Rotation set for maps of degree 1 on sun graphs

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Abstract

For a continuous map on a topological graph containing a unique loop S, it is possible to define the degree and, for a map of degree 1, rotation numbers. It is known that the set of rotation numbers of points in S is a compact interval and for every rational r in this interval there exists a periodic point of rotation number r. The whole rotation set (i.e. the set of all rotation numbers) may not be connected and it is not known in general whether it is closed.

A sun graph is the space consisting in finitely many segments attached by one of their endpoints to a circle. We show that, for a map of degree 1 on a sun graph, the rotation set is closed and has finitely many connected components. Moreover, for all but finitely many rational numbers r in the rotation set, there exists a periodic point of rotation number r.

1 Introduction

In [2], a rotation theory is developed for continuous self maps of degree 1 of topological graphs having a unique loop. A rotation theory is usually developed in the universal covering space by using the liftings of the maps under consideration. The universal covering of a graph containing a unique loop is an "infinite tree modulo 1" (see Figure 1). It turns out that the rotation theory on the universal covering of a graph with a unique loop can be easily extended to the setting of infinite graphs that look like the space $\widehat{G}$ on Figure 2. These spaces are defined in detail in Section 2.1 and called *lifted graphs*. Each lifted graph T has a subset $\widehat{T}$ homeomorphic to the real line $\mathbb{R}$ that corresponds to an "unwinding" of a distinguished loop of the original graph. In the sequel, we identify $\widehat{T}$ with $\mathbb{R}$.

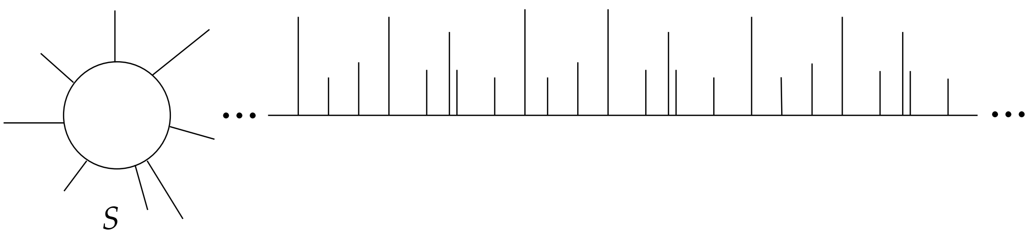

Figure 1: A sun graph on the left, and its universal covering on the right.

Given a lifted graph T and a map F from T to itself of degree one, there is no difficulty to extend the definition of rotation number to this setting in such a way that every periodic point has a rational rotation number as in the circle case. However, the obtained rotation set $\mathrm{Rot}(F)$ may not be connected (see [2, Example 1.12]). Despite of this fact, it is proved in [2] that the set $\mathrm{Rot}_{\mathbb{R}}(F)$ corresponding to the rotation numbers of all points belonging to $\mathbb{R}$, has properties which are similar to (although weaker than) those of the rotation interval for a circle map of degree one. Indeed, this set is a compact non empty interval, if $p/q \in \mathrm{Rot}_{\mathbb{R}}(F)$ then there exists

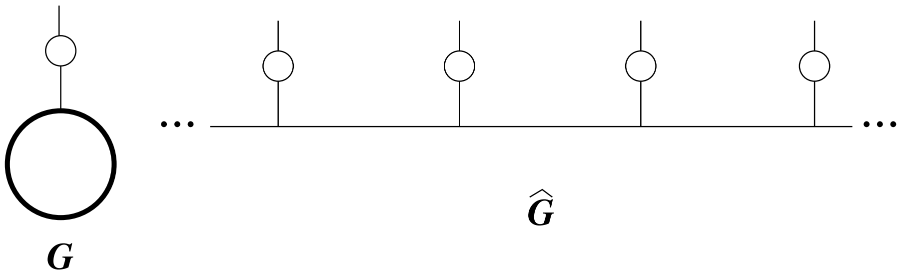

Figure 2: The graph G is unwound with respect to the bold loop to obtain $\widehat{G}$, which is a lifted graph.

a periodic point of rotation number p/q, and if $p/q \in \operatorname{Int}(\operatorname{Rot}_{\mathbb{R}}(F))$ then, for all large enough positive integers n, there exists a periodic point of period nq of rotation number p/q.

We conjecture that the whole rotation set $\operatorname{Rot}(F)$ is closed. In this paper, we prove that, when the space T is the universal covering of a sun graph (consisting in finitely many disjoint segments attached by one of their endpoints to a circle, see Figure 1), then the rotation set is the union of finitely many compact intervals. Moreover, all but finitely many rational points r in $\operatorname{Rot}(F)$ are rotation numbers of periodic (mod 1) points. It turns out that the proofs extend to a class of maps on graphs that we call sun-like maps, which are defined in Section 2.3.

This paper is the sequel of [5], which deals with the graph σ, i.e., a sun graph with a unique branch. The results obtained for σ in [5] are stronger but the methods cannot be generalised to sun graphs. Here the main tool is the construction of a countable oriented graph, and the symbolic dynamics on this graph reflects enough of the dynamics of the original map to compute rotation numbers and find periodic points. The idea is inspired by the Markov diagram introduced by Hofbauer [4] to study piecewise monotone interval maps, although the goals are very different (the Markov diagram was used to study measures of maximal entropy).

The paper is organised as follows. In Section 2, we give the definitions of the objects we deal with: lifted graphs, maps of degree 1, sun graphs and sun-like maps, rotation numbers and rotation sets; we also recall the main results on the rotation set of $\mathbb{R} \subset T$ when T is a lifted graph. In Section 3, we recall the notion of positive covering, which is a key tool to find periodic points. In Section 4, we define a partition $\mathcal{P}$ of the branches of T (where T is the universal covering of a sun graph) according to some dynamical properties and we state that the rotation number of a point can be computed using its itinerary according to the partition $\mathcal{P}$. Then, in Section 5, we define the covering graph $\mathcal{G}$ associated to this partition $\mathcal{P}$ ($\mathcal{G}$ is a countable oriented graph), which gives a relation between itineraries of points and infinite paths in this graph, and we study the structure of the graph. Finally, in Section 6, we study the rotation set of the covering graph and, in Section 7, we pull back these results on the space T and we prove the main result about the rotation set of sun-like maps.

2 Definitions and first properties

2.1 Lifted graphs

A *topological finite graph* is a compact connected set G containing a finite subset V such that each connected component of $G \setminus V$ is homeomorphic to an open interval. The aim of this section is to define in detail the class of *lifted graphs* where we develop the rotation theory. They are obtained from a topological finite graph by unwinding one of its loops. This gives a new space that contains a subset homeomorphic to the real line and that is "invariant by a translation" (see Figures 1 and 2). In [2], a larger class of spaces called *lifted spaces* is defined.

Definition 2.1 Let T be a connected topological space. We say that T is a *lifted graph* if there exist a homeomorphism $h\colon \mathbb{R} \to h(\mathbb{R}) \subset T$, and a homeomorphism $\tau\colon T \to T$ such that

i) $\tau(h(x)) = h(x+1)$ for all $x \in \mathbb{R}$,

ii) the closure of each connected component of $T \setminus h(\mathbb{R})$ is a topological finite graph that intersects $h(\mathbb{R})$ at a single point,

iii) the number of connected components C of $T \setminus h(\mathbb{R})$ such that $\overline{C} \cap h([0,1]) \neq \emptyset$ is finite.

The class of all lifted graphs will be denoted by $\mathbf{T}^\circ$.

To simplify the notation, in the rest of the paper we identify $h(\mathbb{R})$ with $\mathbb{R}$ itself. In this setting, the map τ can be interpreted as a translation by 1. So, for all $x \in T$, we write $x+1$ to denote $\tau(x)$. Since τ is a homeomorphism, this notation can be extended by denoting $\tau^m(x)$ by $x+m$ for all $m \in \mathbb{Z}$.

We endow a lifted graph T with a distance d invariant by the translation τ, i.e., $\forall x, y \in T$, $d(x+1, y+1) = d(x, y)$.

A loop is a subset homeomorphic to a circle. If G is a topological finite graph with a unique loop, then its universal covering is an infinite tree (i.e., it has no loop) and belongs to $\mathbf{T}^\circ$. Figure 1 illustrates this situation. Because of (ii) in the previous definition, if the topological finite graph G has several loops, the infinite graph obtained by unwinding a distinguished loop may or may not be a lifted graph. The essential property of the class $\mathbf{T}^\circ$ is the existence of a natural retraction from T to $\mathbb{R}$.

Definition 2.2 Let $T \in \mathbf{T}^\circ$. The retraction $r_{\mathbb{R}}\colon T \to \mathbb{R}$ is the continuous map defined as follows. When $x \in \mathbb{R}$, then $r_{\mathbb{R}}(x) := x$. When $x \notin \mathbb{R}$, there exists a connected component C of $T \setminus \mathbb{R}$ such that $x \in C$ and $\overline{C}$ intersects $\mathbb{R}$ at a single point z, and we let $r_{\mathbb{R}}(x) := z$.

2.2 Maps of degree 1 and rotation numbers

A standard approach to study the periodic points and orbits of a graph map is to work at lifting level with the periodic (mod 1) points. The results on the lifted graph can obviously been pulled back to the original graph (see [2]). Moreover, the rotation numbers have a signification only for maps of degree 1, as in the case of circle maps (see, e.g., [1] for the rotation theory for circle maps). In this paper, we deal only with maps of degree 1 on lifted graphs.

Definition 2.3 Let $T \in \mathbf{T}^\circ$. A continuous map $F\colon T \to T$ is of degree 1 if $F(x+1) = F(x)+1$ for all $x \in T$.

A point $x \in T$ is called *periodic* (mod 1) for F if there exists a positive integer n such that $F^n(x) \in x + \mathbb{Z}$. The *period* of x is the least integer n satisfying this property.

Definition 2.4 Let $T \in \mathbf{T}^\circ$, $F\colon T \to T$ a continuous map of degree 1 and $x \in T$. When the limit exists, the *rotation number* of x is

$$\rho_F(x) := \lim \frac{r_{\mathbb{R}} \circ F^n(x) - r_{\mathbb{R}}(x)}{n}.$$

The next, easy lemma states that two points in the same orbit have the same rotation number, as well as two point equal (mod 1).

Lemma 2.5 *Let $T \in \mathbf{T}^\circ$, $F\colon T \to T$ a continuous map of degree 1, and $x \in T$ such that $\rho_F(x)$ exists.*

i) $\forall k \in \mathbb{Z}$, $\rho_F(x+k) = \rho_F(x)$.

ii) $\forall n \geq 1$, $\rho_F(F^n(x)) = \rho_F(x)$.

Remark 2.6 If $F^q(x) = x + p$ with $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, then $\rho_F(x) = p/q$. Therefore all periodic (mod 1) points have rational rotation numbers.

An important object that synthesises the information about rotation numbers is the *rotation set*.

Definition 2.7 Let $T \in \mathbf{T}^\circ$ and $F\colon T \to T$ a continuous map of degree 1. For $E \subset T$, the *rotation set* of E is:

$$\mathrm{Rot}_E(F) := \{\rho_F(x) \mid x \in E \text{ and } \rho_F(x) \text{ exists}\}.$$

When $E = T$, we omit the subscript and we write $\mathrm{Rot}(F)$ instead of $\mathrm{Rot}_T(F)$.

We define

$$T_\mathbb{R} := \overline{\bigcup_{n \geq 0} F^n(\mathbb{R})}.$$

The next theorem summarises the properties of $T_\mathbb{R}$ and $\mathrm{Rot}_\mathbb{R}(F)$ (see Lemma 5.2 and Theorems 3.1, 5.7, 5.18 in [2]).

Theorem 2.8 *Let $T \in \mathbf{T}^\circ$ and $F\colon T \to T$ a continuous map of degree 1. Then $T_\mathbb{R} \in \mathbf{T}^\circ$, $\mathrm{Rot}_{T_\mathbb{R}}(F) = \mathrm{Rot}_\mathbb{R}(F)$ and the set $\mathrm{Rot}_\mathbb{R}(F)$ is a non empty compact interval. Moreover, if $r \in \mathrm{Rot}_\mathbb{R}(F) \cap \mathbb{Q}$, then there exists a periodic (*mod 1*) point $x \in T_\mathbb{R}$ such that $\rho(x) = r$.*

2.3 Sun graphs and sun-like maps

A *sun graph* is a topological finite graph that looks like the graph S on Figure 1. It is composed of a circle and finitely many disjoint compact intervals, each interval being attached by one of its endpoints to the circle.

Let $T \in \mathbf{T}^\circ$ and $F\colon T \to T$ a continuous map of degree 1. We define

$$X := \overline{T \setminus T_\mathbb{R}} \cap r_\mathbb{R}^{-1}([0,1)).$$

Then X is composed of finitely many finite graphs and $T = T_\mathbb{R} \cup (X + \mathbb{Z})$. Note that $T_\mathbb{R}$ and X implicitly depend on F.

If T is the lifting of a sun graph, then X is either empty, or composed of finitely many disjoint intervals, each intersecting $T_\mathbb{R}$ at one of its endpoint. Maps with the same properties will be called *sun-like maps*.

Definition 2.9 Let $T \in \mathbf{T}^\circ$ and $F\colon T \to T$ a continuous map of degree 1. If $T \setminus T_\mathbb{R} \cap r_\mathbb{R}^{-1}([0,1))$ is composed of finitely many intervals whose closures are disjoint, we say that F is a *sun-like* map. The intervals

$$\{\overline{C} \mid C \text{ is a connected component of } T \setminus T_\mathbb{R} \cap r_\mathbb{R}^{-1}([0,1))\}$$

are called the *branches* of F and denoted by $(X^i)_{i\in\Lambda}$, where Λ is some finite set of indices. The set of all sun-like maps of degree 1 on T is denoted by $\mathcal{S}_1(T)$.

Remark 2.10 In a sun graph, two different segments do not meet the circle at the same point because the segments are compact and disjoint. Similarly, two branches of a sun-like map are not allowed to have a common endpoint in $T_{\mathbb{R}}$. This property prevents F from oscillating infinitely many times between two branches. On the contrary, F may oscillate between a branch and $T_{\mathbb{R}}$.

Definition 2.11 Let F be a sun-like map and $(X^i)_{i\in\Lambda}$ its branches. Each branch X^i may be endowed with two opposite orders. We choose the one such that $\min X^i$ is the one-point intersection $X^i \cap T_{\mathbb{R}}$.

Consider a sun-like map F. Because of the definition of sun-like maps, all the paths starting in $T_{\mathbb{R}}$ and ending in some branch X^i must pass through the one-point intersection $T_{\mathbb{R}} \cap X^i$. Thus, if E is a connected set in T containing one point of $T_{\mathbb{R}}$ and one point of X^i, then E contains $T_{\mathbb{R}} \cap X^i = \{\min X^i\}$. We shall use this property several times.

3 Positive covering

Definition 3.1 Let F be a sun-like map and $(X^i)_{i\in\Lambda}$ its branches. For every $i \in \Lambda$, the retraction map $r_i\colon T \to X^i$ can be defined in a natural way by $r_i(x) := x$ if $x \in X^i$ and $r_i(x) := \min X^i$ otherwise.

The notion of positive covering for subintervals of $\mathbb{R}$ has been introduced in [2]. It can be extended for subintervals of any subset of T on which a retraction can be defined, as in [5]. In this paper, we shall use positive covering in the branches of F. All properties of positive covering remain valid in this context. In particular, if a compact interval I positively F-covers itself, then F has a fixed point in I (Proposition 3.5).

Definition 3.2 Let $T \in \mathbf{T}^\circ$, $F \in \mathcal{S}_1(T)$ and I, J two non empty compact subintervals with $I \subset X^i$ and $J \subset X^j$, where X^i, X^j are two branches of F (X^i and X^j may be equal). Let n be a positive integer and $p \in \mathbb{Z}$. We say that I *positively* F^n*-covers* $J+p$ and we write $I \xrightarrow[F^n]{+} J+p$ if there exist $x, y \in I$ with $x \le y$ (with respect to the order in X^i) such that $r_j(F^n(x)-p) \le \min J$ and $\max J \le r_j(F^n(y)-p)$ (with respect to the order in X^j). In this situation, we also say that $I+q$ positively F^n-covers $J+p+q$ for all $q \in \mathbb{Z}$.

Remark 3.3 If $F^n(x) \in T_{\mathbb{R}}$ and $J \subset X^j$ for some j, then the inequality $r_j(F^n(x)-p) \le \min J$ is automatically satisfied. We shall often use this remark to prove that an interval positively covers another.

The next lemma is [2, Lemma 2.2(c)]. It states that positive coverings can be concatenated.

Lemma 3.4 *Let I, J, K be non empty compact intervals, each one included in some branch of F. Let n, m be positive integers and $p, q \in \mathbb{Z}$. If $I \xrightarrow[F^n]{+} J+p$ and $J \xrightarrow[F^m]{+} K+q$, then $I \xrightarrow[F^{n+m}]{+} K+p+q$.*

The next proposition is [2, Proposition 2.3], rewritten in some less general form.

Proposition 3.5 *Let $T \in \mathbf{T}^\circ$ and $F \in \mathcal{S}_1(T)$. Let $I_0, \dots, I_{k-1}$ be non empty compact intervals, each one included in some branch of F, and $p_1, \dots, p_k \in \mathbb{Z}$. Suppose that we have a chain of positive coverings:*

$$I_0 \xrightarrow[F]{+} I_1 + p_1 \xrightarrow[F]{+} I_2 + p_2 \xrightarrow[F]{+} \cdots \qquad \cdots I_{k-1} + p_{k-1} \xrightarrow[F]{+} I_0 + p_k.$$

Then there exists a point $x_0 \in I_0$ such that $F^k(x_0) = x_0 + p_k$ and $F^i(x_0) \in I_i + p_i$ for all $i \in [\![1, n-1]\!]$.

4 Partition of the branches and itineraries

Since $T_{\mathbb{R}}$ is F-invariant, if $F^{n_0}(x) \in T_{\mathbb{R}}$ for some $n_0 \geq 0$, then $F^n(x) \in T_{\mathbb{R}}$ for all $n \geq n_0$ and $\rho_F(x) \in \mathrm{Rot}_{T_{\mathbb{R}}}(F)$. The properties of the rotation set $\mathrm{Rot}_{T_{\mathbb{R}}}(F)$ has been recalled in Theorem 2.8. Consequently, it remains to consider the points whose orbits do not fall in $T_{\mathbb{R}}$. Since $T = T_{\mathbb{R}} \cup (X + \mathbb{Z})$ and $\rho_F(x+1) = \rho_F(x)$ (Lemma 2.5(i)), we have

$$\mathrm{Rot}(F) = \mathrm{Rot}_{T_{\mathbb{R}}}(F) \cup \mathrm{Rot}_{X^\infty}(F), \text{ where } X^\infty := \{x \in X \mid \forall n \geq 0, F^n(x) \in X + \mathbb{Z}\}. \quad (1)$$

Our first step consists in dividing each branch of X according to the location of the images in one of the sets $X^i + p, i \in \Lambda, p \in \mathbb{Z}$.

Lemma 4.1 *Let $T \in \mathbf{T}^\circ$, $F \in \mathcal{S}_1(T)$ and $(X^i)_{i\in\Lambda}$ the branches of F. For every branch X^i, there exist an integer $N_i \geq 0$, disjoint non empty compact intervals $X^i_1, \ldots, X^i_{N_i} \subset X^i$ and, for every $j \in [\![1, N_i]\!]$, there exist $\ell(X^i_j) \in \Lambda$ and $p(X^i_j) \in \mathbb{Z}$ such that*

i) $X^i_1 < X^i_2 < \cdots < X^i_{N_i}$ *(with respect to the order in X^i),*

ii) $F(X^i_j) \subset \left(X^{\ell(X^i_j)} + p(X^i_j)\right) \cup \mathrm{Int}\,(T_{\mathbb{R}})$,

iii) $F(\min X^i_j) = \min X^{\ell(X^i_j)} + p(X^i_j) \in T_{\mathbb{R}}$,

iv) $F\left(X \setminus \bigcup_{i\in\Lambda, j\in[\![1,N_i]\!]} X^i_j\right) \cap (X + \mathbb{Z}) = \emptyset$.

Proof. We fix $i \in \Lambda$. If $F(X^i) \cap (X + \mathbb{Z}) = \emptyset$, we take $N_i = 0$ and there is nothing to do. Otherwise, we can define $a_1 := \min\{x \in X^i \mid F(x) \in X + \mathbb{Z}\}$ and $p_1 \in \mathbb{Z}$ such that $F(a_1) \in X + p_1$. Since F is a sun-like map, there is a unique $\ell_1 \in \Lambda$ such that $a_1 \in X^{\ell_1} + p_1$. We define

$$b_1 := \max\{x \in [a_1, \max X^i] \mid F(x) \in X^{\ell_1} + p_1 \text{ and } F([a_1, x]) \subset (X^{\ell_1} + p_1) \cup \mathrm{Int}\,(T_{\mathbb{R}})\},$$

and $X^i_1 := [a_1, b_1]$. Then X^i_1 satisfies (ii) with $p(X^i_1) := p_1$ and $\ell(X^i_1) := \ell_1$. Moreover, $F(\min X^i) \in T_{\mathbb{R}}$ because $\min X^i \in T_{\mathbb{R}}$, which implies that $F([\min X^i, a_1])$ contains $\min X^{l_1} + p_1$ by connectedness. Thus $F(a_1) = \min X^{l_1} + p_1$ by minimality of a_1, which is (iii) for X^i_1.

We define $X^i_2, \ldots X^i_{N_i}$ inductively. Suppose that $X^i_j = [a_j, b_j]$, $p_j = p(X^i_j)$ and $\ell_j = \ell(X^i_j)$ are already defined and that b_j satisfies:

$$b_j = \max\{x \in [a_j, \max X^i] \mid F(x) \in X^{\ell_j} + p_j \text{ and } F([a_j, x]) \subset (X^{\ell_j} + p_j) \cup \mathrm{Int}\,(T_{\mathbb{R}})\}.$$

If $F((b_j, \max X^i]) \subset \mathrm{Int}\,(T_{\mathbb{R}})$, we take $N_i := j$ and the construction is over. Otherwise, we define

$$a_{j+1} := \inf\{x \in (b_j, \max X^i] \mid F(x) \in X + \mathbb{Z}\}. \quad (2)$$

We first show that $a_{j+1} > b_j$. By definition, there exists a sequence of points $(x_n)_{n\geq 0}$ in $(b_j, \max X^i]$ such that $\lim_{n\to+\infty} x_n = a_{j+1}$ and $F(x_n) \in X+\mathbb{Z}$ for all $n \geq 0$. Since the number of branches is finite, we may assume (by taking a subsequence if necessary) that there exists $\ell_{j+1} \in \Lambda$ such that $F(x_n) \in X^{\ell_{j+1}} + \mathbb{Z}$ for all $n \geq 0$. Let $m_n \in \mathbb{Z}$ be such that $F(x_n) \in X^{\ell_{j+1}} + m_n$. By continuity, $\lim_{n\to+\infty} r_{\mathbb{R}} \circ F(x_n) = r_{\mathbb{R}} \circ F(a_{j+1})$. Since $r_{\mathbb{R}} \circ F(x_n) = r_{\mathbb{R}}(\min X^{\ell_{j+1}}) + m_n$, this implies that the sequence of integers $(m_n)_{n\geq 0}$ is ultimately constant, and equal to some integer p_{j+1}. Then $F(a_{j+1}) = \lim_{n\to+\infty} F(x_n) \in X^{\ell_{j+1}} + p_{j+1}$. By continuity, $F([a_{j+1}, x_n]) \subset$

$(X^{\ell_{j+1}}+p_{j+1})\cup\mathrm{Int}\,(T_{\mathbb{R}})$ for all large enough n. Moreover, $F((b_j,a_{j+1}))\subset\mathrm{Int}\,(T_{\mathbb{R}})$ by definition of a_{j+1}. If $p_{j+1}=p_j$ and $\ell_{j+1}=\ell_j$ then, for all large enough n, we would have

$$F(x_n)\in X^{\ell_j}+p_j \text{ and } F([b_j,x_n])\subset(X^{\ell_j}+p_j)\cup\mathrm{Int}\,(T_{\mathbb{R}}),$$

which would contradict the definition of b_j because $x_n>b_j$. Hence $(p_{j+1},\ell_{j+1})\neq(p_j,\ell_j)$. This implies that $a_{j+1}>b_j$ and, consequently, a_{j+1} is actually a minimum in (2). Since $F((b_i,a_{j+1}))$ is non empty and included in $\mathrm{Int}\,(T_{\mathbb{R}})$, necessarily $F(a_{j+1})$ is equal to $\min X^{\ell_{j+1}}+p_{j+1}$ by minimality of a_{j+1}.

Finally, we define

$$b_{j+1}:=$$
$$\max\{x\in[a_{j+1},\max X^i]\mid F(x)\in X^{\ell_{j+1}}+p_{j+1} \text{ and } F([a_{j+1},x])\subset(X^{\ell_{j+1}}+p_{j+1})\cup\mathrm{Int}\,(T_{\mathbb{R}})\},$$

and $X^i_{j+1}:=[a_{j+1},b_{j+1}]$. Then $X^i_{j+1}>X^i_j$, and (ii) and (iii) are satisfied with $p(X^i_{j+1}):=p_{j+1}$ and $\ell(X^i_{j+1}):=\ell_{j+1}$.

Let δ be the infimum of $d(x,y)$ where x,y belong to two different sets of the form $X^\ell+p,\ell\in\Lambda,p\in\mathbb{Z}$. This is actually a minimum because the sets X^ℓ are compact, Λ is finite and the distance d is invariant by translation by 1. Moreover, $\delta>0$ because the branches are pairwise disjoint.

By uniform continuity of F on the compact set X, there exists $\eta>0$ such that, if x,y belong to X with $d(x,y)<\eta$, then $|F(x)-F(y)|<\delta$. This implies that $|a_{j+1}-b_j|\geq\eta$, otherwise $F(a_{j+1})$ and $F(b_j)$ would be in the same set $X^\ell+p$. This ensures that for a given $i\in\Lambda$, the number of intervals X^i_j is finite, and the construction ultimately ends. By construction, (iv) is satisfied. □

Remark 4.2 The fact that the sets X^i_j are intervals is very important because it will allow us to use positive coverings. In an ideal situation, we would like to define the sets $(X^i_j)_{1\leq j\leq N_i}$ as the connected components of $F^{-1}(X^i)\cap(X+\mathbb{Z})$. This is not possible in general because the number of connected components may be infinite: this occurs when F oscillates infinitely many times between a branch and $T_{\mathbb{R}}$.

We call $\mathcal{P}:=\{X^i_j\mid i\in\Lambda,j\in[\![1,N_i]\!]\}$ the *basic partition* of X (although the true partition of X is $\mathcal{P}\cup\{X\setminus\bigcup X^i_j\}$). The set $X\setminus\bigcup X^i_j$, as well as $T_{\mathbb{R}}$, plays the role of "dustbin": we do not need to care about points whose orbit falls in these sets because their rotation numbers belong to $\mathrm{Rot}_{\mathbb{R}}(F)$.

According to Lemma 4.1(iv), every point $x\in X$ such that $F(x)\in X+\mathbb{Z}$ belongs to some $A\in\mathcal{P}$, and this A is unique because the elements of $\mathcal{P}$ are pairwise disjoint. This allows us to code the orbits of the points of X^∞ with respect to the partition $\mathcal{P}$.

Definition 4.3 Let $X^\infty:=\{x\in X\mid\forall n\geq0,F^n(x)\in X+\mathbb{Z}\}$. If $x\in X^\infty$ then, for every $n\geq0$, there is a unique $A_n\in\mathcal{P}$ such that $F^n(x)\in A_n+\mathbb{Z}$. The sequence $(A_n)_{n\geq0}$ is called the *itinerary* of x. Let $\Sigma\subset\mathcal{P}^{\mathbb{Z}^+}$ be the set of all itineraries of points $x\in X^\infty$.

The next lemma is straightforward.

Lemma 4.4 *If $(A_n)_{n\geq0}$ is the itinerary of $x\in X^\infty$, then*

$$\forall n\geq0,\ F^n(x)\in A_n+p(A_0)+p(A_1)+\cdots+p(A_{n-1}),$$

and, if $\rho_F(x)$ exists,

$$\rho_F(x)=\lim_{n\to+\infty}\frac{p(A_0)+\cdots+p(A_{n-1})}{n}.$$

5 The covering graph associated to $\mathcal{P}$

Knowing the itinerary of a point $x \in X^\infty$ is enough to compute the rotation number of x. Therefore we can focus on the set Σ of all itineraries. If

$$\forall A, B \in \mathcal{P},\ F(A) \cap (B + p(A)) \Longrightarrow F(A) \supset B + p(A),$$

then it can be shown that Σ is a Markov shift on the finite alphabet $\mathcal{P}$. In this case, the rotation set of X^∞ can be easily computed by the use of the Markov graph of Σ. In [7] this is done for transitive subshifts of finite type, and it is shown that in this case the rotation set is a compact interval. When the Markov shift is not transitive, or equivalently when its Markov graph is not strongly connected (see Definition 5.17 below), one has to look at the different connected components of the graph, each of which giving an interval.

In general, Σ may not be a Markov shift. We are going to build a countable oriented graph, called the covering graph associated to $\mathcal{P}$, that plays the role of the Markov graph: the symbolic itineraries can be read in the graph and the structure of the graph (in particular its connected components) will give the structure of the rotation set.

5.1 Definitions and first properties

The construction of the covering graph is inspired by the Markov diagram of a (non Markov) interval map, first introduced by Hofbauer for piecewise monotone maps [4]. Our definition is closer to the Buzzi's version of the Markov diagram [3], although the basis of our covering graph is always finite, as in Hofbauer's graph. In Hofbauer's and Buzzi's constructions, the basis consists in monotone intervals, whereas our basis will be the basic partition $\mathcal{P}$ (no monotonicity is involved here).

Definition 5.1 If $A_i \in \mathcal{P}$ for all $i \in [\![0, n]\!]$, we define

$$\langle A_0 A_1 \dots A_n \rangle := F^n \left(\{x \in T \mid \forall i \in [\![0, n]\!], F^i(x) \in A_i + \mathbb{Z}\} \right) \cap X.$$

Remark 5.2 If the itinerary of x begins with $A_0 \dots A_n$, then $F^n(x) \in \langle A_0 \dots A_n \rangle + \mathbb{Z}$. In this sense, $\langle A_0 \dots A_n \rangle$ is the set of points whose "past itinerary" is $A_0 \dots A_n$.

The next lemma gives an alternative definition of $\langle A_0 \dots A_n \rangle$ and states that this set is actually an interval.

Lemma 5.3 *If $A_i \in \mathcal{P}$ for all $i \in [\![0, n]\!]$, then*

i) $$\begin{aligned} \langle A_0 \dots A_n \rangle &= F^n(A_0 + \mathbb{Z}) \cap F^{n-1}(A_1 + \mathbb{Z}) \cap \dots \cap F(A_{n-1} + \mathbb{Z}) \cap A_n \\ &= F(\langle A_0 \dots A_{n-1} \rangle - p(A_{n-1})) \cap A_n \end{aligned}$$

ii) $\langle A_0 A_1 \dots A_n \rangle$ *is either empty, or a closed subinterval of A_n containing* $\min A_n$.

Proof. By definition,

$$\langle A_0 \dots A_n \rangle = F^n \left((A_0 + \mathbb{Z}) \cap F^{-1}(A_1 + \mathbb{Z}) \cap \dots \cap F^{-n}(A_n + \mathbb{Z}) \right) \cap X.$$

Thus

$$\langle A_0 \dots A_n \rangle = F^n(A_0 + \mathbb{Z}) \cap F^{n-1}(A_1 + \mathbb{Z}) \cap \dots \cap F(A_{n-1} + \mathbb{Z}) \cap (A_n + \mathbb{Z}) \cap X.$$

Since $(A_n + \mathbb{Z}) \cap X = A_n$, this gives the first equality of (i). If we write this equality for $\langle A_0 \dots A_{n-1} \rangle$, we see that $\langle A_0 \dots A_n \rangle = F(\langle A_0 \dots A_{n-1} \rangle + \mathbb{Z}) \cap A_n$. Since $\langle A_0 \dots A_{n-1} \rangle \subset A_{n-1}$, Lemma 4.1(ii) implies that $F(\langle A_0 \dots A_{n-1}\rangle) \subset X + p(A_{n-1})$, and hence $F(\langle A_0 \dots A_{n-1}\rangle + \mathbb{Z}) \cap A_n = F(\langle A_0 \dots A_{n-1}\rangle - p(A_{n-1})) \cap A_n$. This is the second equality of (i).

We show (ii) by induction on n. If $n = 0$, then $\langle A_0\rangle = A_n$ and there is nothing to prove.

Suppose that $\langle A_0 \dots A_n\rangle \neq \emptyset$ and that $\langle A_0 \dots A_{n-1}\rangle$ is a closed subinterval of A_{n-1} containing $\min A_{n-1}$ (note that $\langle A_0 \dots A_{n-1}\rangle$ is not empty if $\langle A_0 \dots A_n\rangle \neq \emptyset$). By (i),

$$\langle A_0 \dots A_n\rangle = (F(\langle A_0 \dots A_{n-1}\rangle) - p(A_{n-1})) \cap A_n.$$

By continuity, $F(\langle A_0 \dots A_{n-1}\rangle)$ is compact and connected, and thus $(F(\langle A_0 \dots A_{n-1}\rangle) - p(A_{n-1})) \cap A_n$ is a closed subinterval of A_n, which is non empty by assumption. Moreover, $\langle A_0 \dots A_{n-1}\rangle$ contains $\min A_{n-1}$ by the induction hypothesis, and $F(\min A_{n-1}) \in T_{\mathbb{R}}$ by Lemma 4.1(iii). This implies that the interval $F(\langle A_0 \dots A_{n-1}\rangle) - p(A_{n-1})$ contains a point of $T_{\mathbb{R}}$ and a point of A_n, and thus it contains $\min A_n$ by connectedness. Therefore (ii) holds for $\langle A_0 \dots A_n\rangle$. This ends the induction. $\square$

We define an equivalence relation between the finite sequences of elements of $\mathcal{P}$.

Definition 5.4 Let $A_0, \dots, A_n, B_0, \dots, B_m \in \mathcal{P}$. We set $A_0 \dots A_n \sim B_0 \dots B_m$ if there is $k \in [\![0, \min(n, m)]\!]$ such that

$$\begin{cases} A_{n-i} = B_{m-i} \quad \text{for all} \quad i \in [\![0, k]\!] \\ \langle A_0 \dots A_{n-k}\rangle = A_{n-k} = B_{m-k} = \langle B_0 \dots B_{m-k}\rangle \end{cases} \tag{3}$$

Remark 5.5 It follows from Lemma 5.3(i) that

$$\text{if } A_0 \dots A_n \sim B_0 \dots B_m, \text{ then } \langle A_0 \dots A_n\rangle = \langle B_0 \dots B_m\rangle.$$

This means that, although the two sets come from different "past itineraries", their futures are indistinguishable. If $\alpha = A_0 \dots A_n/\sim$ is an equivalence class, then $\langle\alpha\rangle$ denotes $\langle A_0 \dots A_n\rangle$, which is well defined according to what precedes.

The next result follows straightforwardly from Lemma 5.3 and the fact that the elements of $\mathcal{P}$ are disjoint.

Lemma 5.6 *If $\alpha = A_0 \dots A_n/\sim$ and $\langle\alpha\rangle \neq \emptyset$, then A_n is the unique element $A \in \mathcal{P}$ such that $\langle\alpha\rangle \subset A$.*

Now we have all the notations to define the covering graph.

Definition 5.7 We define the oriented graph $\mathcal{G}$ as follows:

- the set of vertices is the set of equivalence classes $\alpha = A_0 \dots A_n/\sim$, where $n \geq 0$, $A_0, \dots, A_n \in \mathcal{P}$ and $\langle A_0 \dots A_n\rangle \neq \emptyset$,
- if α, β are two vertices, there is an arrow $\alpha \to \beta$ iff there exist $A_0, \dots, A_n, A_{n+1} \in \mathcal{P}$ such that $\alpha = A_0 \dots A_n/\sim$ and $\beta = A_0 \dots A_n A_{n+1}/\sim$.

$\mathcal{G}$ is called the *covering graph* associated to $\mathcal{P}$.

The next lemma justifies the name "covering graph".

Lemma 5.8 *If $\alpha \to \beta$ in $\mathcal{G}$ and if $A \in \mathcal{P}$ is such that $\langle\alpha\rangle \subset A$, then $\langle\alpha\rangle \xrightarrow[F]{+} \langle\beta\rangle + p(A)$.*

Proof. Let $A_0, \dots, A_n, A_{n+1} \in \mathcal{P}$ be such that $\alpha = A_0 \dots A_n/\sim$ and $\beta = A_0 \dots A_n A_{n+1}/\sim$. Necessarily, $A_n = A$ (Lemma 5.6). According to Lemma 5.3(ii), $\langle\alpha\rangle$ is a closed subinterval of A and $\min\langle\alpha\rangle = \min A$. Thus Lemma 4.1(iii) implies that

$$F(\min\langle\alpha\rangle) \in T_{\mathbb{R}}. \tag{4}$$

Moreover, $\langle\beta\rangle = F(\langle\alpha\rangle - p(A)) \cap A_{n+1}$ by Lemma 5.3(i). Thus

$$\exists x_0 \in \langle\alpha\rangle \text{ such that } \max\langle\beta\rangle = F(x_0) - p(A). \tag{5}$$

Since $\langle\beta\rangle$ is a closed subinterval of A_{n+1}, Equations (4) and (5) imply that $\langle\alpha\rangle \xrightarrow[F]{+} \langle\beta\rangle + p(A)$. □

Definition 5.9 The *significant part* of $A_0 \dots A_n$ is $A_i \dots A_n$, where $i \in [\![0, n]\!]$ is the greatest integer such that $A_0 \dots A_n \sim A_i \dots A_n$. If α is the equivalence class of $A_0 \dots A_n$, the significant part of α is defined as the significant part of $A_0 \dots A_n$. This does not depend on the representative of α.

If $A_0 \dots A_n$ is the significant part of a vertex α, the *height* of α is $H(\alpha) = n$. The *basis* of $\mathcal{G}$ is the set of vertices of height 0, that is, $\{A/\sim \mid A \in \mathcal{P}\}$. We identify it with $\mathcal{P}$.

The next result, quite natural, will simplify the handling of arrows.

Lemma 5.10 *Let $\alpha = A_0 \dots A_n/\sim$ be a vertex of $\mathcal{G}$ and let $\alpha \to \beta$ be an arrow in $\mathcal{G}$. Then there exists $A_{n+1} \in \mathcal{P}$ such that $\beta = A_0 \dots A_n A_{n+1}/\sim$.*

Proof. By definition, there exist $B_0, \dots, B_m, B_{m+1} \in \mathcal{P}$ such that $\alpha = B_0 \dots B_m/\sim$ and $\beta = B_0 \dots B_m B_{m+1}/\sim$. Let $A_{n-k} \dots A_0$ be the significant part of α (with $k = H(\alpha) \in [\![0, n]\!]$). According to the definitions, we have $m \geq k$ and

$$A_{n-i} = B_{m-i} \quad \text{for all } i \in [\![0, k]\!],$$
$$\langle B_0 \dots B_{m-k}\rangle = B_{m-k} = A_{n-k} = \langle A_0 \dots A_{n-k}\rangle.$$

This implies that $B_0 \dots B_m B_{m+1} \sim A_0 \dots A_n B_{m+1}$. This proves the lemma with $A_{n+1} := B_{m+1}$. □

We shall need some notions about paths in oriented graphs.

Definition 5.11 A *(finite) path* in $\mathcal{G}$ is a sequence of vertices $\alpha_0 \dots \alpha_n$ such that $\alpha_i \to \alpha_{i+1}$ is an arrow in $\mathcal{G}$ for all $i \in [\![0, n-1]\!]$. A *loop* is a path $\alpha_0 \dots \alpha_n$ such that $\alpha_n = \alpha_0$. An *infinite path* in $\mathcal{G}$ is an infinite sequence of vertices $\bar{\alpha} = (\alpha_n)_{n\geq 0}$ such that $\alpha_i \to \alpha_{i+1}$ for all $i \geq 0$.

If K is a subgraph of $\mathcal{G}$, let $\Gamma(K)$ be the set of all infinite paths in K.

In the following, the infinite paths in $\mathcal{G}$ will be denoted with a bar (e.g. $\bar{\alpha}$) to distinguish them from vertices (e.g. α_n, β).

Remark 5.12 Endowed with the shift map $\sigma\colon (\alpha_n)_{n\geq 0} \mapsto (\alpha_n)_{n\geq 1}$, $\Gamma(\mathcal{G})$ is a topological Markov chain on a countable graph (see e.g. [6]). We shall not explicitly use this structure of dynamical system, although it will underlie the definition of rotation numbers of elements of $\Gamma(\mathcal{G})$ and the relation between $\Gamma(\mathcal{G})$ and X^∞.

5.2 Relation between itineraries of points of X^∞ and infinite paths in $\mathcal{G}$

The next result states that there is a correspondence between itineraries of points of X^∞ and infinite paths in $\mathcal{G}$. This is a key property of the covering graph: it will allow us to pull back on X^∞ the results obtained for $\mathcal{G}$.

Proposition 5.13 *If $(A_n)_{n\geq 0}$ is the itinerary of some point $x \in X^\infty$ and $\alpha_n = A_0 \dots A_n/\sim$, then $(\alpha_n)_{n\geq 0}$ is an infinite path in $\mathcal{G}$. Reciprocally, if $(\alpha_n)_{n\geq 0}$ is an infinite path in $\mathcal{G}$ with $H(\alpha_0) = k$, then there exists a point $x \in X^\infty$ of itinerary $(A_n)_{n\geq 0}$ such that, for all $n \geq 0$, $\alpha_n = A_0 \dots A_{n+k}/\sim$.*

Proof. Suppose that $(A_n)_{n\geq 0}$ is the itinerary of $x \in X^\infty$. Then $\langle A_0 \dots A_n\rangle$ contains $F^n(x) - (p(A_0) + \cdots + p(A_{n-1}))$. Thus $\langle A_0 \dots A_n\rangle \neq \emptyset$ and $\alpha_n = A_0 \dots A_n/\sim$ is a vertex of $\mathcal{G}$. It follows from the definition that $\alpha_n \to \alpha_{n+1}$ is an arrow in $\mathcal{G}$, that is, $(\alpha_n)_{n\geq 0}$ is an infinite path in $\mathcal{G}$.

Reciprocally, suppose that $(\alpha_n)_{n\geq 0}$ is an infinite path in $\mathcal{G}$. Let $A_0 \dots A_k$ be the significant part of α_0, with $k = H(\alpha_0)$. According to Lemma 5.10, we can find inductively $A_{n+k} \in \mathcal{P}$ such that $\alpha_n = A_0 \dots A_{n+k}/\sim$ for all $n \geq 0$. For every $n \geq 0$, let

$$\begin{aligned} E_n &:= \{x \in X \mid \forall i \in [\![0, n]\!], F^i(x) \in A_i + \mathbb{Z}\} \\ &= \{x \in X \mid \forall i \in [\![0, n]\!], F^i(x) \in A_i + p(A_0) + \cdots + p(A_{i-1})\} \end{aligned}$$

Then E_n is a compact set and $E_{n+1} \subset E_n$. Moreover, $(F^n(E_n + \mathbb{Z})) \cap X = \langle A_0 \dots A_n\rangle \neq \emptyset$ (Lemma 5.3(i)). Hence $E_n \neq \emptyset$. Therefore, the set $\bigcap_{n\geq 0} E_n$ is non empty and every point x in this set satisfies: $x \in X$ and $\forall n \geq 0, F^n(x) \in A_i + \mathbb{Z}$, that is, $x \in X^\infty$ and its itinerary is $(A_n)_{n\geq 0}$. $\square$

5.3 Structure of the covering graph

The oriented graph $\mathcal{G}$ is usually infinite. However its infinite part is "small" and we shall exploit the particular structure of the covering graph. Proposition 5.15 gives the main properties of the structure of $\mathcal{G}$. It implies that "most" infinite paths come back infinitely many times to the basis, which is rigorously stated in Proposition 5.19.

Lemma 5.14 *Let α, β be vertices of $\mathcal{G}$ such that there is an arrow $\alpha \to \beta$. Then there exist $A_0, \cdots, A_n, A_{n+1} \in \mathcal{P}$ and $k \in [\![1, N_{\ell(A_n)}]\!]$ such that $\alpha = A_0 \dots A_n/\sim$, $\beta = A_0 \dots A_n A_{n+1}/\sim$ and $A_{n+1} = X_k^{\ell(A_n)}$. Moreover, for all $j \in [\![1, k-1]\!]$, $\langle\alpha\rangle \xrightarrow[F]{+} X_i^{\ell(A_n)} + p(A_n)$ and $\alpha \to X_i^{\ell(A_n)}$ is an arrow in $\mathcal{G}$.*

Proof. We write $\alpha = A_0 \dots A_n/\sim$, $p := p(A_n)$ and $\ell := \ell(A_n)$. Lemma 5.10 states that there is $A_{n+1} \in \mathcal{P}$ such that $\beta = A_0 \dots A_n A_{n+1}/\sim$. The set $\langle A_0 \dots A_n A_{n+1}\rangle$ is non empty and satisfies

$$\begin{aligned} \langle A_0 \dots A_n A_{n+1}\rangle &= F(\langle A_0 \dots A_n\rangle - p) \cap A_{n+1} \quad \text{(Lemma 5.3(i))} \\ &\subset F(A_n - p) \cap A_{n+1} \\ &\subset X^\ell \cap A_{n+1} \end{aligned}$$

Thus A_{n+1} is necessarily of the form X_k^ℓ for some $k \in [\![1, N_\ell]\!]$. According to Lemma 5.8, $\langle\alpha\rangle \xrightarrow[F]{+} \langle A_0 \dots A_n X_k^\ell\rangle + p$. Thus there exists $x \in \langle\alpha\rangle$ such that $F(x) = \min\langle A_0 \dots A_n X_k^\ell\rangle + p = \min X_k^\ell + p$ (the second equality comes from Lemma 5.3(ii)). Moreover, $\min\langle\alpha\rangle = \min A_n$

(by Lemma 5.3(ii) again), and thus $F(\min\langle\alpha\rangle) \in T_{\mathbb{R}}$ by Lemma 4.1(iii). This implies that $\langle\alpha\rangle \xrightarrow[F]{+} [\min X^\ell, \min X_k^\ell] + p$. Thus, for all $j \in [\![1, k-1]\!]$, we have

$$\langle\alpha\rangle \xrightarrow[F]{+} X_j^\ell + p,$$

and hence $F(\langle\alpha\rangle + \mathbb{Z}) \cap X_j^\ell = X_j^\ell$. According to Lemma 5.3(i), $F(\langle\alpha\rangle + \mathbb{Z}) \cap X_j^\ell = \langle A_0 \dots A_n X_j^\ell\rangle$. Consequently, $A_0 \dots A_n X_j^\ell \sim X_j^\ell$ and $\alpha \to X_j^\ell$ is an arrow in $\mathcal{G}$. □

Proposition 5.15 *Let α, β be vertices of $\mathcal{G}$.*

i) *All but at most one arrows starting from α end at a vertex in the basis.*

ii) *If $\alpha \to \beta$ with $H(\beta) > 0$, then $H(\beta) = H(\alpha) + 1$.*

Proof. We write $\alpha = A_0 \dots A_n / \sim$ and $p = p(A_n)$.

i) If $\alpha \to \beta$, Lemma 5.14 states that there exists $k \in [\![1, N_\ell]\!]$ such that $\beta = A_0 \dots A_n X_k^\ell / \sim$ and, for all $j \in [\![1, k-1]\!]$, $\alpha \to X_j^\ell$ is an arrow in $\mathcal{G}$. This implies that there is at most one vertex of the form $A_0 \dots A_n X_j^\ell / \sim$ of height different from 0. This proves (i).

ii) Suppose that $\alpha \to \beta$ with $H(\beta) > 0$. Let $A_{n+1} \in \mathcal{P}$ be such that $\beta = A_0 \dots A_n A_{n+1} / \sim$ (Lemma 5.10). The significant part of β is $A_i \dots A_n A_{n+1}$ with $i := n + 1 - H(\beta) \le n$ because $H(\beta) \ge 1$. Then by definition $\langle A_0 \dots A_i\rangle = A_i$, and the significant part of α is $A_i \dots A_n$. Hence $H(\alpha) = n - i = H(\beta) - 1$. This proves (ii). □

Remark 5.16 The structure of the graph $\mathcal{G}$ can be deduced from Proposition 5.15: if we start from a vertex A in the basis and go up into the heights, there is a unique, finite or infinite, path (α_n) starting at A and such that $H(\alpha_i) = i$. Two such paths starting at two different vertices in the basis are disjoint because if α is a vertex of height $n > 0$ and significant part $A_0 \dots A_n$ then α belongs only to the path starting at $A_0 \in \mathcal{P}$ (this paths begins with $A_0, A_0A_1/\sim, \dots, A_0 \dots A_n/\sim = \alpha$). The only other arrows in $\mathcal{G}$ end in the basis. This is illustrated in Figure 3.

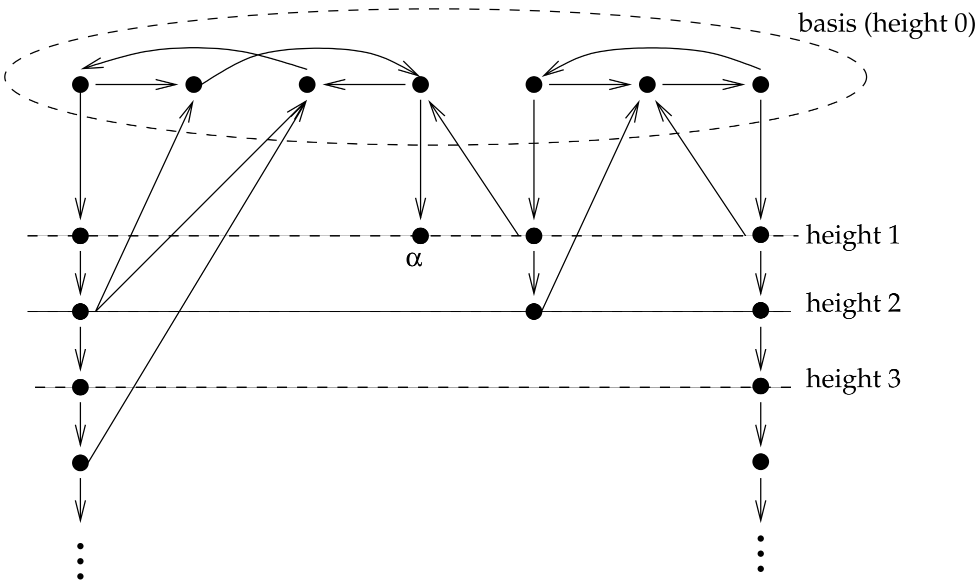

Figure 3: The structure of the covering graph. At the top: the finite basis.

Definition 5.17 An oriented graph is *strongly connected* if for every pair of vertices (u, v), there exists an oriented path of positive length from u to v.

The *connected components* of an oriented graph are the maximal strongly connected subgraphs.

Remark 5.18 Two connected components are either disjoint or equal.

Some vertices (called inessential vertices) may belong to no connected component, see e.g. the vertex α in the middle of Figure 3.

Proposition 5.19 i) *Every connected component of $\mathcal{G}$ meets the basis and the number of connected components is finite and bounded by $\#\mathcal{P}$.*

ii) *Let*

$$\mathcal{I} := \{(\alpha_n)_{n\geq 0} \in \Gamma(\mathcal{G}) \mid H(\alpha_0) = 0 \text{ and } \forall n \geq 1, H(\alpha_n) > 0\}.$$

If $(\alpha_n)_{n\geq 0} \in \mathcal{I}$, then $H(\alpha_n) = n$ for all $n \geq 0$. Moreover, $\mathcal{I}$ is a finite set with $\#\mathcal{I} \leq \#\mathcal{P}$.

iii) *If $(\alpha_n)_{n\geq 0}$ is an infinite path in $\mathcal{G}$ then, either there exists a connected component C of $\mathcal{G}$ such that all vertices α_n belong to C for all great enough n, or there exist $(\beta_n)_{n\geq 0} \in \mathcal{I}$ and $N, M \geq 0$ such that, $\forall n \geq 0$, $\alpha_{N+n} = \beta_{M+n}$.*

Proof. i) If the vertex v belongs to some connected component, there exists a loop starting at v. According to the structure of $\mathcal{G}$ (see Remark 5.16), every loop goes through the basis, which is finite. Thus every connected component meets the basis and the number of connected components of $\mathcal{G}$ is at most $\#\mathcal{P} < +\infty$.

ii) Let $(\alpha_n)_{n\geq 0} \in \mathcal{I}$. By Proposition 5.15(ii), $H(\alpha_{n+1}) = H(\alpha_n) + 1$ for all $n \geq 0$, and thus $H(\alpha_n) = n$ for all $n \geq 0$. According to Proposition 5.15(i), each vertex α_{n+1} is uniquely determined by the properties that $\alpha_n \to \alpha_{n+1}$ and $H(\alpha_{n+1}) \neq 0$. Since $\alpha_0 \in \mathcal{P}$, the number of such infinite paths is less than or equal to $\#\mathcal{P} < +\infty$.

iii) Let $(\alpha_n)_{n\geq 0}$ be an infinite path in $\mathcal{G}$. There are two cases.

• Suppose that there exists N such that $\forall n \geq N, \alpha_n \notin \mathcal{P}$. Then, by Proposition 5.15(ii), $\forall n \geq N, H(\alpha_{n+1}) = H(\alpha_n)+1$, and thus $\forall n \geq 0$, $H(\alpha_{N+n}) = H(\alpha_N)+n$. Let $A_0 \dots A_M$ be the significant part of α_N, with $M = H(\alpha_N)$. By Lemma 5.10, there exists a sequence $(A_n)_{n\geq M+1}$ of elements of $\mathcal{P}$ such that, for all $n \geq 0$, $\alpha_{N+n} = A_0 \dots A_{M+n}/\sim$. We define $\beta_n := A_0 \dots A_n/\sim$ for all $n \geq 0$. Since $H(\beta_M) = H(\alpha_N) = M$, we have $H(\beta_n) = n$ for all $n \in [\![0, M]\!]$, and $H(\beta_{M+n}) = H(\alpha_{N+n}) = M + n$ for all $n \geq 0$. Hence $(\beta_n)_{n\geq 0} \in \mathcal{I}$ and $\alpha_{N+n} = \beta_{M+n}$ for all $n \geq 0$.

• Otherwise, there exist infinitely many n such that $\alpha_n \in \mathcal{P}$. Since $\mathcal{P}$ is finite, there exists $A \in \mathcal{P}$ such that $\alpha_n = A$ for infinitely many n. Consequently, A belongs to some connected component C and α_n belongs to C for all great enough n. □

6 The rotation set of the covering graph

6.1 Rotation numbers of infinite paths

Definition 6.1 Let $\alpha \to \beta$ be an arrow in $\mathcal{G}$ and let A be the unique element of $\mathcal{P}$ such that $\langle\alpha\rangle \subset A \in \mathcal{P}$. The *weight* of the arrow $\alpha \to \beta$ is defined as $W(\alpha\beta) := p(A)$.

This naturally leads to the following definition of rotation numbers for infinite paths (see [7] for a similar notion in the case of subshifts of finite type).

Definition 6.2 If $\alpha_0 \dots \alpha_n$ is a finite path in $\mathcal{G}$, its *length* is $L(\alpha_0 \dots \alpha_n) := n$ and its *weight* is $W(\alpha_0 \dots \alpha_n) := \sum_{i=0}^{n-1} W(\alpha_i \alpha_{i+1})$.

If $\bar{\alpha} = (\alpha_n)_{n \geq 0} \in \Gamma(\mathcal{G})$, then its *rotation number* is

$$\rho(\bar{\alpha}) := \lim_{n \to +\infty} \frac{W(\alpha_0 \dots \alpha_n)}{n}$$

when this limit exists.

If $\Gamma' \subset \Gamma(\mathcal{G})$, let $\mathrm{Rot}_{\Gamma'} := \{\rho(\bar{\alpha}) \mid \bar{\alpha} \in \Gamma' \text{ and } \rho(\bar{\alpha}) \text{ exists}\}$.

Definition 6.3 If $\gamma = \alpha_0 \dots \alpha_n$ and $\gamma' = \beta_0 \dots \beta_m$ are two paths in $\mathcal{G}$ with $\beta_0 = \alpha_n$, let $\gamma \cdot \gamma'$ denote the concatenation of the two paths, that is,

$$\gamma \cdot \gamma' := \alpha_0 \dots \alpha_n \beta_1 \dots \beta_m.$$

If γ is a loop then $\gamma^n := \gamma \cdot \gamma \dots \gamma$ is the n-time concatenation of γ. We define similarly the concatenation of infinitely many finite paths, and the concatenation of a finite path with an infinite path; in these two cases, the resulting paths are infinite.

If γ is a loop in $\mathcal{G}$, let $\widetilde{\gamma} := \gamma^\infty$ be the corresponding periodic infinite path.

The next lemma is straightforward.

Lemma 6.4 i) *If γ, γ' are two finite paths that can be concatenated, then $L(\gamma \cdot \gamma') = L(\gamma) + L(\gamma')$ and $W(\gamma \cdot \gamma') = W(\gamma) + W(\gamma')$.*

ii) *If γ is a loop, then $\rho(\widetilde{\gamma}) = \frac{W(\gamma)}{L(\gamma)} \in \mathbb{Q}$.*

6.2 The rotation set of a connected component

We have seen in Proposition 5.19 that an infinite path either ultimately belongs to some connected component of $\mathcal{G}$, or ultimately coincide with some infinite path belonging to some finite set $\mathcal{I}$. In this subsection, we focus on the first case and we study the rotation set of a given connected component of $\mathcal{G}$.

We start with a lemma that uses the concatenation of loops to get rotation numbers.

Lemma 6.5 i) *Let γ_1, γ_2 be two loops in $\mathcal{G}$ starting at the same vertex α. If $r \in [\rho(\widetilde{\gamma}_1), \rho(\widetilde{\gamma}_2)] \cap \mathbb{Q}$, there exists a loop γ starting at α such that $\rho(\widetilde{\gamma}) = r$.*

ii) *Let α be a vertex of $\mathcal{G}$ and, for every $n \geq 0$, let γ_n be a loop in $\mathcal{G}$ starting at α. If $\lim_{n \to +\infty} \rho(\widetilde{\gamma}_n) = s \in \mathbb{R}$, then there exists $\bar{\alpha} = (\alpha_n)_{n \geq 0} \in \Gamma(\mathcal{G})$ such that $\alpha_0 = \alpha$, $\alpha_n = \alpha$ for infinitely many n and $\rho(\bar{\alpha}) = s$.*

Proof. i) We write $r = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Let L be a multiple of $L(\gamma_1), L(\gamma_2)$ and q, and let k_1, k_2, k be such that $L = k_1 L(\gamma_1) = k_2 L(\gamma_2) = kq$. We set $\gamma_i' := (\gamma_i)^{k_i}$ and $W_i := W(\gamma_i')$ for $i \in \{1, 2\}$, and $p' := kp$. In this way, $L(\gamma_i') = L$ and $\rho(\widetilde{\gamma}_i) = \rho(\widetilde{\gamma}_i') = W_i/L$ for $i \in \{1, 2\}$, and $r = p'/L$. Since $\rho(\widetilde{\gamma}_1) \leq r \leq \rho(\widetilde{\gamma}_2)$, we have $W_1 \leq p' \leq W_2$. We define $\gamma := (\gamma_1')^{W_2 - p'} \cdot (\gamma_2')^{p' - W_1}$. This is a loop starting at α, and

$$\rho(\widetilde{\gamma}) = \frac{W(\gamma)}{L(\gamma)} = \frac{W_1(W_2 - p') + W_2(p' - W_1)}{L(W_2 - p') + L(p' - W_1)} = \frac{(W_2 - W_1)p'}{(W_2 - W_1)L} = r.$$

This proves (i).

ii) Let $(i_n)_{n \geq 0}$ be a sequence of positive integers and $\bar{\alpha} := \gamma_0^{i_0} \cdot \gamma_1^{i_1} \cdot \dots \gamma_n^{i_n} \dots$. This is an infinite path starting at α and passing at α infinitely many times. It can be shown that, if the sequence $(i_n)_{n \geq 0}$ increases sufficiently fast, then $\rho(\bar{\alpha}) = \lim_{n \to +\infty} \rho(\widetilde{\gamma}_n) = s$ (see e.g. the proof of [2, Theorem 3.1] for a similar proof expliciting the growth of $(i_n)_{n \geq 0}$). □

Proposition 6.6 *Let C be a connected component of $\mathcal{G}$. Then*

i) $\mathrm{Rot}_{\Gamma(C)}$ *is a non empty compact interval.*

ii) $\forall r \in \mathrm{Int}\left(\mathrm{Rot}_{\Gamma(C)}\right) \cap \mathbb{Q}$, *there exists a loop γ in C such that $\rho(\widetilde{\gamma}) = r$.*

Proof. The set $C \cap \mathcal{P}$ is non empty by Proposition 5.19(i), and thus we can fix $A \in C \cap \mathcal{P}$. For every $B, B' \in C \cap \mathcal{P}$, we choose a path $u(B, B')$ from B to B'. Since $\mathcal{P}$ is finite, we can bound $L(u(B, B'))$ and $W(u(B, B'))$ by some quantities L_0 and W_0 respectively, independently of B, B'. Let $\mathcal{L}_A$ be the set of all loops starting at A. If $\gamma \in \mathcal{L}_A$, then the periodic path $\widetilde{\gamma}$ belongs to $\Gamma(C)$.

Let $\bar{\alpha} = (\alpha_n)_{n\geq 0} \in \Gamma(C)$ such that $\rho(\bar{\alpha})$ exists. We are going to show that

$$\forall \varepsilon > 0, \exists \gamma \in \mathcal{L}_A \text{ such that } |\rho(\bar{\alpha}) - \rho(\widetilde{\gamma})| < \varepsilon. \tag{6}$$

If u is a path from A to α_0 then $u \cdot (\alpha_n)_{n\geq 0} \in \Gamma(C)$ and it has the same rotation number as $\bar{\alpha}$. Thus we can assume that $\alpha_0 = A$.

We first show that there exist a sequence of integers $(n_i)_{i\geq 0}$ increasing to infinity, and finite paths v_i from α_{n_i} to A such that $\forall i \geq 0, W(v_i) \leq W$ and $L(v_i) \leq L$, where

$$W := W_0 + \max\{p(B) \mid B \in \mathcal{P}\} \quad \text{and} \quad L := L_0 + 1.$$

Since $\bar{\alpha}$ is in $\Gamma(C)$, for all integers $N \geq 0$, there exists a path in C from α_N to A. Because of the structure of $\mathcal{G}$ (see Proposition 5.15), this implies that there exists $n \geq N$ and $B \in C \cap \mathcal{P}$ such that $\alpha_n \to B$. Thus we can find a sequence $(n_i)_{i\geq 0}$ increasing to infinity and vertices $(B_i)_{i\geq 0}$ in $C \cap \mathcal{P}$ such that $\alpha_{n_i} \to B_i$ for all $i \geq 0$. We set $v_i := \alpha_{n_i} B_i \cdot u(B_i, A)$. Then $L(v_i) \leq L$ and $W(v_i) \leq W$.

Now we define $\gamma_i \in \mathcal{L}_A$ by concatenating $\alpha_0 \ldots \alpha_{n_i}$ with v_i. Fix $\varepsilon > 0$ and let i be great enough such that

$$\left|\rho(\bar{\alpha}) - \frac{W(\alpha_0 \ldots \alpha_{n_i})}{n_i}\right| < \frac{\varepsilon}{3}, \qquad \frac{|W|}{n_i} < \frac{\varepsilon}{3}, \quad \text{and} \quad (|\rho(\bar{\alpha})| + \varepsilon/3)\,\frac{L}{n_i} < \frac{\varepsilon}{3}.$$

Since $\rho(\widetilde{\gamma}_i) = \frac{W(\alpha_0 \ldots \alpha_{n_i}) + W(v_i)}{n_i + L(v_i)}$, we have

$$\begin{aligned}
|\rho(\bar{\alpha}) - \rho(\widetilde{\gamma}_i)| &\leq \left|\rho(\bar{\alpha}) - \frac{W(\alpha_0 \ldots \alpha_{n_i})}{n_i}\right| + \left|\frac{W(\alpha_0 \ldots \alpha_{n_i}) + W(v_i)}{n_i + L(v_i)} - \frac{W(\alpha_0 \ldots \alpha_{n_i})}{n_i}\right| \\
&< \frac{\varepsilon}{3} + \left|\frac{W(v_i)}{n_i + L(v_i)} - W(\alpha_0 \ldots \alpha_{n_i})\frac{L(v_i)}{n_i(n_i + L(v_i))}\right| \\
&< \frac{\varepsilon}{3} + \frac{|W|}{n_i} + \frac{|W(\alpha_0 \ldots \alpha_{n_i})|}{n_i}\,\frac{L}{n_i} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + (|\rho(\bar{\alpha})| + \varepsilon/3)\,\frac{L}{n_i} \\
&< \varepsilon
\end{aligned}$$

This proves Equation (6). In other words, $\{\rho(\widetilde{\gamma}) \mid \gamma \in \mathcal{L}_A\}$ is dense in $\mathrm{Rot}_{\Gamma(C)}$.

The set $\mathrm{Rot}_{\Gamma(C)}$ is non empty because there exists a loop γ starting at $A \in C \cap \mathcal{P} \neq \emptyset$, and $\rho(\widetilde{\gamma})$ exists. We set $a := \inf \mathrm{Rot}_{\Gamma(C)}$ and $b := \sup \mathrm{Rot}_{\Gamma(C)}$. We suppose $a < b$, otherwise there is nothing to prove. Let $r \in (a, b) \cap \mathbb{Q}$ and let $\varepsilon > 0$ be such that $a + 2\varepsilon < r < b - 2\varepsilon$. Let $\bar{\alpha}, \bar{\beta} \in \Gamma(C)$ be such that $|\rho(\bar{\alpha}) - a| < \varepsilon$ and $|\rho(\bar{\beta}) - b| < \varepsilon$. By Equation (6), there exist $\gamma_1, \gamma_2 \in \mathcal{L}_A$ such that $|\rho(\widetilde{\gamma}_1) - \rho(\bar{\alpha})| < \varepsilon$ and $|\rho(\widetilde{\gamma}_2) - \rho(\bar{\beta})| < \varepsilon$, and hence

$$\rho(\widetilde{\gamma}_1) < r < \rho(\widetilde{\gamma}_2).$$

Then by Lemma 6.5(i) there exists $\gamma \in \mathcal{L}_A$ such that $\rho(\widetilde{\gamma}) = r$. Now let $s \in [a,b]$ and let $(r_n)_{n\geq 0}$ be a sequence in $(a,b)\cap\mathbb{Q}$ such that $\lim_{n\to+\infty} r_n = s$. What precedes implies that, for all $n \geq 0$, there exists $\gamma_n \in \mathcal{L}_A$ such that $\rho(\widetilde{\gamma}_n) = r_n$. Then, according to Lemma 6.5(ii), there exists an infinite path $\bar{\alpha}$ such that $\rho(\bar{\alpha}) = s$ and $\bar{\alpha}$ starts at A and passes infinitely many times at A, which implies that $\bar{\alpha} \in \Gamma(C)$. This ends the proof of the proposition. □

6.3 The rotation numbers of infinite paths not in connected components

Proposition 6.7 *Let*

$$\mathcal{J} := \{\bar{\alpha} = (\alpha_n)_{n\geq 0} \in \Gamma(\mathcal{G}) \mid \exists N(\bar{\alpha}) \geq 0, \forall n \geq N(\bar{\alpha}), \nexists B \in \mathcal{P}, \alpha_n \to B\}.$$

Let $\bar{\alpha} = (\alpha_n)_{n\geq 0} \in \mathcal{J}$ and let $(A_n)_{n\geq 0}$ be the sequence of elements of $\mathcal{P}$ such $\alpha_n = A_0 \dots A_n/\sim$ for all $n \geq 0$. Then $\rho(\bar{\alpha})$ exists and is a rational number. More precisely, there exist $q \geq 1$, $B_0, \cdots, B_{q-1} \in \mathcal{P}$ and $M \geq N(\bar{\alpha})$ such that

$$\forall n \geq 0, \forall r \in [\![0, q-1]\!], A_{M+nq+r} = B_r$$

and $\rho(\bar{\alpha}) = \frac{p}{q}$, where $p := p(B_0) + p(B_1) + \cdots + p(B_{q-1})$. Moreover, there exists $x \in X^\infty$ such that $F^q(x) = x + p$, that is, x is periodic (mod 1) *and $\rho_F(x) = \frac{p}{q} = \rho(\bar{\alpha})$.*

Proof. Let $n \geq N(\bar{\alpha})$. By Lemma 5.14, there exists $i \in [\![1, \ell(A_n)]\!]$ such that $A_{n+1} = X_i^{\ell(A_n)}$ and, if $i \geq 2$, then $\alpha \to X_1^{\ell(A_n)}$. By definition of $\mathcal{J}$ and choice of n, there is no arrow from α to some element of $\mathcal{P}$. Therefore $i = 1$. This implies that, for all $n \geq N(\bar{\alpha})$, A_{n+1} is uniquely determined by A_n. Since $\mathcal{P}$ is finite, there exist $M \geq N(\bar{\alpha})$ and $q \geq 1$ such that $A_{M+q} = A_M$. If we set $B_0 \dots B_{q-1} := A_M \dots A_{M+q-1}$, we get

$$\forall n \geq 0, \forall r \in [\![0, q-1]\!], A_{M+nq+r} = B_r$$

Then, if we set $p := p(B_0) + p(B_1) + \cdots + p(B_{q-1})$, it is clear that $\rho(\bar{\alpha}) = \frac{p}{q}$. This proves the first part of the proposition.

For all $n \geq 1$, we set

$$\beta_n := \underbrace{B_0 \dots B_{q-1} \;\cdots\; B_0 \dots B_{q-1}}_{B_0 \dots B_{q-1} \text{ repeated } n \text{ times}} / \sim .$$

Then $\beta_n = A_M \dots A_{M+nq-1}/\sim$. Since $\langle A_M \dots A_{M+m}\rangle \xrightarrow[F]{+} \langle A_M \dots A_{M+m+1}\rangle + p(A_{M+m})$ for all $m \geq 0$, Lemma 3.4 implies that $\langle\beta_n\rangle \xrightarrow[F^q]{+} \langle\beta_{n+1}\rangle + p$ for all $n \geq 1$. We set $a_0 := \min B_{q-1}$. Then $a_0 = \min\langle\beta_n\rangle$ for all $n \geq 1$ by Lemma 5.3(ii) and $F(a_0) \in T_\mathbb{R}$ by Lemma 4.1(iii). Let $\ell \in \Lambda$ be such that $B_{q-1} \subset X^\ell$. By Proposition 5.13, there exists $y' \in X^\infty$ of itinerary $(A_m)_{m\geq 0}$. Thus the itinerary of $y := F^{M+q-1}(y')$ is $(A_{M+q-1+m})_{m\geq 0}$. Let $G := F^q - p$. It is clear that $G(T_\mathbb{R}) \subset T_\mathbb{R}$. For all $n \geq 0$, $G^n(y) \in B_{q-1} \subset X^\ell$, and in particular $G^n(y) \geq a_0$ for the order in X^ℓ. We define inductively a sequence of points $(a_i)_{i\geq 1}$ such that $a_i \in [a_{i-1}, y]$ and $G^i(a_i) = a_0$ for all $i \geq 1$.

- Since $G(a_0) \in T_\mathbb{R}$ and $G(y) \geq a_0$, we have $a_0 \in G([a_0, y])$ by continuity. Thus there exists $a_1 \in [a_0, y]$ such that $G(a_1) = a_0$.
- Assume that $a_0, \dots, a_i$ are already defined. Since $G^{i+1}(a_i) = G(a_0) \in T_\mathbb{R}$ and $G^{i+1}(y) \geq a_0$, the point a_0 belongs to $G^{i+1}([a_i, y])$ by continuity. Thus there exists $a_{i+1} \in [a_i, y]$ such that $G^{i+1}(a_{i+1}) = a_0$. This concludes the construction of a_{i+1}.

The sequence $(a_i)_{i\geq 0}$ is non decreasing and contained in the compact interval X^ℓ. Therefore $x := \lim_{i\to+\infty} a_i$ exists and belongs to X^ℓ. Since $G(a_{i+1}) = a_i$, we get that $G(x) = x$. In other words, $F^q(x) = x + p$. Thus $F^{nq}(x) \in X$ for all $n \geq 0$, which implies that $x \in X^\infty$. Clearly, x is periodic (mod 1) and $\rho_F(x) = \frac{p}{q}$. □

6.4 The rotation set of $\Gamma(\mathcal{G})$

Proposition 5.19 implies that the rotation set of $\Gamma(\mathcal{G})$ can be decomposed as the finite union of the rotation sets of the connected components plus a finite set (see Equation (7) in the proof below). This decomposition, together with the study of the rotation set of a connected component in the previous subsection, leads to the following theorem.

Theorem 6.8 *The set* $\mathrm{Rot}_{\Gamma(\mathcal{G})}$ *is compact and has finitely many connected components.*

There exists a finite set D such that, for every rational number p/q in $\mathrm{Rot}_{\Gamma(\mathcal{G})}\setminus D$, there exists a loop γ in $\mathcal{G}$ such that $\rho(\widetilde{\gamma}) = p/q$.

Proof. Let $\mathcal{C}$ be the set of connected components of $\mathcal{G}$. Let $\mathcal{I}$ and $\mathcal{J}$ be the sets defined in Proposition 5.19 and 6.7 respectively. Let $\bar{\alpha} = (\alpha_n)_{n\geq 0} \in \Gamma(\mathcal{G})$ such that $\rho(\bar{\alpha})$ exists. By Proposition 5.19(iii):

• Either there exist a connected component $C \in \mathcal{C}$ and an integer N such that, $\forall n \geq N, \alpha_n \in C$. In this case, $\rho(\bar{\alpha}) \in \mathrm{Rot}_{\Gamma(C)}$.

• Or there exist $\bar{\beta} = (\beta_n)_{n\geq 0} \in \mathcal{I} \cap \mathcal{J}$ and integers N, M such that, $\forall n \geq 0$, $\alpha_{N+n} = \beta_{M+n}$. Thus $\rho(\bar{\beta})$ exists and is a rational number (Proposition 6.7) and is equal to $\rho(\bar{\alpha})$, and so $\rho(\bar{\alpha}) \in \mathrm{Rot}_{\mathcal{I}\cap\mathcal{J}}$.

Consequently,

$$\mathrm{Rot}_{\Gamma(\mathcal{G})} = \bigcup_{C\in\mathcal{C}} \mathrm{Rot}_{\Gamma(C)} \cup \mathrm{Rot}_{\mathcal{I}\cap\mathcal{J}}\,. \tag{7}$$

The sets $\mathcal{C}$ and $\mathcal{I}$ are finite by Proposition 5.19(i)-(ii), and thus $\mathrm{Rot}_{\mathcal{I}\cap\mathcal{J}}$ is finite. Moreover, for every $C \in \mathcal{C}$, $\mathrm{Rot}_{\Gamma(C)}$ is a compact interval by Proposition 6.6(i). Therefore $\mathrm{Rot}_{\Gamma(\mathcal{G})}$ is a finite union of compact intervals (some intervals may be reduced to a single point). This proves the first point of the theorem.

According to Proposition 6.6(ii), for every rational number $p/q \in \bigcup_{C\in\mathcal{C}} \mathrm{Int}\left(\mathrm{Rot}_{\Gamma(C)}\right)$, there exists a loop γ such that $\rho(\widetilde{\gamma}) = p/q$. Since the set $D := \bigcup_{C\in\mathcal{C}} \partial\,\mathrm{Rot}_{\Gamma(C)} \cup \mathrm{Rot}_{\mathcal{I}\cap\mathcal{J}}$ is finite, this implies the second point of the theorem. □

Remark 6.9

- The intervals $(\mathrm{Rot}_{\Gamma(C)})_{C\in\mathcal{C}}$ may not be disjoint.
- The number of connected components of $\mathrm{Rot}_{\Gamma(\mathcal{G})}$ is at most $2\#\mathcal{P}$ and at most $\#\mathcal{P}$ are not reduced to a single point. This is because both $\#\mathcal{C}$ and $\#\mathcal{I}$ are bounded by $\#\mathcal{P}$.

7 The rotation set of a sun-like map

Now that we have studied the rotation set of the covering graph, we are ready to prove that the rotation set of a sun-like map is composed of finitely many compact intervals, and that all but finitely many rational numbers in the rotation set are rotation numbers of some periodic (mod 1) points.

According to Equation (1), the rotation set of F has the following decomposition:

$$\mathrm{Rot}(F) = \mathrm{Rot}_{T_{\mathbb{R}}}(F) \cup \mathrm{Rot}_{X^\infty}(F).$$

First, we pull back the results from $\mathrm{Rot}_{\Gamma(\mathcal{G})}$ to $\mathrm{Rot}_{X^\infty}(F)$.

Proposition 7.1 *Let $F \in \mathcal{S}_1(T)$ and $\mathcal{G}$ its covering graph. Then* $\mathrm{Rot}_{X^\infty}(F) = \mathrm{Rot}_{\Gamma(\mathcal{G})}$.

If γ is a loop in $\mathcal{G}$, then $\rho(\widetilde{\gamma}) \in \mathbb{Q}$ and there exists a periodic (mod 1) *point $x \in X^\infty$ such that $\rho_F(x) = \rho(\widetilde{\gamma})$. If $\bar{\alpha} \in \mathcal{J}$, then $\rho(\bar{\alpha}) \in \mathbb{Q}$ and there exists a periodic* (mod 1) *point $x \in X^\infty$ such that $\rho_F(x) = \rho(\bar{\alpha})$*

Proof. If we put together Lemma 4.4, Proposition 5.13 and the definition of the rotation numbers in $\Gamma(\mathcal{G})$, then it is clear that

$$\exists x \in X^\infty \text{ such that } \rho_F(x) = r \Longleftrightarrow \exists \bar{\alpha} \in \Gamma(\mathcal{G}) \text{ such that } \rho(\bar{\alpha}) = r.$$

Hence $\mathrm{Rot}_{X^\infty}(F) = \mathrm{Rot}_{\Gamma(\mathcal{G})}$.

Suppose that $\gamma = \alpha_0 \dots \alpha_n$ is a loop in $\mathcal{G}$. According to Lemma 5.8, we have the following chain of positive coverings:

$$\langle \alpha_0 \rangle \xrightarrow[F]{+} \langle \alpha_1 \rangle + W(\alpha_0\alpha_1) \xrightarrow[F]{+} \langle \alpha_2 \rangle + W(\alpha_0\alpha_1\alpha_2) \cdots \xrightarrow[F]{+} \langle \alpha_n \rangle + W(\alpha_0 \dots \alpha_n).$$

Since $\alpha_0 = \alpha_n$, Proposition 3.5 implies that there exists $x \in \langle \alpha_0 \rangle$ such that $F^n(x) = x + W(\alpha_0 \dots \alpha_n)$ and $F^i(x) \in \langle \alpha_i \rangle + W(\alpha_0 \dots \alpha_i)$ for all $i \in [\![1, n-1]\!]$. This implies that x is a periodic (mod 1) point, $x \in X^\infty$ because $\langle \alpha_i \rangle \subset X$ for all $i \in [\![0, n]\!]$, and

$$\rho_F(x) = \frac{W(\alpha_0 \dots \alpha_n)}{n} = \frac{W(\gamma)}{L(\gamma)} = \rho(\widetilde{\gamma}).$$

If $\bar{\alpha} \in \mathcal{J}$, the conclusion is given by Proposition 6.7. □

The main result of this paper is now a mere consequence of Proposition 7.1, Proposition 6.7, Theorem 6.8 and Theorem 2.8.

Theorem 7.2 *Let $F \in \mathcal{S}_1(T)$. Then* $\mathrm{Rot}(F)$ *is a nonempty compact set and has finitely many connected components. There exists a finite set E such that, for every rational number p/q in* $\mathrm{Rot}(F) \setminus E$, *there exists a periodic* (mod 1) *point $x \in T$ such that $\rho_F(x) = p/q$. More precisely, $\#E \leq 2\#\mathcal{P}$ and $E \subset \bigcup_{C \in \mathcal{C}} \partial\, \mathrm{Rot}(\Gamma(C))$, where $\mathcal{C}$ is the set of connected components of $\mathcal{G}$.*

We conjecture that the exceptional set E is empty. This conjecture is true for the graph σ [5]. In the general case of a degree-one map on a graph with a unique loop, it is not known if the rotation set is closed, if the number of connected components is finite or if (almost) all rational rotation numbers are rotation numbers of periodic points.

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