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ROTATION SET FOR MAPS OF DEGREE 1 ON THE GRAPH SIGMA

BY

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ABSTRACT

For a continuous map on a topological graph containing a unique loop S it is possible to define the degree and, for a map of degree 1, rotation numbers. It is known that the set of rotation numbers of points in S is a compact interval and for every rational r in this interval there exists a periodic point of rotation number r. The whole rotation set (i.e., the set of all rotation numbers) may not be connected and it is not known in general whether it is closed.

The graph sigma is the space consisting in an interval attached by one of its endpoints to a circle. We show that, for a map of degree 1 on the graph sigma, the rotation set is closed and has finitely many connected components. Moreover, for all rational numbers r in the rotation set, there exists a periodic point of rotation number r.

1. Introduction

In [2] a rotation theory is developed for continuous self maps of degree 1 of topological graphs having a unique loop, using the ideas and techniques of [4, 3]. A rotation theory is usually developed in the universal covering space by using the liftings of the maps under consideration. The universal covering of a graph containing a unique loop is an "infinite tree invariant by translation" (see Figure 1). It turns out that the rotation theory on the universal covering

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of a graph with a unique loop can be easily extended to the setting of infinite graphs that look like the space $\widehat{G}$ from Figure 2. These spaces are defined in detail in Section 2.1 and called lifted graphs. Each lifted graph T has a subset $\widehat{T}$ homeomorphic to the real line $\mathbb{R}$ that corresponds to an "unwinding" of a distinguished loop of the original graph. In the sequel, we identify $\widehat{T}$ with $\mathbb{R}$.

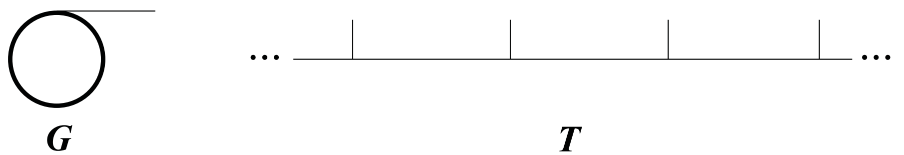

Figure 1. G is the graph σ, its universal covering is T.

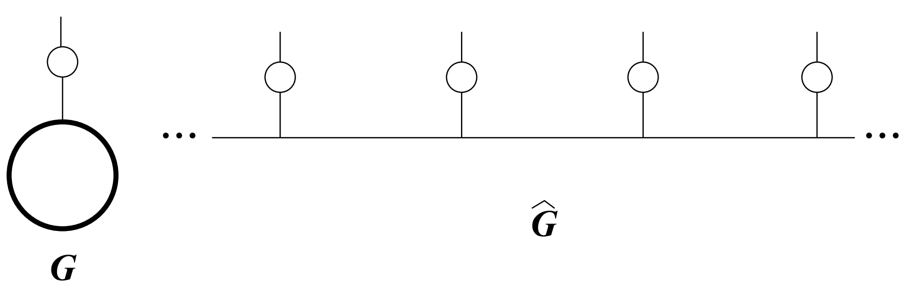

Figure 2. The graph G is unwound with respect to the bold loop to obtain $\widehat{G}$, which is a lifted graph.

Given a lifted graph T and a map F from T to itself of degree one, there is no difficulty to extend the definition of rotation number to this setting in such a way that every periodic point still has a rational rotation number as in the circle case. However, the obtained rotation set $\mathrm{Rot}(F)$ may not be connected. Despite this fact, it is proven in [2] that the set $\mathrm{Rot}_{\mathbb{R}}(F)$ corresponding to the rotation numbers of all points belonging to $\mathbb{R}$ has properties which are similar to (although weaker than) those of the rotation interval for a circle map of degree one. Indeed, this set is a compact non-empty interval; if $p/q \in \mathrm{Rot}_{\mathbb{R}}(F)$ then there exists a periodic point of rotation number p/q, and if in addition

$p/q \in \mathrm{Int}\,(\mathrm{Rot}_{\mathbb{R}}(F))$ then for all large enough positive integers n there exists a periodic point of period nq of rotation number p/q.

We conjecture that the whole rotation set $\mathrm{Rot}(F)$ is closed. In this paper, we prove that, when the space T is the universal covering of the graph σ consisting in an interval attached by one of its endpoint to a circle (see Figure 1), then the rotation set is the union of finitely many compact intervals. Moreover, all rational points r in $\mathrm{Rot}(F)$ are rotation numbers of periodic points. It turns out that the proofs extend to a class of maps on graphs that we call σ-like maps, which are defined in Section 2.3.

The paper is organised as follows. In Section 2, we give the definitions of the objects we deal with: lifted graphs, maps of degree 1, σ-like maps, rotation numbers and rotation sets. In Section 3, we recall the notion of positive covering and state some of its properties, which are key tools to find periodic points. In Section 4, we first partition the space T according to some dynamical properties, then we prove the main result, which is done is several steps.

2. Definitions and elementary properties

2.1. Lifted graphs. A **(topological) finite graph** is a compact connected set G containing a finite subset V such that each connected component of $G \setminus V$ is homeomorphic to an open interval.

The aim of this section is to define in detail the class of lifted graphs where we develop the rotation theory. They are obtained from a topological graph by unwinding one of its loops. This gives a new space that contains a subset homeomorphic to the real line and that is "invariant by a translation" (see Figures 1 and 2). In [2], a larger class of spaces called lifted spaces is defined.

Definition 2.1: Let T be a connected closed topological space. We say that T is a **lifted graph** if there exist a homeomorphism h from $\mathbb{R}$ into T, and a homeomorphism $\tau\colon T \to T$ such that

(i) $\tau(h(x)) = h(x+1)$ for all $x \in \mathbb{R}$,
(ii) the closure of each connected component of $T \setminus h(\mathbb{R})$ is a finite graph that intersects $h(\mathbb{R})$ at a single point,
(iii) the number of connected components C of $T \setminus h(\mathbb{R})$ such that $\overline{C} \cap h([0,1]) \neq \emptyset$ is finite.

The class of all lifted graphs is denoted by $\mathbf{T}^{\circ}$.

To simplify the notation, in the rest of the paper we identify $h(\mathbb{R})$ with $\mathbb{R}$ itself. In this setting, the map τ can be interpreted as a translation by 1. So, for all $x \in T$ we write $x+1$ to denote $\tau(x)$. Since τ is a homeomorphism, this notation can be extended by denoting $\tau^m(x)$ by $x+m$ for all $m \in \mathbb{Z}$.

Because of (ii), not all infinite graphs obtained by unwinding a finite graph with a distinguished loop are lifted graphs. The essential property of this class is the existence of a natural retraction from T to $\mathbb{R}$.

Definition 2.2: Let $T \in \mathbf{T}^\circ$. The **retraction** $r\colon T \to \mathbb{R}$ is the continuous map defined as follows. When $x \in \mathbb{R}$, then $r(x) = x$. When $x \notin \mathbb{R}$, there exists a connected component C of $T \setminus \mathbb{R}$ such that $x \in C$ and $\overline{C}$ intersects $\mathbb{R}$ at a single point z, and we let $r(x) = z$.

It can be easily shown that the retraction r is a continuous map.

2.2. Maps of degree 1 and rotation numbers. A standard approach to study the periodic points and orbits of a graph map is to work at lifting level with the periodic (mod 1) points. The results on the lifted graph can obviously be pulled back to the original graph. Moreover, the rotation numbers have a signification only for maps of degree 1, as in the case of circle maps. In this paper, we deal only with maps of degree 1 on lifted graphs.

Definition 2.3: Let $T \in \mathbf{T}^\circ$. A continuous map $F\colon T \to T$ is **of degree** 1 if $F(x+1) = F(x)+1$ for all $x \in T$.

A point $x \in T$ is called **periodic** (mod 1) for F if there exists a positive integer n such that $F^n(x) \in x + \mathbb{Z}$. The **period** of x is the least integer n satisfying this property, that is, $F^n(x) \in x+\mathbb{Z}$ and $F^i(x) \notin x + \mathbb{Z}$ for all $1 \le i \le n-1$.

The next easy lemma summarises the basic properties of maps of degree 1 (see, for instance, [1, Section 3.1]).

Lemma 2.4: *Let $T \in \mathbf{T}^\circ$ and $F\colon T \to T$ be a continuous map of degree 1. The following statements hold for $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $x \in T$:*

(i) $F^n(x+k) = F^n(x)+k$.
(ii) $(F+k)^n(x) = F^n(x)+kn$.
(iii) *If $G\colon T \to T$ is another continuous map of degree 1, then $F \circ G$ is a map of degree 1. In particular, F^n is of degree 1 for all $n \ge 1$.*

We define three types of rotation numbers.

Definition 2.5: Let $T \in \mathbf{T}^\circ$, $F\colon T \to T$ be a continuous map of degree 1 and $x \in T$. We set

$$\underline{\rho}_F(x) = \liminf_{n\to+\infty} \frac{r \circ F^n(x) - r(x)}{n}$$

and

$$\overline{\rho}_F(x) = \limsup_{n\to+\infty} \frac{r \circ F^n(x) - r(x)}{n}.$$

When $\underline{\rho}_F(x) = \overline{\rho}_F(x)$, then this number is denoted by $\rho_F F(x)$ and called the **rotation number of** x.

We now give some elementary properties of rotation numbers (see [2, Lemma 1.10]).

LEMMA 2.6: *Let $T \in \mathbf{T}^\circ$, $F\colon T \to T$ be a continuous map of degree 1, $x \in T$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.*

(i) $\overline{\rho}_F(x + k) = \overline{\rho}_F(x)$.
(ii) $\overline{\rho}_{(F+k)}(x) = \overline{\rho}_F(x) + k$.
(iii) $\overline{\rho}_{F^n}(x) = n\overline{\rho}_F(x)$.

The same statements hold with $\underline{\rho}$ instead of $\overline{\rho}$.

An important object that synthesises all the information about rotation numbers is the rotation set (i.e., the set of all rotation numbers). Since we have three types of rotation numbers, we have several kinds of rotation sets.

Definition 2.7: Let $T \in \mathbf{T}^\circ$ and $F\colon T \to T$ be a continuous map of degree 1. For $S \subset T$ we define the following **rotation sets**:

$$\mathrm{Rot}_S^+(F) = \{\overline{\rho}_F(x) \mid x \in S\},$$
$$\mathrm{Rot}_S^-(F) = \{\underline{\rho}_F(x) \mid x \in S\},$$
$$\mathrm{Rot}_S(F) = \{\rho_F(x) \mid x \in S \text{ and } \rho_F(x) \text{ exists}\}.$$

When $S = T$, we omit the subscript and we write $\mathrm{Rot}^+(F)$, $\mathrm{Rot}^-(F)$ and $\mathrm{Rot}(F)$ instead of $\mathrm{Rot}_T^+(F)$, $\mathrm{Rot}_T^-(F)$ and $\mathrm{Rot}_T(F)$, respectively.

2.3. SIGMA-LIKE MAPS. Let $T \in \mathbf{T}^\circ$ and $F\colon T \to T$ be a continuous map of degree 1. Define

$$T_{\mathbb{R}} = \overline{\bigcup_{n\geq 0} F^n(\mathbb{R})}$$

and $X_F = \overline{T \setminus T_{\mathbb{R}}} \cap r^{-1}([0,1))$ (see Figure 3). Then $T_{\mathbb{R}} \in \mathbf{T}^\circ$ (by Lemma 5.2 in [2]), X_F is composed of finitely many finite graphs and $T = T_{\mathbb{R}} \cup (X_F + \mathbb{Z})$.

If T is the lifting of the graph σ (see Figure 1), then X_F is either empty, or an interval with an endpoint in $T_{\mathbb{R}}$. Maps with the same property will be called σ-like maps.

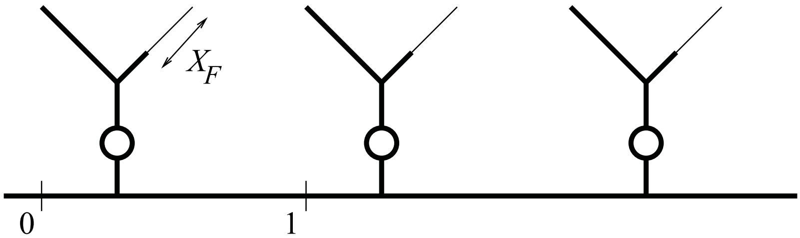

Figure 3. Illustration of the sets $T_{\mathbb{R}}$ (in bold) and X_F (thin line). This is a σ-like map.

Definition 2.8: Let $T \in \mathbf{T}^\circ$ and $F\colon T \to T$ be a continuous map of degree 1. If X_F is either empty, or an interval such that $X_F \cap T_{\mathbb{R}}$ is reduced to an endpoint of X_F, we say that F is a **σ-like map** and we write $F \in \mathcal{C}_1^\sigma(T)$.

Remark 2.9: If F is a σ-like map, then so is F^n, because $X_{F^n} \subset X_F$.

This paper is devoted to the study of the rotation set of σ-like maps when $X_F \neq \emptyset$. The study of the rotation set $\mathrm{Rot}_{T_{\mathbb{R}}}(F)$ has already been done in [2].

3. Positive covering

Let $F \in \mathcal{C}_1^\sigma(T)$. The interval X_F, when it is not empty, may be endowed with two opposite orders. We choose the one such that $\min X_F$ is the one-point intersection $X_F \cap T_{\mathbb{R}}$. The retraction map $r_X\colon T \to X_F$ can be defined in a natural way by $r_X(x) = x$ if $x \in X_F$ and $r_X(x) = \min X_F$ if $x \in T_{\mathbb{R}}$.

The notion of positive covering for subintervals of $\mathbb{R}$ has been introduced in [2]. It can be extended for subintervals of any subset of T on which a retraction can be defined. In this paper, we shall use positive covering on X_F. All properties of positive covering remain valid in this context. In particular, if a compact interval I positively F-covers itself, then F has a fixed point in I (Proposition 3.5).

Definition 3.1: Let $T \in \mathbf{T}^\circ$, $F \in \mathcal{C}_1^\sigma(T)$, I, J be two non-empty compact subintervals of X_F, n a positive integer and $p \in \mathbb{Z}$. We say that I **positively F^n-covers** $J+p$ and we write $I \xrightarrow[F^n]{+} J+p$ if there exist $x \leq y$ in I such that $r_X(F^n(x)-p) \leq \min J$ and $r_X(F^n(y)-p) \geq \max J$. In this situation, $I+q$ positively F^n-covers $J+p+q$ for all $q \in \mathbb{Z}$.

Remark 3.2: If $F^n(x) \in T_\mathbb{R}$ and $J \subset X_F$, then the inequality $r_X(F^n(x)-p) \leq \min J$ is automatically satisfied. We shall often use this remark to prove that an interval positively covers another.

We introduce some definitions in order to handle sequences of positive coverings.

Definition 3.3: Let $T \in \mathbf{T}^\circ$ and $F \in \mathcal{C}_1^\sigma(T)$. If we have the following sequence of positive coverings:

$$\mathcal{C}\colon I_0+p_0 \xrightarrow[F^{n_1}]{+} I_1+p_1 \xrightarrow[F^{n_2}]{+} I_2+p_2 \cdots \qquad \cdots I_{k-1}+p_{k-1} \xrightarrow[F^{n_k}]{+} I_k+p_k$$

(where $I_0, \ldots, I_k$ are non-empty compact subintervals of X_F, $n_1, \ldots, n_k$ are positive integers and $p_0, \ldots, p_k \in \mathbb{Z}$), then $\mathcal{C}$ is called a **chain** of intervals for F. Its **length** is $L_F(\mathcal{C}) = n_1+\cdots+n_k$, and its **weight** is $W_F(\mathcal{C}) = p_k - p_0$ (notice that a weight can be negative). A point x **follows** the chain $\mathcal{C}$ if $F^{n_1+\cdots+n_i}(x) \in I_i + p_i$ for all $0 \leq i \leq k$.

If $i \in \mathbb{Z}$, the chain $\mathcal{C}+i$ is the translation of $\mathcal{C}$, that is

$$\mathcal{C}+i\colon I_0+p_0+i \xrightarrow[F^{n_1}]{+} I_1+p_1+i \xrightarrow[F^{n_2}]{+} \cdots$$
$$\cdots I_{k-1}+p_{k-1}+i \xrightarrow[F^{n_k}]{+} I_k+p_k+i.$$

If $\mathcal{C}'$ is another chain of intervals beginning with I_k+p for some $p \in \mathbb{Z}$, then $\mathcal{C}\mathcal{C}'$ is the concatenation of $\mathcal{C}$ and $(\mathcal{C}'-p+p_k)$. If $I_k = I_0$, then $\mathcal{C}^n$ is the n-times concatenation $\mathcal{C}\cdots\mathcal{C}$ if $n \geq 1$ and $\mathcal{C}^0$ is the empty chain.

The next properties are straightforward.

Lemma 3.4: *Let $T \in \mathbf{T}^\circ$ and $F \in \mathcal{C}_1^\sigma(T)$.*

- *If $\mathcal{C}$ is a chain of intervals for F^n, then it is also a chain of intervals for F and $L_F(\mathcal{C}) = nL_{F^n}(\mathcal{C})$ and $W_F(\mathcal{C}) = W_{F^n}(\mathcal{C})$. Since the weight is independent of the power of the map, we shall denote it by $W(\mathcal{C})$.*

- *If $\mathcal{C}, \mathcal{C}'$ are two chains of intervals for F that can be concatenated, then $L_F(\mathcal{C}\mathcal{C}) = L_F(\mathcal{C}) + L_F(\mathcal{C}')$ and $W(\mathcal{C}\mathcal{C}) = W(\mathcal{C}) + W(\mathcal{C}')$.*

The next proposition is [2, Proposition 2.3] (rewritten in some less general form).

Proposition 3.5: *Let $T \in \mathbf{T}^\circ$, $F \in \mathcal{C}_1^\sigma(T)$ and $\mathcal{C}$ be a chain of subintervals of X_F such that $\mathcal{C}$ starts with some interval I_0 and ends with a translation of I_0 (i.e., $I_0 + p$ for some $p \in \mathbb{Z}$). Then there exists a point x_0 following the chain $\mathcal{C}$ such that $F^{L_F(\mathcal{C})}(x_0) = x_0 + W(\mathcal{C})$.*

The next lemma says that if two intervals I, J both positively cover translations of I and J, then every rational number in the rotation interval corresponding to this "horseshoe" can be obtained as a rotation number of a periodic (mod 1) point. This will be a key tool.

Lemma 3.6: *Let $T \in \mathbf{T}^\circ$, $G \in \mathcal{C}_1^\sigma(T)$, I, J be two non-empty compact subintervals of X_G and $m_1, m_2 \in \mathbb{Z}$ such that*

$$I \xrightarrow[G]{+} I + m_1 \quad \text{and} \quad I \xrightarrow[G]{+} J + m_1,$$
$$J \xrightarrow[G]{+} I + m_2 \quad \text{and} \quad J \xrightarrow[G]{+} J + m_2.$$

Suppose that $m_1 \leq m_2$. For every $p/q \in [m_1, m_2]$, there exists $\mathcal{C}$ a chain of intervals for G in which all the intervals are translations of I and J, and $p/q = W(\mathcal{C})/L_G(\mathcal{C})$. Moreover, there exists a periodic (mod 1) *point $x \in I \cup J$ such that $\rho_G(x) = p/q$.*

If $p/q \neq m_2$, then $\mathcal{C}$ can be chosen such that the first interval is I and the last interval is a translation of I, and the periodic (mod 1) *point x can be chosen in I.*

Proof. By considering $G - m_1$ instead of G, we may suppose that $m_1 = 0$ (use Lemma 2.6). Since $p/q \in [0, m_2]$, we have $0 \leq p \leq m_2 q$. If $p/q = m_2$, we take $\mathcal{C}\colon J \xrightarrow[G]{+} J + m_2$. By Proposition 3.5, there exists a point $x \in J$ such that $F(x) = x + m_2$, and hence $\rho_G(x) = m_2$.

If $p = 0$, we take $\mathcal{C}\colon I \xrightarrow[G]{+} I$. If $1 \leq p \leq m_2 q - 1$, we take

$$\mathcal{C}\colon (I \xrightarrow[G]{+} I)^{m_2 q - 1 - p} (I \xrightarrow[G]{+} J)(J \xrightarrow[G]{+} J + m_2)^{p-1} (J \xrightarrow[G]{+} I + m_2).$$

In these two cases, it is straightforward that $W(\mathcal{C})/L_G(\mathcal{C}) = p/q$. By Proposition 3.5, there exists a point $x \in I$ such that $G^{L_G(\mathcal{C})}(x) = x + W(\mathcal{C})$, and so x is periodic (mod 1) and $\rho_G(x) = W(\mathcal{C})/L_G(\mathcal{C}) = p/q$. ∎

4. Study of the rotation set of F

Let $T \in \mathbf{T}^\circ$ and $F \in \mathcal{C}_1^\sigma(T)$. Since $T = T_\mathbb{R} \cup (X_F + \mathbb{Z})$, it is clear that $\mathrm{Rot}(F) = \mathrm{Rot}_{T_\mathbb{R}}(F) \cup \mathrm{Rot}_{X_F}(F)$, and the same holds with Rot^+ and Rot^-. The rotation set $\mathrm{Rot}_{T_\mathbb{R}}(F)$ has been studied in [2]. Consequently, it remains to study $\mathrm{Rot}_{X_F}(F)$. The next theorem summarises the main properties of $\mathrm{Rot}_\mathbb{R}(F)$ (see Theorems 3.1, 3.11, 5.7 and 5.18 in [2]).

Theorem 4.1: *Let $T \in \mathbf{T}^\circ$ and $F\colon T \to T$ be a continuous map of degree 1. Then $\mathrm{Rot}_\mathbb{R}(F)$ is a non-empty compact interval and, if $T_\mathbb{R}$ is defined as above, $\mathrm{Rot}_{T_\mathbb{R}}(F) = \mathrm{Rot}^+_{T_\mathbb{R}}(F) = \mathrm{Rot}^-_{T_\mathbb{R}}(F) = \mathrm{Rot}_\mathbb{R}(F)$. Moreover, if $r \in \mathrm{Rot}_\mathbb{R}(F) \cap \mathbb{Q}$, then there exists a periodic* (mod 1) *point $x \in T_\mathbb{R}$ such that $\rho_F(x) = r$.*

4.1. Partition of X_F. If $F^n(x) \in T_\mathbb{R}$ for some n, then $\rho_F(x) \in \mathrm{Rot}_{T_\mathbb{R}}(F)$. Therefore, it is sufficient to consider the points $x \in X_F$ whose orbit does not fall in $T_\mathbb{R}$, or equivalently the points in $X_\infty = \{x \in X_F \mid \forall n \geq 1, F^n(x) \in X_F + \mathbb{Z}\}$.

Our first step consists in dividing X_F according to the translations of the images with respect to $X_F + \mathbb{Z}$. If $F(x) \in X_F + p$ and $F(y) \in X_F + p'$ with $p \neq p'$, then necessarily there is a gap between x and y by continuity. Thus we can include the points $\{x \in X_F \mid F(x) \in X_F + \mathbb{Z}\}$ in a finite union of disjoint compact intervals such that, for each I among these intervals, there is a unique integer p satisfying $F(I) \cap (X_F + p) \neq \emptyset$.

Lemma 4.2: *Let $T \in \mathbf{T}^\circ$ and $F \in \mathcal{C}_1^\sigma(T)$. There exist an integer $N \geq 0$, disjoint non-empty compact subintervals $X_1, \ldots, X_N$ of X_F and integers $p_1, \ldots, p_N$ in $\mathbb{Z}$ such that*

- (i) $X_1 < X_2 < \cdots < X_N$ *(for the order on X_F),*
- (ii) $F(X_i) \subset (X_F + p_i) \cup T_\mathbb{R}$ *for all* $1 \leq i \leq N$,
- (iii) $F(\min X_i) = \min X_F + p_i$ *for all* $1 \leq i \leq N$,
- (iv) $p_{i+1} \neq p_i$ *for all* $1 \leq i \leq N-1$,
- (v) $F(X_F \setminus (X_1 \cup \cdots \cup X_N)) \cap (X_F + \mathbb{Z}) = \emptyset$.

Proof. If $F(X_F) \cap (X_F + \mathbb{Z}) = \emptyset$, we take $N = 0$ and there is nothing to do. Otherwise, we can define $a_1 = \min\{x \in X_F \mid F(x) \in X_F + \mathbb{Z}\}$ and $p_1 \in \mathbb{Z}$ such that $F(a_1) \in X_F + p_1$. We define

$$b_1 = \max\{x \in [a_1, \max X_F] \mid F(x) \in X_F + p_1 \text{ and } F([a_1, x]) \subset (X_F + p_1) \cup T_{\mathbb{R}}\},$$

and $X_1 = [a_1, b_1]$. Then X_1 satisfies (ii). Moreover, $F(\min X_F) \in T_{\mathbb{R}}$ because $\min X_F \in T_{\mathbb{R}}$, which implies that $F([\min X_F, a_1])$ contains $\min X_F + p_1$. Thus $F(a_1) = \min X_F + p_1$ by minimality of a_1, which is (iii) for X_1.

We define $X_2, \ldots, X_N$ inductively. Suppose that $X_i = [a_i, b_i]$ and p_i are already defined and that b_i verifies

$$b_i = \max\{x \in [a_i, \max X_F] \mid F(x) \in X + p_i \text{ and } F([a_i, x]) \subset (X_F + p_i) \cup T_{\mathbb{R}}\}.$$

If $F((b_i, \max X_F]) \cap (X_F + \mathbb{Z}) = \emptyset$, then we take $N = i$ and the construction is over. Otherwise, we define

$$a_{i+1} = \inf\{x \in (b_i, \max X_F] \mid F(x) \in X_F + \mathbb{Z}\}. \tag{1}$$

We first show that a_{i+1} is actually defined by a minimum in (1). By definition, there exists a sequence of points $x_n \in (b_i, \max X_F]$ tending to a_{i+1} and such that $F(x_n) \in X_F + \mathbb{Z}$. Let $m_n \in \mathbb{Z}$ such that $F(x_n) \in X_F + m_n$. By continuity of $r \circ F$, $\lim_{n\to+\infty} r \circ F(x_n) = r \circ F(a_{i+1})$. Since $r \circ F(x_n) = r(\min X_F) + m_n$, this implies that the sequence of integers $(m_n)_{n\geq 0}$ is ultimately constant, and equal to some integer p_{i+1}. Then $F(a_{i+1}) = \lim_{n\to+\infty} F(x_n) \in X_F + p_{i+1}$. By continuity, $F([a_{i+1}, x_n]) \subset (X_F + p_{i+1}) \cup T_{\mathbb{R}}$ for all n large enough. Moreover, $F((b_i, a_{i+1})) \cap (X_F + \mathbb{Z}) = \emptyset$ by definition of a_{i+1}. If $p_{i+1} = p_i$, then, for n large enough, we would have

$$F(x_n) \in X_F + p_i \quad \text{and} \quad F([b_i, x_n]) \in (X + p_i) \cup T_{\mathbb{R}},$$

which would contradict the definition of b_i because $x_n > b_i$. Hence $p_{i+1} \neq p_i$. This implies that $a_{i+1} > b_i$. Since $F((b_i, a_{i+1}))$ is non-empty and included in $T_{\mathbb{R}}$, necessarily $F(a_{i+1})$ is equal to $\min X_F + p_{i+1}$ by minimality of a_{i+1}.

Finally, we define

$$b_{i+1} = \max\{x \in [a_{i+1}, \max X_F] : F(x) \in X_F + p_{i+1} \text{ and } F([a_{i+1}, x]) \subset (X_F + p_{i+1}) \cup T_{\mathbb{R}}\},$$

and $X_{i+1} = [a_{i+1}, b_{i+1}]$. Then $X_{i+1} > X_i$ and (ii), (iii) and (iv) are satisfied.

By uniform continuity of $r \circ F$ on the compact set X_F, there exists $\delta > 0$ such that, if $x, y \in X_F$ with $|x - y| < \delta$, then $|r \circ F(x) - r \circ F(y)| < 1$. This implies that $|a_{i+1} - b_i| \geq \delta$, which ensures that the number of intervals X_i is finite, and the construction ultimately stops. By construction, (v) is satisfied. ∎

Remark 4.3:

- The fact that the sets $X_1, \dots, X_N$ are intervals is very important because it will allow us to use positive coverings. Notice that we cannot ask that $F(X_i) \subset (X_F + p_i)$, even if we do not require that $p_{i+1} \neq p_i$. Indeed, if $\min X_F$ is a fixed point, the map F may oscillate infinitely many times between X_F and $T_{\mathbb{R}}$ in any neighbourhood of $\min X_F$, and in this case the number of connected components of $F(X_F) \cap X_F$ is infinite.
- In the partition of X_F into $X_1, \dots, X_N, X_F \setminus (X_1 \cup \dots \cup X_N)$, the set $X_F \setminus (X_1 \cup \dots \cup X_N)$ plays the role of "dustbin", and we can code the itinerary of every point in X_∞ with respect to $X_1, \dots, X_N$. More precisely, if $F^n(x) \in X_F \setminus (X_1 \cup \dots \cup X_N) + \mathbb{Z}$, then $x \notin X_\infty$. Therefore, for every $x \in X_\infty$, $\forall n \geq 0, \exists! \omega_n \in \{1, \dots, N\}$ such that $F^n(x) \in X_{\omega_n} + \mathbb{Z}$. The rotation number of x can be deduced from this coding sequence because $\forall n \geq 0$, $F^n(x) \in X_{\omega_n} + p_{\omega_0} + \dots + p_{\omega_{n-1}}$.
- It can additionally be shown that $F(\max X_i) = \min X_F + p_i$ for all $1 \leq i \leq N-1$ and, for $i = N$, either $F(\max X_N) = \min X_F + p_N$, or $\max X_N = \max X_F$.

4.2. Periodic (mod 1) points associated to the endpoints of rotation sets. When proving that every rational number in the rotation set is the rotation number of a periodic (mod 1) point, we shall make a distinction between the interior and the boundary of the rotation sets (the same distinction is necessary to deal with $\mathrm{Rot}_{\mathbb{R}}(F)$ [2]). For rational numbers in the boundary, harder to handle, we shall need Lemma 4.5, which is analogous to [2, Lemma 5.16] in our context. We first prove a technical but key lemma.

Lemma 4.4: *Let $T \in \mathbf{T}^\circ$ and $F \in \mathcal{C}_1^\sigma(T)$. Let Y be a compact interval included in X_F and $Y_\infty = \{x \in Y \mid \forall i \geq 1, F^i(x) \in Y + \mathbb{Z}\}$. Suppose that for all integers $i \geq 1$,*

$$\text{(2)} \qquad \text{if } x \in Y_\infty \text{ and } F^i(x) \in Y + k \text{ with } k \leq 0, \text{ then } F^i(x) - k < x.$$

Then there exists an integer M_1 such that, if $x \in Y_\infty$ verifies $\forall\, 0 \leq i \leq M$, $\exists k_i \leq 0$, $F^i(x) \in X + k_i$, then $M \leq M_1$.

Proof. Let $X_1, \dots, X_N$ and $p_1, \dots, p_N$ be the intervals and integers given by Lemma 4.2, and let d denote a distance on T. We are going to prove by induction on n decreasing from N to 1 that there exists an integer M_n such that

$$\text{(3)} \qquad \text{if } x \in Y_\infty \text{ verifies } \forall\, 0 \le i \le M, \exists k_i \le 0, F^i(x) \in \bigcup_{j=n}^{N} X_j + k_i,$$

then $M \le M_n$. This property for $n = 1$ is the statement of the lemma (notice that $F^i(x) \in Y + k_i$ implies that $F^i(x) \in \bigcup_{j=1}^N X_j + k_i$ because $x \in Y_\infty \subset X_\infty$).

First, let us prove the induction property for $n = N$. Let $x \in Y_\infty$ such that $\forall\, 0 \le i \le M$, $F^i(x) \in X_N + k_i$ with $k_i \le 0$. According to the definition of X_N and p_N, this implies that for all $0 \le i \le M$, $k_i = ip_N$ and $p_N \le 0$. Because of (2), $(F^i(x) - k_i)_{0 \le i \le M}$ is a decreasing sequence in Y. Then the induction property for N is given by the following fact.

Fact: *There exists an integer M_N such that, if $(F^i(x) - k_i)_{0 \le i \le M}$ is a decreasing sequence in Y with $x \in Y_\infty$ and $k_{i+1} \le k_i$ for all $0 \le i \le M-1$, then $M \le M_N$.*

Proof of the Fact. For all integers $k \le 0$, we define

$$\delta_k = \inf\{d(x, F(x) - k) \mid x \in Y_\infty, F(x) \in Y + k\}.$$

According to (2), for all $x \in Y_\infty$, $F(x) - k \ne x$. Since F is continuous and Y_∞ is compact, this implies that $\forall k \le 0$, $\delta_k > 0$. Moreover, the set of integers k such that $F(Y) \cap (Y + k) \ne \emptyset$ is finite, and thus

$$\delta = \inf\{\delta_k \mid k \le 0, F(Y) \cap (Y + k) \ne \emptyset\} > 0.$$

Consequently, $(F^i(x) - k_i)_{0 \le i \le M}$ is a decreasing sequence in Y and, for all $0 \le i \le M - 1$, $d(F^i(x) - k_i, F^{i+1}(x) - k_{i+1}) \ge \delta$. Since

$$d(x - k_0, F^M(x) - k_M) = \sum_{i=0}^{M-1} d(F^i(x) - k_i, F^{i+1}(x) - k_{i+1}),$$

this implies that $\operatorname{diam}(Y) \ge M\delta$. This proves the fact if we take $M_N \ge \operatorname{diam}(Y)/\delta$. ∎

Now, suppose that the induction property holds for $n+1$ with $2 \le n+1 \le N$, and let x satisfy (3) for n. We can assume that there exists $i \in \{0, \dots, M\}$ such that $F^i(x) \in X_n + k_i$ (otherwise x already satisfies (3) for $n + 1$). Let

$$i_0 = \min\{i \in \{0, \dots, M\} \mid F^i(x) - k_i \in X_n\}.$$

By choice of i_0, $F^{i_0} - k_{i_0} \in X_n$ and $\forall 0 \le i \le i_0 - 1, F^i(x) \in \bigcup_{j=n+1}^N X_j + k_i$. Thus $i_0 - 1 \le M_{n+1}$ by the induction property for $n+1$. We split the proof into two cases depending on p_n.

Case 1: $p_n \le 0$. If $F^i(x) \in X_n + k_i$, then $F^{i+1}(x) \in X + k_{i+1}$ with $k_{i+1} = k_i + p_n \le k_i$. According to (2), $F^{i+1}(x) - k_{i+1} < F^i(x) - k_i$, and thus $F^{i+1}(x) - k_{i+1} \in X_n$ because $X_j > X_n$ if $j > n$. This implies that, for all $i_0 \le i \le M$, $F^i(x) - k_i$ belongs to X_n, and $(F^i(x) - k_i)_{i_0 \le i \le M}$ is a decreasing sequence in Y with $k_{i+1} \le k_i$ for all $i_0 \le i \le M-1$. Then the fact above says that $M - i_0$ is bounded by M_N. Hence $M = i_0 + (M - i_0) \le M_{n+1} + M_N + 1$. This proves the induction property for n in this case if we take $M_n \ge M_{n+1} + M_N + 1$.

Case 2: $p_n \ge 1$. Let $K = \max\{k \ge 0 \mid F(Y) \cap (Y - k) \neq \emptyset\}$. First we show by induction on i that for all $i \in \{i_0, \ldots, M\}$,

$$k_i \ge k_{i_0} - K. \tag{4}$$

This is trivially true for $i = i_0$. Suppose that (4) is true for i. If $k_i \ge k_{i_0}$, then $k_{i+1} \ge k_i - K \ge k_{i_0} - K$. If $k_i < k_{i_0}$, then $F^i(x) - k_i < F^{i_0}(x) - k_{i_0}$ by (2), and thus $F^i(x) - k_i \in X_n$ by definition of n. Hence $k_{i+1} = k_i + p_n > k_i \ge k_{i_0} - K$. Therefore, (4) is true for all $i_0 \le i \le M$.

Let A be the number of integers $k \le 0$ such that $Y + k$ contains some point of $(F^i(x))_{i_0 \le i \le M}$. Equation (4) implies that $A \le -k_{i_0} + K + 1$. Moreover, $-k_{i_0} \le (i_0 - 1)K$ and $i_0 - 1 \le M_{n+1}$. Thus, if we set $A_0 = (M_{n+1} + 1)K + 1$, we have $A \le A_0$, with A_0 a constant independent of x, i_0, M. By Dirichlet's drawer principle, there exists an integer k such that $Y + k$ contains q points points among $(F^i(x))_{i_0 \le i \le M}$ with $q \ge (M - i_0)/A_0$.

We are going to show that this implies that $M - i_0$ is bounded, using an argument taken from the proof of [2, Lemma 5.17]. Let $i_1 < i_2 < \cdots < i_q$ be the integers among $\{i_0, \ldots, M\}$ such that $F^{i_j}(x) \in Y + k$ for all $1 \le j \le q$. According to (2), $(F^{i_j}(x))_{1 \le j \le q}$ is a decreasing sequence in $Y + k$. This implies that

$$d(F^{i_q}(x), F^{i_1}(x)) = \sum_{j=1}^{q-1} d(F^{i_j}(x), F^{i_{j+1}}(x)). \tag{5}$$

For $j = 1, \ldots, q-1$, we set $\Delta_j = i_{j+1} - i_j$. We also set

$$c = \#\{j \in \{1, \ldots, q-1\} \mid \Delta_j \le 2A_0\}.$$

There are $q-1-c$ integers j such that $\Delta_j \geq 2A_0+1$, and for the rest we have $\Delta_j \geq 1$. Thus we have

$$M - i_0 \geq i_q - i_1 = \Delta_1 + \cdots + \Delta_{q-1} \geq c + (2A_0+1)(q-1-c).$$

Hence

$$c \geq \frac{(2A_0+1)(q-1)-(M-i_0)}{2A_0} \geq q-1-\frac{M-i_0}{2A_0}.$$

Since $q \geq \frac{M-i_0}{A_0}$, we get that

$$c \geq \frac{M-i_0}{2A_0} - 1. \tag{6}$$

Let $d(i) = \min\{d(F^i(x),x) \mid x \in Y_\infty, F^i(x) \in Y\}$. For every $i \geq 1$, $d(i) > 0$ because $\forall x \in Y_\infty, F^i(x) \neq x$ by (2). We have $d(F^{i_{j+1}}(x), F^{i_j}(x)) \geq d(\Delta_j)$. In Equation (5), there are c integers j such that $\Delta_j \leq 2A_0$. Thus we get that $d(F^{i_q}(x), F^{i_1}(x)) \geq cD$, where $D = \min\{d(1),\ldots,d(2A_0)\} > 0$. It follows that $\mathrm{diam}\,(X) \geq cD$ and thus, by (6),

$$M - i_0 \leq 2A_0\left(\frac{\mathrm{diam}\,(X)}{D}+1\right).$$

Finally,

$$M = i_0 + (M-i_0) \leq M_N + 2A_0\left(\frac{\mathrm{diam}\,(X)}{D}+1\right)+1.$$

This proves the induction property for n in case 2, and this concludes the proof of the lemma. ∎

The next lemma is aimed to be applied first with $T' = T_{\mathbb{R}}$, $Y = X_F$ and $Z = X_1$, where X_1 is defined in Lemma 4.2. After dealing with X_1, an induction will be done to deal with $X_2,\ldots,X_N$, that is why the lemma is stated with general notations.

LEMMA 4.5: *Let $T \in \mathbf{T}^\circ$ and $F \in \mathcal{C}_1^\sigma(T)$. Let T' be a closed connected subset of T such that $T_{\mathbb{R}} \subset T'$, $T'+1 = T'$ and $F(T') \subset T'$. Let Y denote the compact subinterval of X_F equal to $\overline{T \setminus T'} \cap r^{-1}([0,1))$ and define*

$$Y_\infty = \{x \in Y \mid \forall n \geq 1, F^n(x) \in Y + \mathbb{Z}\}.$$

Let Z be a compact subinterval of Y such that $F(\min Z) \in T'$ and $F(Z) \cap (Z+\mathbb{Z}) \neq \emptyset$. Assume that $\inf \mathrm{Rot}_{Z\cap Y_\infty}(F) \geq p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, and

$$\forall x \in \overline{\bigcup_{n\geq 0}(F^n(Z)+\mathbb{Z})} \cap Y_\infty, \quad \forall n \geq 1,\ F^{nq}(x) \neq x + np.$$

Then $\inf \mathrm{Rot}_{Z\cap Y_\infty}(F) > p/q$.

Proof. We note $x_0 = \min Y$ and $Y' = \overline{\bigcup_{n\geq 0}(F^n(Z)+\mathbb{Z})} \cap Y$. For all $n \geq 1$, $F^n(\min Z) \in T'$ (because T' is F-invariant), which implies that $(F^n(Z)+\mathbb{Z})\cap Y$ is either empty, or an interval containing x_0. In addition, $Z \cap (F(Z)+\mathbb{Z})$ is non-empty by assumption, and thus Y' is a non-empty compact subinterval of Y containing x_0. Moreover, $(F(Y')+\mathbb{Z})\cap Y \subset Y'$, and thus

$$Y' \cap Y_\infty = \{x \in Y' \mid \forall n \geq 1, F^n(x) \in Y' + \mathbb{Z}\}.$$

Let $Y'' = \bigcup_{n\geq 0}(F^n(Z)+\mathbb{Z})\cap Y$. Since $\rho_F(x) = \rho_F(F^n(x)+k)$ for all $n \geq 0$ and $k \in \mathbb{Z}$, it is clear that $\mathrm{Rot}_{Z\cap Y_\infty}(F) = \mathrm{Rot}_{Y''\cap Y_\infty}(F)$. The set Y'' is an interval and $Y' \setminus Y''$ is either empty, or reduced to $\{\max Y'\}$. If $Y' \setminus Y''$ is non-empty, it can be shown that $\max Y'$ is a fixed (mod 1) point and that there exists a point $z \in Z$ whose orbit is attracted by $\max Y'$, and hence $\rho_F(z) = \rho_F(\max Y')$. The proof is not straightforward, but it is identical to the proofs of Lemma 5.3 and Theorem 5.5 in [2], and thus we do not repeat it. As a consequence, we get that $\mathrm{Rot}_{Z\cap Y_\infty}(F) = \mathrm{Rot}_{Y'\cap Y_\infty}(F)$.

Let $x \in Y' \cap Y_\infty$, and for all $i \geq 1$ let $k_i \in \mathbb{Z}$ such that $F^i(x) \in Y + k_i$. Suppose that $F^{nq}(x) - k_{nq} \geq x$. Then, using that $F^i(x_0) \in T'$ and $F^i(x) \in Y$ for all $i \geq 0$, we get

$$[x_0, x] \xrightarrow[F]{+} [x_0, F(x)] + k_1 \xrightarrow[F]{+} \cdots \xrightarrow[F]{+} [x_0, F^{nq-1}(x)] + k_{nq-1} \xrightarrow[F]{+} [x_0, x] + k_{nq}.$$

According to Proposition 3.5, there exists $y \in [x_0, x] \subset Y'$ such that $F^{nq}(y) = y + k_{nq}$ and $\forall\, 0 \leq i \leq nq, F^i(y) \in Y + \mathbb{Z}$. Hence $y \in Y' \cap Y_\infty$. By assumption, k_{nq} cannot be equal to np. Moreover, $\rho_F(y) = k_{nq}/nq \in \mathrm{Rot}_{Z\cap Y_\infty}(F)$, thus $k_{nq} > np$. Therefore, if we set $G = F^q - p$, we have, for all $n \geq 1$,

$$\text{if } x \in Y' \cap Y_\infty \text{ and } G^n(x) \in Y + k \text{ with } k \leq 0 \text{ then } G^n(x) - k < x.$$

We can apply Lemma 4.4 to the map G and the interval Y', and we get that there exists an integer M_1 such that

$$\text{if } x \in Y' \cap Y_\infty \text{ verifies } \forall\, 1 \leq n \leq M,\ \exists\, k_n \leq 0, G^n(x) \in Y + k_n, \text{ then } M < M_1. \tag{7}$$

Now let $x \in Z \cap Y_\infty$. For all $n \geq 0$, $G^n(x) \in Y' + \mathbb{Z}$. According to (7), there exist an increasing sequence of positive integers $(n_i)_{i\geq 1}$ and integers $(k_i)_{i\geq 1}$ such that

$$\forall i \geq 1,\ G^{n_i}(x) \in Y' + k_i,\ n_{i+1} - n_i \leq M_1 \quad \text{and} \quad k_{i+1} \geq k_i + 1.$$

This implies that $\overline{\rho}_G(x) = q\overline{\rho}_F(x) - p \geq 1/M_1 > 0$. This concludes the proof of the lemma. ∎

Let us restate Lemma 4.5 when $T' = T_{\mathbb{R}}$, $Y = X_F$ and $Z = X_1$: if $\inf \mathrm{Rot}_{X_1 \cap X_\infty}(F) = p/q$ and $F(X_1) \cap (X_1 + \mathbb{Z}) \neq \emptyset$, then there exist a point $x \in \overline{\bigcup_{n\geq 0}(F^n(X_1) + \mathbb{Z})} \cap X_\infty$ such that x is periodic (mod 1) and $\rho_F(x) = p/q$. Notice that the assumption $F(X_1) \cap (X_1 + \mathbb{Z}) \neq \emptyset$ is fulfilled as soon as $\mathrm{Rot}_{X_1 \cap X_\infty}(F) \neq \emptyset$. Indeed, if $F(X_1) \cap (X_1 + \mathbb{Z}) = \emptyset$, then $F(X_1) \cap (X_F + \mathbb{Z}) \subset [\min X_F, \min X_1)$, and thus $X_1 \cap X_\infty = \emptyset$.

4.3. Rotation set of X_1. In the sequel, we shall heavily use the fact that X_F is an interval with an endpoint in $T_{\mathbb{R}}$. By definition, $\min X_F$ belongs to $T_{\mathbb{R}}$ and $T_{\mathbb{R}}$ is invariant by F. Hence $F(\min X_F) \in T_{\mathbb{R}}$. Therefore, if I is a subinterval of X_F such that $\min I = \min X_F$ and $F(I) \cap X_F \neq \emptyset$, then necessarily $F(I) \cap X_F$ is an interval containing $\min X_F$. This simple observation allows us to study the rotation set of the interval X_1 defined in Lemma 4.2. This is done in Proposition 4.7, by considering $T' = T_{\mathbb{R}}$ and $Y_1, \ldots, Y_M = X_1, \ldots, X_N$. When this is done for X_1, the idea is to proceed by induction for the rotation sets of $X_2, \ldots, X_N$, which is why the proposition is stated with more general notations.

In the proof of Proposition 4.7, we shall need the next, technical lemma.

Lemma 4.6: *Let $(n_k)_{k\geq 0}$ be a sequence of real numbers bounded from above by some constant C. Let $\varepsilon > 0$,*

$$L = \limsup_{k\to+\infty} \frac{n_0 + \cdots + n_{k-1}}{k}$$

and $l < L - \varepsilon$. Then there exists an integer $k \geq 1$ such that $\frac{n_0+\cdots+n_{k-1}}{k} \geq L - \varepsilon$ and $n_k \neq l$.

Proof. Let K be an integer such that $C/K < \varepsilon/2$. Let

$$E = \left\{ k \in \mathbb{Z}^+ \mid \frac{n_0 + \cdots + n_k}{k+1} \geq L - \varepsilon/2 \right\}.$$

The set E is infinite by definition of L. We are going to do a proof by contradiction. We assume that

$$n_k = l \quad \text{for all } k \in E \text{ such that } k > K. \tag{8}$$

If E contains all integers $n \geq N$ for some N, then $L = l$, which is absurd. Thus there exists an integer $k > K$ such that $k \in E$ and $k-1 \notin E$. We have

$$\frac{n_0 + \cdots + n_k}{k+1} = \frac{k}{k+1} \cdot \frac{n_0 + \cdots + n_{k-1}}{k} + \frac{1}{k+1} n_k.$$

By definition of E,

$$\frac{n_0 + \cdots + n_k}{k+1} \geq L - \varepsilon/2 \quad \text{and} \quad \frac{n_0 + \cdots + n_{k-1}}{k} < L - \varepsilon/2.$$

Moreover, $n_k = l < L - \varepsilon/2$ by (8). Thus

$$\frac{n_0 + \cdots + n_k}{k+1} < \frac{k}{k+1}(L - \varepsilon/2) + \frac{1}{k+1}(L - \varepsilon/2) = L - \varepsilon/2,$$

which is a contradiction. Therefore, (8) does not hold, and there exists $k \in E$ such that $k > K$ and $n_k \neq l$. Moreover,

$$\begin{aligned}\frac{n_0 + \cdots + n_{k-1}}{k} &= \frac{k+1}{k} \cdot \frac{n_0 + \cdots + n_k}{k+1} - \frac{n_k}{k} \\ &> \frac{n_0 + \cdots + n_k}{k+1} - \frac{C}{K} \\ &> L - \varepsilon/2 - \varepsilon/2 = L - \varepsilon.\end{aligned}$$

Such an integer k is suitable. ∎

Proposition 4.7: *Let $T \in \mathbf{T}^\circ$ and $F \in \mathcal{C}_1^\sigma(T)$. Let T' be a connected subset of T such that $T_\mathbb{R} \subset T'$, $T' + 1 = T'$ and $F(T') \subset T'$. Let Y denote the compact subinterval of X_F equal to $\overline{T \setminus T'} \cap r^{-1}([0,1))$ and define $Y_\infty = \{x \in Y \mid \forall n \geq 1, F^n(x) \in Y + \mathbb{Z}\}$. Let $Y_1, \ldots, Y_M$ be disjoint compact subintervals of Y and $q_1, \ldots, q_M \in \mathbb{Z}$ such that:*

(a) $Y_1 < \cdots < Y_M$,
(b) $F(Y_i) \subset (Y + q_i) \cup T'$ *for all* $1 \leq i \leq M$,
(c) $F(\min Y_1) = \min Y \pmod 1$,
(d) $F(Y \setminus (Y_1 \cup \cdots \cup Y_M)) \cap (Y + \mathbb{Z}) = \emptyset$.

Assume that $Y_1 \cap Y_\infty \neq \emptyset$. Then there exists a compact interval $I \subset \mathbb{R}$ such that:

(i) $\mathrm{Rot}_{Y_1 \cap Y_\infty}(F) = \mathrm{Rot}^+_{Y_1 \cap Y_\infty}(F) = \mathrm{Rot}^-_{Y_1 \cap Y_\infty}(F) = I$,
(ii) *I contains q_1,*
(iii) *there exists $a \in Y_1 \cap Y_\infty$ such that $F(a) = a + q_1$ and $[\min Y_1, a) \cap Y_\infty = \emptyset$,*
(iv) *if $r \in \mathrm{Int}(I) \cap \mathbb{Q}$ then there exists a periodic* (mod 1) *point $x \in Y_1 \cap Y_\infty$ with $\rho_F(x) = r$.*

(v) *if* $r \in \partial I \cap \mathbb{Q}$ *then there exists a periodic* (mod 1) *point* $x \in \overline{\bigcup_{n\geq 0}(F^n(Y_1)+\mathbb{Z})} \cap Y_\infty$ *with* $\rho_F(x) = r$.

Proof. We first prove (iii) under an additional assumption:

(9) If $\exists y_0 \in Y_1$ such that $\forall n \geq 1, F^n(y_0) \in Y_1 + \mathbb{Z}$, then (iii) holds.

Let y_0 satisfy (9), $G = F - q_1$ and $a_0 = \min Y_1$. Then $G(Y_1) \subset Y \cup T'$. For all $n \geq 0$, $G^n(y_0) \in Y_1$, and in particular $G^n(y_0) \geq a_0$. We define inductively a sequence of points $(a_i)_{i\geq 1}$ such that $a_i \in [a_{i-1}, y_0]$, $G^i(a_i) = a_0$ and $[a_0, a_i) \cap Y_\infty = \emptyset$ for all $i \geq 1$.

- Since $G(a_0) = \min Y$ (by assumption (c)) and $G(y_0) \geq a_0$, we have $a_0 \in G([a_0, y_0])$ by continuity. Thus there exists $a_1 \in [a_0, y_0]$ such that $G(a_1) = a_0$. We choose a_1 minimum with this property, which implies that $G([a_0, a_1)) \cap Y_1 = \emptyset$. Hence $[a_0, a_1) \cap Y_\infty = \emptyset$.
- Assume that $a_0, \ldots, a_i$ are already defined. Since $G^{i+1}(a_i) = G(a_0) = \min Y$ and $G^{i+1}(y_0) \geq a_0$, the point a_0 belongs to $G^{i+1}([a_i, y_0])$ by continuity. Thus there exists $a_{i+1} \in [a_i, y_0]$ such that $G^{i+1}(a_{i+1}) = a_0$. We choose a_{i+1} minimum with this property, which implies that $G^{i+1}([a_i, a_{i+1})) \cap Y_1 = \emptyset$. Hence $[a_i, a_{i+1}) \cap Y_\infty = \emptyset$. This concludes the construction of a_{i+1}.

The sequence $(a_i)_{i\geq 0}$ is non-decreasing and contained in the compact interval Y_1. Therefore, $a = \lim_{i\to+\infty} a_i$ exists and belongs to Y_1. Since $G(a_{i+1}) = a_i$, we get that $G(a) = a$. In other words, $F(a) = a + q_1$. This implies that $a \in Y_\infty$. Moreover, $[a_0, a) = \bigcup_{i\geq 0}[a_i, a_{i+1})$, and thus $[a_0, a) \cap Y_\infty = \emptyset$. This proves (9).

We split the rest of the proof into two cases.

Case 1: $F(Y_1) \cap (Y_i + \mathbb{Z}) = \emptyset$ for all $i \geq 2$ (this includes the case $M = 1$). Then $F(Y_1 \cap Y_\infty) \subset Y_1 + q_1$ and $F^n(Y_1 \cap Y_\infty) \subset Y_1 + nq_1$. Thus, for all $x \in Y_1 \cap Y_\infty$, the rotation number $\rho_F(x)$ exists and is equal to q_1. We take $I = \{q_1\}$ and we get (i) and (ii).

Since $Y_1 \cap Y_\infty$ is not empty, there exists a point y such that $\forall n \geq 0$, $F^n(y) \in Y_1 + \mathbb{Z}$. Then (9) gives (iii), which implies (v) in the present case, and (iv) is empty.

Case 2: there exists $i \geq 2$ such that $F(Y_1) \cap (Y_i + \mathbb{Z}) \neq \emptyset$.
Since $F(Y_1) \subset (Y + q_1) \cup T'$, this implies that there exists $x \in Y_1$ such that $F(x) \in Y_i + q_1$, and thus $F(x) \geq \max(Y_1 + q_1)$ because $Y_1 < Y_i$. Moreover,

$F(\min Y_1) = \min Y + q_1$ by assumption. Hence

$$Y_1 \xrightarrow[F]{+} Y_1 + q_1. \tag{10}$$

By Proposition 3.5, there exists $y \in Y_1$ such that $F(y) = y + q_1$. Thus we can use (9) to get (iii).

Let $x \in Y_\infty$. By assumption (d), for all $n \geq 0$, there exists $\omega_n \in \{1, \dots, M\}$ such that $F^n(x) \in Y_{\omega_n} + \mathbb{Z}$. The sequence $(\omega_n)_{n\geq 0}$ is called the **itinerary** of x. The next two results are straightforward.

$$\forall n \geq 0,\ F^n(x) \in Y_{\omega_n} + q_{\omega_0} + q_{\omega_1} + \cdots + q_{\omega_{n-1}}. \tag{11}$$

$$\begin{aligned} \overline{\rho}_F(x) &= \limsup_{n\to+\infty} \frac{q_{\omega_0} + q_{\omega_1} + \cdots + q_{\omega_{n-1}}}{n} \quad \text{and} \\ \underline{\rho}_F(x) &= \liminf_{n\to+\infty} \frac{q_{\omega_0} + q_{\omega_1} + \cdots + q_{\omega_{n-1}}}{n}; \end{aligned} \tag{12}$$

if the limit exists, it is $\rho_F(x)$.

Let $S = \sup \mathrm{Rot}^+_{Y_1\cap Y_\infty}(F)$. Necessarily, $S \geq q_1$ because $q_1 \in \mathrm{Rot}_{Y_1\cap Y_\infty}(F)$ by (iii). We are going to show that $[q_1, S] \subset \mathrm{Rot}_{Y_1\cap Y_\infty}(F)$. If $S = q_1$ there is nothing to prove, and so we suppose that $S > q_1$. Let k be an integer such that $S > q_1 + 1/k$. Let y_k be a point in $Y_1 \cap Y_\infty$ such that $\overline{\rho}_F(y_k) \geq S - 1/2k$, and let $(\omega_n)_{n\geq 0}$ denote the itinerary of y_k. By (12),

$$\limsup_{n\to+\infty} \frac{q_{\omega_0} + q_{\omega_1} + \cdots + q_{\omega_{n-1}}}{n} = \overline{\rho}_F(y_k).$$

Applying Lemma 4.6 with $L = \overline{\rho}_F(y_k)$, $l = q_1$ and $\varepsilon = 1/2k$, we get that there exists an integer n such that

$$\frac{q_{\omega_0} + q_{\omega_1} + \cdots + q_{\omega_{n-1}}}{n} \geq \overline{\rho}_F(y_k) - 1/2k \geq S - 1/k > q_1 \quad \text{and} \quad \omega_n \neq 1. \tag{13}$$

By (11), $F^n(y_k) \in Y_{\omega_n} + q_{\omega_0} + q_{\omega_1} + \cdots + q_{\omega_{n-1}}$. Since $\omega_n \neq 1$, we have $Y_{\omega_n} > Y_1$, and thus $F^n(y_k) - (q_{\omega_0} + q_{\omega_1} + \cdots + q_{\omega_{n-1}}) > \max Y_1$. Moreover, $F^n(\min Y_1) \in F^{n-1}(\overline{T'}) \subset \overline{T'}$ by assumption (c) and invariance of T'. If we let $I_k = [\min Y_1, y_k]$, $N_k = q_{\omega_0} + \cdots + q_{\omega_{n-1}}$ and $n_k = n$, we have then

$$I_k \xrightarrow[F^{n_k}]{+} Y_1 + N_k. \tag{14}$$

Since $I_k \subset Y_1$, Equations (10) and (14) give

$$\begin{aligned} I_k \xrightarrow[F^{n_k}]{+} I_k + N_k \quad &\text{and} \quad I_k \xrightarrow[F^{n_k}]{+} Y_1 + N_k, \\ Y_1 \xrightarrow[F^{n_k}]{+} I_k + n_k q_1 \quad &\text{and} \quad Y_1 \xrightarrow[F^{n_k}]{+} Y_1 + n_k q_1. \end{aligned} \tag{15}$$

Let $r \in [q_1, S) \cap \mathbb{Q}$. By (13), there exists an integer k such that $N_k/n_k > r$. We apply Lemma 3.6 with $I = Y_1$, $J = I_k$, $G = F^{n_k}$, $m_1 = n_k q_1$, $m_2 = N_k$ and $p/q = rn_k \in [q_1 n_k, N_k)$:

(16) $\exists\, \mathcal{C}_r$ a chain of intervals for F whose first and last intervals are translations of Y_1, such that $r = W(\mathcal{C}_r)/L_F(\mathcal{C}_r)$,

and

(17) there exists a periodic (mod 1) point $x \in Y_1$ with
$$\rho_F(x) = \frac{1}{n_k}\rho_{F^{n_k}}(x) = r.$$

We need to show that $x \in Y_\infty$. This is a consequence of the following fact.

FACT: *Let $x \in Y_1$ such that $F^n(x) \in Y_1 + \mathbb{Z}$ for infinitely many n. Then either $x \in Y_\infty$, or there exists n such that $F^n(x)$ is a fixed* (mod 1) *point in $Y_1 \cap Y_\infty \cap T'$.*

Proof of the Fact. If $\forall n \geq 0$, $F^n(x) \notin T'$, then $x \in Y_\infty$ by assumption (d) of the proposition. Suppose on the contrary that there exists n_0 such that $F^{n_0}(x) \in T'$. Hence $F^n(x) \in T'$ for all $n \geq n_0$. Let $e = \min Y$. By definition of Y, the set $Y \cap T'$ is included in $\{e\}$ (we have not supposed that T' is closed, and thus $Y \cap T'$ may be empty). Notice that $Y_1 \cap T'$ is empty if $\min Y_1 > e$. By assumption, there exists $n_1 \geq n_0$ such that $F^{n_1}(x) \in Y_1 + \mathbb{Z}$. Hence $F^{n_1}(x) \in (Y_1 + \mathbb{Z}) \cap T'$. This implies that $F^{n_1}(x)$ is equal to e (mod 1), $\min Y_1 = e$, and $e \in T'$. By assumption (c) of the proposition, $F(\min Y_1) = e$ (mod 1). Thus, e is a fixed (mod 1) point in Y_1, and so $e \in Y_\infty$. This ends the proof of the fact. ∎

Now, let $\alpha \in [q_1, S]$. To show that there exists $x \in Y_1 \cap Y_\infty$ with $\rho_F(y) = \alpha$, we use the same method as in the proof of [2, Theorem 3.7]. We choose a sequence of rational numbers r_i in $[q_1, S) \cap \mathbb{Q}$ such that $\lim_{i \to +\infty} r_i = \alpha$. For all $i \geq 1$, let $\mathcal{C}_{r_i}$ be the chain of intervals given by (16). We define
$$\mathcal{D}_n = (\mathcal{C}r_1)^{i_1}(\mathcal{C}_{r_2})^{i_2} \cdots (\mathcal{C}_{r_n})^{i_n}.$$

Let A_n be the set of points that follow the chain $\mathcal{D}_n$. This set is compact by definition, and it is not empty because it contains at least a periodic (mod 1) point by Proposition 3.5. Moreover, $A_{n+1} \subset A_n$. Therefore, $A = \bigcap_{n \geq 1} A_n \neq \emptyset$. In the proof of [2, Theorem 3.7], it is shown that if the sequence $(i_n)_{n \geq 1}$ increases sufficiently fast and $(|r_{i_n} - \alpha|)_{n \geq 1}$ is non-decreasing, then for all $x \in A$, $\rho_F(x) = \alpha$.

Moreover, the fact above implies that for every $x \in A$ there exists n such that $F^n(x) \in Y_1 \cap Y_\infty$. Obviously, $\rho_F(x) = \rho_F(F^n(x))$. This proves that $[q_1, \sup \mathrm{Rot}^+_{Y_1\cap Y_\infty}(F)]$ is included in $\mathrm{Rot}_{Y_1\cap Y_\infty}(F)$; in addition, (iv) holds for all rational numbers $r \in [q_1, \sup \mathrm{Rot}^+_{Y_1\cap Y_\infty}(F))$ by (17). We can apply the same method to $[\inf \mathrm{Rot}^- Y_1 \cap Y_\infty(F), q_1]$. Finally, if we define $I = [\inf \mathrm{Rot}^-_{Y_1\cap Y_\infty}(F), \sup \mathrm{Rot}^+_{Y_1\cap Y_\infty}(F)]$, we get that $I \subset \mathrm{Rot}_{Y_1\cap Y_\infty}(F)$, $q_1 \in I$ (which is (ii)) and (iv) holds for all $r \in \mathrm{Int}\,(I) \cap \mathbb{Q}$. Since $\mathrm{Rot}^+_{Y_1\cap Y_\infty}(F)$ and $\mathrm{Rot}^-_{Y_1\cap Y_\infty}(F)$ both contain $\mathrm{Rot}_{Y_1\cap Y_\infty}(F)$ and are included in I, this gives (i).

Now we prove (v) for $\min I$ (the case with the maximum is symmetric, and ∂I is reduced to two points). Suppose that $\min \mathrm{Rot}_{Y_1\cap Y_\infty}(F) = p/q$. We apply Lemma 4.5 with $Z = Y_1$ and $\overline{T'}$. Since (iii) is fulfilled, the set $F(Z) \cap (Z + \mathbb{Z})$ is not empty. By refutation of Lemma 4.5, we get that there exists $x \in \overline{\bigcup_{n\geq 0}(F^n(Y_1) + \mathbb{Z})} \cap Y_\infty$ such that x is periodic (mod 1) for F and $\rho_F(x) = p/q$. This gives (v) and concludes the proof of the proposition. ∎

Example 4.8: The periodic (mod 1) point x given by Proposition 4.7(v) may not be in Y_1.

Let T be the universal covering of the graph σ, and let $F\colon T \to T$ be the continuous map of degree 1 such that $F|_{\mathbb{R}} = Id$ and F is defined on the branch of T by:

- $F(a) = F(c) = a$, $F(b) = e$, $F(d) = a + 1$, $F(e) = e + 1$,
- F is affine on each of the intervals $[a, b]$, $[b, c]$, $[c, d]$, $[d, e]$,

where $[a, e]$ is a branch of T with $a \in \mathbb{R}$ and $a < b < c < d < e$. See Figure 4 for the picture of the map F. This entirely determines F because it is of degree 1.

X_F is equal to $[a, e]$ and the intervals given by Lemma 4.2 are $X_1 = [a, c]$ (with $p_1 = 0$) and $X_2 = [d, e]$ (with $p_2 = 1$). F is an affine Markov map and the restriction of its Markov graph to X_1, X_2 is given in Figure 5. See [2, Section 6.1] for general results on Markov maps in this context, and in particular how it is possible to deduce periodic (mod 1) points and rotation numbers from the Markov graph.

It can easily be deduced from the Markov graph of F that $\mathrm{Rot}_{X_1\cap X_\infty}(F) = [0, 1]$ and the unique periodic (mod 1) point $x \in X_F$ such that $\rho_F(x) = 1$ is $x = e$, which does not belong to X_1.

In addition, we notice that $\mathrm{Rot}_{\mathbb{R}}(F) = \{0\}$. Thus $\mathrm{Rot}(F) = [0, 1]$ and $\mathrm{Rot}_{\mathbb{R}}(F)$ is not a connected component of $\mathrm{Rot}(F)$. ∎

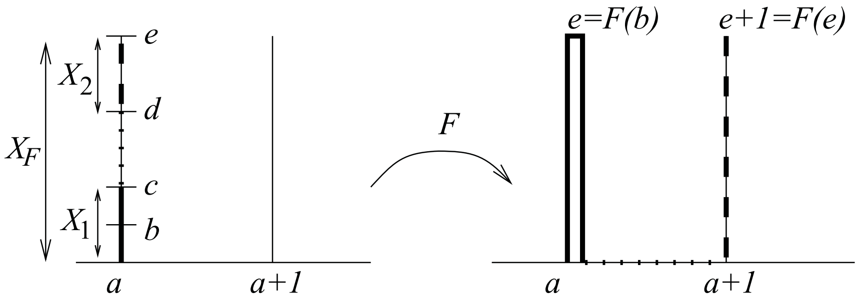

Figure 4. The action of F on the branch X_F.

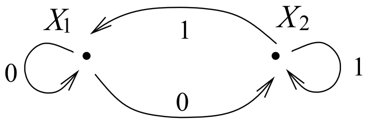

Figure 5. The Markov graph of F restricted to the vertices X_1 and X_2 (actually, X_1 represents the two vertices $[a, b]$ and $[b, c]$). An arrow $A \xrightarrow{i} B$ means than $F(A) \supset B + i$.

4.4. Rotation set of F. Now, we are ready to prove that the set $\mathrm{Rot}(F)$ is closed and has finitely many connected components, and that every rational number in $\mathrm{Rot}(F)$ is the rotation number of some periodic (mod 1) point. Notice that in the following theorem, the intervals $I_0, \dots, I_k$ may be not disjoint; in particular, $I_0 = \mathrm{Rot}_{\mathbb{R}}(F)$ may not be a connected component of $\mathrm{Rot}(F)$ (see Example 4.8).

Theorem 4.9: *Let $T \in \mathsf{T}^\circ$, $F \in \mathcal{C}_1^\sigma(T)$ and*

$$X_\infty = \{x \in X_F \mid \forall n \geq 1, F^n(x) \in X_F + \mathbb{Z}\}.$$

Then there exist an integer $k \geq 0$ and compact non empty intervals $I_0, \dots, I_k$ in $\mathbb{R}$ such that:

- $\mathrm{Rot}(F) = \mathrm{Rot}^+(F) = \mathrm{Rot}^-(F) = I_0 \cup \cdots \cup I_k$,
- $I_0 = \mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}_{T_{\mathbb{R}}}(F)$,
- $\forall\, 1 \leq i \leq k, \forall r \in I_i \cap \mathbb{Q}$, *there exists a periodic* (mod 1) *point* $x \in X_\infty$ *with* $\rho_F(x) = r$,

- $\forall\, 1 \le i \le k,\ I_i \cap \mathbb{Z} \neq \emptyset$.

Moreover, if N is the integer given by Lemma 4.2, then $k \le N$.

Proof. Consider $X_1, \ldots, X_N$ and $p_1, \ldots, p_N$ given by Lemma 4.2. We define inductively $T_1, \ldots, T_N \subset T$ and $I_1, \ldots, I_N \subset \mathbb{R}$ such that:

(a) T_i is a connected subset of T such that $T_i + 1 = T_i$, $F(T_i) \subset T_i$ and, for all $2 \le i \le N$, $T_{i-1} \cup X_{i-1} \subset T_i$;
(b) either I_i is empty, or I_i is a compact interval containing p_i such that, for all $r \in I_i \cap \mathbb{Q}$, there exists a periodic (mod 1) point $x \in X_\infty$ with $\rho_F(x) = r$;
(c) $\mathrm{Rot}_{T_i \cup X_i}(F) = \mathrm{Rot}_{T_i}(F) \cup I_i$, and the same equality is valid with Rot^+ and Rot^-;
(d) if $i \ge 2$, $\mathrm{Rot}_{T_i}(F) = \mathrm{Rot}_{T_{i-1} \cup X_{i-1}}(F)$, and the same equality is valid with Rot^+ and Rot^-.

Let $T_1 = T_{\mathbb{R}}$. It satisfies (a). If $X_1 \cap X_\infty = \emptyset$, we take $I_1 = \emptyset$. Otherwise, we apply Proposition 4.7 with $T' = T_1$, $Y = X_F$ and $X_1, \ldots, X_N$ in place of $Y_1, \ldots, Y_M$. It provides a compact interval $I_1 = I = \mathrm{Rot}_{X_1 \cap X_\infty}$ that satisfies (b). Moreover, $\mathrm{Rot}_{T_1 \cup X_1}(F) = \mathrm{Rot}_{T_1}(F) \cup \mathrm{Rot}_{X_1 \cap X_\infty}(F)$, and the same equality is valid with Rot^+ and Rot^-. Hence (c) is satisfied for $i = 1$.

Let $i \ge 2$. Suppose that T_j and I_j are already defined for all $1 \le j \le i-1$, and satisfy (a)–(d). Define

$$A_i = [\min X_F, \min X_{i-1}) \cup X_{i-1} \cup \left(\left(\bigcup_{n \ge 1} F^n(X_{i-1}) + \mathbb{Z} \right) \cap X_F \right).$$

For all $n \ge 1$, $F^n(\min X_{i-1}) \in T_{\mathbb{R}}$, and thus $(F^n(X_{i-1}) + \mathbb{Z}) \cap X_F$ is either empty, or a compact subinterval of X_F containing $\min X_F$. Therefore, A_i is a subinterval of X_F containing $\min X_F$ and X_{i-1}. Let $T_i = T_{i-1} \cup (A_i + \mathbb{Z})$. It is a connected subset of T, $T_i + 1 = T_i$ and $T_{i-1} \cup X_{i-1} \subset T_i$. Let us show that $F(T_i) \subset T_i$. Let $x \in T_{i-1} \cup A_i$. We distinguish 3 cases.

- If $x \in T_{i-1}$ then $F(x) \in T_{i-1}$ by invariance of T_{i-1}.
- If $x \in [\min X_F, \min X_{i-1})$, then either $x \in X_1 \cup \cdots \cup X_{i-2} \subset T_{i-1}$ and $F(x) \in T_{i-1}$, or $x \in X_F \setminus (X_1 \cup \cdots \cup X_N)$ and $F(x) \in T_{\mathbb{R}}$.
- If $x \in \left(\bigcup_{n \ge 0} F^n(X_{i-1}) + \mathbb{Z} \right) \cap X_F$, then either

$$F(x) \in \Big(\bigcup_{n \ge 0} F^n(X_{i-1}) + \mathbb{Z} \Big) \cap (X_F + \mathbb{Z}) \subset A_i + \mathbb{Z}, \quad \text{or} \quad F(x) \in T_{\mathbb{R}}.$$

Consequently, $F(T_i) \subset T_i$, and (a) is satisfied. Moreover, what precedes also shows that

$$\mathrm{Rot}_{T_i}(F) = \mathrm{Rot}_{T_{i-1}}(F) \cup \mathrm{Rot}_{X_{i-1}}(F) = \mathrm{Rot}_{T_{i-1}\cup X_{i-1}}(F),$$

and the same equality holds with Rot^+ and Rot^-, which is (d) for i.

If $F(X_i) \subset T_i$, we take $I_i = \emptyset$ and (b)–(c) are clearly satisfied. Otherwise, let $b \in X_i$ such that $F(b) \notin T_i$. Let $Y = \overline{T \setminus T_i} \cap X_F$ and $Y_\infty = \{x \in Y \mid \forall n \geq 1, F^n(x) \in Y + \mathbb{Z}\}$. The set Y is a compact subinterval of X_F and $Y \cap \overline{T_i} = \{\min Y\}$. Since $F(b) \notin T_i$, we have $b \in Y$ by invariance of T_i, and $F(b) \in Y + p_i$ because $F(X_i) \subset (X_F + p_i) \cup T_{\mathbb{R}}$.

Let $a = \max(\min X_i, \min Y)$. We can define $c = \min\{x \in [a,b] \mid F(x) \in Y + \mathbb{Z}\}$ because $b \geq a$. Moreover, $F(a) \in \overline{T_i}$ because $F(\min X_i) \in T_{\mathbb{R}}$ and $F(\min Y) \in F(\overline{T_i}) \subset \overline{T_i}$. Therefore, $F(c) = \min Y \pmod 1$ by minimality. Let $X_i' = [c, \max X_i] \subset X_i$. We apply Proposition 4.7 with $T' = T_i$ and $X_i', X_{i+1}, \ldots, X_N$ in place of $Y_1, \ldots, Y_M$. We obtain a compact interval $I_i = I = \mathrm{Rot}_{X_i' \cap Y_\infty}(F)$ that satisfies (b) for i.

We have

$$X_i' \cap Y_\infty = \{x \in X_i \mid \forall n \geq 0, F^n(x) \notin \mathrm{Int}\,(T_i)\}.$$

Therefore, $\mathrm{Rot}_{T_i \cup X_i}(F) = \mathrm{Rot}_{T_i}(F) \cup \mathrm{Rot}_{X_i' \cap Y_\infty}(F) = \mathrm{Rot}_{T_i}(F) \cup I_i$, and the same equality holds with Rot^+ and Rot^-. Hence (c) is satisfied for i. This concludes the construction of T_i and I_i.

Now, we end the proof of the theorem. Since $X_1 \cup \cdots \cup X_N \subset T_N \cup X_N$, we have $F(T \setminus (T_N \cup X_N)) \subset T_{\mathbb{R}} \subset T_N$, and thus it is clear that $\mathrm{Rot}(F) = \mathrm{Rot}_{T_N \cup X_N}(F)$. Combining this with (c) and (d), we get that

$$\begin{aligned}
\mathrm{Rot}(F) &= \mathrm{Rot}_{T_N \cup X_N}(F) \\
&= \mathrm{Rot}_{T_N}(F) \cup I_N \\
&= \mathrm{Rot}_{T_{N-1} \cup X_{N-1}}(F) \cup I_N \\
&= \mathrm{Rot}_{T_{N-1}}(F) \cup I_{N-1} \cup I_N \\
&= \cdots \\
&= \mathrm{Rot}_{T_1}(F) \cup I_1 \cup \cdots \cup I_N
\end{aligned}$$

and the same equalities hold with Rot^+ and Rot^-. Let $I_0 = \mathrm{Rot}_{\mathbb{R}}(F)$. By Theorem 4.1, I_0 is a non-empty compact interval and $I_0 = \mathrm{Rot}_{T_{\mathbb{R}}}(F) = \mathrm{Rot}^+_{T_{\mathbb{R}}}(F) = \mathrm{Rot}^-_{\mathbb{R}}(F)$. To conclude, it remains to remove the empty intervals among $I_1, \ldots, I_N$. ∎

If the empty rotation intervals are not removed in the proof of Theorem 4.9, then the theorem can be stated as follows:

THEOREM 4.9′: *Let* $T \in \mathbf{T}^\circ$ *and* $F \in \mathcal{C}_1^\sigma(T)$. *Let* $X_1, \ldots, X_N$, $p_1, \ldots, p_N$ *be given by Lemma 4.2 and* $X_\infty = \{x \in X_F \mid \forall n \geq 1, F^n(x) \in X_F + \mathbb{Z}\}$. *Then there exist compact intervals* $I_0, \ldots, I_N$ *in* $\mathbb{R}$ *such that:*

- $\mathrm{Rot}(F) = \mathrm{Rot}^+(F) = \mathrm{Rot}^-(F) = I_0 \cup \cdots \cup I_k$,
- $I_0 = \mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}_{T_{\mathbb{R}}}(F)$,
- *for all* $1 \leq i \leq N$, *either* $I_i = \emptyset$ *or* $p_i \in I_i$,
- $\forall\ 1 \leq i \leq N, \forall r \in I_i \cap \mathbb{Q}$, *there exists a periodic* (mod 1) *point* $x \in \overline{\bigcup_{n\geq 0}(F^n(X_i) + \mathbb{Z})} \cap X_\infty$ *with* $\rho_F(x) = r$; *if, in addition,* $r \in \mathrm{Int}\,(I_i)$, *then* x *can be chosen in* $X_i \cap X_\infty$.

References

[1] Ll. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, Advanced Series in Nonlinear Dynamics, Vol. 5, World Scientific Publications, River Edge, NJ, 1993.

[2] Ll. Alsedà and S. Ruette, *Rotation sets for graph maps of degree 1*, Annales de l'Institut Fourier **58** (2008), 1233–1294.

[3] L. Block, J. Guckenheimer, M. Misiurewicz and L. S. Young, *Periodic points and topological entropy of one dimensional maps* in *Global Theory of Dynamical Systems*, Lecture Notes in Mathematics, Vol. 819, Springer-Verlag, Berlin, 1980, 18–34.

[4] M. Misiurewicz, *Horseshoes for mappings of the interval*, L'Académie Polonaise des Sciences. Bulletin. Série des Sciences Mathématiques **27** (1979), 167–169.