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PERIODIC ORBITS OF LARGE DIAMETER FOR CIRCLE MAPS

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Abstract. Let f be a continuous circle map and let F be a lifting of f. In this paper we study how the existence of a large orbit for F affects its set of periods. More precisely, we show that, if F is of degree $d \geq 1$ and has a periodic orbit of diameter larger than 1, then F has periodic points of period n for all integers $n \geq 1$, and thus so has f. We also give examples showing that this result does not hold when the degree is nonpositive.

1. Introduction

One of the basic problems in topological dynamics in one dimension is the characterization of the sets of periods of all periodic points. This problem has its roots and motivation in Sharkovskiĭ's theorem [7]. A lot of effort has been spent in generalizing Sharkovskiĭ's theorem for more and more general classes of continuous self-maps on trees, and finally the characterization of the set of periods of general tree maps is given in [1]. While the set of periods of tree maps can be described with a finite number of orderings, circle maps display new features. The set of periods of a continuous circle map depends on the degree of the map (see, e.g., [2]). Consider a continuous map $f\colon \mathbb{S} \to \mathbb{S}$, where $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, and F is a lifting of f, that is, a continuous map $F\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ \pi = \pi \circ F$, where $\pi\colon \mathbb{R} \to \mathbb{S}$ is the canonical projection (F is uniquely defined up to the addition of an integer). The degree of f (or F) is the integer $d \in \mathbb{Z}$ such that $F(x+1) = F(x) + d$ for all $x \in \mathbb{R}$. If $|d| \geq 2$, then the set of periods is $\mathbb{N}$ (the case $\mathbb{N} \setminus \{2\}$ is also possible when $d = -2$). If $d = 0$ or $d = -1$, then the possible sets of periods are ruled by the Sharkovskiĭ order, as for continuous interval maps. The case $d = 1$ is the most complex one and requires rotation theory. Let F be a lifting of a degree 1 circle map f. The rotation number of a point $x \in \mathbb{R}$ is $\rho_{_F}(x) = \lim_{n\to+\infty} \frac{F^n(x)-x}{n}$, when the limit exists. The set of all rotation numbers is a compact interval $[a, b]$, and the set of periods of f contains

$$\{q \in \mathbb{N} \mid \exists\, p \in \mathbb{Z},\ a < \frac{p}{q} < b\}.$$

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This comes from the knowledge of the set of periods of periodic points with a given rotation number, which can be reduced from the study of periods of points of rotation number 0.

In this paper, we show that the set of periods of a lifting F of a circle map f of degree $d \geq 1$ is $\mathbb{N}$ if F has a periodic orbit of diameter larger than 1. This result obviously projects on the circle: if such a periodic orbit exists for F (for f, this means that the periodic orbit "spreads" on more than one turn on the circle), then the set of periods of f is $\mathbb{N}$. Our result improves the well known fact that the set of periods of f is $\mathbb{N}$ for $d \geq 2$. Indeed, when a large orbit exists, it shows that there is a subclass of orbits of f (namely those that come from a true periodic orbit of a lifting F) whose set of periods already contains $\mathbb{N}$. This study, in addition to its own interest, is mainly motivated by the case $d = 1$ because it might shed some light on the characterization of the set of periods of maps of degree 1 on topological graphs containing a loop, in particular, the graph shaped like the letter σ (an interval glued to a circle). For liftings of maps of the graph σ, it seems that periodic orbits of rotation number 0 of "large" diameter force all periods greater than or equal to 2. When the branching point of σ is fixed, the possible sets of periods are known [4]. On the other hand, a rotation theory has been developed for continuous self-maps on topological graphs with a unique loop in [3], and the rotation set of a σ map is studied in [6], which is a first step in the comprehension of the case of graph maps of degree 1.

2. Statement and proof of the result

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a continuous map. A point $x \in \mathbb{R}$ is *periodic* (for F) if there exists an integer $n \geq 1$ such that $F^n(x) = x$. The *period* of x is the least integer n with this property, that is, $F^n(x) = x$ and $F^i(x) \neq x$ for all $1 \leq i \leq n-1$. A *periodic orbit* is the orbit of some periodic point x, that is, $\{F^i(x) \mid i \geq 0\}$, which is a finite set. A set $A \subset \mathbb{R}$ is F-*invariant* if $F(A) \subset A$. Clearly, the only nonempty F-invariant subset of a periodic orbit P is P itself.

Remark 2.1. Let F be a lifting of a circle map $f\colon \mathbb{S} \to \mathbb{S}$. Then a point $\pi(x) \in \mathbb{S}$ is periodic for f if and only if x is periodic (mod 1) for F, that is, $\exists n \geq 1, k \in \mathbb{Z}, F^n(x) = x + k$. If in addition f is of degree 1, then $\rho_F(x) = k/n$, and thus the periodic points of F are exactly the periodic (mod 1) points of rotation number 0.

Now we state the main result of this paper.

Theorem 2.2. *Let $F\colon \mathbb{R} \to \mathbb{R}$ be a continuous map which is the lifting of a circle map of degree $d \geq 1$. Assume that F has a periodic orbit P of period n such that* $\max P - \min P > 1$. *If $d = 1$, then the rotation interval of F contains the interval $[-\frac{1}{n}, \frac{1}{n}]$ and, consequently, F has periodic points of all periods. If $d \geq 2$, then F also has periodic points of all periods.*

Proof. We consider separately the cases $d = 1$ and $d \geq 2$.

Assume first that $d = 1$. Set $p := \min P$ and let $k < n$ be the positive integer such that $F^k(p) = \max P > p + 1$. Let

$$F_u(x) := \sup\{f(y)\colon y \leq x\}.$$

From [2, Proposition 3.7.7(d)] it follows that F_u is continuous, nondecreasing and has degree one (that is, $F_u(x+1) = F_u(x) + 1$ for every $x \in \mathbb{R}$). Moreover, if we use [2, Proposition 3.7.7(a)] and the fact that F_u is nondecreasing, we get $F_u^i(x) \geq F^i(x)$ for all $x \in \mathbb{R}$ and $i \geq 1$, and hence $F_u^k(p) > p + 1$.

Assume that $F_u^{k\ell}(p) > p + \ell$ for some $\ell \in \mathbb{N}$. Then, by [2, Proposition 3.1.7(c)],

$$F_u^{k(\ell+1)}(p) = F_u^k(F_u^{k\ell}(p)) \geq F_u^k(p + \ell) = F_u^k(p) + \ell > p + (\ell + 1).$$

Hence, $F_u^{k\ell}(p) > p + \ell$ for every $\ell > 0$ and, consequently,

$$\limsup_{j\to+\infty} \frac{F_u^j(p) - p}{j} \geq \limsup_{\ell\to+\infty} \frac{F_u^{k\ell}(p) - p}{k\ell} \geq \tfrac{1}{k} > \tfrac{1}{n}.$$

On the other hand, since F_u is nondecreasing, [5, Theorem 1] implies that $\rho_{F_u}(x)$ exists for each $x \in \mathbb{R}$ and is independent of the choice of the point x. This number is called the *rotation number of* F_u and is denoted by $\rho(F_u)$. From the above it follows that $\rho(F_u) = \rho_{F_u}(p) > \frac{1}{n}$. Then, in view of [2, Theorem 3.7.20(a)] it follows that the right endpoint of the rotation interval of F is larger than $\frac{1}{n}$. In a similar way (using $\max P$ instead of $\min P$) it follows that the left endpoint of the rotation interval of F is smaller than $-\frac{1}{n}$. Thus, the theorem in the case $d = 1$ follows from [2, Lemma 3.9.1].

Now we consider the case $d \geq 2$. As above we set $p := \min P$ and $q := \max P > p + 1$. Since the F-orbit of P is periodic, $F^j(p) \geq p$ for every $j \geq 0$. So, by [2, Proposition 3.1.7(c)], the sequence $\{F^j(p+1)\}_{j=0}^{\infty}$ is contained in $(p, +\infty)$ and diverges to $+\infty$ (in particular, $F^j(p + 1) \neq q$ for every j). Since $p + 1 < q$ there exists $m \geq 0$ such that $p < r := F^m(p+1) < q$ but $F^j(r) > q$ for every $j > 0$. Since $P \not\subset [r, +\infty)$, there exists $s \in P$ such that $q \geq s > r$ but $F(s) < r$. Set $I = [r, s]$ and $J = [s, F(r)]$. Then $F(I) \supset I \cup J$ and $F(J) \supset I$. It is well known that in this situation, there exist periodic points of period ℓ for every integer $\ell \geq 1$. To give a precise proof, we use [2, Corollary 1.2.8]: for $\ell = 1$, we use $F(I) \supset I$; and for all $\ell \geq 2$, we get that there exists $x \in J$ such that $F^\ell(x) = x$ and $F^i(x) \in I$ for all $1 \leq i \leq \ell - 1$, which implies that x is periodic of period ℓ (indeed, if $F^i(x) = x$ for some $1 \leq i \leq \ell-1$, then $x \in I \cap J$, which is impossible because $F(s) \notin I \cup J$). This ends the proof of the theorem. □

Remark 2.3. A simple generalization of the above theorem and its proof for the case $d = 1$ is the following. Assume that F has periodic orbits $P_1, P_2, \ldots, P_j$ such that the set $\bigcup_{i=1}^{j} \langle P_i \rangle$ is connected and has diameter larger than one, where $\langle P_i \rangle$ denotes the *convex hull of* P_i (that is, the smallest closed interval containing P_i). By ordering the periodic orbits (and possibly withdrawing some of them), we may assume that $\min P_{i+1} \leq \max P_i$ for all $1 \leq i \leq j - 1$. Let $|P_i|$ denote the period of P_i. For each $1 \leq i \leq j$, there exists a positive integer $k_i < |P_i|$ such that $F^{k_i}(\min P_i) = \max P_i$. Let $p := \min P_1 = \min \bigcup_{i=1}^{j} P_i$. Using the facts that F_u is nondecreasing and $F_u^k(x) \geq F^k(x)$ for all $x \in \mathbb{R}$ and all $k \geq 1$ (see the proof of Theorem 2.2), we get

$$\begin{aligned}
F_u^{k_1}(p) &\geq \max P_1 \geq \min P_2, \\
F_u^{k_1+k_2}(p) &\geq F_u^{k_2}(\min P_2) \geq \max P_2 \geq \min P_3, \\
&\vdots \\
F_u^{k_1+\ldots+k_j}(p) &\geq \max P_j = \max \bigcup_{i=1}^{j} P_i > p + 1.
\end{aligned}$$

Then, in a similar way as in the proof of Theorem 2.2, it is possible to show that, for every $\ell > 0$,

$$F_u^{m\ell}(p) > p + \ell,$$

where $m = k_1 + \cdots + k_j < n := \sum_{i=1}^{j} |P_i|$. Consequently, the rotation interval of F contains the nondegenerate interval $[-\frac{1}{n}, \frac{1}{n}]$. Thus, F has periodic points of all periods.

The next corollary is a straightforward consequence of Theorem 2.2.

Corollary 2.4. *Let $f\colon \mathbb{S} \to \mathbb{S}$ be a continuous circle map of degree $d \geq 1$ and let F be a lifting of f. If there exists a periodic orbit P for F such that $\max P - \min P > 1$, then f has periodic points of all periods.*

The conclusion of Theorem 2.2 does not hold when the degree d is nonpositive. For $d = -1$, $F(x) = -x$ gives a trivial counterexample. The cases $d = 0$ and $d \leq -2$ are treated in Examples 2.6 and 2.5, respectively.

Example 2.5. Let d be an integer, $d \geq 2$, and let $\widetilde{F}\colon [0,1] \to \mathbb{R}$ be the map defined by (see Figure 1):

$$\widetilde{F}(x) := \begin{cases} (3-4d)x & \text{if } x \in [0, 1/4], \\ (2d-3)x + \frac{3(1-d)}{2} & \text{if } x \in [1/4, 3/4], \\ (3-4d)x + 3(d-1) & \text{if } x \in [3/4, 1]. \end{cases}$$

Observe that

$$(2d-3)\tfrac{1}{4} + \tfrac{3(1-d)}{2} = \tfrac{3}{4} - d = (3-4d)\tfrac{1}{4}, \quad \text{and}$$
$$(2d-3)\tfrac{3}{4} + \tfrac{3(1-d)}{2} = -\tfrac{3}{4} = (3-4d)\tfrac{3}{4} + 3(d-1).$$

Therefore, $\widetilde{F}$ is continuous, $\widetilde{F}(0) = 0$, $\widetilde{F}(\frac{1}{4}) = \frac{3}{4} - d$, $\widetilde{F}(\frac{3}{4}) = -\frac{3}{4}$ and $\widetilde{F}(1) = -d$. Observe that $\widetilde{F}(x) = -x$ for $x \in \{0, \frac{3}{4}\}$ and $\widetilde{F}(x) < -x$ for $x \in [0,1] \setminus \{0, \frac{3}{4}\}$. Moreover, it is a straightforward computation to show that $\widetilde{F}(x) + d = -\widetilde{F}(1-x)$ by considering separately the cases $x \in [0, 1/4] \cup [3/4, 1]$ and $x \in [1/4, 3/4]$.

Now consider the map $F\colon \mathbb{R} \to \mathbb{R}$ defined by $F(x) := \widetilde{F}(x - \lfloor x \rfloor) - d\lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x (see Figure 1).

Clearly F is a lifting of a continuous map of the circle of degree $-d$ (in particular, $F(x+k) = F(x) - dk$ for every $x \in \mathbb{R}$ and $k \in \mathbb{Z}$). Moreover, F is odd. To see this, take $x \in \mathbb{R}$ and write $x = \lfloor x \rfloor + \widetilde{x}$ with $\widetilde{x} \in [0, 1)$. Then,

$$\begin{aligned} -F(-x) &= -F(-\widetilde{x} - \lfloor x \rfloor) = -F(-(\lfloor x \rfloor + 1) + (1 - \widetilde{x})) \\ &= -F(1-\widetilde{x}) - d(\lfloor x \rfloor + 1) = -\widetilde{F}(1-\widetilde{x}) - d(\lfloor x \rfloor + 1) \\ &= \widetilde{F}(\widetilde{x}) + d - d(\lfloor x \rfloor + 1) = F(\widetilde{x}) - d\lfloor x \rfloor = F(\widetilde{x} + \lfloor x \rfloor) = F(x). \end{aligned}$$

From the above it follows that $F(0) = 0$, $F(\frac{3}{4}) = -\frac{3}{4}$ and $F(-\frac{3}{4}) = \frac{3}{4}$. Hence, 0 is a fixed point of F whereas $\{-\frac{3}{4}, \frac{3}{4}\}$ is a periodic orbit of F of period 2 with diameter larger than one. To end this example we will show that F has no other periodic points.

We claim that $|F(x)| > |x|$ for all $x \in \mathbb{R} \setminus \{-\frac{3}{4}, 0, \frac{3}{4}\}$. When $x \in [0,1]$ this amounts to showing that $F(x) < -x$ whenever $x \notin \{0, \frac{3}{4}\}$, and this follows from our remarks on $\widetilde{F}$. When $x \geq 1$ we have $\lfloor x \rfloor + 1 > x \geq \lfloor x \rfloor \geq 1$ and, hence,

$$F(x) = F(x - \lfloor x \rfloor) - d\lfloor x \rfloor \leq -\lfloor x \rfloor - 1 < -x.$$

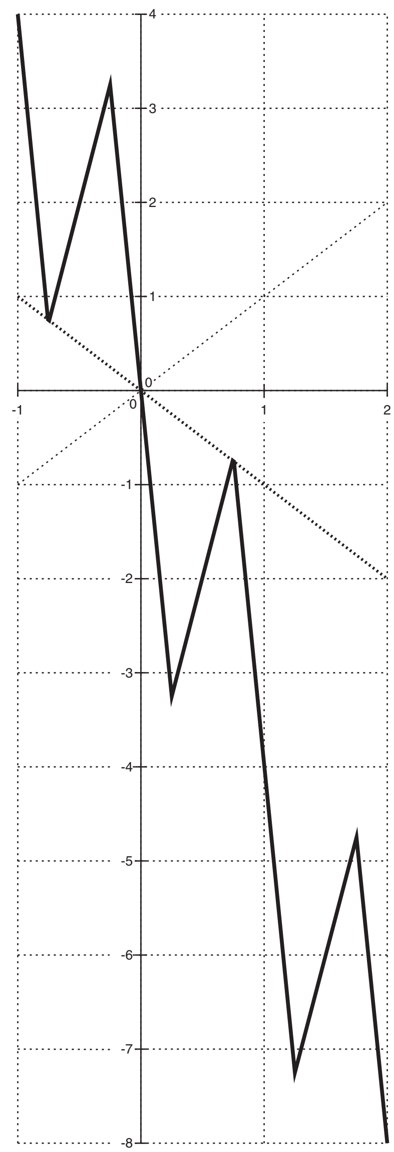

FIGURE 1. The graphs of the map F with $d = 4$ and $y = -x$ (dotted line) in the interval $[-1, 2]$.

The case $x < 0$ follows from the fact that F is odd. This ends the proof of the claim.

From the above claim it follows that if $x \in \mathbb{R}$ is not a preimage of 0 or $\frac{3}{4}$ under some iterate of F, then $|x| < |F(x)| < |F^2(x)| < \cdots$ and thus it cannot be periodic. Hence, F has no periodic points other than $\{-\frac{3}{4}, 0, \frac{3}{4}\}$.

Example 2.6. We define $F\colon \mathbb{R} \to \mathbb{R}$, which is a (continuous) lifting of a circle map of degree 0, as follows. First we choose $p \geq 3$ odd and points $x_0, x_1, \ldots, x_p$ and

$z_0, z_1, \dots, z_p$ in $\mathbb{R}$ such that

$$x_0 < z_0 - 1 < x_p < z_{p-1} < x_{p-2} < z_{p-3} < \cdots < x_3 < z_2 < x_1 \\ < z_1 < x_2 < z_3 < \cdots < x_{p-3} < z_{p-2} < x_{p-1} < z_p < x_0 + 1 < z_0.$$

Set $P := \{x_0, x_1, \dots, x_p, z_0, z_1, \dots, z_p\}$ and $\widetilde{P} := P \cup \{z_0 - 1, x_0 + 1\}$. Then we define F so that $F(x_i) = x_{i+1}$ and $F(z_i) = z_{i+1}$ for $i = 0, 1, \dots, p-1$, $F(x_p) = z_0$, $F(z_p) = x_0$, F is affine in the closure of every connected component of $[x_0, z_0] \setminus \widetilde{P}$ and furthermore we impose that $F(x+1) = F(x)$ for every $x \in \mathbb{R}$ (in particular, $F(z_0 - 1) = F(z_0) = z_1$ and $F(x_0 + 1) = F(x_0) = x_1$) (see Figure 2 for an example with $p = 3$).

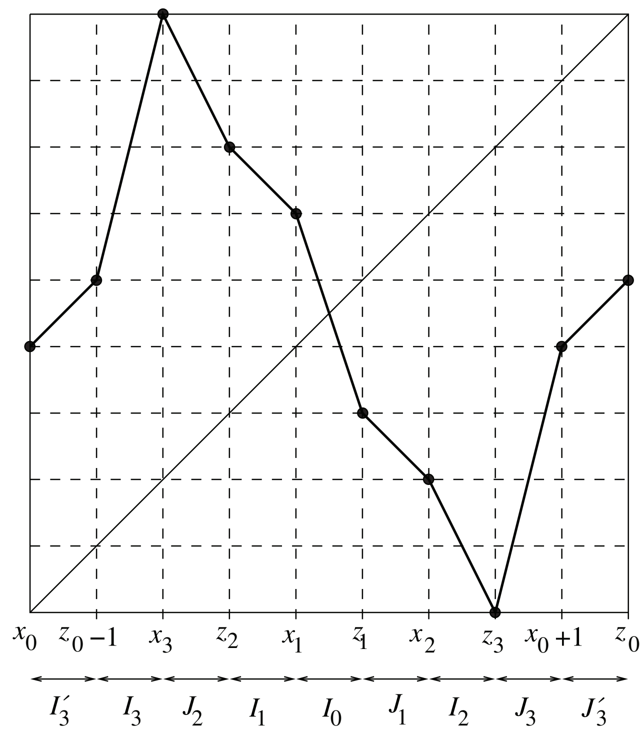

FIGURE 2. Graph of F for $p = 3$.

Clearly, the above conditions define a continuous function from $\mathbb{R}$ to itself that is the lifting of a circle map of degree 0. Moreover, P is a periodic orbit of F of period $2p + 2$, and this orbit is large since $\max P = z_0 > x_0 + 1 = \min P + 1$. We will show that F has no periodic orbits of periods $3, 5, \dots, p$. Thus, F does not have periodic points of all periods.

To show our claim we will compute the Markov graph of the map F and show that it has no loops of the specified length. We observe that, by definition, $F(\mathbb{R}) = [x_0, z_0]$. So we only have to consider the graph on the finitely many vertices contained in $[x_0, z_0]$. To this end, we define the intervals $I_0 := [x_1, z_1]$, $I_i := \langle x_i, z_{i+1} \rangle$ and $J_i = \langle z_i, x_{i+1} \rangle$ for $i = 1, 2, \dots, p-1$ (where $\langle a, b \rangle$ denotes either $[a, b]$ or $[b, a]$ depending on the order of a, b), $I_p := [z_0 - 1, x_p]$, $J_p := [z_p, x_0 + 1]$, $I_p' := [x_0, z_0 - 1]$,

$J'_p := [x_0+1, z_0]$. Then F is a Markov map with respect to this partition and its Markov graph has exactly the following arrows:

- $I_0 \longrightarrow I_0$,
- $I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \dots \longrightarrow I_{p-1} \begin{smallmatrix} \nearrow & I'_p \longrightarrow I_0 \\ \searrow & I_p \end{smallmatrix}$,
- $I_0 \longrightarrow J_1 \longrightarrow J_2 \longrightarrow \dots \longrightarrow J_{p-1} \begin{smallmatrix} \nearrow & J'_p \longrightarrow I_0 \\ \searrow & J_p \end{smallmatrix}$,
- $I_p \longrightarrow K$ for all $K \in \{J_1, J_3, \dots, J_p, J'_p, I_2, I_4, \dots, I_{p-1}\}$,
- $J_p \longrightarrow K$ for all $K \in \{I_1, I_3, \dots, I_p, I'_p, J_2, J_4, \dots, J_{p-1}\}$.

By direct inspection one can see that in the above graph any loop contains either I_0, I_p or J_p. Moreover the loops not containing I_0 are all of even length. The shorter simple loops of odd length greater than 1 are exactly the following four loops of length $p+2$:

- $I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \dots \longrightarrow I_{p-1} \longrightarrow I'_p \longrightarrow I_0 \longrightarrow I_0$,
- $I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \dots \longrightarrow I_{p-1} \longrightarrow I_p \longrightarrow J'_p \longrightarrow I_0$,
- $I_0 \longrightarrow J_1 \longrightarrow J_2 \longrightarrow \dots \longrightarrow J_{p-1} \longrightarrow J'_p \longrightarrow I_0 \longrightarrow I_0$,
- $I_0 \longrightarrow J_1 \longrightarrow J_2 \longrightarrow \dots \longrightarrow J_{p-1} \longrightarrow J_p \longrightarrow I'_p \longrightarrow I_0$.

Consequently the Markov graph of F has no loops of lengths $3, 5, \dots, p$, and, by [2, Lemma 1.2.12], the map F cannot have periodic points of any of these periods.

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