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Dense chaos for continuous interval maps

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Abstract
A continuous map f from a compact interval I into itself is densely (resp. generically) chaotic if the set of points (x, y) such that $\limsup_{n\to+\infty} |f^n(x) - f^n(y)| > 0$ and $\liminf_{n\to+\infty} |f^n(x) - f^n(y)| = 0$ is dense (resp. residual) in $I \times I$. We prove that if the interval map f is densely but not generically chaotic then there is a descending sequence of invariant intervals, each of which contains a horseshoe for f^2. It implies that every densely chaotic interval map is of type at most 6 for Sharkovskiĭ's order (i.e. there exists a periodic point of period 6), and its topological entropy is at least $(\log 2)/2$. We show that equalities can be obtained.

Mathematics Subject Classification: 37E05, 37B40

1. Introduction

This paper deals with the dynamics of interval maps, that is, continuous maps $f: I \to I$, where I is a compact interval in $\mathbb{R}$. We first give some notation used in this paper. An *invariant* set (for the map f) is a closed non-empty subset A such that $f(A) \subset A$. A *transitive* subset (for f) is an invariant set A such that $f|_A$ is transitive (see, e.g., [6] for the definition of transitivity). The length of an interval J is denoted by $|J|$. An interval J is *non-degenerate* if $|J| > 0$, that is, J is neither empty nor reduced to a single point.

Li and Yorke [10] called some kind of behaviour of interval maps *chaotic*, although without formal definition. The notions of *Li–Yorke pairs* and *Li–Yorke chaos* in the following were derived from this paper.

Definition 1.1. *Let $T: X \to X$ be a continuous map on a metric space X. If $x, y \in X$ and $\delta > 0$, (x, y) is called a* Li-Yorke pair of modulus δ *if*

$$\limsup_{n\to+\infty} d(T^n(x), T^n(y)) > \delta \qquad \textit{and} \qquad \liminf_{n\to+\infty} d(T^n(x), T^n(y)) = 0$$

(x, y) *is a* Li–Yorke pair *if it is a Li–Yorke pair of modulus* δ *for some* $\delta > 0$. *The set of Li–Yorke pairs of modulus* δ *is denoted by* $\mathrm{LY}(T, \delta)$ *and the set of Li–Yorke pairs by* $\mathrm{LY}(T)$.

Definition 1.2. *Let* $T : X \to X$ *be a continuous map on a metric space* X. *The system* (X, T) *is said to be* chaotic in the sense of Li–Yorke *if there exists an uncountable set* $S \subset X$ *such that for all* $x, y \in S$, $x \neq y$, (x, y) *is a Li–Yorke pair.*

The definition of *generic chaos* is due to Lasota (see [13]). Being inspired by this definition, Snoha defined *generic* δ*-chaos*, *dense chaos* and *dense* δ*-chaos* [15].

Definition 1.3. *Let* $T : X \to X$ *be a continuous map on a metric space* X *and* $\delta > 0$.

- T *is generically chaotic if* $\mathrm{LY}(T)$ *is residual in* X^2,
- T *is generically* δ*-chaotic if* $\mathrm{LY}(T, \delta)$ *is residual in* X^2,
- T *is densely chaotic if* $\mathrm{LY}(T)$ *is dense in* X^2,
- T *is densely* δ*-chaotic if* $\mathrm{LY}(T, \delta)$ *is dense in* X^2.

Generic δ-chaos obviously implies both generic chaos and dense δ-chaos, which in turn imply dense chaos.

Snoha [15] proved that for an interval map generic chaos implies generic δ-chaos for some $\delta > 0$ and the notions of generic δ-chaos and dense δ-chaos coincide, but a densely chaotic interval map may not be generically chaotic. Snoha [16] gave a characterization of densely chaotic interval maps and proved that for piecewise monotone interval maps the notions of dense chaos and generic chaos coincide. He asked what the infimum of topological entropy and the type for Sharkovskiĭ's order are for densely chaotic interval maps. We recall Sharkovskiĭ's theorem (see [17]).

Theorem 1.4 (Sharkovskiĭ [14]). *Consider the following order:*

$$3 \lhd 5 \lhd 7 \lhd 9 \lhd \cdots \lhd 2\cdot 3 \lhd 2\cdot 5 \lhd 2\cdot 7 \lhd \cdots \lhd 2^2\cdot 3 \lhd 2^3\cdot 5 \lhd \cdots \lhd 2^n \lhd \cdots 2^3 \lhd 2^2 \lhd 2 \lhd 1.$$

If the interval map f *has a periodic point of period* n *then it has periodic points of period* m *for all* $m \rhd n$.

According to theorem 1.4, the set of periods of periodic points of an interval map f is either $\{m \mid m \trianglerighteq n\}$ for some positive integer n, and in this case f is said to be of *type* n, or $\{2^n \mid n \geqslant 0\}$, and f is said to be of *type* 2^∞. Note that there exist interval maps of all types [5, 17].

Our first motivation was to answer Snoha's questions. In theorem 3.1, we actually obtain a result on the structure of interval maps that are densely chaotic but not generically chaotic: for such a map f, there exists a descending sequence of invariant intervals, with lengths tending to 0, and each of them contains a horseshoe for the map f^2 (i.e. two closed non-degenerate intervals J, K with disjoint interiors such that $f^2(J) \cap f^2(K) \supset J \cup K$). On the other hand, Snoha gave a characterization of generic chaos in term of transitive subintervals.

Theorem 1.5 (Snoha [15]). *Let* f *be an interval map. The following conditions are equivalent:*

- f *is generically chaotic,*
- *either there exists a unique transitive non-degenerate subinterval or there exist two transitive non-degenerate subintervals with a common endpoint. Moreover for every non-degenerate interval* J, $f^n(J)$ *is non-degenerate and there exist a transitive subinterval* T *and an integer* $n \geqslant 0$ *such that* $f^n(J) \cap \mathrm{Int}(T) \neq \emptyset$.

We deduce from the structure of densely chaotic interval maps that such maps are at most of type 6 for Sharkovskiĭ's order and their topological entropy is greater than or equal to $(\log 2)/2$ (corollary 3.6). Example 3.7 shows that equalities are possible, and in addition they can be realized by generically chaotic interval maps.

Murinová [11] generalized Snoha's work and showed that for a complete metric space X, generic δ-chaos and dense δ-chaos are equivalent. She also exhibited a generically chaotic system which is not generically δ-chaotic for any $\delta > 0$.

If X is a complete metric space and $G \subset X \times X$ is a dense G_δ-set, then using Kuratowski's theorem (see, e.g., [12]), one can find an uncountable set S such that $S \times S$ deprived of the diagonal of $X \times X$ is included in G (see, e.g., [7, lemma 3.1]). Therefore a generically chaotic map on a complete metric space is chaotic in the sense of Li–Yorke. Kuchta and Smítal [8] showed that on the interval the existence of one Li–Yorke pair is enough to imply chaos in the sense of Li–Yorke, and consequently dense chaos implies Li–Yorke chaos for interval maps. However, it is not known whether dense chaos implies Li–Yorke chaos in general.

Section 3 contains our main results. Some preliminary lemmas are needed; they are stated in section 2

2. Preliminary results

Lemma 2.1. *Let f be a densely chaotic interval map.*

(i) If J is a non-degenerate interval then $f^n(J)$ is non-degenerate for all $n \geqslant 0$.
(ii) Consider disjoint non-degenerate intervals $J_1, \ldots, J_p$ such that $f(J_i) \subset J_{i+1 \bmod p}$. Then either $p = 1$, or $p = 2$ and J_1, J_2 have a common endpoint. If the intervals J_i are closed, then $p = 1$.
(iii) If J, J' are invariant non-degenerate intervals, then $J \cap J' \neq \emptyset$.

Proof.

(i) If J is a non-degenerate interval then there exists $(x, y) \in J \times J$ such that (x, y) is a Li–Yorke pair, and thus $\limsup_{n\to+\infty} |f^n(J)| > 0$ and for every $n \geqslant 0$ the interval $f^n(J)$ is not reduced to a point.
(ii) Let $J_1, \ldots, J_p$ be disjoint non-degenerate intervals such that $f(J_i) \subset J_{i+1 \bmod p}$. Suppose that there exist $0 \leqslant i, j \leqslant p$ such that the distance D between J_i and J_j is positive. By continuity there exists $\eta > 0$ such that if $|x - y| < \eta$ then $|f^k(x) - f^k(y)| < D$ for all $0 \leqslant k \leqslant p$. If $(x, y) \in J_i \times J_j$ then for all $l \geqslant 0$ one has $|f^{lp}(x) - f^{lp}(y)| \geqslant D$ and thus for all $n \geqslant 0$ $|f^n(x) - f^n(y)| \geqslant \eta$, which contradicts the assumption that f is densely chaotic. If the intervals J_i are closed it implies that $p = 1$; otherwise it implies that $p = 1$ or $p = 2$, and if $p = 2$ then J_1 and J_2 have a common endpoint.
(iii) Let J, J' be two invariant non-degenerate intervals. Then there exists a Li–Yorke pair (x, x') in $J \times J'$, and in particular there exists an increasing sequence (n_i) such that $\lim_{i\to+\infty} |f^{n_i}(x) - f^{n_i}(x')| = 0$. By compactness there exist (m_i) a subsequence of (n_i) and a point z such that $\lim_{i\to+\infty} f^{m_i}(x) = \lim_{i\to+\infty} f^{m_i}(x') = z$ and the point z belongs to $J \cap J'$. □

Lemma 2.2. *Let f be a densely chaotic interval map. Suppose that there exists a sequence of non-degenerate invariant intervals $(J_n)_{n\geqslant 0}$ such that $\lim_{n\to+\infty} |J_n| = 0$. Then there exists a point $z \in \bigcap_{n\geqslant 0} J_n$ and $f(z) = z$.*

Moreover there exists a subsequence of closed non-degenerate intervals $(J'_n)_{n\geqslant 0}$ such that $f(J'_n) \subset J'_n$, $\lim_{n\to+\infty} |J'_n| = 0$ and for all $n \geqslant 0$, J'_{n+1} is included in the interior of J'_n with respect to the induced topology on J'_0.

Proof. First we show that $\bigcap_{n=0}^{+\infty} J_n \neq \emptyset$. If $\bigcap_{n=0}^{N} J_n$ is non-degenerate for all $N \geqslant 0$ then $\bigcap_{n=0}^{+\infty} J_n$ is not empty. Otherwise let $N \geqslant 0$ be the greatest integer such that $\bigcap_{n=0}^{N} J_n$ is non-degenerate. The interval $K = \bigcap_{n=0}^{N} J_n$ is closed and non-degenerate and $f(K) \subset K$. By lemma 2.1(iii), $J_{N+1} \cap K \neq \emptyset$ and thus the set $J_{N+1} \cap K$ is reduced to one point z. For every $n \geqslant 0$ one has $J_n \cap K \neq \emptyset$ and $J_n \cap J_{N+1} \neq \emptyset$ by lemma 2.1(iii) and thus by connectedness $z \in J_n$. Consequently $z \in \bigcap_{n=0}^{+\infty} J_n$.

The set $\bigcap_{n=0}^{+\infty} J_n$ is reduced to a single point z because $|J_n| \to 0$. One has $f(z) = z$ because $f(J_n) \subset J_n$ for all $n \geqslant 0$.

There exists an increasing sequence $(n_i)_{i\geqslant 0}$ such that either $J_{n_i} \cap (z, +\infty) \neq \emptyset$ for all $i \geqslant 0$ or $J_{n_i} \cap (-\infty, z) \neq \emptyset$ for all $i \geqslant 0$. Define $K_n = \bigcap_{i=0}^{n} J_{n_i}$; this is a closed non-degenerate interval, $f(K_n) \subset K_n$ and $K_{n+1} \subset K_n$.

Case 1. There exists an increasing sequence $(m_i)_{i\geqslant 0}$ such that $K_{m_{i+1}} \subset \mathrm{Int}(K_{m_i})$ for all $i \geqslant 0$. Take then $J'_i = K_{m_i}$.

Case 2. If the assumption of case 1 is not satisfied then there exists $N \geqslant 0$ such that for all $n \geqslant N$, $K_n \not\subset \mathrm{Int}(K_N)$, that is, either $\min K_n = \min K_N$ for all $n \geqslant N$ or $\max K_n = \max K_N$ for all $n \geqslant N$. Since $|K_n| \to 0$ one can find an increasing sequence $(m_i)_{i\geqslant 0}$ with $m_0 = N$ such that $|K_{m_{i+1}}| < |K_{m_i}|$ for all $i \geqslant 0$. In this case $J'_i = K_{m_i}$ is a suitable subsequence of intervals. □

Lemma 2.3. *Let J be a bounded interval and $f : J \to J$ a continuous map with no fixed point. If $K \subset J$ is a compact interval then* $\lim_{n\to+\infty} |f^n(K)| = 0$.

Proof. Suppose that $f(x) < x$ for all $x \in J$, the case $f(x) > x$ for all $x \in J$ being similar. Write $f^n(K) = [a_n, b_n]$. For every $n \geqslant 0$ there exists $x \in [a_n, b_n]$ such that $f(x) = b_{n+1}$, and thus $b_{n+1} < x \leqslant b_n$. The sequence $(b_n)_{n\geqslant 0}$ is decreasing and thus has a limit in $\bar{J}$; let $z = \lim_{n\to+\infty} b_n$. Suppose that $z \in J$. Let $\varepsilon > 0$ such that $f(z) + \varepsilon < z$. By continuity there exists $\eta > 0$ such that if $|x - z| < \eta$ then $|f(x) - f(z)| < \varepsilon$. Let $n \geqslant 0$ such that $|b_n - z| < \eta$. Then for all $x \in [z, b_n]$ one has $f(x) < f(z) + \varepsilon < z$ and for all $x \in [a_n, z)$ one has $f(x) < x < z$. This implies that $b_{n+1} < z$, which is absurd. Hence $z = \inf J$ and $z < a_n \leqslant b_n$ for all $n \geqslant 0$. This implies that $|f^n(K)| \leqslant |b_n - z| \to 0$ when n goes to infinity. □

Proposition 2.4 (Snoha [15]). *Let f be an interval map and $\delta > 0$. The following conditions are equivalent:*

- *f is generically δ-chaotic,*
- *for all non-degenerate intervals J_1, J_2 one has* $\limsup_{n\to+\infty} |f^n(J_1)| > \delta$ *and* $\liminf_{n\to+\infty} \mathrm{dist}(f^n(J_1), f^n(J_2)) = 0$ *(where* $\mathrm{dist}(\cdot,\cdot)$ *denotes the distance between two sets).*

Lemma 2.5. *Let f be a densely chaotic interval map. Suppose that there exists $\varepsilon > 0$ such that, for every non-degenerate invariant interval J, $|J| \geqslant \varepsilon$. Then f is generically chaotic.*

Proof. Suppose that

$$\forall \delta > 0, \quad \exists\, J \text{closed non-degenerate interval}, \forall n \geqslant 0,\ |f^n(J)| \leqslant \delta. \tag{1}$$

We are going to show that is not possible. Let $0 < \delta < \varepsilon/4$ and let J be a closed non-degenerate interval such that $|f^n(J)| \leqslant \delta$ for all $n \geqslant 0$. There exists a Li–Yorke pair $(x, y) \in J \times J$

because f is densely chaotic, and thus

$$\limsup_{n\to+\infty} |f^n(J)| > 0. \tag{2}$$

This implies that there exist N, p such that $f^N(J) \cap f^{N+p}(J) \neq \emptyset$, and thus $f^n(J) \cap f^{n+p}(J) \neq \emptyset$ for all $n \geqslant N$. Since $f^n(J)$ is an interval, this implies that, for every $0 \leqslant i \leqslant p-1$, the set $Z_i = \bigcup_{k\geqslant 0} f^{N+i+kp}(J)$ is an interval too. Consequently, the set $Z = \bigcup_{n\geqslant N} f^n(J)$ has at most p connected components, which are non-degenerate by lemma 2.1(i). The image of a connected component is connected and $f(Z_i) \subset Z_{i+1 \bmod p}$, and thus the connected components of Z are necessarily cyclically mapped into each other and lemma 2.1(ii) applies: Z has either one connected component or two connected components with a common endpoint, and $\bar{Z}$ is a closed interval.

If there exist a point z and an integer $n_0 \geqslant N$ such that $f^2(z) = z$ and $z \in f^{n_0}(J)$ then $z \in f^{n_0+2k}(J)$ for all $k \geqslant 0$. Since $|f^n(J)| \leqslant \delta$ for all $n \geqslant 0$ one gets the results that $|\bigcup_{k\geqslant 0} f^{n_0+2k}(J)| \leqslant 2\delta$ and $|\bigcup_{k\geqslant 0} f^{n_0+2k+1}(J)| \leqslant 2\delta$. Let $L = \overline{\bigcup_{n\geqslant n_0} f^n(J)} = f^{n_0-N}(\bar{Z})$. Then L is a closed non-degenerate interval, $f(L) \subset L$ and $|L| \leqslant 4\delta$. Moreover $|L| \geqslant \varepsilon$ according to the hypothesis of the lemma, which is a contradiction because we have chosen $\delta < \varepsilon/4$. We deduce that Z contains no point z such that $f^2(z) = z$.

Let Z_0 be the connected component of Z containing $f^N(J)$ and put $g = f^2$. Then $g(Z_0) \subset Z_0$ and $g|_{Z_0}$ has no fixed point. The interval $K = f^N(J)$ is compact because J is compact and f^N is continuous, and so lemma 2.3 applies and we get the relation that $\lim_{n\to+\infty} |f^{N+2n}(J)| = 0$. By continuity of f we get that $|f^n(J)| \to 0$ when n goes to infinity, which contradicts equation (2). We conclude that equation (1) is false, and consequently there exists $\delta > 0$ such that for all closed non-degenerate intervals J, $\limsup_{n\to+\infty} |f^n(J)| \geqslant \delta$. The map f is densely chaotic and thus for every non-degenerate interval J_1, J_2 there is a Li–Yorke pair in $J_1 \times J_2$, and hence $\liminf_{n\to+\infty} \operatorname{dist}(f^n(J_1), f^n(J_2)) = 0$. Then proposition 2.4 implies that f is generically chaotic. □

3. Structure of densely chaotic interval maps

Recall that the interval map g has a horseshoe if there exist two closed non-degenerate subintervals J, K with disjoint interiors such that $g(J) \cap g(K) \supset J \cup K$.

Theorem 3.1. *Let f be an interval map. If f is densely chaotic but not generically chaotic then there exists a sequence of invariant non-degenerate subintervals $(J_n)_{n\geqslant 0}$ such that $J_{n+1} \subset J_n$, $\lim_{n\to+\infty} |J_n| = 0$ and $f^2|_{J_n}$ has a horseshoe for all $n \geqslant 0$.*

We need two lemmas in the proof of this theorem. Lemma 3.2 is proven in [9] under slightly weaker hypotheses (see also [4, p 28]). Lemma 3.3 can be found in [4, p 31].

Lemma 3.2. *Let f be an interval map with no horseshoe and x a point. Write $x_n = f^n(x)$ for all $n \geqslant 0$. Suppose that $x_{n+1} \geqslant x_n$ and $x_{m+1} \leqslant x_m$. Then $x_n \leqslant x_m$.*

Lemma 3.3. *Let f be an interval map such that f^2 has no horseshoe. Let x be a point which is not ultimately periodic and write $x_n = f^n(x)$ for $n \geqslant 0$. Suppose that there exists $k_0 \geqslant 2$ such that either $x_{k_0} < x_0 < x_1$ or $x_{k_0} > x_0 > x_1$. Then there exist a fixed point z and an integer N such that, for all $n \geqslant N$, $x_n > z \Leftrightarrow x_{n+1} < z$.*

Proof of theorem 3.1. By assumption the map f is not generically chaotic. Thus, by lemma 2.5,

$$\forall \varepsilon > 0, \quad \text{there exists an invariant non-degenerate interval } J \text{ such that } |J| < \varepsilon.$$

Let $(I_n)_{n\geqslant 0}$ be a sequence of invariant non-degenerate intervals I_n such that $|I_n| \to 0$. By lemma 2.2 there exists a sequence of invariant non-degenerate intervals $(J_n)_{n\geqslant 0}$ such that $\lim_{n\to+\infty} |J_n| = 0$, and $J_{n+1} \subset \mathrm{Int}(J_n)$ with respect to the induced topology on J_0. From now on we fix $n_0 \geqslant 0$ and we restrict ourselves to the interval J_{n_0}. The map $f|_{J_{n_0}}$ is densely chaotic, the set $\bigcap_{n\geqslant n_0} J_n$ is reduced to a single point z and $f(z) = z$. Let

$$\mathcal{P} = \left\{x \in J_{n_0} \mid \exists p \geqslant 1, \lim_{n\to+\infty} f^{np}(x) \text{ exists}\right\}.$$

If $x, y \in \mathcal{P}$ then (x, y) is not a Li–Yorke pair, and thus the set $J_{n_0}\setminus\mathcal{P}$ is not empty.

Assume that $f^2|_{J_{n_0}}$ has no horseshoe; we are going to prove that this is absurd. Let $x_0 \in J_{n_0}\setminus\mathcal{P}$ and write $x_n = f^n(x_0)$ for all $n \geqslant 0$. According to lemma 3.3 there exist a fixed point c and an integer N such that, for all $n \geqslant 0$, $x_{N+2n} < c < x_{N+2n+1}$. Suppose for instance that $c \leqslant z$, the case with reverse inequality being symmetric. Since $x_0 \notin \mathcal{P}$, the sequence $(x_{N+2n})_{n\geqslant 0}$ is not ultimately monotone; thus there exists $i \geqslant 0$ such that

$$x_{N+2i+2} < x_{N+2i} < c \leqslant z.$$

By continuity there exists a closed non-degenerate interval K containing x_{N+2i} such that $z \notin K$ and for all $y \in K$, $f^2(y) < y$. Let $k \geqslant n_0$ such that $K < J_k$.

The set $K \times K$ contains a Li–Yorke pair because f is densely chaotic, and thus $\limsup_{n\to+\infty} |f^n(K)| > 0$ and there exist p, q such that $f^{q+p}(K) \cap f^q(K) \neq \emptyset$. Let $L = \overline{\bigcup_{n\geqslant q} f^n(K)}$. One has $f(L) \subset L$. The same argument as for $\bar{Z}$ in the proof of lemma 2.5 implies that L is an invariant non-degenerate interval. Moreover, lemma 2.1(iii) implies that $L \cap J_n \neq \emptyset$ for all $n \geqslant n_0$. Since $J_{k+1} \subset \mathrm{Int}(J_k)$, this implies that there exists an integer $n \geqslant 0$ such that $f^n(K) \cap \mathrm{Int}(J_k) \neq \emptyset$, and thus there exists a closed non-degenerate subinterval $K' \subset K$ such that $f^n(K') \subset J_k$.

Let $m_0 \geqslant n/2$ and $g = f^2$. For all $y \in K'$ and all $m \geqslant m_0$ one has $g^m(y) \in J_k$ because $f(J_k) \subset J_k$; thus

$$g(y) < y < g^m(y).$$

This implies that there exists $0 < j < m_0$ such that $g^j(y) < g^{j+1}(y)$. By assumption g has no horseshoe and thus $g^j(y) \leqslant y$ by lemma 3.2. For all $m \geqslant m_0$, one has $y \leqslant g^m(y)$, and thus the same lemma implies that $g^{m+1}(y) \leqslant g^m(y)$. Consequently, $(g^m(y))_{m\geqslant m_0}$ is a non-increasing sequence, and thus it converges. But this implies that $K' \times K'$ contains no Li–Yorke pair, which contradicts the fact that f is densely chaotic. This concludes the proof. □

The next theorem sums up two results on horseshoes; the first point is due to Block and Coppel [3], and the second one derives from [2] (see also [4, p 196]).

Theorem 3.4. *Let f be an interval map with a horseshoe. Then*

- *f is of type 3 for Sharkovskiĭ's order,*
- $h_{top}(f) \geqslant \log 2$.

According to theorem 1.5, a generically chaotic interval map f admits a transitive subinterval, and thus the next theorem implies that f^2 has a horseshoe.

Theorem 3.5 (Block–Coven [1]). *Let f be a transitive interval map. Then f^2 has a horseshoe.*

Corollary 3.6. *Let f be a densely chaotic interval map. Then f^2 has a horseshoe, $h_{top}(f) \geqslant (\log 2)/2$ and f is of type at most 6 for Sharkovskiĭ's order.*

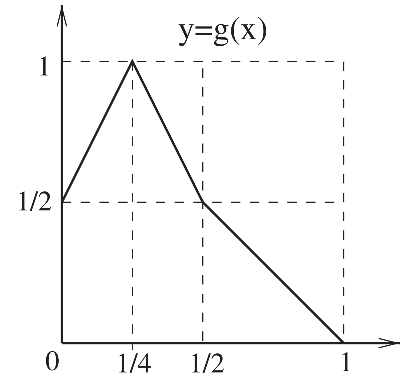

Figure 1. Densely chaotic map of entropy $(\log 2)/2$ and type 6.

Proof. If f is generically chaotic then f^2 has a horseshoe by theorems 1.5 and 3.5, otherwise f^2 has a horseshoe by theorem 3.1. By theorem 3.4, $h_{top}(f) \geqslant (\log 2)/2$ and f^2 has a periodic point of period 3; thus f has a periodic point of period 3 or 6. □

Example 3.7. In corollary 3.6 equalities are possible. Consider the 'square-root' of the tent map, pictured in figure 1. The map g swaps the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Thus every periodic point $x \neq \frac{1}{2}$ has an even period. Moreover the intervals $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$ form a horseshoe for g^2, and so theorem 3.4 implies that g is of type 6 for Sharkovskiĭ's order. The map g^2 restricted to either $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ is the classical tent map (upside down on $[0, \frac{1}{2}]$), which is known to be mixing (see, e.g., [4, p 159]), so g is transitive (thus generically chaotic). Finally, the topological entropy of g^2 is equal to $(\log 2)$ (use either the fact that it is Markov or the combination of theorem 3.4 and [6, proposition (14.20)]), and hence $h_{top}(g) = (\log 2)/2$.

This example shows that the infimum of the topological entropy of densely (respectively, generically) chaotic interval maps is reached and is equal to $(\log 2)/2$.

There also exist transitive (thus generically chaotic) interval maps of type $2k + 1$ for all $k \geqslant 1$ [1]. It may be derived from [2] that the topological entropy of a map of type $2k + 1$ is greater than $(\log 2)/2$.

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