Interval maps of given topological entropy and Sharkovskii’s type

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Abstract

It is known that the topological entropy of a continuous interval map $f$ is positive if and only if the type of $f$ for Sharkovskii’s order is $2^d p$ for some odd integer $p \geq 3$ and some $d \geq 0$; and in this case the topological entropy of $f$ is greater than or equal to $\log \frac{\lambda_p}{2^d}$, where $\lambda_p$ is the unique positive root of $X^p - 2X^{p-2} - 1$. For every odd $p \geq 3$, every $d \geq 0$ and every $\lambda \geq \lambda_p$, we build a piecewise monotone continuous interval map that is of type $2^d p$ for Sharkovskii’s order and whose topological entropy is $\log \frac{\lambda}{2^d}$. This shows that, for a given type, every possible finite entropy above the minimum can be reached provided the type allows the map to have positive entropy. Moreover, if $d = 0$ the map we build is topologically mixing.

1 Introduction

In this paper, an interval map is a continuous map $f: I \to I$ where $I$ is a compact nondegenerate interval. A point $x \in I$ is periodic of period $n$ if $f^n(x) = x$ and $n$ is the least positive integer with this property, i.e. $f^k(x) \neq x$ for all $k \in [1, n - 1]$.

Let us recall Sharkovskii’s theorem and the definitions of Sharkovskii’s order and type [6] (see e.g. [5, Section 3.3]).

Definition 1.1 Sharkovskii’s order is the total ordering on $\mathbb{N}$ defined by:

$$3 < 5 < 7 < 9 < \cdots < 2 \cdot 3 < 2 \cdot 5 < \cdots < 2^2 \cdot 3 < 2^2 \cdot 5 < \cdots < 2^3 < 2^2 < 2 < 1$$

(first, all odd integers $n > 1$, then 2 times the odd integers $n > 1$, then successively $2^2 \times$, $2^3 \times$, $\ldots$, $2^k \times \ldots$ the odd integers $n > 1$, and finally all the powers of 2 by decreasing order).

$a \triangleright b$ means $b \triangleleft a$. The notation $\leq, \geq$ will denote the order with possible equality.

Theorem 1.2 (Sharkovskii) If an interval map $f$ has a periodic point of period $n$ then, for all integers $m \triangleright n$, $f$ has periodic points of period $m$.

Definition 1.3 Let $n \in \mathbb{N} \cup \{2^\infty\}$. An interval map $f$ is of type $n$ (for Sharkovskii’s order) if the periods of the periodic points of $f$ form exactly the set $\{m \in \mathbb{N} | m \triangleright n\}$, where the notation $\{m \in \mathbb{N} | m \triangleright 2^\infty\}$ stands for $\{2^k | k \geq 0\}$.

It is well known that an interval map $f$ has positive topological entropy if and only if its type is $2^d p$ for some odd integer $p \geq 3$ and some $d \geq 0$ (see e.g. [5, Theorem 4.58]). The entropy of such a map is bounded from below (see theorem 4.57 in [5]).

Theorem 1.4 (Ștefan, Block-Guckenheimer-Misiurewicz-Young) Let $f$ be an interval map of type $2^d p$ for some odd integer $p \geq 3$ and some $d \geq 0$. Let $\lambda_p$ be the unique positive root of $X^p - 2X^{p-2} - 1$. Then $\lambda_p > \sqrt{2}$ and $h_{\text{top}}(f) \geq \frac{\log \lambda_p}{2^d}$. 

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This bound is sharp: for every $p, d$, there exists an interval map of type $2^d p$ and topological entropy $\frac{\log \lambda_p}{2^d}$. These examples were first introduced by Štefan, although the entropy of these maps was computed later [7, 2].

Moreover, it is known that the type of a topological mixing interval map is $p$ for some odd integer $p \geq 3$ (see e.g. [5, Proposition 3.36]). The Štefan maps of type $p$ are topologically mixing [5, Example 3.21].

We want to show that the topological entropy of a piecewise monotone map can be equal to any real number, the lower bound of Theorem 1.4 being the only restriction. First, for every odd integer $p \geq 3$ and every real number $\lambda \geq \lambda_p$, we are going to build a piecewise monotone map $f_{p, \lambda} : [0, 1] \to [0, 1]$ such that its type is $p$ for Sharkovskii’s order, its topological entropy is $\log \lambda$, and the map is topologically mixing. Then we will show that for every odd integer $p \geq 3$, every integer $d \geq 0$ and every real number $\lambda \geq \lambda_p$, there exists a piecewise monotone interval map $f$ such that its type is $2^d p$ for Sharkovskii’s order and its topological entropy is $\frac{\log \lambda}{2^d}$.

1.1 Notations

We say that an interval is degenerate if it is either empty or reduced to one point, and nondegenerate otherwise. When we consider an interval map $f : I \to I$, every interval is implicitly a subinterval of $I$.

Let $J$ be a nonempty interval. Then $\partial J := \{\inf J, \sup J\}$ are the endpoints of $J$ (they may be equal if $J$ is reduced to one point) and $|J|$ denotes the length of $J$ (i.e. $|J| := \sup J - \inf J$). Let $\text{mid}(J)$ denote the middle point of $J$, that is, $\text{mid}(J) := \frac{\inf J + \sup J}{2}$.

An interval map $f : I \to I$ is piecewise monotone if there exists a finite partition of $I$ into intervals such that $f$ is monotone on each element of this partition.

An interval map $f$ has a constant slope $\lambda$ if $f$ is piecewise monotone and if on each of its pieces of monotonicity $f$ is linear and the absolute value of the slope coefficient is $\lambda$.

2 Štefan maps

We recall the definition of the Štefan maps of odd type $p \geq 3$.

Let $n \geq 1$ and $p := 2n + 1$. The Štefan map $f_p : [0, 2n] \to [0, 2n]$, represented in Figure 1, is defined as follows: it is linear on $[0, n - 1], [n - 1, n], [n, 2n - 1]$ and $[2n - 1, 2n]$, and $f_p(0) := 2n, f_p(n - 1) := n + 1, f_p(n) := n - 1, f_p(2n - 1) := 0, f_p(2n) := n$.

Note that $n = 1$ is a particular case because $0 = n - 1$ and $n = 2n - 1$.

Next proposition summarises the properties of $f_p$, see [5, Example 3.21] for the proof.

Proposition 2.1 The map $f_p$ is topologically mixing, its type for Sharkovskii’s order is $p$ and $h_{\text{top}}(f) = \log \lambda_p$. Moreover, the point $n$ is periodic of period $p$, and $f_p^{2k - 1}(n) = n - k$ and $f_p^{2k}(n) = n + k$ for all $k \in \{1, n\}$.

3 Mixing map of given entropy and odd type

For every odd integer $p \geq 3$ and every real number $\lambda \geq \lambda_p$, we are going to build a piecewise monotone continuous map $f_{p, \lambda} : [0, 1] \to [0, 1]$ such that its type is $p$ for Sharkovskii’s order, its topological entropy is $\log \lambda$, and the map is topologically mixing. We will write $f$ instead of $f_{p, \lambda}$ when there is no ambiguity on $p, \lambda$. 

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The idea is the following: we start with the Štefan map \( f_p \), we blow up the minimum into an interval and we define the map of this interval in such a way that the added dynamics increases the entropy without changing the type. At the same time, we make the slope constant and equal to \( \lambda \), so that the entropy is \( \log \lambda \) according to the following theorem [1, Corollary 4.3.13], which is due to Misiurewicz-Szlenk [4], Young [8] and Milnor-Thurston [3].

**Theorem 3.1** Let \( f \) be a piecewise monotone interval map. Suppose that \( f \) has a constant slope \( \lambda \geq 1 \). Then \( h_{\text{top}}(f) = \log \lambda \).

We will also need the next result (see the proof of Lemma 4.56 in [5]).

**Lemma 3.2** Let \( p \geq 3 \) be an odd integer and \( P_p(X) := X^p - 2X^{p-2} - 1 \). Then \( P_p(X) \) has a unique positive root, denoted by \( \lambda_p \). Moreover, \( P_p(x) < 0 \) if \( x \in [0, \lambda_p] \) and \( P_p(x) > 0 \) if \( x > \lambda_p \).

Let \( \chi_p(X) := X^{p-1} - X^{p-2} - \sum_{i=0}^{p-2} (-X)^i \). Then \( P_p(X) = (X+1)\chi_p(X) \), and thus \( \chi_p(x) < 0 \) if \( x \in [0, \lambda_p] \) and \( \chi_p(x) > 0 \) if \( x > \lambda_p \).

### 3.1 Definition of the map

We fix an odd integer \( p \geq 3 \) and a real \( \lambda \geq \lambda_p \). Recall that \( \lambda_p > \sqrt{2} > 1 \) (Theorem 1.4).

We are going to define points ordered as follows:

\[
x_{p-2} < x_{p-4} < \cdots < x_1 < x_0 < x_2 < x_4 < \cdots < x_{p-3} \leq t < x_{p-1},
\]

with \( x_{p-2} = 0, \ x_{p-3} = \frac{1}{\lambda} \) and \( x_{p-1} = 1 \).

The points \( x_0, \ldots, x_{p-1} \) will form a periodic orbit of period \( p \), that is, \( f(x_i) = x_{i+1 \mod p} \) for all \( i \in [0, p-1] \).

**Remark 3.3** In the following construction, the case \( p = 3 \) is degenerate. The periodic orbit is reduced to \( x_1 < x_0 < x_2 \) with \( x_1 = 0, x_0 = \frac{1}{\lambda}, x_2 = 1 \). We only have to determine the value of \( t \).

The map \( f: [0,1] \to [0,1] \) is defined as follows (see Figure 2):

- \( f(x) := 1 - \lambda x \) for all \( x \in [0, \frac{1}{\lambda}] = [x_{p-2}, x_{p-3}] \) (so that \( f(0) = 1 \) and \( f(\frac{1}{\lambda}) = 0 \),

Figure 1: On the left: the map \( f_3 \). On the right: the map \( f_p \) with \( p = 2n + 1 > 3 \).
\begin{itemize}

- \( f(x) := \lambda(x-t) \) for all \( x \in [t, 1] \) (so that \( f(t) = 0 \) and \( f(1) = \lambda(1-t) \)),

- definition on \([\frac{1}{\lambda}, t]\): we want to have \( f([\frac{1}{\lambda}, t]) \subset [0, x_{p-4}] \) (note that \( x_{p-4} \) is the least positive point among \( x_0, \ldots, x_{p-1} \)), with \( f \) of constant slope \( \lambda \), in such a way that all the critical points except at most one are sent by \( f \) on either 0 or \( x_{p-4} \). If \( t = \frac{1}{\lambda} \), there is nothing to do. If \( t > \frac{1}{\lambda} \), we set \( \ell := t - \frac{1}{\lambda} \) (length of the interval), \( k := \left\lfloor \frac{\lambda t}{2x_{p-4}} \right\rfloor \),

\[
J_i := \left[ \frac{1}{\lambda} + (i-1) \frac{2x_{p-4}}{\lambda}, \frac{1}{\lambda} + i \frac{2x_{p-4}}{\lambda} \right] \quad \text{for all } i \in [1, k],
\]

\[
K := \left[ \frac{1}{\lambda} + k \frac{2x_{p-4}}{\lambda}, t \right].
\]

If \( p = 3 \), we replace \( x_{p-4} \) (not defined) by 1 in the above definitions.

It is possible that there is no interval \( J_1, \ldots, J_k \) (if \( k = 0 \)) or that \( K \) is reduced to the point \( \{t\} \). On each interval \( J_1, \ldots, J_k \), \( f \) is defined as the tent map of summit \( x_{p-4} \): \( f(\min J_i) = 0 \), \( f \) is increasing of slope \( \lambda \) on \( [\min J_i, \mid \text{mid}(J_i) \mid] \) (thus \( f(\mid \text{mid}(J_i) \mid) = x_{p-4} \) because of the length of \( J_i \)), then \( f \) is decreasing of slope \(-\lambda \) on \( [\mid \text{mid}(J_i) \mid, \max J_i] \) and \( f(\max J_i) = 0 \). On \( K \), \( f \) is defined as a tent map with a summit \( < x_{p-4} \): \( f(\min K) = 0 \), \( f \) is increasing of slope \( \lambda \) on \( [\min K, \mid \text{mid}(K) \mid] \), then \( f \) is decreasing of slope \(-\lambda \) on \( [\mid \text{mid}(K) \mid, \max K] \) and \( f(\max K) = 0 \).

In this way, we get a map \( f \) that is continuous on \([0, 1]\), piecewise monotone, of constant slope \( \lambda \). It remains to define \( t \) and the points \( \{x_i\}_{0 \leq i \leq p-4} \) (recall that \( x_{p-3} = \frac{1}{\lambda}, x_{p-2} = 0 \) and \( x_{p-1} = 1 \)).

We want these points to satisfy:

\[
x_0 = \lambda(1-t)
\]

and

\[
\begin{aligned}
  x_1 &= 1 - \lambda x_0 \\
  x_2 &= 1 - \lambda x_1 \\
  \vdots \\
  x_{p-3} &= 1 - \lambda x_{p-4}
\end{aligned}
\]

and to be ordered as follows:

\[
x_{p-2} < x_{p-4} < \cdots < x_1 < x_0 < x_2 < x_4 < \cdots < x_{p-3}
\]

\[
\frac{1}{\lambda} \leq t < x_{p-1}.
\]

If \( p = 3 \), the system (4) is empty, and equation (5) is satisfied because it reduces to \( 0 = x_1 < x_0 = \frac{1}{\lambda} \).

According to the definition of \( f \), the equations (3), (4), (5), (6) imply that \( f(x_i) = x_{i+1} \) for all \( i \in [0, p-2] \) and \( f(x_{p-1}) = x_0 \).

We are going to show that the system (4) is equivalent to:

\[
\forall i \in [0, p-4], \quad x_i = \frac{(-1)^i}{\lambda^{p-1}} \sum_{j=0}^{p-i-3} (-\lambda)^j.
\]

We use a descending induction on \( i \).

- According to the last line of (4), \( x_{p-4} = \frac{1}{\lambda}(1-x_{p-3}) = \frac{1}{\lambda}(\lambda-1) \). This is (7) for \( i = p-4 \).
• Suppose that (7) holds for \( i \) with \( i \in \mathbb{I}_{1, p - 4} \). By (4), \( x_i = 1 - \lambda x_{i-1} \), thus

\[
x_{i-1} = \frac{1}{\lambda}(x_i - 1)
= \frac{(-1)^i}{\lambda^{p-i-1}} \left( \sum_{j=0}^{p-i-3} (-\lambda)^j - (-1)^i \lambda^{p-i-2} \right)
\]

Since \( p \) is odd, \( -(-1)^i \lambda^{p-i-2} = (-\lambda)^{p-i-2} \). Hence

\[
x_{i-1} = \frac{(-1)^{i-1}}{\lambda^{p-i-1}} \sum_{j=0}^{p-i-2} (-\lambda)^j,
\]

which gives (7) for \( i - 1 \). This ends the proof of (7).
Equation (3) is equivalent to $t = 1 - \frac{1}{\lambda} x_0$. Thus, using (7), we get

$$ t = \frac{1}{\lambda^{p-1}} \left( \lambda^{p-1} - \sum_{j=0}^{p-3} (-\lambda)^j \right). \quad (8) $$

Conclusion: with the values of $x_0, \ldots, x_{p-4}$ and $t$ given by (7) and (8), the system of equations (3)-(4) is satisfied (and there is a unique solution). It remains to show that these points are ordered as stated in (5) and (6).

Let $i$ be in $\{0, p-6\}$. By (7), we have

$$ x_{i+2} - x_i = \frac{(-1)^i}{\lambda^{p-2-2}} \left( \lambda^2 \sum_{j=0}^{p-i-5} (-\lambda)^j - \sum_{j=0}^{p-i-3} (-\lambda)^j \right) $$

$$ = \frac{(-1)^i}{\lambda^{p-2}} \left( \sum_{j=2}^{p-i-3} (-\lambda)^j - \sum_{j=0}^{p-i-3} (-\lambda)^j \right) $$

$$ = \frac{(-1)^i}{\lambda^{p-2}} (\lambda - 1) $$

Since $\lambda - 1 > 0$, we have, for all $i \in \{0, p-6\}$,

- $x_i < x_{i+2}$ if $i$ is even,
- $x_{i+2} < x_i$ if $i$ is odd.

By (7), $x_{p-4} = \frac{\lambda^2 - \lambda}{\lambda^2}$. Since $\lambda > 1$, $x_{p-4} > 0 = x_{p-2}$. Again by (7),

$$ x_0 - x_1 = \frac{1}{\lambda^{p-2}} \left( \sum_{j=0}^{p-3} (-\lambda)^j + \lambda \sum_{j=0}^{p-4} (-\lambda)^j \right) $$

$$ = \frac{1}{\lambda^{p-2}} \left( \sum_{j=0}^{p-3} (-\lambda)^j - \sum_{j=1}^{p-3} (-\lambda)^j \right) $$

$$ = \frac{1}{\lambda^{p-2}} \cdot \chi_{p-1}(\lambda) > 0 $$

thus $x_0 < x_1$. Moreover,

$$ x_{p-3} - x_{p-5} = \frac{1}{\lambda} - \frac{\lambda^2 - \lambda + 1}{\lambda^3} = \frac{\lambda - 1}{\lambda^3} > 0 $$

thus $x_{p-5} < x_{p-3}$. This several inequalities imply (5).

By (8), we have

$$ t - \frac{1}{\lambda} = \frac{1}{\lambda^{p-1}} \left( \lambda^{p-1} - \lambda^{p-2} - \sum_{j=0}^{p-3} (-\lambda)^j \right) = \frac{1}{\lambda^{p-1}} \cdot \chi_p(\lambda), $$

where $\chi_p$ is defined in Lemma 3.2. According to this lemma, $\chi_p(\lambda) \geq 0$ (with equality iff $\lambda = \lambda_p$) because $\lambda \geq \lambda_p$. This implies that $t \geq \frac{1}{\lambda}$ (with equality iff $\lambda = \lambda_p$). Moreover, if $t \geq 1$, then $x_0 = \lambda(1-t) \leq 0$, which is impossible by (5); thus $t < 1$. Therefore, the inequalities (6) hold.

Finally, we have shown that the map $f_{p,\lambda} = f$ is defined as wanted.
3.2 Entropy

Corollary 3.4 \( h_{\text{top}}(f_{p,\lambda}) = \log \lambda \).

Proof. This result is given by Theorem 3.1 because, by definition, \( f_{p,\lambda} \) is piecewise monotone of constant slope \( \lambda \) with \( \lambda > 1 \). \( \square \)

3.3 Type

Lemma 3.5 Let \( g: [0,1] \to [0,1] \) be a continuous map. Let \( A \) be a finite family of closed intervals that form a pseudo-partition of \( [0,1] \), that is,

\[
\bigcup_{A \in \mathcal{A}} A = [0,1] \quad \text{and} \quad \forall A, B \in \mathcal{A}, A \neq B \Rightarrow \text{Int } (A) \cap \text{Int } (B) = \emptyset.
\]

We set \( \partial \mathcal{A} = \bigcup_{A \in \mathcal{A}} \partial A \). Let \( \mathcal{G} \) be the oriented graph whose vertices are the elements of \( \mathcal{A} \) and in which there is an arrow \( A \rightarrow B \) iff \( g(A) \cap \text{Int } (B) \neq \emptyset \). Let \( x \) be a periodic point of period \( q \) for \( g \) such that \( \{g^n(x) \mid n \geq 0\} \cap \partial \mathcal{A} = \emptyset \). Then there exist \( A_0, \ldots, A_{q-1} \in \mathcal{A} \) such that \( A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_{q-1} \rightarrow A_0 \) is a cycle in the graph \( \mathcal{G} \).

Proof. For every \( n \geq 0 \), there exists a unique element \( A_n \in \mathcal{G} \) such that \( g^n(x) \in \text{Int } (A_n) \) because \( \{g^n(x) \mid n \geq 0\} \cap \partial \mathcal{A} = \emptyset \). We have \( g^n(x) \in A_n \) and \( g^{n+1}(x) \in \text{Int } (A_{n+1}) \), thus \( g(A_n) \cap \text{Int } (A_{n+1}) \neq \emptyset \); in other words, there is an arrow \( A_n \rightarrow A_{n+1} \) in \( \mathcal{G} \). Finally, \( A_q = A_0 \) because \( g^q(x) = x \). \( \square \)

Proposition 3.6 The map \( f_{p,\lambda} \) is of type \( p \) for Sharkovskii’s order.

Proof. According to the definition of \( f = f_{p,\lambda} \), \( x_0 \) is a periodic point of period \( p \). It remains to show that \( f \) has no periodic point of period \( q \) with \( q \) odd and \( 3 \leq q < p \).

We set \( I_1 := (x_0, x_1) \), \( I_i := (x_{i-2}, x_i) \) for all \( i \in \mathbb{Z} \) and \( I_{p-1} := [t, 1] \), where \( (a,b) \) denotes the convex hull of \( \{a,b\} \) (i.e. \( \langle a,b \rangle = [a,b] \) or \( [b,a] \)). The intervals \( J_i, K \) have been defined in (1) and (2). The family \( A := \{I_1, \ldots, I_{p-1}, J_1, \ldots, J_k, K\} \) is a pseudo-partition of \( [0,1] \). Let \( \mathcal{G} \) be the oriented graph associated to \( \mathcal{A} \) for the map \( g = f \) as defined in Lemma 3.5. If \( f(A) \supset B \), the arrow \( A \rightarrow B \) is replaced by \( A \to B \) (full covering). The graph \( \mathcal{G} \) is represented in Figure 3; a dotted arrow \( A \rightarrow B \) means that \( f(A) \cap \text{Int } (B) \neq \emptyset \) but \( f(A) \not\supset B \) (partial covering).

Figure 3: Covering graph \( \mathcal{G} \) associated to \( f \).

The subgraph associated to the intervals \( I_1, \ldots, I_{p-1} \) is the graph associated to a Štefán cycle of period \( p \) (see [5, Lemma 3.16]). The only additional arrows with respect to the Štefán graph are between the intervals \( J_1, \ldots, J_k, K \) on the one hand and \( I_{p-2} \) on the other hand. There is only one partial covering, which is \( K \rightarrow I_{p-2} \).

Let \( q \) be an odd integer with \( 3 \leq q < p \). We easily see that, in this graph, there is no primitive cycle of length \( q \) (a cycle is primitive if it is not the repetition of a shorter cycle): the cycles not passing through \( I_1 \) have an even length, whereas the cycles passing through \( I_1 \) have...
a length either equal to 1, or greater than or equal to \(p - 1\). Moreover, if \(x\) is a periodic point of period \(q\), then \(\{f^n(x) \mid n \geq 0\} \cap \partial A = \emptyset\) (because the periodic points in \(\partial A\) are of period \(p\)). According to Lemma 3.5, \(f\) has no periodic point of period \(q\). Conclusion: \(f\) is of type \(p\) for Sharkovskii’s order. \(\square\)

3.4 Mixing

**Proposition 3.7** The map \(f_{p, \lambda}\) is topologically mixing.

**Proof.** This proof is inspired by [5, Lemmas 2.10, 2.11] and their use in [5, Example 2.13].

We will use several times that the image by \(f = f_{p, \lambda}\) of a nondegenerate interval is a nondegenerate interval (and thus all its iterates are nondegenerate).

Let \(A\) be a nondegenerate closed interval included in \([0, 1]\). We are going to show that there exists an integer \(n \geq 0\) such that \(f^n(A) = [0, 1]\).

We set \(A_0 := A\). This proof is inspired by [5, Lemmas 2.10, 2.11] and their use in [5, Example 2.13].

*Step 1:* there exists \(i_0 \geq 0\) such that \(f^{i_0}(A) \cap (C_0 \cup C_1) \neq \emptyset\) and there exists \(n_0 \geq 0\) such that \(0 \in f^{n_0}(A)\).

Let \(J_i := [\min J_i, \text{mid}(J_i)]\) and \(J_i^0 := [\text{mid}(J_i), \max J_i]\) for all \(i \in [1, k]\),

\[\mathcal{F} := \left\{\left[0, \frac{1}{\lambda}\right], [t, 1], K\right\} \cup \{J_i', J_i'' \mid i \in [1, k]\}\].

If \(A \subset B\) for some \(B \in \mathcal{F}\) and \(B \neq K\), then \(|f(A)| = \lambda|A|\). If \(A \subset K\), then \(|f(A)| \geq \frac{\lambda|A|}{2}\) and \(f(A) \subset I_{p-2}\), thus \(|f^2(A)| = \lambda|f(A)| \geq \frac{\lambda^2}{2}|A|\). We have \(\lambda > 1\) and \(\frac{\lambda^2}{2} > 1\) because \(\lambda > \sqrt{2}\) (Theorem 1.4). If for all \(i \geq 0\), there exists \(A_i \in \mathcal{F}\) such that \(f^i(A) \subset A_i\), then what precedes implies that \(\lim_{i \to +\infty} |f^i(A)| = +\infty\). This is impossible because \(f^i(A) \subset [0, 1]\). Thus there exist \(i_0 \geq 0\) and \(c \in C_0 \cup C_1\) such that \(c \in f^{i_0}(A)\). If \(c \in C_0\), then \(f(c) = 0\), and hence \(0 \in f^{i_0+1}(A)\). If \(c \in C_1\), then \(f(c) = x_{p-4}\) and hence \(0 \in f^{i_0+3}(A)\). This ends step 1.

*Step 2:* there exist \(n_1 \geq n_0\) and \(j \in [1, p-1]\) such that \(f^{n_1}(A) \supset I_j\).

Recall that \(I_1 = [x_1, x_0]\), \(I_i = (x_{i-2}, x_i)\) for all \(2 \leq i \leq p-2\) and \(I_{p-1} = [t, 1] = [t, x_{p-1}]\).

We set \(I_0 := I_1\). By definition, for all \(0 \leq i \leq p-1\), there exists \(\delta_i > 0\) such that \(I_i = (x_i, x_i + (-1)^{i+1}\delta_i)\). Moreover, \(f\) is linear of slope \(-\lambda\) on each of the intervals \(I_0, \ldots, I_{p-2}\) and of slope \(+\lambda\) on \(I_{p-1}\).

We set \(B_{-2} := f^{n_0}(A)\). This is a nondegenerate closed interval containing 0, thus there exists \(b > 0\) such that \(B_{-2} = [0, b] \cap 0 = x_{p-2}\). We set \(B_i := f^{i+2}(B_{-2})\) for all \(i \geq -2\), and we define \(m \geq -2\) as the least integer such that \(B_m\) is not included in a interval of the form \(I_j\) (such an integer \(m\) exists by step 1).

If \(b > x_{p-4}\), then \(B_{-2} \subset I_{p-2}\) and \(m = -2\). Otherwise, \(B_{-2} \subset I_{p-2}\) and \(B_{-1} = [1 - \lambda b, 1] = [x_{p-1} - \lambda b, x_{p-1}]\) because \(f|_{I_{p-2}}\) is of slope \(-\lambda\). If \(1 - \lambda b < t\), then \(B_{-1} \subset I_{p-1}\) and \(m = -1\). Otherwise, \(B_{-1} \subset I_{p-1}\) and \(B_0 = [x_0 - \lambda^2 b, x_0]\) because \(f|_{I_{p-1}}\) is of slope \(+\lambda\). We go on in a similar way.

- If \(m > 0\), then \(B_0 \subset I_0\) and \(B_1 = [x_1, x_1 + \lambda^3 b]\).
• If $m > 1$, then $B_1 \subset I_1$ and $B_2 = [x_2 - \lambda^4 b, x_2]$.

• If $m > p - 3$, then $B_{p-3} \subset I_{p-3}$ and $B_{p-2} = \langle x_{p-2}, x_{p-2} + (-1)^{p+1} \lambda^p b \rangle = [0, \lambda^p b]$.

Notice that $B_{p-2}$ is of the same form as $B_{-2}$. What precedes implies that

$\forall i \in [-2, m], B_i = \langle x_{i \mod p}, x_{i \mod p} + (-1)^{i+2} \lambda^i b \rangle, \forall i \in [-2, m - 1], B_i \subset I_{i \mod p}$,

$B_m \supset I_{m \mod p}$.

This ends step 2 with $n_1 := n_0 + m + 2$ and $j := m$.

**Step 3:** there exists $n_2 \geq n_1$ such that $f^{n_2}(A) = [0, 1]$.

Let $n_1 \geq 0$ and let $j \in [1, p - 1]$ be such that $f^{n_1}(A) \supset I_j$ (step 2). In the covering graph of Figure 3, we see that there exists an integer $q \geq 0$ such that, for every vertex $C$ of the graph, there exists a path of length $q$, with only arrows of type $\to$, starting from $I_j$ and ending at $C$. This implies that $f^q(I_j) = [0, 1]$, that is, $f^{n_1+q}(A) = [0, 1]$.

We have shown that, for every nondegenerate closed interval $A \subset [0, 1]$, there exists $n$ such that $f^n(A) = [0, 1]$. We conclude that $f$ is topologically mixing. □

4 General case

4.1 Square root of a map

We first recall the definition of the so-called square root of an interval map. If $f : [0, b] \to [0, b]$ is an interval map, the square root of $f$ is the continuous map $g : [0, 3b] \to [0, 3b]$ defined by

• $\forall x \in [0, b], g(x) := f(x) + 2b$,

• $\forall x \in [2b, 3b], g(x) := x - 2b$,

• $g$ is linear on $[b, 2b]$.

The graphs of $g$ and $g^2$ are represented in Figure 4.

The square root map has the following properties, see e.g. [5, Examples 3.22 and 4.62].

**Proposition 4.1** Let $f$ be an interval map of type $n$, and let $g$ be the square root of $f$. Then $g$ is of type $2n$ and $h_{top}(g) = \frac{h_{top}(f)}{2}$. If $f$ is piecewise monotone, then $g$ is piecewise monotone too.

4.2 Piecewise monotone map of given entropy and type

**Theorem 4.2** Let $p \geq 3$ be an odd integer, let $d$ be a non negative integer and $\lambda$ a real number such that $\lambda \geq \lambda_p$. Then there exists a piecewise monotone map $f$ whose type is $2^d p$ for Sharkovskii’s order and such that $h_{top}(f) = \frac{\log \lambda}{2^d}$. If $d = 0$, the map $f$ can be built in such a way that it is topologically mixing.

**Proof.** If $d = 0$, we take $f = f_{p, \lambda}$ defined in Section 3.

If $d > 0$, we start with the map $f_{p, \lambda}$, then we build the square root of $f_{p, \lambda}$, then the square root of the square root, etc. According to Proposition 4.1, after $d$ steps we get a piecewise monotone interval map $f$ of type $2^d p$ and such that $h_{top}(f) = \frac{h_{top}(f_{p, \lambda})}{2^d} = \frac{\log \lambda}{2^d}$. □
Corollary 4.3 For every positive real number \( h \), there exists a piecewise monotone interval map \( f \) such that \( h_{\text{top}}(f) = h \).

Proof. Let \( d \geq 0 \) be an integer such that \( \frac{\log \lambda_3}{2^d} \leq h \) and set \( \lambda := \exp(2^d h) \). Then \( \lambda \geq \lambda_3 \) and, according to Theorem 4.2, there exists a piecewise monotone interval map \( f \) of type \( 2^d \lambda_3 \) such that \( h_{\text{top}}(f) = \frac{\log \lambda}{2^d} = h \). \( \square \)

References


