \bmod 2^n}(J_n^0) \cap Y$ *for all* $i \geq 0$ *and all* $n \geq 0$.

PROOF. Since $f|_Y$ is transitive, there exists $y_0 \in Y$ such that $\omega(y_0, h) = Y$ by Proposition 2.3; in particular, the set $Y' := \mathcal{O}_h(y_0)$ is dense in Y and $y \in \omega(y, h)$ for all $y \in Y'$. Note that Y' is infinite, otherwise Y would be a finite set and would contain isolated points.

Let $n \geq 0$. By Lemma 7.15(iii), $h^{2^{n+1}}([1, 1 + x_n]) = I_{n+1}^1 \cup [1, 1 + x_{n+1}]$. Thus Lemma 7.15(v) implies that, for all $k \in \mathbb{N}$, $h^{k2^{n+1}}([1, 1+x_n]) \subset I_{n+1}^1 \cup [1, 1+x_{n+1}]$ and $h^i([1, 1 + x_n]) \subset [0, 1]$ for all $i > 2^{n+1}$ such that $i \notin 2^{n+1}\mathbb{N}$. This implies that

$$h^i((1 + x_{n+1}, 1 + x_n]) \subset [0, 1 + x_{n+1}] \quad \text{for all } i \geq 2^{n+1}.$$

Consequently, there is no $y \in (1, 3/2] = \bigcup_{n\geq 0}(1 + x_{n+1}, 1 + x_n]$ such that y is in $\omega(y, h)$. So $Y' \cap (1, 3/2] = \emptyset$, and thus $Y \cap (1, 3/2] = \emptyset$ because Y' is dense in Y.

Since $h^{2^n-1}(0) = a_{2n}$ by Lemma 7.14(ii)+(iii), the point 0 is not periodic, so $h^k(0) \notin [a, 1]$ for all $k \geq 1$. If $y \in (a, 1)$, then $h(y) = 0$ and $h^k(y) \notin [a, 1]$ for all $k \geq 1$, which implies that $y \notin \omega(y, h)$. Consequently, $Y \cap (a, 1) = \emptyset$. We have shown that $Y \subset [0, a] \cup \{1\}$; in addition, $1 \notin Y$ because Y has no isolated point; this proves (i).

Let $n \geq 1$. Since $\min L_n = \max I_n^0$ and $\max L_n = \min I_{n+1}^0$, it follows that $h(\min L_n) = \max J_n^1$ and $h(\max L_n) = \min J_{n+1}^1$ according to Lemma 7.14(i), and thus $h(\max L_n) < h(\min L_n)$. Moreover, $h|_{[\min L_n, \mathsf{mid}(L_n)]}$ is $\uparrow$ and $h|_{[\mathsf{mid}(L_n), \max L_n]}$ is linear $\downarrow$, so there exists c_n in $[\mathsf{mid}(L_n), \max L_n]$ such that $h(c_n) = h(\min L_n)$.

Since $h([c_n, \max L_n]) = [\min J_{n+1}^1, \max J_n^1]$ is included in the interval J_{n-1}^0, the map $h^{2^{n-1}}|_{[c_n, \max L_n]}$ is linear $\downarrow$ by Lemma 7.14(ii). Moreover, M_n is included in $h([c_n, \max L_n])$, so $h^{2^{n-1}}([c_n, \max L_n])$ contains L_n by Lemma 7.14(vii). Thus there exists a point z_n in the interval $[c_n, \max L_n]$ such that $h^{2^{n-1}}(z_n) = z_n$ (by Lemma 1.11) and we have $\mathsf{slope}(h^{2^{n-1}}|_{[c_n, \max L_n]}) \leq -2$. Then for every $x \in [c_n, \max L_n]$ with $x \neq z_n$, there exists $k \geq 1$ such that $h^{k2^{n-1}}(x) \notin [c_n, \max L_n]$. By Lemma 7.15(v), we have $h^{2^{n-1}}(I_{n-1}^1 \cup [1, 1+x_{n-1}]) \subset I_{n-1}^1 \cup [1, 1+x_{n-1}]$, which implies that

$$\forall x \in [c_n, \max L_n],\ x \neq z_n,\ \exists k \geq 1,$$
$$h^{k2^{n-1}}(x) \in I_n^0 \cup [\min L_n, c_n] \cup I_n^1 \cup [1, 1 + x_{n-1}]. \tag{7.18}$$

We show by induction on n that

$$\forall n \geq 0, \quad Y' \cap I_n^1 \neq \emptyset. \tag{7.19}$$

This is true for $n = 0$ because $Y \subset [0, 1] = I_0^1$ by (i). Suppose that there exists $y \in Y' \cap I_{n-1}^1$ for some $n \geq 1$. We write $I_{n-1}^1 = I_n^0 \cup L_n \cup I_n^1$; to prove that $Y' \cap I_n^1 \neq \emptyset$, we split into four cases according to the position of y.

- If $y \in I_n^1$, there is nothing to do.
- If $y \in I_n^0$, then $h^{2^{n-1}}(y) \in I_n^1$ by Lemma 7.14(vi) and $h^{2^{n-1}}(y) \in Y'$.

• If $y \in [\min L_n, c_n]$, then $h(y) \in h([\min L_n, \mathsf{mid}(L_n)]$ and $h^{2^{n-1}}(y) \in [1, 1+x_n]$ by Lemma 7.15(ii), which is impossible because $Y \subset [0,a]$ by (i).
• If $y \in [c_n, \max L_n]$, then $y \neq z_n$ because Y' is infinite. In addition $h^j(y) \in [0,1]$ for all $j \geq 0$ according to (i). Then (7.18) states that there exists $j \geq 1$ such that $h^j(y)$ belongs to $I_n^0 \cup [\min L_n, c_n] \cup I_n^1$ and one of the first three cases applies with $y' := h^j(y) \in Y'$.

We have $h(I_n^1) \subset J_n^0 \cup [\min M_n, \mathsf{mid}(M_n)]$ by Lemma 7.15(iv), and also

$$h^{2^n-1}([\min M_n, \mathsf{mid}(M_n)]) = h^{2^{n-1}}([\min L_n, \mathsf{mid}(L_n)]) = [1, 1+x_n]$$

by Lemmas 7.14(vii) and 7.15(ii). Combined with (i) and the f-invariance of Y, this implies that

$$h(Y \cap I_n^1) \subset J_n^0. \tag{7.20}$$

We have $Y \subset \mathcal{O}_h(I_n^1)$ by (7.19). Combined with (7.20) and Lemma 7.14(i)+(iii), we get

$$Y \subset \bigcup_{i=0}^{2^n-1} h^i(J_n^0) \quad \text{for all } n \geq 1;$$

this is (ii).

Furthermore, $Y \cap h^i(J_n^0) = Y \cap h^{i \bmod 2^n}(J_n^0)$ for all $i \geq 0$. Since $h(Y) = Y$, it is clear that $h^i(J_n^0 \cap Y) \subset h^i(J_n^0) \cap Y$ and that $h^{2^n}(h^i(J_n^0) \cap Y) \subset h^{2^n+i}(J_n^0) \cap Y$. Thus

$$h^i(J_n^0 \cap Y) = h^i(J_n^0) \cap Y = h^{i \bmod 2^n}(J_n^0) \cap Y \quad \text{for all } i \geq 0,$$

which is (iii). □

The next lemma is the key tool in the proof of Proposition 7.20. It relies on the knowledge of the precise location of $h^i(J_n^0)$ in $\bigcup_{k=1}^n I_n^0$.

Lemma 7.19. *Let h be the map defined above. Then* $\mathsf{slope}\left(h^{2^n-1-k}|_{h^k(J_n^0)}\right) \geq 1$ *for all $n \geq 1$ and all $k \in [\![0, 2^n-1]\!]$.*

Proof. A *(finite) word* B is an element of $\mathbb{N}^n$ for some $n \in \mathbb{N}$. If $B \in \mathbb{N}^n$, the length of B is $|B| := n$. If $B = b_1 \dots b_n$ and $B' = b'_1 \dots b'_m$ are two words, then BB' denotes the word obtained by concatenation, that is,

$$BB' := b_1 \dots b_n b'_1 \dots b'_m \in \mathbb{N}^{m+n}.$$

An *infinite word* is an element of $\mathbb{N}^{\mathbb{N}}$.

We define inductively a sequence of words $(B_n)_{n\geq 1}$ by:

- $B_1 := 1$,
- $B_n := nB_1B_2 \dots B_{n-1}$,

and we define the infinite word $\bar\alpha = (\alpha(i))_{i\geq 1}$ by concatenating the B_n's:

$$\bar\alpha := B_1B_2B_3 \dots B_n \dots.$$

A straightforward induction shows that $|B_n| = 2^{n-1}$; thus $|B_1|+|B_2|+\dots+|B_k| = 2^k-1$ and, in $\bar\alpha$, the word B_{k+1} starts at the index 2^k, which gives

$$\alpha(2^k) = k+1, \tag{7.21}$$

$$\alpha(2^k+1)\dots\alpha(2^{k+1}-1) = B_1 \dots B_k = \alpha(1)\dots\alpha(2^k-1). \tag{7.22}$$

We prove by induction on $k \geq 1$ that

$$h^{i-1}(J_n^0) \subset I_{\alpha(i)}^0 \quad \text{for all } n \geq k \text{ and all } i \in [\![1, 2^k-1]\!]. \tag{7.23}$$

• Case $k=1$: $J_n^0 \subset I_1^0 = I_{\alpha(1)}^0$ for all $n \geq 1$.
• Suppose that (7.23) holds for k and let $n \geq k+1$. Since $J_n^0 \subset J_{k+1}^0$, Lemma 7.14(iii) implies that $h^{2^k-1}(J_n^0) \subset I_{k+1}^0$, and thus $h^{2^k}(J_n^0) \subset J_{k+1}^1 \subset J_k^0$ by Lemma 7.14(i). According to the induction hypothesis, we have $h^{i-1}(J_k^0) \subset I_{\alpha(i)}^0$ for all $i \in [\![1, 2^k - 1]\!]$, and (7.22) yields $\alpha(i) = \alpha(2^k + i)$ for all $i \in [\![1, 2^k - 1]\!]$. Consequently, $h^{2^k+i-1}(J_n^0) \subset I_{\alpha(2^k+i)}^0$ for all $i \in [\![1, 2^k - 1]\!]$. Together with the induction hypothesis, this gives (7.23) for $k+1$.

Let $\mu_n := \mathsf{slope}(h|_{I_n^0})$. By definition of h, we have

$$\mu_n = \frac{\mathsf{slope}(\varphi_n)}{\prod_{i=1}^{n-1} \mathsf{slope}(\varphi_i)}.$$

It is straightforward from (7.23) that

$$\forall k \in [\![2, 2^n - 1]\!],\ \mathsf{slope}(h^{k-1}|_{J_n^0}) = \prod_{i=1}^{k-1} \mu_{\alpha(i)}. \tag{7.24}$$

By Lemma 7.14(ii)+(iii), the map $h^{2^n-1}|_{J_n^0}$ is linear and $h^{2^n-1}(J_n^0) = I_n^1$. Thus

$$\mathsf{slope}(h^{2^n-1}|_{J_n^0}) = \frac{|I_n^1|}{|J_n^0|} = \prod_{i=1}^{n} \frac{1 - \frac{2}{3^i}}{\frac{1}{3^i}}.$$

Since $\mathsf{slope}(\varphi_i) = \frac{|I_i^1|}{|I_i^0|} = \frac{1-\frac{2}{3^i}}{\frac{1}{3^i}}$, we get

$$\mathsf{slope}(h^{2^n-1}|_{J_n^0}) = \prod_{i=1}^{2^n-1} \mu_{\alpha(i)} = \prod_{i=1}^{n} \mathsf{slope}(\varphi_i). \tag{7.25}$$

We show by induction on $n \geq 1$ that for all $k \in [\![1, 2^n - 1]\!]$

$$\prod_{i=1}^{k} \mu_{\alpha(i)} = \prod_{i=1}^{n} \mathsf{slope}(\varphi_i)^{\varepsilon(i,k,n)} \quad \text{for some } \varepsilon(i,k,n) \in \{0,1\}. \tag{7.26}$$

• $\mu_{\alpha(1)} = \mu_1 = \mathsf{slope}(\varphi_1)$; this gives the case $n=1$.
• Suppose that (7.26) holds for some $n \geq 1$. Since $\mu_{\alpha(2^n)} = \mu_{n+1}$ by (7.21), we have

$$\begin{aligned}
\prod_{i=1}^{2^n} \mu_{\alpha(i)} &= \prod_{i=1}^{2^n-1} \mu_{\alpha(i)} \cdot \mu_{n+1} \\
&= \prod_{i=1}^{n} \mathsf{slope}(\varphi_i) \cdot \frac{\mathsf{slope}(\varphi_{n+1})}{\prod_{i=1}^{n} \mathsf{slope}(\varphi_i)} \quad \text{by (7.25)} \\
&= \mathsf{slope}(\varphi_{n+1}).
\end{aligned}$$

This is (7.26) for $n+1$ and $k = 2^n$ with $\varepsilon(i,k,n+1) = 0$ for all $i \in [\![1, n]\!]$ and $\varepsilon(n+1,k,n+1) = 1$.

Next, $\alpha(2^n+1)\dots\alpha(2^{n+1}-1) = \alpha(1)\dots\alpha(2^n-1)$ by (7.22); so, if k is in $[\![2^n+1, 2^{n+1}-1]\!]$, then

$$\begin{aligned}\prod_{i=1}^{k}\mu_{\alpha(i)} &= \prod_{i=1}^{2^n}\mu_{\alpha(i)}\prod_{i=2^n+1}^{k}\mu_{\alpha(i)} = \mathsf{slope}(\varphi_{n+1})\prod_{i=1}^{k-2^n}\mu_{\alpha(i)}\\ &= \mathsf{slope}(\varphi_{n+1})\prod_{i=1}^{n}\mathsf{slope}(\varphi_i)^{\varepsilon(i,k-2^n,n)}.\end{aligned}$$

That is, (7.26) holds with $\varepsilon(i,k,n+1) = \varepsilon(i,k-2^n,n)$ for all $i \in [\![1,n]\!]$ and $\varepsilon(n+1,k,n+1) = 1$. This concludes the induction.

Equations (7.24), (7.25) and (7.26) imply that, for all $k \in [\![1, 2^n-1]\!]$,

$$\mathsf{slope}(h^k|_{J_n^0}) = \prod_{i=1}^{k}\mu_{\alpha(i)} = \prod_{i=1}^{n}\mathsf{slope}(\varphi_i)^{\varepsilon(i,k,n)} \quad \text{for some } \varepsilon(i,k,n) \in \{0,1\}. \tag{7.27}$$

Since

$$\mathsf{slope}\left(h^{2^n-1-k}|_{h^k(J_n^0)}\right) = \frac{\mathsf{slope}(h^{2^n-1}|_{J_n^0})}{\mathsf{slope}(h^k|_{J_n^0})},$$

(7.25) and (7.27) imply that $\mathsf{slope}\left(h^{2^n-1-k}|_{h^k(J_n)}\right)$ is a product of at most n terms of the form $\mathsf{slope}(\varphi_i)$. This concludes the proof of the lemma because $\mathsf{slope}(\varphi_i) \geq 1$ for all $i \geq 1$. $\square$

PROPOSITION 7.20. *Let h be the map defined above. Then there exists no invariant set Y such that $f|_Y$ is transitive and sensitive to initial conditions.*

PROOF. Let Y be an invariant set such that $h|_Y$ is transitive. If Y has an isolated point, it is easy to see that $f|_Y$ is not sensitive to initial conditions. We assume that Y has no isolated point.

The sets $\left(h^i(J_n^0 \cap Y)\right)_{0\leq i\leq 2^n-1}$ are closed and, by Lemma 7.14(v), they are pairwise disjoint; let $\delta_n > 0$ be the minimal distance between two of these sets. If $x, x' \in Y$ and $|x-x'| < \delta_n$, then there is $i \in [\![0, 2^n-1]\!]$ such that $x, x' \in h^i(J_n^0)$ and $h^k(x), h^k(x') \in h^{i+k \bmod 2^n}(J_n^0)$ for all $k \geq 0$ by Lemma 7.18(ii)+(iii). We set

$$\delta_n := \max\{\mathrm{diam}(h^i(J_n^0)\cap Y) \mid i \in [\![0, 2^n-1]\!]\}.$$

Lemma 7.19 implies that $\mathrm{diam}(h^k(J_n^0)\cap Y) \leq \mathrm{diam}(h^{2^n-1}(J_n^0)\cap Y)$ for all integers k in $[\![0, 2^n-1]\!]$. By Lemma 7.14(iii), $h^{2^n-1}(J_n^0) = I_n^1$; and by Lemma 7.18(i), $I_n^1 \cap Y \subset [a_{2n}, a]$. Thus $\delta_n \leq \mathrm{diam}(I_n^1\cap Y) \leq a - a_{2n}$. This implies that

$$\lim_{n\to+\infty}\delta_n = 0.$$

This shows that $h|_Y$ is not sensitive to initial conditions. $\square$

Propositions 7.17 and 7.20 show that the map h is chaotic in the sense of Li-Yorke but has no transitive sensitive subsystem. At last this example is completed.

CHAPTER 8

Appendix: Some background in topology

The aim of this appendix is to recall succinctly some definitions and results in topology. For details, one can refer to [**57, 58, 106, 134, 140**] (and also [**145**] for topological notions related to analysis).

8.1. Complement of a set, product of sets

Definition 8.1 (complement of a set). Let X be a set and $Y \subset X$. The *complement* of Y in X is $X \setminus Y := \{x \in X \mid x \notin Y\}$.

Lemma 8.2. *Let X be a set and $A, B \subset X$.*

- $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$,
- $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

These two properties remain valid for a countable union or intersection.

Definition 8.3 (product of sets). Let X_1, X_2 be two sets. The (Cartesian) product of X_1 and X_2 is the set $X_1 \times X_2 := \{(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\}$. One can define similarly the product $X_1 \times X_2 \times \cdots \times X_n$. When $X_1 = X_2 = \cdots = X_n = X$, let X^n denote $\underbrace{X \times \cdots \times X}_{n \text{ times}}$.

The set $X^{\mathbb{Z}^+}$ is the countable product of copies of X, that is,

$$X^{\mathbb{Z}^+} := \{(x_n)_{n \geq 0} \mid \forall n \in \mathbb{Z}^+, x_n \in X\}.$$

8.2. Definitions in topology

8.2.1. Distance, limit.

Definition 8.4 (distance, metric space). Let X be a set. A *distance* on X is a map $d \colon X \times X \to [0, +\infty)$ such that, for all $x, y, z \in X$:

- $d(x, y) = d(y, x)$,
- $d(x, y) = 0 \Leftrightarrow x = y$,
- $d(x, z) \leq d(x, y) + d(y, z)$ *(triangular inequality).*

The set X endowed with a distance is called a *metric space*.

The distance will be denoted by d in any metric space, except when several distances are involved.

Example 8.5. In $\mathbb{R}$, the usual distance is given by $d(x, y) := |y - x|$.

In $\mathbb{R}^n$ $(n \geq 2)$, there are several usual distances. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are elements of $\mathbb{R}^n$,

$$d_\infty(x, y) := \max\{|y_i - x_i| \mid i \in [\![1, n]\!]\},$$

$$d_1(x, y) := \sum_{i=1}^{n} |y_i - x_i|,$$

$$d_2(x, y) := \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} \quad \text{(Euclidean distance)}.$$

d_∞, d_1 and d_2 are three distances in $\mathbb{R}^n$. They are said to be *equivalent* because, for all $i, j \in \{1, 2, \infty\}$, there exist positive real numbers m, M such that

$$\forall x, y \in \mathbb{R}^n,\ m d_i(x, y) \leq d_j(x, y) \leq M d_i(x, y).$$

DEFINITION 8.6 (limit). Let X be a metric space. A sequence $(x_n)_{n \geq 0}$ of points of X converges (or tends) to $x \in X$ if $\lim_{n \to +\infty} d(x_n, x) = 0$, that is,

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ d(x_n, x) < \varepsilon.$$

Then x is called the limit of $(x_n)_{n \geq 0}$, and one writes $\lim_{n \to +\infty} x_n = x$.

8.2.2. Open and closed sets, topology; limit point of a set.

DEFINITION 8.7 (open and closed balls). Let X be a metric space. If $x \in X$ and $r > 0$, the open ball of center x and radius r is $B(x, r) := \{y \in X \mid d(x, y) < r\}$, and the closed ball of center x and radius r is $\overline{B}(x, r) := \{y \in X \mid d(x, y) \leq r\}$.

DEFINITION 8.8 (open and closed sets). Let X be a metric space and $Y \subset X$. The set Y is *open* if

$$\forall x \in Y,\ \exists r > 0,\ B(x, r) \subset Y.$$

The set Y is *closed* if $X \setminus Y$ is open.

The family of all open sets of X defines the *topology* of X.

EXAMPLE 8.9. In $\mathbb{R}^n$, the three distances d_∞, d_1, d_2 define the same topology, that is, the same open and closed sets. The notion of convergence of a sequence of points is also the same for these three distances.

DEFINITION 8.10 (discrete topology). Let E be a set endowed with the distance:

$$\forall x, y \in E,\ d(x, y) := \begin{cases} 1 \text{ if } x \neq y, \\ 0 \text{ if } x = 0. \end{cases}$$

The topology corresponding to this distance is called the *discrete topology*. This topology is the usual topology for finite or countable sets (e.g. $\{0, 1\}$ or $\mathbb{Z}$). For the discrete topology, every singleton $\{x\}$ is both open and closed.

PROPOSITION 8.11. *Let X be a metric space.*

- *Any (finite or not) union of open sets is open.*
- *Any finite intersection of open sets is open.*
- *Any (finite or not) intersection of closed sets is closed.*
- *Any finite union of closed sets is closed.*

DEFINITION 8.12 (limit point of a set). Let X be a metric space and $Y \subset X$. A point $x \in X$ is a *limit point* of Y if for every $r > 0$, $B(x, r) \cap Y \neq \emptyset$. Equivalently, x is a limit point of Y if there exists a sequence of points of Y that converges to x.

PROPOSITION 8.13. *Let X be a metric space and $Y \subset X$. The following assertions are equivalent:*

- *the set Y is closed,*
- *all limit points of Y belong to Y,*
- *for every sequence $(y_n)_{n \geq 0}$ of points of Y, if there exists $x \in X$ such that $\lim_{n \to +\infty} y_n = x$, then $x \in Y$.*

8.2.3. Neighborhoods; interior, closure and boundary of a set.

DEFINITION 8.14 (neighborhood). Let X be a metric space and $x \in X$. A *neighborhood* of x is a set U containing an open set V such that $x \in V$. Equivalently, U is a neighborhood of x if there exists $r > 0$ such that $B(x, r) \subset U$.

DEFINITION 8.15 (interior, closure, boundary of a set). Let X be a metric space and $Y \subset X$.

- The *interior* of Y, denoted by $\operatorname{Int}(Y)$, is the set of points x such that there exists a neighborhood of x included in Y. It is the largest open set contained in Y.
- The *closure* of Y, denoted by $\overline{Y}$, is the set of points x such that every neighborhood of x meets Y. It is the smallest closed set containing Y. Equivalently, $\overline{Y}$ is the set of all limit points of Y.
- The *boundary* of Y is $\operatorname{Bd}(Y) := \overline{Y} \setminus \operatorname{Int}(Y)$.

PROPOSITION 8.16. *Let X be a metric space and $A, B \subset X$ such that $A \subset B$. Then $\operatorname{Int}(A) \subset \operatorname{Int}(B)$ and $\overline{A} \subset \overline{B}$.*

PROPOSITION 8.17. *Let X be a metric space and $Y \subset X$.*

- $X \setminus \overline{Y} = \operatorname{Int}(X \setminus Y)$,
- $X \setminus \operatorname{Int}(Y) = \overline{X \setminus Y}$.

8.2.4. Basis of open sets.

DEFINITION 8.18 (basis of open sets). Let X be a metric space. A *basis of open sets* of X is a family $\mathcal{B}$ of nonempty open sets of X such that every open set can be written as a (finite or not) union of elements of $\mathcal{B}$. It is also called a *basis of the topology* of X.

EXAMPLE 8.19.
• In a metric space X, the open balls form a basis of open sets.
• In $\mathbb{R}$, the family $\{(a, b) \mid a, b \in \mathbb{Q}, a < b\}$ is a countable basis of open sets.
• In a set E endowed with the discrete topology, the family $(\{x\})_{x \in E}$ is a basis of open sets.

8.2.5. Distance between two sets, diameter.

DEFINITION 8.20 (distance between two sets). Let X be a metric space and $A, B \subset X$. The *distance* between the sets A and B is

$$d(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\}.$$

DEFINITION 8.21 (diameter, bounded set). Let X be a metric space and Y a nonempty subset of X. The *diameter* of Y is $\operatorname{diam}(Y) := \sup\{d(x, y) \mid x, y \in Y\}$.

The set Y is *bounded* if there exist $x \in X$ and $r > 0$ such that $Y \subset \overline{B}(x, r)$. Equivalently, Y is bounded if $\operatorname{diam}(Y) < +\infty$.

8.2.6. Dense sets, G_δ-sets.

DEFINITION 8.22 (dense set). Let X be a metric space. A set $Y \subset X$ is *dense* in X if $\overline{Y} = X$. Equivalently, Y is dense in X if

$$\forall x \in X,\ \forall \varepsilon > 0,\ \exists y \in Y,\ d(x, y) \leq \varepsilon.$$

DEFINITION 8.23 (G_δ-set). A *G_δ-set* is a countable intersection of open sets.

PROPOSITION 8.24. *A countable union of G_δ-sets is a G_δ-set.*

8.2.7. Borel sets.

DEFINITION 8.25 (σ-algebra). A σ-algebra of a set X is a collection $\mathcal{A}$ of subsets of X such that:

- $\emptyset \in \mathcal{A}$,
- if $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A}$,
- if $A_n \in \mathcal{A}$ for all $n \geq 0$, then

$$\bigcup_{n\geq 0} A_n \in \mathcal{A} \quad \text{and} \bigcap_{n\geq 0} A_n \in \mathcal{A}.$$

DEFINITION 8.26 (Borel set). Let X be a metric space. A *Borel set* is any subset of X that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable unions, countable intersection and complement. Equivalently, the family of all Borel sets is the smallest σ-algebra containing all open sets of X.

8.3. Topology derived from the topology on X

DEFINITION 8.27 (induced topology). Let X be a metric space and $Y \subset X$. The restriction of the distance d to $Y \times Y$ is a distance on Y, and the topology given by this distance is called the *induced topology* on Y. Equivalently, a set $A \subset Y$ is open (resp. closed) for the induced topology on Y if there exists an open (resp. closed) set $A' \subset X$ such that $A = A' \cap Y$.

DEFINITION 8.28 (product topology). Let X_1, X_2 be two metric spaces. The *product topology* on $X_1 \times X_2$ is generated by the basis of open sets of the form $U_1 \times U_2$, where U_i is a nonempty open set of X_i for $i \in \{1, 2\}$.

If the distances in X_1, X_2 are respectively d_1, d_2, one can define a distance d_∞ on $X_1 \times X_2$ by

$$d_\infty((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2)),$$

and the product topology on $X_1 \times X_2$ is the topology given by this distance.

One can define similarly the product topology on $X_1 \times X_2 \times \cdots \times X_n$.

DEFINITION 8.29 (product topology on $X^{\mathbb{Z}^+}$). Let X be a metric space. The *product topology* on $X^{\mathbb{Z}^+}$ is generated by the basis of open sets of the form

$$U_0 \times U_1 \times \cdots \times U_{k-1} \times X^{n\geq k} := \{(x_n)_{n\geq 0} \in X^{\mathbb{Z}^+} \mid \forall n \in [\![0, k-1]\!], x_n \in U_n\},$$

where $U_0, \ldots, U_k$ are nonempty open sets of X and $k \in \mathbb{N}$. If d_X denotes the distance on X, one can define a distance d on $X^{\mathbb{Z}^+}$ by

$$d((x_n)_{n\geq 0}, (y_n)_{n\geq 0}) := \sum_{n=0}^{+\infty} \frac{d_X(x_n, y_n)}{2^n}$$

(if $\mathrm{diam}(X) = +\infty$, one should replace $d_X(x_n, y_n)$ by $\min(d_X(x_n, y_n), 1)$).

The product topology on $X^{\mathbb{Z}^+}$ is the topology given by this distance.

EXAMPLE 8.30. Let $E := \{0, 1\}$ endowed with the discrete topology. The set $\{0, 1\}^{\mathbb{Z}^+}$ is the set of all infinite sequences of 0 and 1. This is a metric space. The family

$$\{(x_n)_{n\geq 0} \in \{0,1\}^{\mathbb{Z}^+} \mid \forall n \in [\![0, k-1]\!], x_n = a_n\}, \text{ where } k \in \mathbb{N}, a_0, \dots, a_{k-1} \in \{0,1\},$$

is a countable basis of open sets of $\{0, 1\}^{\mathbb{Z}^+}$.

8.4. Connectedness, intervals

DEFINITION 8.31 (connected set). Let X be a metric space. A set $Y \subset X$ is *disconnected* if there exist disjoint open sets $U, V \subset X$ such that $Y \subset U \cup V$, $Y \cap U \neq \emptyset$ and $Y \cap V \neq \emptyset$. Otherwise Y is called *connected*.

PROPOSITION 8.32. *Let X be a metric space and let $(C_i)_{i\in\mathcal{I}}$ be a (finite or infinite) family of connected sets in X. If there exists a point x such that $x \in C_i$ for all $i \in \mathcal{I}$, then $\bigcup_{i\in\mathcal{I}} C_i$ is a connected set.*

DEFINITION 8.33 (connected component). Let X be metric space, $Y \subset X$ and $y \in Y$. The *connected component* of y in Y is the largest (for inclusion) connected set C containing y such that $C \subset Y$. The connected components of two points are either equal or disjoint. The connected components of all points of Y are called the *connected components of Y*.

DEFINITION 8.34 (interval). A *(real) interval* I is a subset of $\mathbb{R}$ of one of the following forms:

- $I = [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ with $a, b \in \mathbb{R}$, $a \leq b$ (if $a = b$, then $I = \{a\}$),
- $I = (a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ with $a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R} \cup \{+\infty\}$, $a \leq b$ (if $a = -\infty$ and $b = +\infty$, then $I = \mathbb{R}$; if $a = b$, then $I = \emptyset$),
- $I = [a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$ with $a \in \mathbb{R}, b \in \mathbb{R} \cup \{+\infty\}$, $a < b$,
- $I = (a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$ with $a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R}$, $a < b$.

If I is an interval, a *subinterval* of I is an interval included in I.

THEOREM 8.35. *When X is a real interval, the connected sets in X are exactly the subintervals of X. In particular, the connected sets in $\mathbb{R}$ are exactly the intervals.*

PROPOSITION 8.36. *Let $(I_n)_{n\geq 0}$ be a (finite or infinite) sequence of intervals in $\mathbb{R}$. Then*

- *$\bigcap_{n\geq 0} I_n$ is an interval (maybe empty).*
- *If there exists a point x such that $x \in I_n$ for all $n \geq 0$, then $\bigcup_{n\geq 0} I_n$ is an interval containing x (this is a particular case of Proposition 8.32).*

8.5. Compactness

8.5.1. Definition and equivalent conditions.

DEFINITION 8.37 (open cover). An *open cover* of a metric space X is a family of open sets $(U_i)_{i\in\mathcal{I}}$ such that $X = \bigcup_{i\in\mathcal{I}} U_i$.

Notice that the set of indices is arbitrary in the previous definition. For example, if $r > 0$, $(B(x, r))_{x \in X}$ is an open cover of X.

DEFINITION 8.38 (compact set). A metric space X is *compact* if every open cover $(U_i)_{i \in \mathcal{I}}$ of X admits a finite subcover, that is, there is a finite set of indices $J \subset \mathcal{I}$ such that $X = \bigcup_{i \in J} U_i$.

A subset $Y \subset X$ is *compact* if Y is compact for the induced topology on Y.

DEFINITION 8.39 (subsequence, limit point of a sequence). Let X be a metric space and $(x_n)_{n \geq 0}$ a sequence of points of X. A *subsequence* of $(x_n)_{n \geq 0}$ is a sequence of the form $(x_{n_i})_{i \geq 0}$, where $(n_i)_{i \geq 0}$ is an increasing sequence of non negative numbers. A point $x \in X$ is a *limit point* of $(x_n)_{n \geq 0}$ if there exist a subsequence of $(x_n)_{n \geq 0}$ that converges to x.

THEOREM 8.40 (Bolzano-Weierstrass theorem). *A metric space X is compact if and only if every sequence $(x_n)_{n \geq 0}$ of points of X admits a limit point.*

THEOREM 8.41. *Let X be a compact metric space. A subset $Y \subset X$ is compact if and only if Y is closed in X.*

PROPOSITION 8.42. *A set $X \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is compact if and only if X is closed and bounded for the distance d_∞ (or equivalently for d_1 or d_2).*

8.5.2. product, intersection of compact sets.

THEOREM 8.43. *Let X_1, X_2 be compact metric spaces. Then $X_1 \times X_2$ is a compact metric space.*

PROPOSITION 8.44. *Let X be a metric space. Let $(Y_n)_{n \geq 0}$ be a sequence of nonempty compact subsets of X such that $Y_{n+1} \subset Y_n$ for all $n \geq 0$. Then $\bigcap_{n=0}^{+\infty} Y_n$ is a nonempty compact set. If in addition $\lim_{n \to +\infty} \operatorname{diam}(Y_n) = 0$, then $\bigcap_{n=0}^{+\infty} Y_n$ is a singleton (i.e., it contains a single point).*

8.5.3. Cauchy sequence, complete space.

DEFINITION 8.45 (Cauchy sequence). Let X be a metric space and $(x_n)_{n \geq 0}$ be a sequence of points of X. Then $(x_n)_{n \geq 0}$ is a *Cauchy sequence* if

$$\forall \varepsilon > 0,\ \exists N \geq 0,\ \forall n > m \geq N,\ d(x_n, x_m) < \varepsilon.$$

PROPOSITION 8.46. *Let X be a metric space. If $(x_n)_{n \geq 0}$ is a sequence of points of X that converges, then it is a Cauchy sequence.*

DEFINITION 8.47 (complete space). A metric space X is *complete* if every Cauchy sequence converges.

PROPOSITION 8.48. *A compact metric space is complete.*

8.5.4. Countable basis of open sets.

PROPOSITION 8.49. *A compact metric space admits a countable basis of open sets, that is, there exists a family $(U_n)_{n \in \mathbb{N}}$ of nonempty open sets of X such that, for every open set $U \subset X$, there exists $\mathcal{I} \subset \mathbb{N}$ such that $U = \bigcup_{n \in \mathcal{I}} U_n$. The sets U_n can be chosen to be open balls.*

8.5.5. Lebesgue number.

Proposition 8.50 (Lebesgue's number Lemma). *Let X be a compact metric space and $(U_i)_{i\in\mathcal{I}}$ an open cover of X. There exists $\delta > 0$ such that*

$$\forall x \in X,\ \exists i \in \mathcal{I},\ B(x,\delta) \subset U_i.$$

Such a number δ is called a Lebesgue number *of this cover.*

8.5.6. Baire category theorem.

Theorem 8.51 (Baire category theorem). *Let X be a complete metric space and $(U_n)_{n\geq 0}$ a sequence of dense open sets. Then $\bigcap_{n\geq 0} U_n$ is a dense G_δ-set.*

Corollary 8.52. *Let X be a nonempty complete metric space and $(F_n)_{n\geq 0}$ a sequence of closed sets such that $X = \bigcup_{n\geq 0} F_n$. Then there exists an integer $n \geq 0$ such that* $\operatorname{Int}(F_n) \neq \emptyset$.

Proof. Suppose on the contrary that $\operatorname{Int}(F_n) = \emptyset$ for all $n \geq 0$. We set $U_n := X \setminus F_n$. By Lemma 8.2,

$$X \setminus \bigcap_{n\geq 0} U_n = \bigcup_{n\geq 0} (X \setminus U_n) = \bigcup_{n\geq 0} F_n = X,$$

which implies that $\bigcap_{n\geq 0} U_n$ is empty. On the other hand, $\overline{U_n} = X \setminus \operatorname{Int}(F_n) = X$ by Proposition 8.17, and thus U_n is a dense open set for every $n \geq 0$. Therefore $\bigcap_{n\geq 0} U_n$ is dense according to the Baire category Theorem 8.51, which is a contradiction. We conclude that there exists $n \geq 0$ such that $\operatorname{Int}(F_n) \neq \emptyset$. $\square$

Corollary 8.53. *Let X be a complete metric space and $(G_n)_{n\geq 0}$ a sequence of dense G_δ-sets. Then $\bigcap_{n\geq 0} G_n$ is a dense G_δ-set.*

Proof. For every $n \geq 0$, one can write $G_n = \bigcap_{k\geq 0} U_n^k$, where U_n^k is an open set. Since G_n is dense and $G_n \subset U_n^k$ for all $k \geq 0$, the sets $(U_n^k)_{n,k\geq 0}$ are dense open sets. The Baire category Theorem 8.51 states that $\bigcap_{n,k\geq 0} U_n^k$ is a dense G_δ-set. Finally, we have $\bigcap_{n\geq 0} G_n = \bigcap_{n,k\geq 0} U_n^k$. $\square$

8.6. Cantor set

8.6.1. Definitions.

Definition 8.54 (isolated point). Let X be a metric space and $Y \subset X$. A point $y \in Y$ is an *isolated point* in Y if $B(y,r) \cap Y = \{y\}$ for some $r > 0$. If there is no such point in Y, one says that Y has no isolated point.

Definition 8.55. Let X be a metric space and $Y \subset X$. The set Y is said to be *totally disconnected* if for every $y \in Y$, the connected component of y in Y is reduced to $\{y\}$.

Definition 8.56 (Cantor set). Let X be a metric space. The set X is a *Cantor set* if it is nonempty, compact, totally disconnected and has no isolated point.

Proposition 8.57. *Let X be a complete metric space and $Y \subset X$. If Y is nonempty, closed and has no isolated point, then Y is uncountable. In particular, a Cantor set is uncountable.*

8.6.2. Examples of Cantor sets.

EXAMPLE 8.58. The set $\{0,1\}^{\mathbb{Z}^+}$, endowed with the product topology given by the discrete topology on $\{0,1\}$, is a Cantor set.

EXAMPLE 8.59 (triadic Cantor set). We are going to build by induction on n a family of intervals $(I_{\alpha_0\dots\alpha_{n-1}})_{(\alpha_0,\dots,\alpha_{n-1})\in\{0,1\}^n}$ in $[0,1]$ such that, for all $n\in\mathbb{N}$,

- $(I_{\alpha_0\dots\alpha_{n-1}})_{(\alpha_0\dots\alpha_{n-1})\in\{0,1\}^n}$ are pairwise disjoint closed intervals,
- $|I_{\alpha_0\dots\alpha_{n-1}}| = \frac{1}{3^n}$ for all $\alpha_0,\dots,\alpha_{n-1}\in\{0,1\}$,
- $I_{\alpha_0\dots\alpha_{n-1}\alpha_n} \subset I_{\alpha_0\dots\alpha_{n-1}}$ for all $\alpha_0,\dots,\alpha_n\in\{0,1\}$.

See Figure 1 for the first steps of the construction.

• At step 0, we start with $I=[0,1]$.
• At step 1, we cut I in three equal parts and we remove the open interval $(\frac{1}{3},\frac{2}{3})$ in the middle. There remain two intervals $I_0 := [0,\frac{1}{3}]$ and $I_1 := [\frac{2}{3},1]$.

• At step $n\geq 1$, we cut every interval $I_{\alpha_0\dots\alpha_{n-1}}$ into three equal parts and we remove the open third in the middle. If $I_{\alpha_0\dots\alpha_{n-1}} = [a,b]$, there remain two intervals $I_{\alpha_0\dots\alpha_{n-1}0} := [a,\frac{2a+b}{3}]$ and $I_{\alpha_0\dots\alpha_{n-1}1} := [\frac{a+2b}{3},b]$. Trivially, these two intervals are disjoint, they are included in $I_{\alpha_0\dots\alpha_{n-1}}$, and their length is equal to $\frac{1}{3}|I_{\alpha_0\dots\alpha_{n-1}}|$, and so their length is $\frac{1}{3^n}$ by the induction hypothesis. Moreover, the sets $(I_{\alpha_0\dots\alpha_n})_{(\alpha_0,\dots,\alpha_n)\in\{0,1\}^{n+1}}$ are pairwise disjoint by construction.

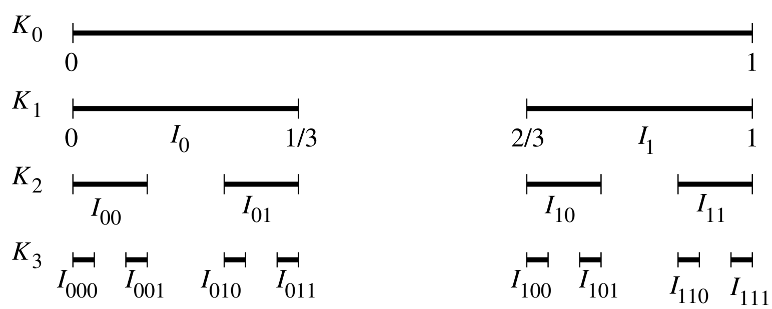

FIGURE 1. The first steps of the construction of the triadic Cantor set.

Let $K_0 := [0,1]$ and, for all $n\geq 1$, let K_n be the union of the intervals $I_{\alpha_0\dots\alpha_{n-1}}$ for all $(\alpha_0,\dots,\alpha_{n-1})\in\{0,1\}^n$. Then K_n is a compact set with 2^n connected components of length $\frac{1}{3^n}$ and $K_{n+1}\subset K_n$. Let $K := \bigcap_{n\geq 0} K_n$. Then K is a nonempty compact set by Proposition 8.44. One can show that K is a Cantor set. The set K is called the *triadic Cantor set*.

EXAMPLE 8.60. On can construct other sets in a similar way as in Example 8.59, by varying the size and/or the number of the gaps. More precisely,
• At step 0, we start with a non degenerate compact interval K_0.
• At step $n\geq 1$, for every connected component C of K_{n-1}, we choose p disjoint non degenerate closed subintervals of C (with $p=p(C)\geq 2$) such that one contains $\min C$ and another one contains $\max C$. We call K_n the union of all these intervals.

Finally, $K := \bigcap_{n\geq 0} K_n$ is a nonempty compact set with uncountably many connected components and no isolated point. All the Cantor sets included in $\mathbb{R}$ can be obtained by this construction. But notice that the sets obtained in this way are not all Cantor sets. Indeed, let $\ell_n := \max\{|C| \mid C$ connected component of $K_n\}$; then K is a Cantor set if and only if $\lim_{n\to+\infty} \ell_n = 0$, otherwise K has a non degenerate connected component (e.g., the set K built in Example 5.59 is not a Cantor set; see Lemma 5.63).

A classical family of Cantor sets in $\mathbb{R}$ is obtained by fixing a ratio $r \in (0, 1/2)$ and constructing the intervals such that all connected components of K_n have the same length ℓ_n with $\ell_n = r\ell_{n-1}$ (e.g., $r = \frac{1}{2p+1}$ in Example 2.42).

PROPOSITION 8.61. *Let $\mathbb{R}$ be the ambient space and X a Cantor set included in $\mathbb{R}$. Then* $\mathrm{Int}\,(X) = \emptyset$.

PROOF. Every nonempty open set in $\mathbb{R}$ contains a nonempty open interval. On the other hand, every connected component of X is reduced to a single point. This implies that $\mathrm{Int}\,(X) = \emptyset$. $\square$

THEOREM 8.62. *Every Cantor set K is homeomorphic to $\{0,1\}^{\mathbb{Z}^+}$, that is, there exists a homeomorphism $\varphi\colon X \to \{0,1\}^{\mathbb{Z}^+}$ (see Definition 8.69 below for the definition of homeomorphism).*

8.7. Continuous maps

8.7.1. Definitions.

DEFINITION 8.63 (image, preimage of a set). Let $f\colon X \to Y$ be a map. The *image* of a set $A \subset X$ under f is $f(A) := \{f(x) \mid x \in A\}$. The *preimage* (or *inverse image*) of a set $B \subset Y$ under f is $f^{-1}(B) := \{x \in X \mid f(x) \in B\}$.

DEFINITION 8.64 (continuity). Let X, Y be metric spaces endowed with the distances d_X, d_Y respectively. A map $f\colon X \to Y$ is *continuous* if one of the following equivalent assertions is satisfied:

i) $\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0,\ \forall x' \in X, d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$,
ii) for all open sets $U \subset Y$, $f^{-1}(U)$ is open,
iii) for all closed sets $F \subset Y$, $f^{-1}(F)$ is closed,
iv) for all $x \in X$ and all sequences $(x_n)_{n\geq 0}$ of points of X converging to x, $\lim\limits_{n\to+\infty} f(x_n) = f(x)$.

REMARK 8.65. Let X, Y be metric spaces and $f\colon X \to Y$ a continuous map. If $X' \subset X$, then $f|_{X'}\colon X' \to Y$ (restriction of f to X') is also a continuous map.

PROPOSITION 8.66. *Let X, Y, Z be metric spaces. If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then $g \circ f\colon X \to Z$ is a continuous map. In particular, if $f\colon X \to X$ is a continuous map, then $f^n\colon X \to X$ is a continuous map for every $n \in \mathbb{N}$, where $f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$.*

DEFINITION 8.67 (one-to-one and onto map, bijection). Let X, Y be metric spaces and let $f\colon X \to Y$ be a map.

- f is *one-to-one* (or *injective*) if, for all $x, x' \in X$, $x \neq x' \Rightarrow f(x) \neq f(x')$.
- f is *onto* (or *surjective*) if $f(X) = Y$.
- f is a *bijection* (or a *bijective map*) if it is one-to-one and onto. In this case, the *inverse map* of f is the map $f^{-1}\colon Y \to X$ satisfying $f(x) = y \Leftrightarrow x = f^{-1}(y)$.

PROPOSITION 8.68. *Let $f\colon I \to J$ be a continuous onto map, where I, J are two nonempty real intervals. Then f is a bijection if and only if*

- *either f is increasing, that is, $\forall x, y \in I$, $x < y \Rightarrow f(x) < f(y)$,*
- *or f is decreasing, that is, $\forall x, y \in I$, $x < y \Rightarrow f(x) > f(y)$.*

DEFINITION 8.69 (homeomorphism). Let X, Y be two metric spaces. A map $f\colon X \to Y$ is a *homeomorphism* if f is continuous, bijective and f^{-1} is continuous.

DEFINITION 8.70 (uniform continuity). Let X, Y be metric spaces endowed with the distances d_X, d_Y. A map $f\colon X \to Y$ is *uniformly continuous* if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, x' \in X, d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon.$$

8.7.2. Inverse image of an intersection.

PROPOSITION 8.71. *Let X, Y be metric spaces and let $f\colon X \to Y$ be a map. If $(Y_n)_{n\geq 0}$ is a family of subsets of Y, then $f^{-1}(\bigcap_{n\geq 0} Y_n) = \bigcap_{n\geq 0} f^{-1}(Y_n)$.*

PROOF. A point x belongs to $f^{-1}(\bigcap_{n\geq 0} Y_n)$ if and only if $f(x) \in \bigcap_{n\geq 0} Y_n$, that is, $f(x) \in Y_n$ for all $n \geq 0$. Since $f(x) \in Y_n \Leftrightarrow x \in f^{-1}(Y_n)$, we get

$$x \in f^{-1}(\bigcap_{n\geq 0} Y_n) \iff x \in \bigcap_{n\geq 0} f^{-1}(Y_n).$$

□

PROPOSITION 8.72. *Let X, Y be metric spaces and let $f\colon X \to Y$ be a continuous map. If $G \subset Y$ is a G_δ-set, then $f^{-1}(G)$ is a G_δ-set in X.*

PROOF. One can write $G = \bigcap_{n\geq 0} U_n$, where U_n is an open set of Y for every $n \geq 0$. Then $f^{-1}(U_n)$ is an open set of X because f is continuous, and $f^{-1}(G) = \bigcap_{n\geq 0} f^{-1}(U_n)$ by Proposition 8.71. Therefore, $f^{-1}(G)$ is a G_δ-set. □

8.7.3. Continuity and denseness.

THEOREM 8.73. *Let X, Y be metric spaces and D a dense subset of X. Let $f\colon X \to Y, g\colon X \to Y$ be two continuous maps. If $f(x) = g(x)$ for all $x \in D$, then $f(x) = g(x)$ for all $x \in X$.*

8.7.4. Continuity and connectedness.

THEOREM 8.74. *Let X, Y be metric spaces and $f\colon X \to Y$ a continuous map. If $C \subset X$ is a connected set, then $f(C)$ is connected.*

The *intermediate value theorem* is a corollary of Theorem 8.74 for real maps. See Theorem 1.9 in Chapter 1.

8.7.5. Continuity and compactness.

THEOREM 8.75. *Let X, Y be metric spaces and $f\colon X \to Y$ a continuous map. If $K \subset X$ is a compact set, then $f(K)$ is compact.*

PROPOSITION 8.76. *Let X, Y be metric spaces with X compact, $f\colon X \to Y$ a continuous map and $A \subset X$. Then $f(\overline{A}) = \overline{f(A)}$.*

PROOF. The set $\overline{A}$ is compact by Theorem 8.41, and thus $f(\overline{A})$ is compact by Theorem 8.75. Trivially, $f(A) \subset f(\overline{A})$, which implies that $\overline{f(A)} \subset f(\overline{A})$.

Let x be a point in $\overline{A}$. Then there exists $(x_n)_{n\geq 0}$ a sequence of points of A that converges to x. Since f is continuous, $\lim_{n\to+\infty} f(x_n) = f(x)$. This implies that $f(x) \in \overline{f(A)}$, and hence $f(\overline{A}) \subset \overline{f(A)}$. We conclude that $f(\overline{A}) = \overline{f(A)}$. □

PROPOSITION 8.77. *Let X, Y be metric spaces and $f\colon X \to Y$ a continuous bijection. If X is compact, then f is a homeomorphism.*

PROPOSITION 8.78. *Let X, Y be metric spaces and $f\colon X \to Y$ a continuous map. If X is compact, then f is uniformly continuous.*

THEOREM 8.79. *Let $f\colon X \to \mathbb{R}$ be a continuous map, where X is a compact metric space. Then f admits a maximum and a minimum, that is,*

$$\exists x_M \in X,\ f(x_M) = \sup\{f(x) \mid x \in X\},$$
$$\exists x_m \in X,\ f(x_m) = \inf\{f(x) \mid x \in X\}.$$

COROLLARY 8.80. *Let X be a metric space and $A, B \subset X$. If A, B are compact, then $d(A,B)$ and $\operatorname{diam}(A)$ are reached, that is,*

- *there exist $a \in A, b \in B$ such that $d(A,B) = d(a,b)$,*
- *there exist $a, a' \in A$ such that $\operatorname{diam}(A) = d(a,a')$.*

PROOF. Let $f\colon X \times X \to \mathbb{R}$ defined by $f(x,y) := d(x,y)$. One can easily show that f is continuous.

The set $A \times B$ is compact by Theorem 8.43. Thus $f|_{A\times B}$ admits a minimum by Theorem 8.79, that is, there exists a couple of points (a,b) in $A \times B$ such that $d(a,b) = \inf\{d(x,y) \mid (x,y) \in A \times B\}$; the last expression is the definition of $d(A,B)$.

Similarly, the set $A \times A$ is compact, and thus $f|_{A\times A}$ admits a maximum, that is, there exist $(a,a') \in A \times A$ such that $d(a,a') = \sup\{d(x,y) \mid (x,y) \in A \times A\}$; the last expression is the definition of $\operatorname{diam}(A)$. □

8.7.6. Uniform convergence of a sequence of real maps.

DEFINITION 8.81. Let $\mathcal{F}$ be the space of all maps $f\colon [0,1] \to [0,1]$. The *uniform distance* on $\mathcal{F}$ is defined by $d_\infty(f,g) := \sup\{|g(x) - f(x)| \mid x \in [0,1]\}$, where $f, g \in \mathcal{F}$.

Let $(f_n)_{n\geq 0}$ be a sequence of maps of $\mathcal{F}$. Then $(f_n)_{n\geq 0}$ *uniformly converges* to $f \in \mathcal{F}$ if it converges to f for the distance d_∞, that is,

$$\forall \varepsilon > 0, \exists N \geq 0, \forall n \geq N, \forall x \in [0,1],\ |f_n(x) - f(x)| < \varepsilon.$$

THEOREM 8.82. *The space $\mathcal{F}$ endowed with the distance d_∞ defined above is a complete space.*

8.8. Zorn's Lemma

A *partially ordered* set is a set endowed with a binary relation that indicates that, for certain pairs of elements, one of the elements precedes the other. Such a relation is called a *partial order* to reflect the fact that not every pair of elements need be related, contrary to a *total order*. The formal definitions are given below.

DEFINITION 8.83 (partial and total order). A *partial order* on the set E is a binary relation $\leq$ such that, for all $a, b, c \in E$,

- $a \leq a$,
- if $a \leq b$ and $b \leq a$, then $a = b$,
- if $a \leq b$ and $b \leq c$, then $a \leq c$.

Such a set E is called *partially ordered*. If $a \leq b$ or $b \leq a$, the elements a, b are said to be *comparable*.

A *total order* on E is a partial order such that all pairs of elements are comparable. Such a set E is called *totally ordered*.

DEFINITION 8.84 (lower and upper bound). Let E be a partially ordered set and $F \subset E$. An element $b \in E$ is a *lower bound* (resp. *upper bound*) of F if $x \geq b$ (resp. $x \leq b$) for all $x \in F$.

DEFINITION 8.85 (minimal and maximal element). Let E be a partially ordered set. A *minimal* (resp. *maximal*) element of E is an element $m \in E$ that is not greater (resp. smaller) than any other element in E, that is, if $m \geq x$ (resp. $m \leq x$) for some $x \in E$, then $m = x$.

THEOREM 8.86 (Zorn's Lemma). *Let E be a nonempty partially ordered set. Suppose that every nonempty family of elements of E that is totally ordered has a lower (resp. upper) bound in E. Then E contains at least one minimal (resp. maximal) element.*

Zorn's Lemma is equivalent to the axiom of choice; it is a result of set theory (see e.g. [**106**]). However it can be used in topology by considering the partial order given by the inclusion: the set E is a family of subsets of some space X, and $A \leq B$ if $A \subset B$, where $A, B \in E$.

Bibliography

[1] R. L. Adler, A. G. Konheim, and M. H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.*, 114:309–319, 1965.

[2] R. L. Adler and M. H. McAndrew. The entropy of Chebyshev polynomials. *Trans. Amer. Math. Soc.*, 121:236–241, 1966.

[3] Ll. Alsedà, S. Baldwin, J. Llibre, and M. Misiurewicz. Entropy of transitive tree maps. *Topology*, 36(2):519–532, 1997.

[4] Ll. Alsedà, M. A. del Río, and J. A. Rodríguez. A splitting theorem for transitive maps. *J. Math. Anal. Appl.*, 232(2):359–375, 1999.

[5] Ll. Alsedà, D. Juher, and P. Mumbrú. A note on the periodic orbits and topological entropy of graph maps. *Proc. Amer. Math. Soc.*, 129(10):2941–2946 (electronic), 2001.

[6] Ll. Alsedà, D. Juher, and P. Mumbrú. Sets of periods for piecewise monotone tree maps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 13(2):311–341, 2003.

[7] Ll. Alsedà, D. Juher, and P. Mumbrú. On the preservation of combinatorial types for maps on trees. *Ann. Inst. Fourier (Grenoble)*, 55(7):2375–2398, 2005.

[8] Ll. Alsedà, D. Juher, and P. Mumbrú. Periodic behavior on trees. *Ergodic Theory Dynam. Systems*, 25(5):1373–1400, 2005.

[9] Ll. Alsedà, D. Juher, and P. Mumbrú. Minimal dynamics for tree maps. *Discrete Contin. Dyn. Syst.*, 20(3):511–541, 2008.

[10] Ll. Alsedà, S. Kolyada, J. Llibre, and L'. Snoha. Entropy and periodic points for transitive maps. *Trans. Amer. Math. Soc.*, 351(4):1551–1573, 1999.

[11] Ll. Alsedà, S. Kolyada, J. Llibre, and L'. Snoha. Axiomatic definition of the topological entropy on the interval. Preprint, 2000.

[12] Ll. Alsedà, J. Llibre, F. Mañosas, and M. Misiurewicz. Lower bounds of the topological entropy for continuous maps of the circle of degree one. *Nonlinearity*, 1(3):463–479, 1988.

[13] Ll. Alsedà, J. Llibre, and M. Misiurewicz. Periodic orbits of maps of Y. *Trans. Amer. Math. Soc.*, 313(2):475–538, 1989.

[14] Ll. Alsedà, J. Llibre, and M. Misiurewicz. *Combinatorial dynamics and entropy in dimension one*, volume 5 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co. Inc., River Edge, NJ, second edition, 2000.

[15] Ll. Alsedà, F. Mañosas, and P. Mumbrú. Minimizing topological entropy for continuous maps on graphs. *Ergodic Theory Dynam. Systems*, 20(6):1559–1576, 2000.

[16] Ll. Alsedà and S. Ruette. Rotation sets for graph maps of degree 1. *Ann. Inst. Fourier (Grenoble)*, 58(4):1233–1294, 2008.

[17] Ll. Alsedà and S. Ruette. Periodic orbits of large diameter for circle maps. *Proc. Amer. Math. Soc.*, 138(9):3211–3217, 2010.

[18] Ll. Alsedà and S. Ruette. On the set of periods of sigma maps of degree 1. *Discrete Contin. Dyn. Syst.*, 35(10):4683–4734, 2015. arXiv:1407.1419.

[19] Ll. Alsedà and X.D. Ye. No division and the set of periods for tree maps. *Ergodic Theory Dynam. Systems*, 15(2):221–237, 1995.

[20] J. Auslander and J. Yorke. Interval maps, factors of maps and chaos. *Tohoku Math. J.*, 32:177–188, 1980.

[21] S. Baldwin. An extension of Šarkovskiĭ's theorem to the n-od. *Ergodic Theory Dynam. Systems*, 11(2):249–271, 1991.

[22] S. Baldwin and J. Llibre. Periods of maps on trees with all branching points fixed. *Ergodic Theory Dynam. Systems*, 15(2):239–246, 1995.

[23] F. Balibrea, J. Smítal, and M. Štefánková. The three versions of distributional chaos. *Chaos Solitons Fractals*, 23(5):1581–1583, 2005.

[24] F. Balibrea, J. Smítal, and M. Štefánková. A triangular map of type 2^∞ with positive topological entropy on a minimal set. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 74(5):1690–1693, 2011.

[25] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey. On Devaney's definition of chaos. *Amer. Math. Monthly*, 99(4):332–334, 1992.

[26] M. Barge and J. Martin. Chaos, periodicity, and snakelike continua. *Trans. Amer. Math. Soc.*, 289(1):355–365, 1985.

[27] M. Barge and J. Martin. Dense periodicity on the interval. *Proc. Amer. Math. Soc.*, 94(4):731–735, 1985.

[28] M. Barge and J. Martin. Dense orbits on the interval. *Michigan Math. J.*, 34(1):3–11, 1987.

[29] C. Bernhardt. Vertex maps for trees: algebra and periods of periodic orbits. *Discrete Contin. Dyn. Syst.*, 14(3):399–408, 2006.

[30] F. Blanchard. Fully positive topological entropy and topological mixing. In *Symbolic dynamics and its applications (New Haven, CT, 1991)*, Contemp. Math., 135, pages 95–105. Amer. Math. Soc., Providence, RI, 1992.

[31] F. Blanchard, F. Durand, and A. Maass. Constant-length substitutions and countable scrambled sets. *Nonlinearity*, 17(3):817–833, 2004.

[32] F. Blanchard, E. Glasner, S. Kolyada, and A. Maass. On Li-Yorke pairs. *J. Reine Angew. Math.*, 547:51–68, 2002.

[33] L. Block. Homoclinic points of mappings of the interval. *Proc. Amer. Math. Soc.*, 72(3):576–580, 1978.

[34] L. Block and E. M. Coven. Topological conjugacy and transitivity for a class of piecewise monotone maps of the interval. *Trans. Amer. Math. Soc.*, 300(1):297–306, 1987.

[35] L. Block and E. M. Coven. Approximating entropy of maps of the interval. In *Dynamical systems and ergodic theory (Warsaw, 1986)*, volume 23 of *Banach Center Publ.*, pages 237–241. PWN, Warsaw, 1989.

[36] L. Block, J. Guckenheimer, M. Misiurewicz, and L. S. Young. Periodic points and topological entropy of one dimensional maps. In *Global Theory of Dynamical Systems*, Lecture Notes in Mathematics, no. 819, pages 18–34. Springer-Verlag, 1980.

[37] L. Block and J. Keesling. A characterization of adding machine maps. *Topology Appl.*, 140(2-3):151–161, 2004.

[38] L. Block and J. Keesling. Topological entropy and adding machine maps. *Houston J. Math.*, 30(4):1103–1113 (electronic), 2004.

[39] L. Block, J. Keesling, and M. Misiurewicz. Strange adding machines. *Ergodic Theory Dynam. Systems*, 26(3):673–682, 2006.

[40] L. S. Block and W. A. Coppel. Stratification of continuous maps of an interval. *Trans. Amer. Math. Soc.*, 297(2):587–604, 1986.

[41] L. S. Block and W. A. Coppel. *Dynamics in one dimension*. Lecture Notes in Mathematics, no. 1513. Springer-Verlag, 1992.

[42] A. M. Blokh. An interpretation of a theorem of A. N. Sharkovskiĭ(Russian). In *Oscillation and stability of solutions of functional-differential equations*, pages 3–8. Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1982.

[43] A. M. Blokh. Limit behavior of one-dimensional dynamic systems (Russian). *Uspekhi Mat. Nauk*, 37(1-223):137–138, 1982. English translation *Russ. Math. Surv.*, 37(1):157–158, 1982.

[44] A. M. Blokh. On sensitive mappings of the interval (Russian). *Uspekhi Mat. Nauk*, 37(2(224)):189–190, 1982. English translation: *Russ. Math. Surv.*, 37:203–204, 1982.

[45] A. M. Blokh. On the limit behavior of one-dimensional dynamical systems 1 (Russian). Preprint 2704-82, VINITI, Moscow, 1982.

[46] A. M. Blokh. "Spectral expansion" for piecewise monotone mappings of an interval (Russian). *Uspekhi Mat. Nauk*, 37(3-225):175–176, 1982. English translation: *Russ. Math. Surv.*, 37(3):198–199, 1982.

[47] A. M. Blokh. Decomposition of dynamical systems on an interval (Russian). *Uspekhi Mat. Nauk*, 38(5-223):179–180, 1983. English translation *Russ. Math. Surv.*, 38(5):133–134, 1983. Erratum (Russian) *Uspekhi Mat. Nauk*, 42, no. 6(258):233, 1987.

[48] A. M. Blokh. On transitive mappings of one-dimensional branched manifolds (Russian). In *Differential-difference equations and problems of mathematical physics*, pages 3–9, 131. Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1984.

[49] A. M. Blokh. Dynamical systems on one-dimensional branched manifolds. I (Russian). *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (46):8–18, 1986. English translation in *J. Soviet Math.*, 48(5):500–508, 1990.

[50] A. M. Blokh. Dynamical systems on one-dimensional branched manifolds. II (Russian). *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (47):67–77, 1987. English translation in *J. Soviet Math.*, 48(6):668–674, 1990.

[51] A. M. Blokh. Dynamical systems on one-dimensional branched manifolds. III (Russian). *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (48):32–46, 1987. English translation in *J. Soviet Math.*, 49(2):875–883, 1990.

[52] A. M. Blokh. On the connection between entropy and transitivity for one-dimensional mappings (Russian). *Uspekhi Mat. Nauk*, 42(5(257)):209–210, 1987. English translation in *Russ. Math. Surv.*, 42(5):165–166, 1987.

[53] A. M. Blokh. The spectral decomposition, periods of cycles and Misiurewicz conjecture for graph maps. In *Ergodic theory and related topics III – Proceedings of the Third International Conference held in Güstrow, October 22–27, 1990*, Lecture Notes in Mathematics 1514, pages 24–31. Springer, 1992.

[54] A. M. Blokh. Trees with snowflakes and zero entropy maps. *Topology*, 33(2):379–396, 1994.

[55] A. M. Blokh. The "spectral" decomposition for one-dimensional maps. In *Dynamics Reported*, Dynam. Report. Expositions Dynam. Systems (N. S.), no. 4, pages 1–59. Springer, Berlin, 1995.

[56] J. Bobok and M. Kuchta. Topological entropy and variation for transitive maps. *Math. Slovaca*, 43(3):345–355, 1993.

[57] N. Bourbaki. *Eléments de mathématique. XVI. Première partie: Les structures fondamentales de l'analyse. Livre III. Topologie générale (fascicule de résultats)*. Actualités Sci. Ind., no. 1196. Hermann & Cie, Paris, 1953.

[58] N. Bourbaki. *General topology. Chapters 1–4*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[59] R. Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.*, 153:401–414, 1971.

[60] R. Bowen. Periodic points and measures for Axiom A diffeomorphisms. *Trans. Amer. Math. Soc.*, 154:377–397, 1971.

[61] R. Bowen and J. Franks. The periodic points of maps of the disk and the interval. *Topology*, 15(4):337–342, 1976.

[62] A. M. Bruckner and T. Hu. On scrambled sets for chaotic functions. *Trans. Amer. Math. Soc.*, 301(1):289–297, 1987.

[63] G. J. Butler and G. Pianigiani. Periodic points and chaotic functions in the unit interval. *Bull. Austral. Math. Soc.*, 18(2):255–265, 1978.

[64] J. S. Cánovas. Distributional chaos on tree maps: the star case. *Comment. Math. Univ. Carolin.*, 42(3):583–590, 2001.

[65] J. S. Cánovas. Topological sequence entropy and chaos of star maps. *Nonlinear Dyn. Syst. Theory*, 5(1):1–8, 2005.

[66] J. S. Cánovas and R. Hric. Distributional chaos of tree maps. *Topology Appl.*, 137(1-3):75–82, 2004. IV Iberoamerican Conference on Topology and its Applications.

[67] P. Collet and J.-P. Eckmann. *Iterated maps on the interval as dynamical systems*. Progress in Physics, 1. Birkhäuser Boston, Mass., 1980.

[68] W. A. Coppel. Continuous maps of an interval. Lectures Notes, IMA, Preprint Series, 26, 1983.

[69] E. M. Coven and M. C. Hidalgo. On the topological entropy of transitive maps of the interval. *Bull. Austral. Math. Soc.*, 44(2):207–213, 1991.

[70] E. M. Coven and I. Mulvey. Transitivity and the centre for maps of the circle. *Ergodic Theory Dynam. Systems*, 6(1):1–8, 1986.

[71] W. de Melo and S. van Strien. *One-dimensional dynamics*, volume 25 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1993.

[72] J.-P. Delahaye. Fonctions admettant des cycles d'ordre n'importe quelle puissance de 2 et aucun autre cycle (french. english summary). *C. R. Acad. Sci. Paris Sér. A-B*, 291(4):A323–A325, 1980. Addendum, 291(13):A671, 1980.

[73] M. Denker, C. Grillenberger, and K. Sigmund. *Ergodic theory on compact spaces*. Lecture Notes in Mathematics, no. 527. Springer-Verlag, 1976.

[74] R. L. Devaney. *An introduction to chaotic dynamical systems*. Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, second edition, 1989.

[75] T. Downarowicz. Survey of odometers and Toeplitz flows. In *Algebraic and topological dynamics*, volume 385 of *Contemp. Math.*, pages 7–37. Amer. Math. Soc., Providence, RI, 2005.

[76] T. Downarowicz. Positive topological entropy implies chaos DC2. *Proc. Amer. Math. Soc.*, 142(1):137–149, 2014.

[77] R. Engelking. *General topology*. PWN—Polish Scientific Publishers, Warsaw, 1977. Translated from the Polish by the author, Monografie Matematyczne, Tom 60. [Mathematical Monographs, Vol. 60].

[78] V. V. Fedorenko, A. N. Sharkovsky, and J. Smítal. Characterizations of weakly chaotic maps of the interval. *Proc. Amer. Math. Soc.*, 110(1):141–148, 1990.

[79] V.V. Fedorenko, A. N. Sharkovsky, and J. Smital. Characterization of some classes of mappings of an interval with zero topological entropy. *Akad. Nauk Ukrain. SSR Inst. Mat. Preprint*, (58):18, 1989.

[80] N. Franzová and J. Smítal. Positive sequence topological entropy characterizes chaotic maps. *Proc. Amer. Math. Soc.*, 112(4):1083–1086, 1991.

[81] X.-C. Fu and J. Duan. Infinite-dimensional linear dynamics systems with chaoticity. *J. Nonlinear Sci.*, 9:197–211, 1999.

[82] H. Furstenberg. Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation. *Math.System Th*, 1:1–55, 1967.

[83] E. Glasner and B. Weiss. Sensitive dependence on initial conditions. *Nonlinearity*, 6(6):1067–1075, 1993.

[84] T. N. T. Goodman. Topological sequence entropy. *Proc. London Math. Soc. (3)*, 29:331–350, 1974.

[85] J. Guckenheimer. On the bifurcation of maps of the interval. *Invent. Math.*, 39(2):165–178, 1977.

[86] J. Guckenheimer. Sensitive dependence to initial conditions for one-dimensional maps. *Comm. Math. Phys.*, 70(2):133–160, 1979.

[87] B. Hasselblatt and A. Katok. *A first course in dynamics*. Cambridge University Press, New York, 2003. With a panorama of recent developments.

[88] R. Hric. Topological sequence entropy for maps of the circle. *Comment. Math. Univ. Carolin.*, 41(1):53–59, 2000.

[89] R. Hric and M. Málek. Omega limit sets and distributional chaos on graphs. *Topology Appl.*, 153(14):2469–2475, 2006.

[90] R. Ito. Minimal entropy for endomorphisms of the circle. *Proc. Amer. Math. Soc.*, 86(2):321–327, 1982.

[91] A. Iwanik. Independent sets of transitive points. In *Dynamical systems and ergodic theory (Warsaw, 1986)*, Banach Center Publ., 23, pages 277–282. PWN, Warsaw, 1989.

[92] A. Iwanik. Independence and scrambled sets for chaotic mappings. In *The mathematical heritage of C. F. Gauss*, pages 372–378. World Sci. Publishing, River Edge, NJ, 1991.

[93] K. Janková and J. Smítal. A characterization of chaos. *Bull. Austral. Math. Soc.*, 34(2):283–292, 1986.

[94] V. Jiménez López. Algunas cuestiones acerca de la estrúctura del caos (spanish). Master's thesis, Universidad de Murcia, 1989. Available on `http://www.um.es/docencia/vjimenez/ficheros/articulos/tesina.pdf`.

[95] V. Jiménez López. *Is Li and Yorke's definition a good measure of chaos?* PhD thesis, Universidad de Murcia, 1992. Available on `http://www.um.es/docencia/vjimenez/ficheros/articulos/tesis.pdf`.

[96] V. Jiménez López. Large chaos in smooth functions of zero topological entropy. *Bull. Austral. Math. Soc.*, 46(2):271–285, 1992.

[97] V. Jiménez López. An explicit description of all scrambled sets of weakly unimodal functions of type 2^∞. *Real Anal. Exchange*, 21(2):664–688, 1995/96.

[98] L. Jonker. Periodic orbits and kneading invariants. *Proc. London Math. Soc. (3)*, 39(3):428–450, 1979.

[99] I. Kan. A chaotic function possessing a scrambled set with positive Lebesgue measure. *Proc. Amer. Math. Soc.*, 92(1):45–49, 1984.

[100] H. Kato. Chaos of continuum-wise expansive homeomorphisms and dynamical properties of sensitive maps of graphs. *Pacific J. Math.*, 175(1):93–116, 1996.

[101] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[102] B. P. Kitchens. *Symbolic dynamics – One-sided, two-sided and countable state Markov shifts*. Universitext. Springer-Verlag, Berlin, 1998.

[103] Z. Kočan, V. Kornecká-Kurková, and M. Málek. Entropy, horseshoes and homoclinic trajectories on trees, graphs and dendrites. *Ergodic Theory Dynam. Systems*, 31(1):165–175, 2011. Erratum: *Ergodic Theory Dynam. Systems*, 31, no. 1:177, 2011.

[104] M. Kuchta and J. Smítal. Two-point scrambled set implies chaos. In *European Conference on Iteration Theory (Caldes de Malavella, 1987)*, pages 427–430. World Sci. Publishing, Teaneck, NJ, 1989.

[105] C. Kuratowski. *Topologie I et II.* Éditions Jacques Gabay, Sceaux, 1992. (French).

[106] K. Kuratowski. *Topology. Vol. I.* New edition, revised and augmented. Translated from the French by J. Jaworowski. Academic Press, New York, 1966.

[107] A. Lasota and J. A. Yorke. On the existence of invariant measures for transformations with strictly turbulent trajectories. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 25(3):233–238, 1977.

[108] M. C. Leseduarte and J. Llibre. On the set of periods for σ maps. *Trans. Amer. Math. Soc.*, 347(12):4899–4942, 1995.

[109] J. Li, T. K. S. Moothathu, and P. Oprocha. Corrigendum to "syndetically proximal pairs". Preprint, 2017.

[110] S. Li. ω-chaos and topological entropy. *Trans. Amer. Math. Soc.*, 339(1):243–249, 1993.

[111] T. Y. Li, M. Misiurewicz, G. Pianigiani, and J. A. Yorke. Odd chaos. *Phys. Lett. A*, 87(6):271–273, 1981/82.

[112] T. Y. Li, M. Misiurewicz, G. Pianigiani, and J. A. Yorke. No division implies chaos. *Trans. Amer. Math. Soc.*, 273(1):191–199, 1982.

[113] T. Y. Li and J. A. Yorke. Period three implies chaos. *Amer. Math. Monthly*, 82(10):985–992, 1975.

[114] J. Llibre and M. Misiurewicz. Horseshoes, entropy and periods for graph maps. *Topology*, 32(3):649–664, 1993.

[115] J. Llibre, J. Paraños, and J. A. Rodríguez. Periods for continuous self-maps of the figure-eight space. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 13(7):1743–1754, 2003. Dynamical systems and functional equations (Murcia, 2000).

[116] E. Lorenz. Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences*, pages 130–141, 1963.

[117] E. Lorenz. The mecanics of vacillation. *Journal of the Atmospheric Sciences*, 20:448–464, 1963.

[118] E. Lorenz. The problem of deducing the climate from the governing equations. *Tellus*, 16:1–11, 1964.

[119] J. Mai and T. Sun. The ω-limit set of a graph map. *Topology Appl.*, 154(11):2306–2311, 2007.

[120] E. N. Makhrova. Homoclinic points and topological entropy of a continuous mapping of a dendrite. *Sovrem. Mat. Prilozh.*, (54, Trudy Mezhdunarodnoi Konferentsii po Dinamicheskim Sistemam i Differentsial'nym Uravneniyam. Ch. 2):79–86, 2008.

[121] M. Málek. Distributional chaos for continuous mappings of the circle. *Ann. Math. Sil.*, (13):205–210, 1999. European Conference on Iteration Theory (Muszyna-Złockie, 1998).

[122] M. Málek. Distributional chaos and spectral decomposition on the circle. *Topology Appl.*, 135(1-3):215–229, 2004.

[123] R. M. May. Simple mathematical models with very complicated dynamics. *Nature*, 261:645–647, 1974.

[124] J. Milnor and W. Thurston. On iterated maps of the interval. In *Dynamical systems (College Park, MD, 1986–87)*, volume 1342 of *Lecture Notes in Math.*, pages 465–563. Springer, Berlin, 1988.

[125] M. Misiurewicz. Horseshoes for mappings of the interval. *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, 27(2):167–169, 1979.

[126] M. Misiurewicz. Invariant measures for continuous transformations of $[0,1]$ with zero topological entropy. In *Ergodic theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1978)*, volume 729 of *Lecture Notes in Math.*, pages 144–152. Springer, Berlin, 1979.

[127] M. Misiurewicz. Horseshoes for continuous mappings of an interval. In *Dynamical systems (Bressanone, 1978)*, pages 125–135. Liguori, Naples, 1980.

[128] M. Misiurewicz. Periodic points of maps of degree one of a circle. *Ergodic Theory Dynamical Systems*, 2(2):221–227, 1982.

[129] M. Misiurewicz and Z. Nitecki. Combinatorial patterns for maps of the interval. *Mem. Amer. Math. Soc.*, 94(456):vi+112, 1991.

[130] M. Misiurewicz and J. Smítal. Smooth chaotic maps with zero topological entropy. *Ergodic Theory Dynam. Systems*, 8(3):421–424, 1988.

[131] M. Misiurewicz and W. Szlenk. Entropy of piecewise monotone mappings. In *Dynamical systems, Vol. II—Warsaw*, Astérisque, no. 50, pages 299–310. Soc. Math. France, Paris, 1977.

[132] M. Misiurewicz and W. Szlenk. Entropy of piecewise monotone mappings. *Studia Math.*, 67(1):45–63, 1980.

[133] T. K. S. Moothathu. Syndetically proximal pairs. *J. Math. Anal. Appl.*, 379(2):656–663, 2011.

[134] J. R. Munkres. *Topology: a first course*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975.

[135] E. Murinová. Generic chaos in metric spaces. *Acta Univ. M. Belii Math.*, 8:43–50, 2000.

[136] J. Mycielski. Independent sets in topological algebras. *Fund. Math.*, 55:139–147, 1964.

[137] M. B. Nathanson. Permutations, periodicity, and chaos. *J. Combinatorial Theory Ser. A*, 22(1):61–68, 1977.

[138] Y. Oono. Period $\neq 2^n$ implies chaos. *Progr. Theoret. Phys.*, 59:1029–1030, 1978.

[139] M. Osikawa and Y. Oono. Chaos in C^0-endomorphism of interval. *Publ. Res. Inst. Math. Sci.*, 17(1):165–177, 1981.

[140] J. Oxtoby. *Measure and category*. Springer-Verlag, 1980.

[141] W. Parry. Symbolic dynamics and transformations of the unit interval. *Trans. Amer. Math. Soc.*, 122:168–378, 1966.

[142] R. Pikuła. On some notions of chaos in dimension zero. *Colloq. Math.*, 107(2):167–177, 2007.

[143] J. Piórek. On the generic chaos in dynamical systems. *Univ. Iagel. Acta Math.*, 25:293–298, 1985.

[144] R. P. Roe. Dense periodicity on finite trees. *Topology Proc.*, 19:237–248, 1994.

[145] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[146] D. J. Rudolph. *Fundamentals of measurable dynamics – Ergodic theory on Lebesgue spaces*. The Clarendon Press Oxford University Press, New York, 1990.

[147] D. Ruelle and F. Takens. On the nature of turbulence. *Comm. Math. Phys.*, 20:167–192, 1971.

[148] S. Ruette. Dense chaos for continuous interval maps. *Nonlinearity*, 18:1691–1698, 2005.

[149] S. Ruette. Transitive sensitive subsystems for interval maps. *Studia Math.*, 169(1):81–104, 2005.

[150] S. Ruette and Ľ. Snoha. For graph maps, one scrambled pair implies Li-Yorke chaos. *Proc. Amer. Math. Soc.*, 142(6):2087–2100, 2014.

[151] B. Schweizer and J. Smítal. Measures of chaos and a spectral decomposition of dynamical systems on the interval. *Trans. Amer. Math. Soc.*, 344(2):737–754, 1994.

[152] E. Seneta. *Nonnegative matrices and Markov chains*. Springer-Verlag, New York, second edition, 1981.

[153] A. N. Sharkovsky. Co-existence of cycles of a continuous mapping of the line into itself (Russian). *Ukrain. Mat. Ž.*, 16:61–71, 1964. English translation, *J. Bifur. Chaos Appl. Sci. Engrg.*, 5:1263–1273, 1995.

[154] A. N. Sharkovsky. Fixed points and the center of a continuous mapping of the line into itself (Ukrainian). *Dopovidi Akad. Nauk Ukraïn. RSR*, 1964(7):865–868, 1964.

[155] A. N. Sharkovsky. On cycles and the structure of a continuous mapping (Russian). *Ukrain. Mat. Ž.*, 17(3):104–111, 1965.

[156] A. N. Sharkovsky. Behavior of a mapping in the neighborhood of an attracting set (Russian). *Ukrain. Mat. Ž.*, 18(2):60–83, 1966. English translation, Amer. Math. Soc. Translations, Series 2, Vol. 97, pages 227–258, 1970.

[157] A. N. Sharkovsky. On a theorem of G. D. Birkhoff (Ukrainian). *Dopovīdī Akad. Nauk Ukraïn. RSR Ser. A*, 1967(5):429–432, 1967.

[158] A. N. Sharkovsky. On some properties of discrete dynamical systems. In *Théorie de l'itération et ses applications (Toulouse, 1982)*, Colloq. Internat. CNRS, 332, pages 153–158. CNRS, Paris, 1982.

[159] A. N. Sharkovsky, Yu. L. Yu. L. Maistrenko, and E. Yu. Romanenko. *Difference equations and their applications*, volume 250 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1993. Translated from the 1986 Russian original by D. V. Malyshev, P. V. Malyshev and Y. M. Pestryakov.

[160] S. Silverman. On maps with dense orbits and the definition of chaos. *Rocky Mountain J. Math.*, 22(1):353–375, 1992.

[161] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.

[162] J. Smítal. Chaotic functions with zero topological entropy. *Trans. Amer. Math. Soc.*, 297(1):269–282, 1986.

[163] L'. Snoha. Generic chaos. *Comment. Math. Univ. Carolin.*, 31(4):793–810, 1990.

[164] L'. Snoha. Dense chaos. *Comment. Math. Univ. Carolin.*, 33(4):747–752, 1992.

[165] P. Štefan. A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line. IHES preprint, ref. 76/154, 1976.

[166] P. Štefan. A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line. *Comm. Math. Phys.*, 54(3):237–248, 1977.

[167] Y. Takahashi. A formula for topological entropy of one-dimensional dynamics. *Sci. Papers College Gen. Ed. Univ. Tokyo*, 30(1):11–22, 1980.

[168] M. Vellekoop and R. Berglund. On intervals, transitivity = chaos. *Amer. Math. Monthly*, 101(4):353–355, 1994.

[169] P. Walters. *An introduction to ergodic theory*. Springer-Verlag, New York, 1982.

[170] J. Xiong. A chaotic map with [zero] topological entropy. *Acta Math. Sci. (English Ed.)*, 6(4):439–443, 1986.

[171] J. Xiong. Chaoticity of interval self-maps with positive entropy. Preprint IC/88/385, International Centre for Theoretical Physics, Trieste, 1988.

[172] X.D. Ye. No division and the entropy of tree maps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 9(9):1859–1865, 1999. Discrete dynamical systems.

[173] K. Yokoi. Dense periodicity on graphs. *Rocky Mountain J. Math.*, 37(6):2077–2088, 2007.

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Notation