PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 4, Pages 1091–1093
S 0002-9939(03)07222-8
Article electronically published on August 20, 2003

C^n INTERVAL MAPS NOT BOREL CONJUGATE TO ANY C^∞ MAP

SYLVIE RUETTE

(Communicated by Michael Handel)

Abstract. We show that there exist interval maps that are not Borel conjugate to any C^∞ map. These examples can be chosen to be topologically mixing and C^n, for any finite, arbitrarily large n.

Introduction

A *topological dynamical system* (X, T) is a compact metric space X endowed with a continuous map $T: X \to X$. An *interval map* is a continuous map $f: I \to I$, where I is a compact interval.

Two topological dynamical systems (X, T) and (Y, S) are said to be *topologically conjugate* if there exists a homeomorphism $\varphi: X \to Y$ such that $\varphi \circ T = S \circ \varphi$, and *Borel conjugate* if there exists a bijection $\varphi: X \to Y$ such that both φ, φ^{-1} are Borel maps (i.e., the inverse image of any Borel set is a Borel set) and $\varphi \circ T = S \circ \varphi$. Topological conjugacies entirely preserve the topological dynamics. For systems on the interval, they are rather rigid since they only consist of a "reparametrization" of the interval. Borel conjugacies give much freedom; they can destroy the topology, yet they carry the dynamics of all invariant measures.

Our aim is to show that, in spite of the freedom allowed by Borel conjugacies, every interval map f cannot be made C^∞ with a Borel conjugacy, even if f is already C^n. For this, we use the fact that a Borel conjugacy preserves the existence of measures of maximal entropy, which is a property shared by all C^∞ maps.

Acknowledgment. I would like to thank Albert Fathi; his comments on my work motivated this paper.

1. Measures of maximal entropy

If (X, T) is a topological dynamical system, let $\mathcal{M}_T(X)$ be the set of all probability Borel measures that are T-invariant. If X is compact, the set $\mathcal{M}_T(X)$ is nonempty [2]. For every T-invariant measure μ, one can define the metric entropy $h_\mu(X, T)$ (see, for example, [6]).

Definition 1.1. Let (X, T) be a topological dynamical system. If μ is a T-invariant measure such that $h_\mu(X, T) = \sup\{h_\nu(X, T) \mid \nu\ T\text{-invariant measure}\}$, then μ is called a *measure of maximal entropy*.

Received by the editors November 28, 2002.
2000 *Mathematics Subject Classification.* Primary 37E05, 37C15; Secondary 37A05.

Measures of maximal entropy do not necessarily exist, even for interval maps, yet the following theorem says that such a measure exists if the map is C^∞ [4] (see also [1] for interval maps).

Theorem 1.2. *If $f\colon M \to M$ is a C^∞ map, where M is a compact manifold of finite dimension, then f admits at least one measure of maximal entropy.*

2. Borel conjugacy

The next proposition can be proven easily from the definition of the metric entropy.

Proposition 2.1. *Let (X, T) and (Y, S) be two topological dynamical systems that are Borel conjugate. Then (X, T) has a measure of maximal entropy if and only if (Y, S) has one.*

This proposition implies that a dynamical system (Y, S) without measure of maximal entropy cannot be Borel conjugate to a system (X, T) having a measure of maximal entropy.

In [3] Gurevich and Zargaryan gave an example of a continuous, transitive interval map that has no measure of maximal entropy. They wondered if this example could be made smooth, but it cannot be topologically conjugate to a C^1 map because of the behaviour in the neighbourhood of the endpoints (see [5] for a more precise explanation). Then Buzzi showed that for every integer n there exist C^n interval maps without measure of maximal entropy [1]. We made this result precise and built a C^n, topologically mixing interval map for every integer n [5]. This example is made of countably many monotone pieces and is C^∞ everywhere except at one point, where oscillations accumulate; it has a positive topological entropy.

Combining Proposition 2.1 and Theorem 1.2, we see that these maps cannot be Borel conjugate to any C^∞ interval map, which leads to the following result:

Theorem 2.2. *For every integer $n \geq 1$ there exists a map $f\colon [0,1] \to [0,1]$ that is C^n, topologically mixing, of positive topological entropy, and that is not Borel conjugate to any C^∞ map $g\colon [0,1] \to [0,1]$.*

Remark 2.3. According to Theorem 1.2 these maps are not Borel conjugate to any C^∞ map on a compact manifold of finite dimension.

References

[1] J. Buzzi, *Intrinsic ergodicity of smooth interval maps*, Israel J. Math. **100** (1997), 125–161. MR **99g:**58071

[2] M. Denker, C. Grillenberger, and K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Mathematics, no. 527, Springer-Verlag, Berlin, 1976. MR **56:**15879

[3] B. M. Gurevich and A. S. Zargaryan, *A continuous one-dimensional mapping without measure with maximal entropy (Russian)*, Funktsional. Anal. i Prilozhen. **20** (1986), no. 2, 60–61, English translation, *Functional Anal. Appl.*, **20** (2), 134–136, 1986. MR **87k:**58151

[4] S. E. Newhouse, *Continuity properties of entropy*, Ann. of Math. (2) **129** (1989), no. 2, 215–235, Corrections, **131** (2):409–410, 1990. MR **90f:**58108

[5] S. Ruette, *Mixing C^r maps of the interval without maximal measure*, Israel J. Math **127** (2002), 253–277. MR **2003f:**37068

[6] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, No. 79, Springer-Verlag, New York, 1982. MR **84e:**28017

Universitat Autònoma de Barcelona– Departament de Matemàtiques– Edifici Cc– 08193 Cerdanyola del Vallès–Barcelona–Spain

E-mail address: ruette@mat.uab.es

Current address: Laboratoire de Mathématiques, Topologie et Dynamique, Université Paris Sud, 91405 Orsay, France

E-mail address: sylvie.ruette@math.u-psud.fr