Mixing C^r maps of the interval without maximal measure

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Abstract

We construct a C^r transformation of the interval (or the torus) which is topologically mixing but has no invariant measure of maximal entropy. Whereas the assumption of C^∞ ensures existence of maximal measures for an interval map, it shows we cannot weaken the smoothness assumption. We also compute the local entropy of the example.

Introduction

We are interested in topological dynamical systems on the interval, that is systems of the form $f: I \to I$ where f is at least continuous and I is a compact interval. One can wonder whether such a system has maximal measures, i.e. invariant measures of maximal entropy.

Hofbauer [15], [16] studied piecewise monotone maps, i.e. interval maps with a finite number of monotone continuous pieces (the whole map is not necessarily continuous). He proved in this case that the system admits a non zero finite number of maximal measures if its topological entropy is positive, and transitivity implies intrinsic ergodicity, that is existence of a unique maximal measure. For this purpose, he built a Markov chain which is isomorphic modulo "small sets" with the first system. Buzzi [9] generalized the construction of the Markov extension to any continuous interval map. He showed that the same conclusions as in the piecewise monotone case hold for C^∞ maps.

One can wonder if these results are still valid under a weaker regularity assumption, at least in the mixing case. Actually, if a topological dynamical system is expansive and satisfies the specification property, then it has a unique maximal measure (Bowen [6], [7]). Specification is a strong property on periodic points, which must closely follow arbitrary pieces of orbits (see e.g. [11] for more details). In the particular case of continuous interval maps, the system is never expansive, but the mixing property implies the specification property (this result is due to

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Blokh [4], see [10] for the proof). More recently, Ruelle [20] worked on positively expansive maps satisfying specification.

In fact, transitivity is not much weaker than mixing since for any transitive continuous interval map $f: I \to I$ either the map is mixing or there exist two subintervals J, K such that $J \cup K = I$, $J \cap K$ is reduced to a single point, $f(J) = K$, $f(K) = J$ and $f^2|_J, f^2|_K$ are mixing [2, p59]. We also recall that the topological entropy of any transitive continuous interval map is positive (it is greater than or equal to $\frac{\log 2}{2}$ [3], see [1] for the proof) and, if in addition the map is Lipschitz, it is finite (this classical result appears in the proof of Proposition 2.4).

Gurevich and Zargaryan [12] built a continuous interval map with finite entropy which is transitive (in fact mixing) and has no maximal measure. This map has countably many intervals of monotonicity. The authors asked is this example can be made smooth on the whole interval. Actually it cannot: the end points 0 and 1 are fixed points and the map is not monotone in a neighbourhood of 0 and 1; on the other hand it is not hard to see that a C^1 transitive interval map must have non zero derivatives at fixed points, hence it is monotone near these points.

In [9, Appendix A] Buzzi built a C^r interval map which has no transitive component of maximal entropy, hence it has no maximal measure. He also sketched without details the construction of a C^r interval map with positive entropy which admits no maximal measure and which is transitive after restriction to its unique transitive component (which may be a Cantor set). His proof of non existence of any maximal measure relies on a result of Salama [21] whose proof turned out to be false (see Theorem 2.3 and Errata in [22]). Nevertheless Buzzi's proof can be modified – using extension graphs instead of subgraphs, as we do in Subsection 2.3 – so as to be based on another theorem of Salama.

The aim of this article is to build for any integer $r \geq 1$ a C^r mixing interval map which has no maximal measure. Transitivity instead of mixing would be enough, yet it is not more difficult to prove directly the mixing property. This family of examples is inspired by Buzzi's [9], the important addition is that the system is transitive on the whole interval. Non existence of maximal measure prevents the metric entropy from being an upper semi-continuous map on the set of invariant measures. This is to be put in parallel with the result of Misiurewicz and Szlenk [17], which shows that the topological entropy, considered as a map on the set of C^r interval transformations, is not upper semi-continuous for the C^r topology.

In Section 1, we define for any $r \geq 1$ a C^r transformation of the interval $[0, 4]$ which is topologically mixing. In fact it is C^∞ everywhere except at one point. The map f_r is made of a countable number of monotone pieces and is Markov with respect to a countable partition. Moreover, it can also be seen as a C^r transformation of the torus by identifying the two end points. In the next section, we study the Markov chain associated with f_r and we conclude it has no maximal measure, thanks to results of Gurevič [13], [14] and Salama [22]. As there is an isomorphism modulo countable sets between the two systems, the interval map has no maximal measure either. In Section 3, we compute the local entropy of our examples. Buzzi [9] showed that this quantity bounds the defect in upper-semicontinuity and he gave an estimate of it depending on the differential order and the spectral radius of the derivative. Our examples show these bounds are sharp since the two are realized. Moreover, it also equals the topological entropy. It may be of some importance: we conjecture that the Markov extension admits a maximal measure when the topological entropy is strictly greater than the local entropy.

In addition to the problem of existence of maximal measure, one can ask the question of uniqueness of such a measure. Recently, Buzzi [8] proved that, if the interval transformation is $C^{1+\alpha}$ (i.e. the map is C^1 and its derivative is α-Hölder), then there is no measure of positive entropy on the non Markov part of the system. Since a transitive Markov chain admits at most one maximal measure, a transitive $C^{1+\alpha}$ transformation has a unique maximal measure if it exists. For transitive non smooth interval maps we still do not know if several maximal measures can exist. It would imply that the topological entropy of the critical points would be equal to the topological entropy of the whole map.

I am indebted to Jérôme Buzzi for many discussions which have led to the ideas of this paper.

1 Construction and proof of mixing property

In this section, we construct a family of C^r maps $f_r\colon I \to I$ for $r \geq 1$, where $I = [0,4]$. We first give a general idea of their aspect (see Figure 2). Then we give some lemmas which will be useful to prove the mixing property. Finally, we define f_r by pieces and check some properties at each step. At the end of the section, the maps f_r are totally defined and are proved to be mixing.

1.1 General description

Let $\lambda \geq 14$ ($\log\lambda$ will be the entropy of f_r). The map f_r is increasing on $[0,1/2]$ and decreasing on $[1/2,1]$. Moreover, $f_r(x) = \lambda^r x$ for $0 \leq x \leq \frac{5}{2}\lambda^{-r}$, $f_r(0) = f_r(1) = 0$, $f_r(1/2) = 4$.

Let $x_n = 1 + \frac{1}{n}$ and $y_n = x_n + \frac{1}{2n^2}$ for every $n \geq 1$, and let M_n be a sequence of odd numbers with $(\log M_n)/n \longrightarrow \log\lambda$. We choose a family of C^∞ maps $s_n\colon [0, M_n] \to [-1,1]$ such that s_n is nearly 2-periodic and has M_n oscillations; $s_n(0) = 0$ and $s_n(M_n) = 1$ (see Figure 1).

Then we define f_r on $[x_n, y_n]$ by

$$f_r(x) = \lambda^{-nr}\left[x_n + (y_n - x_n)s_n\left(M_n\frac{x - x_n}{y_n - x_n}\right)\right].$$

In this way, $f_r(x_n) = \lambda^{-nr}x_n$, $f_r(y_n) = \lambda^{-nr}y_n$ and f_r oscillates M_n times between x_n and y_n like s_n. It is worth mentioning that x_n and y_n are periodic

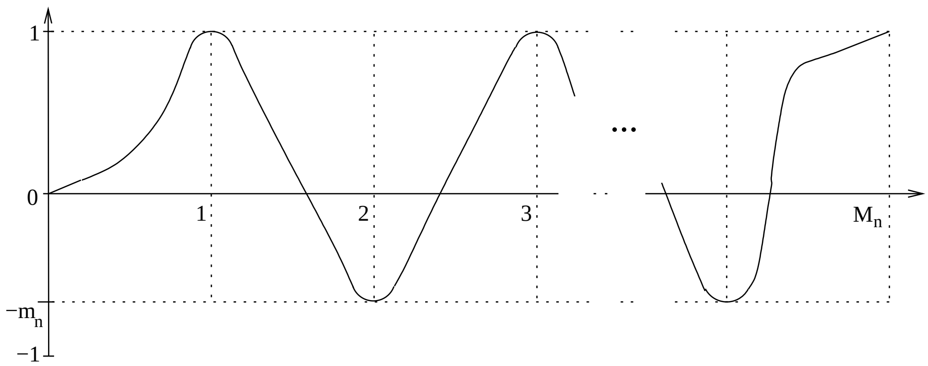

Figure 1: the map s_n

points with period $n+1$, because f_r is linear of slope λ^r on $[0, y_1\lambda^{-r}]$.

On $[y_{n+1}, x_n]$, f_r is increasing.

Finally, f_r is increasing on $[y_1, 4]$, with $f_r(4) = 4$. Figure 2 gives a general idea of f_r.

The map f_r will be built to be mixing and C^r on $[0,4]$, and $\|f_r'\|_\infty = \lambda^r$. Furthermore, the minimum of s_n will be chosen such that $f_r(x) = \lambda^{-nr} y_{n+1}$ if x is a local minimum of f_r in $]x_n, y_n[$ in order to obtain a Markov map.

This brief description is sufficient to build the Markov chain associated with f_r and prove that f_r has no maximal measure, which is done in Section 2. The rest of this section, which may be skipped a first reading, is devoted to prove that maps satisfying these properties do exist.

1.2 Method for the proof of mixing property

We recall the definition of mixing for a topological dynamical system.

Definition 1.1 Let $T: X \to X$ be a continuous map where X is a compact metric space. The system is (topologically) **mixing** if for every non empty open sets U and V, there exists $N \geq 0$ such that for every $n \geq N$, $T^{-n}U \cap V \neq \emptyset$.

In our case, we will show that for any non degenerate subinterval $J \subset I$, there exists $n \geq 0$ such that $f_r^n(J) = I$. So $f_r^k(J) = I$ for every $k \geq n$ and the system is mixing. For this, we will show that, for some constant $\mu_0 > 1$, any non degenerate subinterval J satisfies one of the two following conditions:

(1) $\exists k \geq 0$ such that $|f_r^k(J)| \geq \mu_0 |J|$, where $|J|$ denotes the length of J,

or

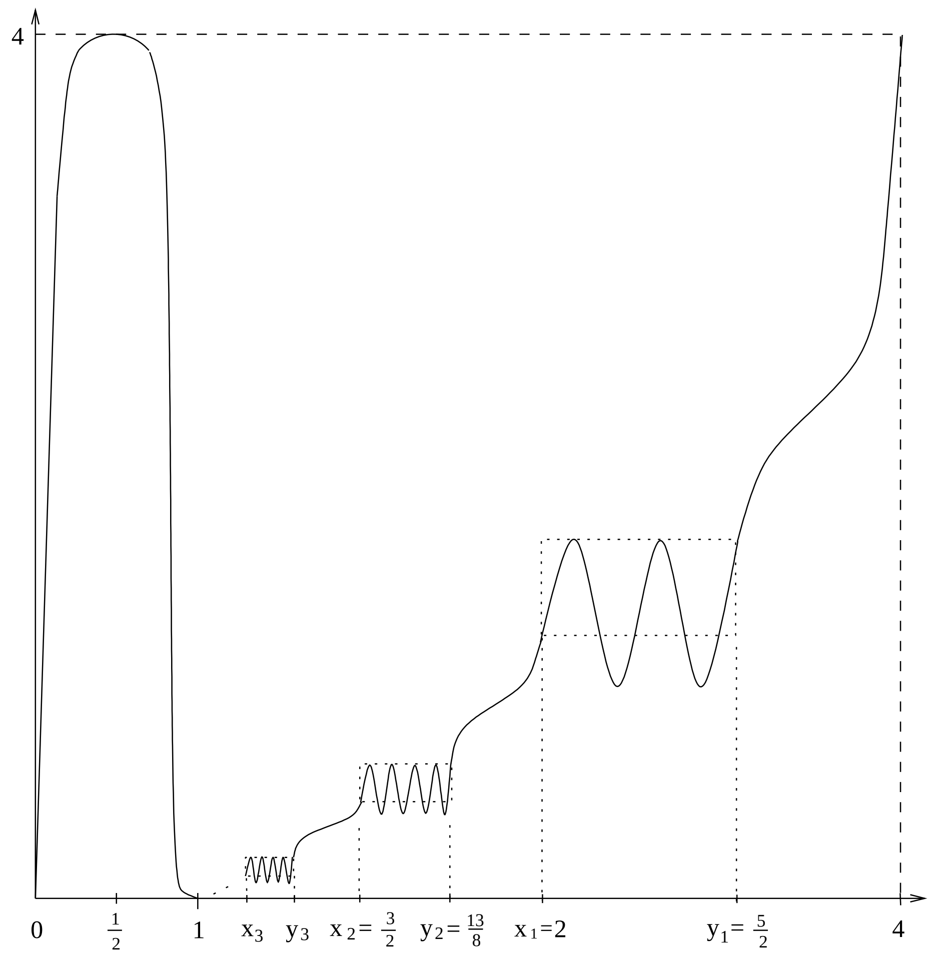

Figure 2: the map f_r (scale is not respected)

(2) $\exists k \geq 0, \exists n \geq 1$ such that either $0 \in f_r^k(J)$ or $\mathrm{Int}\left(f_r^k(J)\right)$ contains x_n or y_n.

Then it will be enough to show that for any non degenerate subinterval J containing 0 or x_n or y_n, there is a k such that $f_r^k(J) = I$.

Lemma 1.3 says that an interval near a suitable extremum satisfies (1) or (2). Lemma 1.2, which is trivial, says how an interval containing a repelling periodic point behaves.

Lemma 1.2 *Let $f\colon I \to I$ where I is a compact interval and let z_0 be a periodic point of period p. Assume $(f^p)'(x) \geq \mu > 1$ for every $x \in [z_0, z_1]$. Then for every $x > z_0$ there exists $n \geq 0$ such that $f^n([z_0, x]) \supset [z_0, z_1]$.*

Lemma 1.3 *Let $f\colon I \to I$ be a C^r map where I is a compact interval and let z_0 be an extremum such that $z_1 = f^k(z_0)$ is a periodic point of period p. Suppose $f^k(x) = z_1 + C(x - z_0)^\alpha$ for $|x - z_0| \leq \delta$, with $C \neq 0$ and α an even integer. Let $z_2 = f^k(z_0 - \delta) = f^k(z_0 + \delta)$. Suppose $f^p|_{[z_1,z_2]}$ is linear of slope $\mu > 1$, and $\frac{\alpha|z_2 - z_1|}{\delta} \geq \mu_0$. Then for every non degenerate interval $J \subset [z_0 - \delta, z_0 + \delta]$, there exists $n \geq 0$ for which one of the following cases holds:*

(i) $|f^n(J)| \geq \mu_0 |J|$.

(ii) $z_2 \in \mathrm{Int}\,(f^n(J))$.

Proof:

Let $J = [a, b]$ be an interval in $[z_0 - \delta, z_0 + \delta]$ with $a < b$. If $z_0 \in J$ then $f^k(J) = [z_1, y]$ for some y. The hypotheses imply that $f^p(z_2) > z_2$, hence z_2 cannot be an end point of I and one can choose $1 < \mu' < \mu$ and $z_3 > z_2$ such that $(f^p)'(x) > \mu'$ for all $x \in [z_1, z_3]$. According to Lemma 1.2 there exists n such that $f^n(J) \supset [z_1, z_3]$, thus $z_2 \in \mathrm{Int}\,(f^n(J))$, which is (ii).

Now assume that $z_0 \notin J$. We restrict to the case $C > 0$ and $z_0 < a < b \leq z_0 + \delta$. Let $J' = f^k(J) = [a', b'] \subset]z_1, z_2]$ and $g = f^p$. The point z_1 is fixed for g and g is linear of slope $\mu > 1$ on $[z_1, z_2]$, so the map g can be iterated on J' as long as $g^m(b') \leq z_2$. Let m be the first integer satisfying $g^m(b') > z_2$. Then there are two cases:

- $g^m(a') < z_2 < g^m(b')$, which implies (ii) with $n = mp + k$.
- $z_2 \leq g^m(a') < g^m(b')$.
  In this case, as $(f^k)'$ is positive and increasing on $[z_0, z_0 + \delta]$ one gets $|J'| \geq \alpha C(a - z_0)^{\alpha - 1}|J|$ and
  $$|f^{mp+k}(J)| \geq \mu^m \alpha C(a - z_0)^{\alpha-1}|J|.$$
  But $g^m(a') - z_1 = \mu^m(a' - z_1) \geq z_2 - z_1$, so
  $$\mu^m \geq \frac{z_2 - z_1}{a' - z_1} = \frac{z_2 - z_1}{C(a - z_0)^\alpha},$$

and

$$|f^{mp+k}(J)| \geq \frac{\alpha|z_2 - z_1|}{|a - z_0|}|J| \geq \frac{\alpha|z_2 - z_1|}{\delta}|J| \geq \mu_0|J|.$$

□

We add a lemma which will be useful for some estimates.

Lemma 1.4 *Let $\lambda \geq 8$ and $[\cdot]$ refer to the entire part of a number. Then for all $n \geq 1$:*

(i) $\dfrac{\lambda^n}{n^2} \geq \lambda$.

(ii) $\dfrac{\lambda^n}{2n^2} \leq 2\left[\dfrac{\lambda^n}{2n^2}\right] - 1 \leq \dfrac{\lambda^n}{n^2}$.

(iii) $2\left[\dfrac{\lambda^n}{2n^2}\right] - 1 \geq \lambda - 3$.

Proof:

(i) is obtained by studying the function $x \mapsto \lambda^{x-1} - x^2$.

For the first inequality of (ii), we write

$$2\left[\frac{\lambda^n}{2n^2}\right] - 1 \geq \frac{\lambda^n}{2n^2} + \left(\frac{\lambda^n}{2n^2} - 3\right) \geq \frac{\lambda^n}{2n^2}$$

thanks to (i). The second inequality is obvious.

(iii) comes from $2\left[\dfrac{\lambda^n}{2n^2}\right] - 1 \geq \dfrac{\lambda^n}{n^2} - 3$ and from (i). □

1.3 Construction of f_r on $[1, y_1]$

Recall that $\lambda \geq 14$, $f_r(1) = 0$, $x_n = 1 + \frac{1}{n}$ and $y_n = x_n + \frac{1}{2n^2}$ for $n \geq 1$; in particular $y_1 = \frac{5}{2}$. In this subsection, we define f_r on $[1, y_1]$ with more details. For this purpose, we define f_r on each $[x_n, y_n]$ and then on each $[y_{n+1}, x_n]$. At each step, we check that the various pieces can be glued together in a C^∞ way and $|f_r'(x)| \leq \lambda^r$ for $x \in [1, y_1]$. In addition, we show that f_r is C^r on the right of 1. Finally, we focus on the mixing property. The map f_r is not totally defined yet, but at this stage we only need to know that $f_r(x) = \lambda^r x$ for $0 \leq x \leq \frac{5}{2}\lambda^{-r}$ and $f_r(\frac{1}{2}) = 4$ in order to prove that any non degenerate subinterval of $[1, y_1]$ satisfies (1) or (2) with $\mu_0 = \frac{4}{3}$. Then we show that for an open interval J containing x_n or y_n there is a k satisfying $f_r^k(J) = [0, 4]$.

1.3.1 On the subintervals $[x_n, y_n]$

Set $M_n = 2\left[\dfrac{\lambda^n}{2n^2}\right] - 1$ (where $[\cdot]$ denotes the entire part), $m_n = 1 - \dfrac{1}{(n+1)^2}$, $\delta = \lambda^{-r}$, $C = \frac{1}{4\delta^2}$ and $k_n = \frac{2\lambda^r}{M_n}$.

First, we choose a sequence of C^∞ functions $s_n\colon [0, M_n] \to [-m_n, 1]$ satisfying:

(3) $s_n(0) = 0$, $s_n(M_n) = 1$, s_n is increasing on each $[2k, 2k+1]$ $(0 \le k \le (M_n - 1)/2)$, s_n is decreasing on each $[2k+1, 2k+2]$ $(0 \le k \le (M_n - 3)/2)$.

(4) $s_n(x) = 1 - C(x-a)^2$ for $|x - a| \le \delta$ if a is a local maximum of s_n, $a \ne M_n$, and $s_n(x) = -m_n + C(x-b)^2$ for $|x - b| \le \delta$ if b is a local minimum of s_n, $b \ne 0$.

(5) $s_n(x) = k_n(x - M_n) + 1$ for $M_n - \delta \le x \le M_n$.

(6) $s_n(x) = k_n x$ for $x \in [0, \delta]$.

(7) $\forall k \ge 1$, $\exists A_k$, $\forall n \ge 1$, $\|s_n^{(k)}\|_\infty \le A_k$.

(8) $\|s_n'\|_\infty \le \lambda^r$ and $|s_n'(x)| \ge \min\{1/2, k_n\}$ if $|x - d| \ge \delta$ for all local extrema $d \in]0, M_n[$.

Property (7) can be fulfilled because m_n and k_n are bounded $(3/4 \le m_n \le 1$, $k_n \le \lambda^r)$ and the maps s_n have a 2-periodic looking.

If d is a local extremum in $]0, M_n[$, then $|s_n(d-\delta) - s_n(d)| = |s_n(d+\delta) - s_n(d)| = 1/4$; moreover $|s_n(\delta) - s_n(0)| \le 1/4$ and $|s_n(M_n - \delta) - s_n(M_n)| \le 1/4$. Thus if d and d' are two successive extrema in $[0, M_n]$ the absolute value of the average slope between $d + \delta$ and $d' - \delta$ is at least $\dfrac{m_n + 1/2}{1 - 2\delta} > \dfrac{1}{2}$ and is less that 2. Since $|s_n'(d+\delta)| = |s_n'(d-\delta)| = \frac{\lambda^r}{2}$ for any extremum $d \in]0, M_n[$, Property (8) can be fulfilled.

Secondly, recall that f_r is defined for $x \in [x_n, y_n]$ by

$$f_r(x) = \lambda^{-nr}\left[x_n + (y_n - x_n)s_n\left(M_n \frac{x - x_n}{y_n - x_n}\right)\right].$$

Now, we look at the C^r character of f_r near 1. The definition of f_r gives

$$f_r^{(k)}(x) = \frac{\lambda^{-nr} M_n^k}{(y_n - x_n)^{k-1}}\, s_n^{(k)}\left(M_n \frac{x - x_n}{y_n - x_n}\right) \quad \text{for } x \in [x_n, y_n],$$

where $f_r^{(k)}(x_n)$ and $f_r^{(k)}(y_n)$ are to be understood as left (resp. right) derivatives at this stage.

Since $M_n \le \frac{\lambda^n}{n^2}$, Property (7) leads to

$$|f_r^{(k)}(x)| \le \lambda^{-n(r-k)} n^{-2} 2^{k-1} A_k.$$

One has $A_1 = \lambda^r$ by (8), thus $|f'(x)| \leq \lambda^r$. Moreover, for $0 \leq k \leq r$,

$$|f_r^{(k)}(x)| \to 0 \text{ when } x \to 1,\ x \in \bigcup_{n\geq 1} [x_n, y_n].$$

Notice that the main factor in this estimate is $\lambda^{-n(r-k)}$. If $k > r$, the k-th derivative $f_r^{(k)}$ does not tend to zero any longer and it can be shown that f_r cannot be C^{r+1} at point 1.

As $f_r(x) = \lambda^r x$ for $x \in [0, y_1\lambda^{-r}]$, the $(n+1)$-th iterate of the map on $[x_n, y_n]$ is given by $f_r^{n+1}(x) = \lambda^{nr} f_r(x)$.

Notice that m_n is chosen such that $\min\{f_r^{n+1}(x) \colon x \in [x_n, y_n]\} = y_{n+1}$. Moreover $f_r^{n+1}(x_n) = x_n$ and $f_r^{n+1}(y_n) = y_n$.

We sum up the previous results in two lemmas, the first one is about derivatives and the second summarizes the behaviour of f_r on $[x_n, y_n]$.

Lemma 1.5

- $|f_r'(x)| \leq \lambda^r$ *for* $x \in [x_n, y_n]$.
- $\lim\limits_{\substack{x \to 1 \\ x \in \cup_{n\geq 1}[x_n, y_n]}} f_r^{(k)}(x) = 0$ *for* $0 \leq k \leq r$.

Lemma 1.6 *Let* $t_i^n = x_n + \dfrac{i(y_n - x_n)}{M_n}$ *for* $i = 0, \cdots, M_n$. *Then*

- f_r *is monotone on* $[t_{i-1}^n, t_i^n]$, $1 \leq i \leq M_n$.
- $f_r(t_i^n) = \lambda^{-nr} y_{n+1}$ *if* i *is even,* $i \neq 0$, *and* $f_r(x_n) = \lambda^{-nr} x_n$.
- $f_r(t_i^n) = \lambda^{-nr} y_n$ *if* i *is odd.*

1.3.2 On the subintervals $[y_{n+1}, x_n]$

We define

$$w_n = y_{n+1} + \frac{n+2}{2n(n+1)^2 M_{n+1} k_{n+1}}.$$

We have $w_n \in]y_{n+1}, x_n[$. On $[y_{n+1}, w_n]$, we define f_r to be affine of slope $\lambda^{-(n+1)r} M_{n+1} k_{n+1}$ (recall that $f_r(y_{n+1}) = \lambda^{-(n+1)r} y_{n+1}$ is already defined). Because of this definition f_r is affine (thus C^∞) in a neighbourhood of y_{n+1}. Moreover

$$f_r(w_n) = \lambda^{-(n+1)r} y_{n+1} + \lambda^{-(n+1)r} \frac{n+2}{2n(n+1)^2} = \lambda^{-(n+1)r} \left(1 + \frac{1}{n}\right),$$

so $f_r(w_n) = \lambda^{-(n+1)r} x_n$ and $f_r^{n+2}(w_n) = x_n$. As we are going to extend f_r in a C^∞ way on $[w_n, x_n]$, we will have

$$f_r'(x_n) = 2\lambda^{-(n-1)r},\ f_r'(w_n) = 2\lambda^{-nr},\ \text{and}\ f_r^{(k)}(x_n) = f_r^{(k)}(w_n) = 0 \text{ for } k \geq 2.$$

Set $h_n = f_r(x_n) - f_r(w_n)$ and $l_n = x_n - w_n$. We compute upper and lower bounds for h_n and l_n. First

$$h_n = \lambda^{-nr}\left(x_n - \lambda^{-r} y_{n+1} - \lambda^{-r}\frac{n+2}{2n(n+1)^2}\right) \leq 2\lambda^{-nr}.$$

We have

$$\frac{n+2}{2n(n+1)^2} = \frac{3}{8} \quad \text{for} \quad n = 1$$

and

$$\frac{n+2}{2n(n+1)^2} = \frac{n^2+2n}{2n^2(n+1)^2} \leq \frac{1}{2n^2} \leq \frac{1}{8} \quad \text{for} \quad n \geq 2.$$

Since $x_n \geq 1, y_{n+1} \leq y_2 = \frac{13}{8}$ and $\frac{n+2}{2n(n+1)^2} \leq \frac{3}{8}$ for all $n \geq 1$ one gets

$$h_n \geq \lambda^{-nr}(1 - 2\lambda^{-r}) \geq \frac{6}{7}\lambda^{-nr}.$$

For l_n one has

$$\begin{aligned} l_n &= 1 + \frac{1}{n} - 1 - \frac{1}{n+1} - \frac{1}{2(n+1)^2} - \frac{n+2}{2n(n+1)^2 M_{n+1} k_{n+1}} \\ &= \frac{n+2}{2n(n+1)^2}\left(1 - \frac{1}{2\lambda^r}\right). \end{aligned}$$

As $\frac{n+2}{2n(n+1)^2} \leq \frac{3}{8}$, one gets $l_n \leq \frac{3}{8}$ too. Moreover

$$\frac{n+2}{2n(n+1)^2} \geq \frac{1}{2(n+1)^2}$$

and $\frac{1}{2\lambda^r} \leq \frac{1}{2}$ thus $l_n \geq \frac{1}{4(n+1)^2}$. Finally we obtain the inequalities

$$\frac{6}{7}\lambda^{-nr} \leq h_n \leq 2\lambda^{-nr} \qquad \text{and} \qquad \frac{1}{4(n+1)^2} \leq l_n \leq \frac{3}{8}.$$

We normalize f_r on $[w_n, x_n]$ as follows: we define $\varphi_n\colon [0,1] \to [0,1]$ by

$$\varphi_n(x) = h_n^{-1}[f_r(w_n + l_n x) - f_r(w_n)].$$

The aim of this normalization is to check that the sequence φ_n can be chosen with uniformly bounded k-th derivatives then to come back to f_r and show that f_r is C^r at the right of 1. We want to have

$$\varphi_n'(1) = 2h_n^{-1} l_n \lambda^{-(n-1)r},\ \varphi_n'(0) = 2h_n^{-1} l_n \lambda^{-nr},$$

and

$$\varphi_n^{(k)}(0) = \varphi_n^{(k)}(1) = 0 \text{ for } k \geq 2,$$

thus $\varphi_n'(1) \leq \frac{7}{8}\lambda^r$ and $\varphi_n'(0) \leq \frac{7}{8}$. Consequently, it is possible to build a sequence of functions φ_n satisfying these conditions and the following additional conditions:

$$\forall k \geq 1,\ \exists B_k,\ \forall n \geq 1,\ \|\varphi_n^{(k)}\|_\infty \leq B_k$$

and

$$\forall x \in [0,1],\ \frac{2}{3}\varphi_n'(0) \leq \varphi_n'(x) \leq \lambda^r.$$

By definition of φ_n, the derivatives of f_r are given by

$$f_r^{(k)}(x) = h_n l_n^{-k} \varphi_n^{(k)}\left(\frac{x - w_n}{l_n}\right) \leq \lambda^{-nr}(n+1)^{2k} 2^{2k+1} B_k \text{ for } w_n \leq x \leq x_n,$$

hence for every $k \geq 0$

$$f_r^{(k)}(x) \to 0 \text{ when } x \to 1, x \in \bigcup_{n \geq 1} [y_{n+1}, x_n].$$

Moreover, $\frac{4}{3}\lambda^{-nr} \leq f_r'(x) \leq \lambda^r h_n l_n^{-1}$ for every $x \in [w_n, x_n]$ and

$$h_n l_n^{-1} \leq \frac{8(n+1)^2}{\lambda^{nr}} \leq 1 \text{ by Lemma 1.4 (i).}$$

The next lemma recalls the behaviour of f_r on $[y_{n+1}, x_n]$.

Lemma 1.7

- $\frac{4}{3}\lambda^{-nr} \leq f_r'(x) \leq \lambda^r$ *for* $x \in [y_{n+1}, x_n]$.
- $f_r(w_n) = \lambda^{-(n+1)r} x_n$.
- $\displaystyle\lim_{\substack{x \to 1 \\ x \in \bigcup_{n \geq 1}[y_{n+1}, x_n]}} f_r^{(k)}(x) = 0$ *for* $0 \leq k \leq r$.

1.3.3 Beginning of the proof of the mixing property

We show that any non degenerate subinterval $J \subset [1, y_1]$ satisfies (1) or (2) with $\mu_0 = \frac{4}{3}$. It is sufficient to consider $J \subset [x_n, y_n]$ or $J \subset [y_{n+1}, x_n]$.

First, we look at $[y_{n+1}, x_n]$. For $x \in [y_{n+1}, x_n]$, $f_r^{n+1}(x) = \lambda^{nr} f_r(x)$ and the derivative of f_r satisfies $f_r'(x) \geq \frac{4}{3}\lambda^{-nr}$ by Lemma 1.7, so $|f_r^{n+1}(J)| \geq \frac{4}{3}|J|$ if $J \subset [y_{n+1}, x_n]$.

Now, we focus on $[x_n, y_n]$. According to Property (8), $s_n'(x) \geq \min\{k_n, 1/2\}$ for all $x \in [M_n - 1 + \delta, M_n]$ thus

$$(f_r^{n+1})'(x) \geq \min\{M_n k_n, M_n/2\} \geq 2 \text{ for all } x \in \left[y_n - \frac{(y_n - x_n)(1-\delta)}{M_n}, y_n\right].$$

Because of Property (4), $s_n(M_n - 1 + \delta) = -m_n + 1/4 < 0$, thus

$$f_r^{n+1}\left(y_n - \frac{(y_n - x_n)(1-\delta)}{M_n}\right) < x_n.$$

Let $t_n = \dfrac{y_n - x_n}{\lambda^r M_n}$, then according to Lemma 1.2, there exists an integer α such that $f_r^{(n+1)\alpha}([y_n - t_n, y_n]) \supset [x_n, y_n]$, so there exists $z \in [y_n - t_n, y_n[$ with $f_r^{(n+1)\alpha}(z) = x_n$. Because of the choice of t_n and Property (5), f_r^{n+1} is affine of slope $k_n M_n = 2\lambda^r$ on $[y_n - t_n, y_n]$. Let $k \geq 0$ be the maximal integer i such that $\lambda^{ri}(y_n - z) \leq t_n$. Then $z_n = y_n - \lambda^{rk}(y_n - z)$ belongs to $\left[y_n - t_n, y_n - \frac{t_n}{2\lambda^r}\right]$ and $f_r^{(n+1)\alpha_n}(z_n) = x_n$ if $\alpha_n = \alpha + k$.

Set $\delta_n = \sqrt{\dfrac{(y_n - z_n)(y_n - x_n)}{CM_n^2}}$, and let a be a local maximum of f_r on $]x_n, y_n[$. If $|t| \leq \delta_n$, then

$$\left|\frac{M_n t}{y_n - x_n}\right|^2 \leq \frac{y_n - z_n}{C(y_n - x_n)} \leq \frac{1}{CM_n\lambda^r} \leq \delta^2.$$

Now we check the hypotheses of Lemma 1.3 for the extremum a:

- $f_r^{n+1}(a) = y_n$ and $f_r^{n+1}(y_n) = y_n$.
- $f_r^{n+1}(a + t) = y_n - \dfrac{C(M_n t)^2}{y_n - x_n}$ if $|t| \leq \delta_n$ (because of Property (4)).
- $f_r^{n+1}(a - \delta_n) = f_r^{n+1}(a + \delta_n) = z_n$.
- f_r^{n+1} is linear on $[z_n, y_n]$, with a slope $k_n M_n \geq 2$.
- $\dfrac{2|z_n - y_n|}{\delta_n} = 2\sqrt{\dfrac{CM_n^2(y_n - z_n)}{y_n - x_n}} \geq 2\sqrt{\dfrac{CM_n^2 t_n}{2\lambda^r(y_n - x_n)}} = 2\sqrt{\dfrac{M_n}{8}}$ and the last quantity is greater than 2 because $M_n \geq \lambda - 3$ by Lemma 1.4 (iii) and $\lambda \geq 14$.

Consequently, we can apply Lemma 1.3 at this maximum: for any non degenerate subinterval $J \subset [a - \delta_n, a + \delta_n]$, there exists k such that either $z_n \in \operatorname{Int}\left(f_r^k(J)\right)$ or $|f_r^k(J)| \geq 2|J|$. Since $f_r^{(n+1)\alpha_n}(z_n) = x_n$ and $f_r^{(n+1)\alpha_n}$ is a local homeomorphism in a neighbourhood of z_n, if $z_n \in \operatorname{Int}\left(f_r^k(J)\right)$ then $x_n \in \operatorname{Int}\left(f_r^{k'}(J)\right)$ with $k' = k + (n+1)\alpha_n$.

Set $\delta'_n = \sqrt{\dfrac{(w_n - y_{n+1})(y_n - x_n)}{CM_n^2}}$ and let b be a local minimum of f_r on

$]x_n, y_n[$. If $|t| \leq \delta'_n$, then

$$\left|\frac{M_n t}{y_n - x_n}\right|^2 \leq \frac{w_n - y_{n+1}}{C(y_n - x_n)} = \frac{2n(n+2)}{(n+1)^2 \lambda^{3r}} \leq \frac{2}{\lambda^{3r}} \leq \delta^2.$$

We check the hypotheses of Lemma 1.3 for the extremum b:

- $f_r^{n+1}(b) = y_{n+1}$ and $f_r^{n+2}(y_{n+1}) = y_{n+1}$.
- $f_r^{n+1}(b+t) = y_{n+1} + \dfrac{C(M_n t)^2}{y_n - x_n}$ if $|t| \leq \delta'_n$ (because of Property (4)).
- $f_r^{n+1}(b - \delta'_n) = f_r^{n+1}(b + \delta'_n) = w_n$ and $f_r^{n+2}(w_n) = x_n$.
- f_r^{n+2} is linear on $[y_{n+1}, w_n]$ of slope $M_{n+1}k_{n+1} \geq 2$.
- $\dfrac{2|y_{n+1} - w_n|}{\delta'_n} \geq 2$.

To prove the last point, define

$$C_n = \left(\frac{w_n - y_{n+1}}{\delta'_n}\right)^2 = \frac{n(n+2)M_n^2 \lambda^r}{8(n+1)^2}.$$

One has $M_n \geq \lambda - 3$ (Lemma 1.4 (iii)), $\lambda \geq 14$ and

$$\frac{2n(n+2)}{(n+1)^2} = \frac{(n+1)^2 + n^2 + 2n - 1}{(n+1)^2} > 1,$$

thus $C_n \geq \frac{14 \times 11^2}{16} > 1$.

Hence we can apply Lemma 1.3 to this extremum: for any non degenerate subinterval $J \subset [b - \delta'_n, b + \delta'_n]$, there exists k such that either $x_n \in \mathrm{Int}\left(f_r^k(J)\right)$ or $|f_r^k(J)| \geq 2|J|$.

If $|x - d| \geq \delta |y_n - x_n|/M_n$ for all local extrema $d \in]x_n, y_n[$, then $|(f_r^{n+1})'(x)| \geq \min\{2\lambda^r, M_n/2\} \geq 2$ according to Property (8). If $a \in]x_n, y_n[$ is a local maximum and $\delta_n \leq |x - a| \leq \dfrac{\delta |y_n - x_n|}{M_n}$, then

$$|(f_r^{n+1})'(x)| \geq |(f_r^{n+1})'(a + \delta_n)| = \frac{2M_n^2 C \delta_n}{y_n - x_n} \geq \sqrt{M_n/2}.$$

If $b \in]x_n, y_n[$ is a local minimum and $\delta'_n \leq |x - b| \leq \dfrac{\delta |y_n - x_n|}{M_n}$, then

$$|(f_r^{n+1})'(x)| \geq |(f_r^{n+1})'(b + \delta'_n)| = \frac{2M_n^2 C \delta'_n}{y_n - x_n} = \lambda^{\frac{r}{2}} \frac{M_n}{2} \sqrt{\frac{2n(n+2)}{(n+1)^2}} \geq \lambda^{\frac{r}{2}} M_n/2.$$

Consequently, $|(f_r^{n+1})'(x)| \geq 2$ if for all local maxima a, $|x-a| \geq \delta_n$ and for all local minima b, $|x-b| \geq \delta'_n$.

Finally, if J is a non degenerate subinterval of $[x_n, y_n]$, there exists k such that either $|f_r^k(J)| \geq 2|J|$ or $\mathrm{Int}\left(f_r^k(J)\right)$ contains x_n. Together with the previous result on $[y_{n+1}, x_n]$ it gives:

Lemma 1.8 *If J is a non degenerate subinterval of $[1, y_1]$, there exist $k \geq 0$ and $n \geq 1$ such that either $|f_r^k(J)| \geq \frac{4}{3}|J|$ or $x_n \in \mathrm{Int}\left(f_r^k(J)\right)$ or $y_n \in \mathrm{Int}\left(f_r^k(J)\right)$.*

The point x_n is periodic of period $n+1$, and $(f_r^{n+1})'(x) \geq 2$ for $x_n \leq x \leq x_n + \frac{y_n - x_n}{2M_n}$. In this situation, we can apply Lemma 1.2. For any interval $J = [x_n, x]$ with $x > x_n$ there exists k such that $f_r^k(J) \supset [x_n, x_n + \frac{y_n - x_n}{2M_n}]$. But

$$f_r^{n+1}\left(x_n + \frac{y_n - x_n}{2M_n}\right) \geq x_n + \frac{y_n - x_n}{M_n} \quad \text{and} \quad f_r^{n+1}\left(x_n + \frac{y_n - x_n}{M_n}\right) = y_n.$$

Hence $f_r^{k+2(n+1)}(J) \supset [x_n, y_n]$.

We do the same thing for y_n: for any interval $J = [y, y_n]$ with $y < y_n$ there exists k such that $f_r^k(J) \supset [x_n, y_n]$.

Moreover

$$f_r^{2(n+1)}([x_n, y_n]) = f_r^{n+1}([y_{n+1}, y_n]) = [\lambda^{-1} y_{n+1}, y_n] \supset [1/2, 1],$$

so $f_r^{2(n+1)+1}([x_n, y_n]) = [0, 4]$. This leads to the next lemma.

Lemma 1.9 *If J is an open subinterval with $x_n \in J$ or $y_n \in J$, then there exists $k \geq 0$ such that $f_r^k(J) = [0, 4]$.*

1.4 Construction of f_r on $[0,1]$ and $[y_1, 4]$ and end of the proof of the mixing property

Recall that $f_r(x) = \lambda^r x$ for $0 \leq x \leq \frac{5}{2}\lambda^{-r}$ and $\delta = \lambda^{-r}$. We define f_r near the points 1/2, 1 and 4 as follows:

- $f_r(x) = 4 - C_0(x - 1/2)^2$ for $|x - 1/2| \leq \delta$, with $C_0 = \frac{3}{2}\delta^{-1}$.
- $f_r(x) = C_1(x-1)^{\alpha_1}$ for $1 - \delta \leq x \leq 1$, with $\alpha_1 = 2r$ and $C_1 = \delta^{1-\alpha_1}$.
- $f(x) = 4 + \lambda^r(x - 4)$ for $4 - \frac{3}{2}\delta \leq x \leq 4$.

The definition of f_r on the left of 1, together with Lemmas 1.5 and 1.7, leads to the next lemma.

Lemma 1.10 *f_r is C^r in a neighbourhood of 1.*

Now we complete the map such that the pieces are glued together in a C^∞ way (except at 1 where f_r is only C^r). As $f_r'(1/2-\delta)=3$ and

$$\frac{f_r(1/2-\delta)-f_r(\frac{5}{2}\delta)}{(1/2-\delta)-\frac{5}{2}\delta}=\frac{3-3\lambda^{-r}}{1-7\lambda^{-r}}\in[2,6],$$

the map can be chosen such that $3/2\le f_r'(x)\le\lambda^r$ for every $x\in[\frac{5}{2}\delta,\frac{1}{2}-\delta]$. In the same way, it is possible to have $-\lambda^r\le f_r'(x)\le -3/2$ for every $x\in[1/2+\delta,1-\delta]$ because $f_r'(1/2+\delta)=-3$, $f_r'(1-\delta)=-2r$ and

$$\frac{f_r(1/2+\delta)-f_r(1-\delta)}{1/2-2\delta}=\frac{8-5\lambda^{-r}}{1-4\lambda^{-r}}\in[7,12].$$

Finally, $f_r'(y_1)=2$ because of the earlier construction of f_r on $[x_1,y_1]$ (see parag. 1.3.1) and

$$\frac{f_r(4-\frac{3}{2}\delta)-f_r(y_1)}{(4-\frac{3}{2}\delta)-y_1}=\frac{4-4\lambda^{-r}}{\frac{3}{2}-\frac{3}{2}\lambda^{-r}}=\frac{8}{3}.$$

Hence it is possible to have $\frac{3}{2}\le f_r'(x)\le\lambda^r$ for $y_1\le x\le 4$.

Consequently, $\frac{3}{2}\le|f_r'(x)|\le\lambda^r$ if $x\in[0,\frac{1}{2}-\delta]\cup[\frac{1}{2}+\delta,1-\delta]\cup[y_1,4]$.

A quick check shows that Lemma 1.3 can be applied to the two extrema 1/2 and 1 (we apply it only to the left of 1). For $z_0=1$, the repulsive periodic point is $z_1=0$, the interval $[z_1,z_2]$ is $[0,\lambda^{-r}]$, and the growth factor is $\frac{\alpha_0\delta}{\delta}=2r$. For $z_0=1/2$, the repulsive periodic point is $z_1=4$, the interval $[z_1,z_2]$ is $[4-\frac{3}{2}\lambda^{-r},4]$, and the growth factor is $\frac{2\delta}{\frac{3}{2}\lambda^{-r}}=3$.

Since $f_r^2(\lambda^{-r})=0$ and $f_r(4-\frac{3}{2}\lambda^{-r})=y_1$, for any non degenerate interval $J\subset[0,1]\cup[y_1,4]$ there exists k such that either $|f_r^k(J)|\ge\frac{3}{2}|J|$ or $f_r^k(J)$ contains one of the points $0,4,y_1$.

Lemma 1.11 *If J is a non degenerate subinterval of $[0,1]\cup[y_1,4]$, there exists $k\ge 0$ such that either $|f_r^k(J)|\ge\frac{3}{2}|J|$ or $0\in f_r^k(J)$ or $4\in f_r^k(J)$ or $y_1\in\mathrm{Int}\left(f_r^k(J)\right)$.*

Since $f_r^2([0,\lambda^{-r}])=[0,4]$ and $f_r^3([4-\frac{3}{2}\lambda^{-r},4])=f_r^2([y_1,4])=[0,4]$, applying Lemma 1.2 we obtain the next lemma.

Lemma 1.12 *If J is a non degenerate subinterval containing either 0 or 4, then there exists $k\ge 0$ such that $f_r^k(J)=[0,4]$.*

The construction of $f_r\colon[0,4]\to[0,4]$ is now finished. The map is C^r on $[0,4]$ (and is C^∞ on $[0,4]\setminus\{1\}$), and $\|f_r'\|_\infty=\lambda^r$. Furthermore, if we put together

Lemmas 1.8, 1.9, 1.11 and 1.12, we see that for any non degenerate subinterval $J \subset [0,4]$, there exists $k \geq 0$ such $f_r^k(J) = [0,4]$.

Proposition 1.13 *$f_r: I \to I$ is C^r, mixing and $\|f_r'\|_\infty = \lambda^r$.*

Remark 1.14 If we identify the two end points 0 and 4, the map f_r can be seen as a mixing C^r map on the torus, since $f_r^{(k)}(0) = f_r^{(k)}(4)$ for every $k \geq 1$.

2 Markov chain associated with f_r

We show that f_r is a Markov map for a suitable countable partition. The associated Markov chain reflects almost all topological properties of the system (I, f_r).

2.1 Definition of the graph

We explicit the Markov partition V_r and the associated graph G_r.

Let $t_0^n = x_n < t_1^n < \cdots < t_{M_n}^n = y_n$ the local extrema of f_r on $[x_n, y_n]$. Let

$$\begin{aligned} V_r &= \{[t_{i-1}^n, t_i^n]: 1 \leq n, 1 \leq i \leq M_n\} \\ &\cup \{[\lambda^{-kr}x_n, \lambda^{-kr}y_n]: 1 \leq k \leq n\} \\ &\cup \{[\lambda^{-kr}y_{n+1}, \lambda^{-kr}x_n]: 1 \leq n, 0 \leq k \leq n\} \\ &\cup \{[\lambda^{-nr}y_n, \lambda^{-(n-1)r}]: 2 \leq n\} \\ &\cup \{[\lambda^{-r}y_1, 1/2], [1/2, 1], [y_1, 4]\}. \end{aligned}$$

The elements of V_r have pairwise disjoint interior and their union is $]0,4]$. We check that the map f_r is monotone on each element of V_r and if $J \in V_r$ then $f_r(J)$ is a union of elements of $V_r \cup \{0\}$.

- By Lemma 1.6, f_r is monotone on $[t_{i-1}^n, t_i^n]$, $f_r([t_0^n, t_1^n]) = [\lambda^{-nr}x_n, \lambda^{-nr}y_n]$ and $f_r([t_{i-1}^n, t_i^n]) = [\lambda^{-nr}y_{n+1}, \lambda^{-nr}x_n] \cup [\lambda^{-nr}x_n, \lambda^{-nr}y_n]$ if $2 \leq i \leq M_n$.
- By Lemmas 1.6 and 1.7, f_r is increasing on $[y_{n+1}, x_n]$ for all $n \geq 1$ and $f_r([y_{n+1}, x_n]) = [\lambda^{-(n+1)r}y_{n+1}, \lambda^{-nr}x_n]$ $= [\lambda^{-(n+1)r}y_{n+1}, \lambda^{-nr}] \cup \bigcup_{k \geq n} [\lambda^{-nr}x_{k+1}, \lambda^{-nr}y_{k+1}] \cup [\lambda^{-nr}y_{k+1}, \lambda^{-nr}x_k]$.
- Since $f_r(x) = \lambda^r x$ for $x \in [0, \lambda^{-r}y_1]$ we have
  – $f_r([\lambda^{-kr}x_n, \lambda^{-kr}y_n]) = [\lambda^{-(k-1)r}x_n, \lambda^{-(k-1)r}y_n]$ for $1 \leq k \leq n$ and this interval is an element of V_r except $[x_n, y_n] = \bigcup_{i=1}^{M_n}[t_{i-1}^n, t_i^n]$ which is a union of elements of V_r.
  – $f_r([\lambda^{-kr}y_{n+1}, \lambda^{-kr}x_n]) = [\lambda^{-(k-1)r}y_{n+1}, \lambda^{-(k-1)r}x_n]$ for $1 \leq k \leq n$.

– $f_r([\lambda^{-(n+1)r}y_{n+1}, \lambda^{-nr}]) = [\lambda^{-nr}y_{n+1}, \lambda^{-(n-1)r}]$
$= [\lambda^{-nr}y_{n+1}, \lambda^{-nr}x_n] \cup [\lambda^{-nr}x_n, \lambda^{-nr}y_n] \cup [\lambda^{-nr}y_n, \lambda^{-(n-1)r}]$ for $n \geq 1$.

- f_r is monotone on $[0, 1/2]$, $[1/2, 1]$ and $[y_1, 4]$ (see Subsection 1.4) and
  – $f_r([\lambda^{-r}y_1, 1/2]) = [y_1, 4]$.
  – $f_r([1/2, 1]) = [0, 4] = \{0\} \cup \bigcup_{J \in V_r} J$.
  – $f_r([y_1, 4]) = [\lambda^{-r}y_1, 4]$
  $= [\lambda^{-r}y_1, 1/2] \cup [1/2, 1] \cup [y_1, 4] \cup \bigcup_{n\geq 1} [y_{n+1}, x_n] \cup \bigcup_{\substack{1 \leq n \\ 1 \leq i \leq M_n}} [t^n_{i-1}, t^n_i]$.

We define the directed graph G_r as follows: the set of vertices of G_r is V_r and there is an arrow from J to K if and only if $K \subset f_r(J)$. The decomposition above of $f_r(J)$ into elements of V_r for all $J \in V_r$ gives an exhaustive description of the arrows in G_r.

Notice that the graphs G_r are identical for all $r \geq 1$. The only difference is the name of the vertices, corresponding to the partition of f_r.

2.2 Isomorphism between f_r and the Markov chain

Let Γ_r^+ be the set of all one-sided infinite sequences $(D_n)_{n\geq 0}$ such that $D_n \in V_r$ and $D_n \to D_{n+1}$ $\forall n \in \mathbb{N}$, and let Γ_r be the set of all two-sided infinite sequences $(D_n)_{n\in\mathbb{Z}}$. We write σ for the shift transformation in both spaces. (Γ_r, σ) is called the **Markov chain associated with** f_r. As the systems (Γ_r, σ) are isomorphic for all $r \geq 1$, we just write (Γ, σ) when we want to talk about one of them without referring to the partition associated with f_r.

We are going to build an isomorphism modulo countable sets between (I, f_r) and (Γ_r^+, σ), that is a map $\phi_r: I \backslash \mathcal{N}_r \longrightarrow \Gamma_r^+ \backslash \mathcal{M}_r$ where $\mathcal{N}_r$, $\mathcal{M}_r$ are countable sets, ϕ_r is bijective bimeasurable (in fact bicontinuous) and $\phi_r \circ f_r = \sigma \circ \phi_r$.

Define

$$\begin{aligned}\mathcal{P}_r \;=\; & \{\lambda^{-kr}x_n, \lambda^{-kr}y_n: 1 \leq k \leq n\} \cup \{t^n_i: 1 \leq n, 0 \leq i \leq M_n\} \\ & \cup \{\lambda^{-nr}: 1 \leq n\} \cup \{0, 1/2, 1, 4\}\end{aligned}$$

and let $\mathcal{N}_r = \bigcup_{n\geq 0} f_r^{-n}(\mathcal{P}_r)$ which is countable. We have $f_r(\mathcal{N}_r) = \mathcal{N}_r$ and $f_r(I \backslash \mathcal{N}_r) = I \backslash \mathcal{N}_r$. If $x \in I \backslash \mathcal{P}_r$ then there is a unique $D \in V_r$ such that $x \in D$ (in fact $x \in \mathrm{Int}\,(D)$). Hence if $x \in I \backslash \mathcal{N}_r$, for every $n \geq 0$ there is a unique $D_n \in V$ such that $f_r^n(x) \in D_n$. Moreover $(D_n)_{n\geq 0} \in \Gamma_r^+$. We define

$$\begin{array}{rccc}\phi_r: & I \backslash \mathcal{N}_r & \longrightarrow & \Gamma_r^+ \\ & x & \mapsto & (D_n)_{n\geq 0}\end{array}$$

This application satisfies $\phi_r \circ f_r(x) = \sigma \circ \phi_r(x)$.

For any $(D_n)_{n\geq 0} \in \Gamma_r^+$, the set $J = \bigcap_{n\geq 0} f_r^{-n}(D_n)$ is a compact interval because f_r is monotone on each D_n. The map f_r is mixing (Proposition 1.13) and $f_r^n(J) \subset D_n$, hence J is necessarily reduced to a single point $\{x\}$. We define

$$\begin{array}{rrcl} \psi_r\colon & \Gamma_r^+ & \longrightarrow & I \\ & (D_n)_{n\geq 0} & \mapsto & x \end{array}$$

Let $\mathcal{M}_r = \psi_r^{-1}(\mathcal{N}_r)$. The application ψ_r, restricted to $\Gamma_r^+ \backslash \mathcal{M}_r$, is the inverse of ϕ_r. Moreover, both ϕ_r and ψ_r are continuous. Indeed, choose $x_0 \in I \backslash \mathcal{N}_r$ and write $(D_n)_{n\geq 0} = \phi_r(x_0)$ and $J_n = \bigcap_{k=0}^{n} f_r^{-k}(D_k)$. The diameters of the compact intervals J_n tend to 0, the point x_0 belongs to $\mathrm{Int}\,(J_n)$ for every n, and for every $x \in J_n \backslash \mathcal{N}_r$ the sequence $\phi_r(x)$ begins with $(D_0, \cdots, D_n)$. Hence ϕ_r is continuous. Inversely, fix $\gamma_0 = (D_n)_{n\geq 0} \in \Gamma_r^+ \backslash \mathcal{M}_r$, then for every sequence $\gamma \in \Gamma_r^+ \backslash \mathcal{M}_r$ beginning with $(D_0, \cdots, D_n)$ the point $\psi_r(\gamma)$ belongs to J_n which is an arbitrarily small neighbourhood of $\psi_r(\gamma_0)$. Hence ψ_r is continuous too.

Now, we are going to show that $\mathcal{M}_r$ is countable. It is sufficient to show that $\psi_r^{-1}(x)$ is finite for any $x \in \mathcal{N}_r$. For any $y \in I$ there are at most two elements of V_r containing y. Let $x \in \mathcal{N}_r$. If there is a k such that $f_r^k(x) = 0$ then $\psi_r^{-1}(x) = \emptyset$. If there is a k such that $f_r^k(x) = 4$ then $\psi^{-1}(x)$ is finite because $\psi^{-1}(4)$ contains only the constant sequence of symbol $[y_1, 4]$. Otherwise there exist k, n such that $f_r^k(x) = x_n$ or $f_r^k(x) = y_n$. Thus it is sufficient to focus on the points x_n and y_n.

We begin with x_n. The intervals $C_0 = [y_{n+1}, x_n]$ and $D_0 = [x_n, t_1^n]$ are the only two elements of V_r containing x_n. If we try to build $(C_k)_{k\geq 0}$ and $(D_k)_{k\geq 0}$ which are elements of $\psi_r^{-1}(x_n)$, we see that there are only two possibilities, which are cycles, namely:

- $C_0 = [y_{n+1}, x_n] \to C_1 = [\lambda^{-nr} y_{n+1}, \lambda^{-nr} x_n] \to \cdots \to C_{n+1} = C_0 \to \cdots$
- $D_0 = [x_n, t_1^n] \to D_1 = [\lambda^{-nr} x_n, \lambda^{-nr} y_n] \to \cdots \to D_{n+1} = D_0 \to \cdots$

Hence, $\mathrm{Card}\,\big(\psi_r^{-1}(x_n)\big) = 2$.

The situation is the same for $y_n, n \geq 2$, with two slightly different cycles, namely:

- $C_0 = [t_{M_n-1}^n, y_n] \to [\lambda^{-nr} x_n, \lambda^{-nr} y_n] \to \cdots \to C_{n+1} = C_0 \to \cdots$
- $D_0 = [y_n, x_{n-1}] \to [\lambda^{-nr} y_n, \lambda^{-(n-1)r}] \to \cdots \to D_{n+1} = D_0 \to \cdots$

A quick look at the map f_r gives the last two cycles for y_1.

Consequently, $\mathrm{Card}\,\big(\psi_r^{-1}(x)\big) < +\infty$ for every $x \in \mathcal{N}_r$, $\mathcal{M}_r$ is countable, and the map $\phi_r\colon I \backslash \mathcal{N}_r \longrightarrow \Gamma_r^+ \backslash \mathcal{M}_r$ is an isomorphism modulo countable sets.

ϕ_r transforms any invariant measure that does not charge $\mathcal{N}_r$ into an invariant measure that does not charge $\mathcal{M}_r$, and inversely. A measure supported by $\mathcal{N}_r$

or $\mathcal{M}_r$ is of zero entropy and the metric entropy $\mu \mapsto h_\mu$ is affine (see e.g. [11]), thus $h_{top}(f_r) = h(\Gamma_r^+, \sigma)$, where

$$h(\Gamma_r^+, \sigma) = sup\{h_\mu : \mu \text{ } \sigma\text{-invariant measure on } \Gamma_r^+\},$$

and ϕ_r establishes a bijection between the sets of maximal measures.

On the other hand, $h(\Gamma_r^+, \sigma) = h(\Gamma_r, \sigma)$ and there is a bijection between the maximal measures of (Γ_r^+, σ) and those of (Γ_r, σ), because the latter is the natural extension of the former (see e.g. [19]). Recall that all systems (Γ_r, σ) are identical and (Γ, σ) represents equally one of them. Hence the question of existence of maximal measure for (I, f_r) can be studied by looking at (Γ, σ).

Proposition 2.1 *$h_{top}(f_r) = h(\Gamma, \sigma)$ and (I, f_r) admits a maximal measure if and only if (Γ, σ) admits one.*

2.3 Non existence of maximal measure

Following the terminology of Vere-Jones [23] a transitive Markov chain is either transient, positive recurrent or null recurrent. According to a result of Gurevič [14], a transitive Markov chain has a maximal measure if and only if its graph is positive recurrent. We do not give the definitions of transience, positive recurrence and null recurrence because we will only need a criterion due to Salama (Theorem 2.1(i) in [22]), which is stated below.

If H is a (strongly) connected directed graph and (Γ_H, σ) is the associated Markov chain, i.e. the set of all sequences $(h_n)_{n\in\mathbb{Z}}$ with $h_n \to h_{n+1}$ in H, we define $h(H) = h(\Gamma_H, \sigma) = \sup\{h_\mu : \mu \text{ } \sigma\text{-invariant probability on } \Gamma_H\}$.

Theorem 2.2 **(Gurevič)** *Let H be a connected directed graph and (Γ_H, σ) be the associated Markov chain. If its entropy $h(H)$ is finite then (Γ_H, σ) admits a maximal measure if and only if H is positive recurrent. In this case, the measure is unique.*

Theorem 2.3 **(Salama)** *Let H be a connected directed graph. If there exists a graph H' such that $H \subsetneq H'$ and $h(H) = h(H')$ then H is transient.*

Next, we compute $h(G_r)$ then we show that G_r is transient, which is enough to conclude that f_r has no maximal measure by Proposition 2.1. As all graphs G_r are identical, it is sufficient to focus on G_1.

Proposition 2.4 *$h_{top}(f_r) = h(G_r) = \log \lambda$.*

Proof:

It is already known that $h_{top}(f_r) = h(G_r) = h(G_1)$ by Proposition 2.1.

A subset $E \subset I$ is called (n, ε)-separated for f_1 if for any two distinct points x, y in E there exists k, $0 \le k < n$, with $|f_1^k(x) - f_1^k(y)| > \varepsilon$. Let $s_n(f_1, \varepsilon)$ be the maximal cardinality of an (n, ε)-separated set. Then the topological entropy of f_1 is given by the following formula (see e.g. [11]):

$$h_{top}(f_1) = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log s_n(f_1, \varepsilon).$$

Let E be an (n, ε)-separated set of I of maximal cardinality. As $\|f_1'\|_\infty = \lambda$ (Proposition 1.13), we have $|f_1(x) - f_1(y)| \leq \lambda |x - y|$ for all $x, y \in I$. If x, y are two distinct points of E, there exists $k < n$ such that $|f_1^k(x) - f_1^k(y)| > \varepsilon$. But $|f_1^k(x) - f_1^k(y)| \leq \lambda^n |x - y|$, hence $|x - y| \geq \lambda^{-n} \varepsilon$ and

$$\mathrm{Card}\,(E) = s_n(f_1, \varepsilon) \leq \frac{\lambda^n}{\varepsilon} + 1.$$

Consequently, $h_{top}(f_1) = h(G_1) \leq \log \lambda$.

Now, let $H_n \subset G_1$ be the subgraph whose vertices are:

$$\{[t_{i-1}^n, t_i^n] \colon 1 \leq i \leq M_n\} \cup \{[\lambda^{-k} x_n, \lambda^{-k} y_n] \colon 1 \leq k \leq n\}.$$

The edges of H_n are all possible edges of G_1 between two vertices, namely:

- $[t_{i-1}^n, t_i^n] \to [\lambda^{-n} x_n, \lambda^{-n} y_n]$ for $1 \leq i \leq M_n$,
- $[\lambda^{-k} x_n, \lambda^{-k} y_n] \to [\lambda^{-k+1} x_n, \lambda^{-k+1} y_n]$ for $2 \leq k \leq n$,
- $[\lambda^{-1} x_n, \lambda^{-1} y_n] \to [t_{i-1}^n, t_i^n]$ for $1 \leq i \leq M_n$.

The graph H_n is represented in Figure 3.

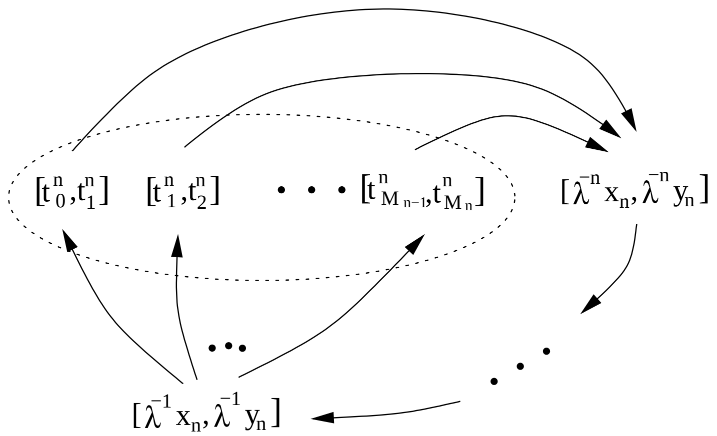

Figure 3: the graph H_n; σ^{n+1} is a full shift on the set of vertices inside the dots.

The system (H_n, σ^{n+1}) is a full shift on M_n symbols, plus n fixed points, thus $h(H_n, \sigma^{n+1}) = \log M_n$ (see e.g. [11, p111]) and $h(H_n) = \dfrac{\log M_n}{n+1}$.

By definition of M_n,

$$\lim_{n\to+\infty} \frac{\log M_n}{n+1} = \log \lambda.$$

As H_n is a subgraph of G_1, $h(H_n) \le h(G_1)$. Therefore $h(G_1) = \log \lambda$. □

Proposition 2.5 *The graph G_1 is transient.*

Proof:

We are going to build a Markov map g, very similar to f_1, such that $\|g'\|_\infty \le \lambda$ and the Markov graph H associated with g expands strictly G_1. Suppose g is already built. The same argument as in the proof of Proposition 2.4 shows that $h(H) \le \log \|g'\|_\infty \le \log \lambda$. As $G_1 \subset H$ we have $h(H) \ge h(G_1)$, thus $h(H) = h(G_1)$ by Proposition 2.4. This is enough to conclude that G_1 is transient by Theorem 2.3.

The map $g: I \to I$ is defined as $g(x) = f_1(x)$ for all $x \in I \setminus [x_2, y_2]$. Let

$$\widetilde{M_2} = M_2 + 2 \text{ and } \widetilde{k}_2 = \frac{2\lambda}{\widetilde{M_2}}$$

and choose $\widetilde{s}_2: [0, \widetilde{M_2}] \to [-m_2, 1]$ satisfying Properties (3)-(8) except that M_2 and k_2 are replaced respectively by $\widetilde{M_2}$ and $\widetilde{k}_2$. Then we define g on $[x_2, y_2]$ by

$$g(x) = \lambda^{-2} \left[x_2 + (y_2 - x_2)\widetilde{s}_2 \left(\widetilde{M_2} \frac{x - x_2}{y_2 - x_2} \right) \right].$$

By Properties (5) and (6), $g'(x_2) = g'(y_2) = \lambda^{-2}\widetilde{M_2}\widetilde{k}_2 = 2\lambda^{-1}$, thus $g'(x_2) = f_1'(x_2)$, $g'(y_2) = f_1'(y_2)$ and g is C^1. Moreover for all $x \in [x_2, y_2]$,

$$|g'(x)| \le \lambda^{-2}\widetilde{M_2}\|\widetilde{s}_2'\|_\infty \le \lambda^{-1}\widetilde{M_2}$$

thus $|g'(x)| < \lambda$ because $\widetilde{M_2} = M_2 + 2 = 2\left[\frac{\lambda}{8}\right] + 1 < \lambda^2$. Since $\|f_1'\|_\infty = \lambda$ by Proposition 1.13, one concludes that $\|g'\|_\infty \le \lambda$

Define the Markov graph H associated with g as in Subsection 2.1, and denote by W the set of vertices of H. Compared to V_1, W has two additional vertices because f_1 has M_2 monotone pieces between x_2 and y_2 and g has $M_2 + 2$. If

$$\widetilde{t}_i = x_2 + \frac{i(y_2 - x_2)}{M_2 + 2}$$

for $0 \le i \le M_2 + 2$ then it is not hard to check that the graph G_1 is equal to H deprived of the vertices $[\widetilde{t}_{M_2}, \widetilde{t}_{M_2+1}]$ and $[\widetilde{t}_{M_2+1}, \widetilde{t}_{M_2+2}]$ and all the edges that begin or end at one of them. Consequently $G_1 \subsetneq H$, which ends the proof. □

Remark 2.6 We can see intuitively what happens for an f_r-invariant measure when its entropy tends to $\log\lambda$. On each finite subgraph H_n, there is a measure of entropy $\frac{\log M_n}{n+1}$. This measure has a corresponding measure μ_n on the interval, the support of which is contained in $\bigcup_{k=0}^{n}[\lambda^{-kr}x_n, \lambda^{-kr}y_n]$ (in fact, the support of μ_n is exactly the Cantor set of all points which never escape from that set). We have of course $h_{\mu_n}(f_r) \to \log\lambda$. But if we consider what happens near 0, we see that μ_n converges to δ_0, the Dirac measure at 0, whose entropy is null.

3 Local entropy

We recall first some definitions due to Bowen [5] and then we define the local entropy. There exist different definitions of local entropy, we give here that of Buzzi [9].

Definition 3.1 Let $T\colon X \to X$ be a continuous map on a compact metric space X.

The **Bowen ball** of radius r and order n, centered at x is $B_n(x,r) = \{y \in X\colon d(T^k(y), T^k(x)) < r, \forall\, k = 0, \cdots, n-1\}$.

An (ε, n)**-separated set** of $Y \subset X$ is a subset $E \subset Y$ such that $\forall\, y \neq y'$ in E, $\exists\, 0 \leq k < n$, $d(T^k(y), T^k(y')) > \varepsilon$. The maximal cardinality of an (ε, n)-separated set of Y is denoted by $s_n(T, \varepsilon, Y)$.

Definition 3.2 The **local entropy** of T, $h_{loc}(T)$, is defined as

$$h_{loc}(T) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\sup_{x\in X}\log s_n(T, \delta, B_n(x, \varepsilon)).$$

Remark 3.3 An (ε, n)-cover of $Y \subset X$ is a subset $S \subset X$ such that $Y \subset \bigcup_{x\in S} B_n(x, \varepsilon)$. Some people use (ε, n)-covers instead of (ε, n)-separated sets: it leads to the same definition of the local entropy.

Local entropy is interesting because it bounds the defect of upper semicontinuity of the metric entropy $\mu \mapsto h_\mu(f)$. On a compact Riemannian m-dimensional manifold, local entropy itself is bounded by $\frac{m\log R(f)}{r}$, where $R(f)$ is the spectral radius of the differential and r is the differential order. These results are stated by Buzzi [9] and follow works of Yomdin [24] and Newhouse [18]. In particular, they directly imply that a C^∞ map on a compact Riemannian manifold always has a maximal measure (this result can be found in Newhouse's work [18]). These results are given in the next two theorems, the second one is stated for interval maps only.

Theorem 3.4 *Let $T\colon X \to X$ be a continuous map on a compact metric space.*

Assume that μ_n is a sequence of T-invariant measures on X, converging to a measure μ. Then

$$\limsup_{n\to\infty} h_{\mu_n}(T) \le h_\mu(T) + h_{loc}(T).$$

Theorem 3.5 *Let $f: I \to I$ be a C^r map on a compact interval I, $r \ge 1$, and let $R(f) = \inf_{k\ge1} \sqrt[k]{\|(f^k)'\|_\infty}$. Then the local entropy satisfies*

$$h_{loc}(f) \le \frac{\log R(f)}{r}.$$

In our family of examples, the local entropy can be computed explicitly.

Proposition 3.6 *For every $n \ge 1$ the local entropy of f_r is*

$$h_{loc}(f_r) = \frac{\log R(f_r)}{r} = \log\lambda.$$

Proof:

The map f_r is such that $\|f_r'\|_\infty \le \lambda^r$ (Proposition 1.13) and 0 is a fixed point with $f_r'(0) = \lambda^r$. Hence $R(f_r) = \lambda^r$ and

$$h_{loc}(f_r) \le \frac{\log R(f_r)}{r} = \log\lambda$$

according to Theorem 3.5.

We are going to show the reverse inequality.

Fix $\varepsilon > 0$ and choose n such that $\frac{1}{2n^2} < \varepsilon$. Put $\delta_0 = \frac{1}{2n^2 M_n}$. If $x \in [x_n, y_n]$ satisfies $f^{n+1}(x) \in [x_n, y_n]$ then $|f^i(x) - f^i(x_n)| < \varepsilon$ for $0 \le i \le n+1$. We write $I_i = [t_{i-1}^n, t_i^n]$ for $1 \le i \le M_n$. The length of each I_i is δ_0.

Choose a finite sequence $\omega = (\omega_0, \cdots, \omega_{p-1})$ with $1 \le \omega_i \le M_n$. Thanks to the isomorphism between (I, f_r) and its Markov extension (Section 2), there is a point $x_\omega \in [x_n, y_n]$ with $f^{(n+1)i}(x_\omega) \in I_{\omega_i}$ for $0 \le i \le p-1$. Consider the set $E_{n,p} = \{x_\omega : \omega = (\omega_0, \cdots, \omega_{p-1}), \omega_i \text{ odd}\}$. The cardinality of $E_{n,p}$ is

$$\left(\frac{M_n+1}{2}\right)^p \ge \left(\frac{\lambda^n}{4n^2}\right)^p$$

by Lemma 1.4 (ii). If $x \in E_{n,p}$ then $|f^k(x_n) - f^k(x)| < \varepsilon$ for $0 \le k < (n+1)p$. Moreover, if $x_\omega, x_{\omega'}$ are two distinct elements of $E_{n,p}$, then there exists $0 \le i \le p-1$ with $|\omega_i - \omega_i'| \ge 2$, hence $|f^{(n+1)i}(x_\omega) - f^{(n+1)i}(x_{\omega'})| \ge \delta_0$. Consequently, $E_{n,p}$ is an $((n+1)p, \delta, B_{(n+1)p}(x_n, \varepsilon))$-separated set for every $\delta < \delta_0$, and

$$h_{loc}(f_r) \ge \lim_{n\to+\infty} \limsup_{p\to\infty} \frac{\log \mathrm{Card}\,(E_{n,p})}{(n+1)p} \ge \log\lambda.$$

□

This computation shows that the bound $\frac{\log R(f)}{r}$ is a sharp one to estimate the local entropy. Moreover, we remarked (Remark 2.6) that there exists a sequence of measures μ_n converging to the Dirac measure δ_0, with $h_{\mu_n}(f_r) \to h_{top}(f_r)$. Hence, the local entropy is exactly the defect of upper semicontinuity of the metric entropy in this case.

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