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**TRANSITIVE TOPOLOGICAL MARKOV CHAINS
OF GIVEN ENTROPY AND PERIOD
WITH OR WITHOUT MEASURE OF MAXIMAL ENTROPY**

SYLVIE RUETTE

TRANSITIVE TOPOLOGICAL MARKOV CHAINS OF GIVEN ENTROPY AND PERIOD WITH OR WITHOUT MEASURE OF MAXIMAL ENTROPY

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We show that, for every positive real number h and every positive integer p , there exist oriented graphs G, G' (with countably many vertices) that are strongly connected, of period p , of Gurevich entropy h , and such that G is positive recurrent (thus the topological Markov chain on G admits a measure of maximal entropy) and G' is transient (thus the topological Markov chain on G' admits no measure of maximal entropy).

1. Vere-Jones classification of graphs

In this paper, all the graphs are oriented and have a finite or countable set of vertices, and if u, v are two vertices, there is at most one arrow $u \rightarrow v$. A *path* of length n in the graph G is a sequence of vertices (u_0, u_1, \dots, u_n) such that $u_i \rightarrow u_{i+1}$ in G for all $i \in \llbracket 0, n-1 \rrbracket$. This path is called a *loop* if $u_0 = u_n$.

Definition 1. Let G be an oriented graph, and let u, v be two vertices in G . We define the following quantities:

- $p_{uv}^G(n)$ is the number of paths (u_0, u_1, \dots, u_n) such that $u_0 = u$ and $u_n = v$; $R_{uv}(G)$ is the radius of convergence of the series $\sum p_{uv}^G(n)z^n$.
- $f_{uv}^G(n)$ is the number of paths (u_0, u_1, \dots, u_n) such that $u_0 = u, u_n = v$, and $u_i \neq v$ for all $0 < i < n$; $L_{uv}(G)$ is the radius of convergence of the series $\sum f_{uv}^G(n)z^n$.

Definition 2. Let G be an oriented graph and V its set of vertices. The graph G is *strongly connected* if, for all $u, v \in V$, there exists a path from u to v in G . The *period* of a strongly connected graph G is the greatest common divisor of $(p_{uu}^G(n))_{u \in V, n \geq 0}$. The graph G is *aperiodic* if its period is 1.

Proposition 3 [Vere-Jones 1962]. *Let G be an oriented graph. If G is strongly connected, $R_{uv}(G)$ does not depend on u and v ; it is denoted by $R(G)$.*

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	transient	null recurrent	positive recurrent
$\sum_{n>0} f_{uu}^G(n)R^n$	< 1	1	1
$\sum_{n>0} n f_{uu}^G(n)R^n$	$\leq +\infty$	$+\infty$	$< +\infty$
$\sum_{n\geq 0} p_{uv}^G(n)R^n$	$< +\infty$	$+\infty$	$+\infty$
$\lim_{n \rightarrow +\infty} p_{uv}^G(n)R^n$	0 $R = L_{uu}$	0 $R = L_{uu}$	$\lambda_{uv} > 0$ $R \leq L_{uu}$

Table 1. Properties of the series associated to a transient, null recurrent or positive recurrent graph G (G is strongly connected); these properties do not depend on the vertices u, v .

If there is no confusion, $R(G)$ and $L_{uv}(G)$ will be written R and L_{uv} .

Vere-Jones [1962] gives a classification of strongly connected graphs as transient, null recurrent, or positive recurrent. These definitions are lines 1 and 2 in Table 1. The other lines of Table 1 state properties of the series $\sum p_{uv}^G(n)z^n$, which give alternative definitions (lines 3 and 4 are in [Vere-Jones 1962], and the last line is Proposition 4).

Proposition 4 [Salama 1992]. *Let G be a strongly connected oriented graph. If G is transient or null recurrent, then $R = L_{uu}$ for all vertices u . Equivalently, if there exists a vertex u such that $R < L_{uu}$, then G is positive recurrent.*

2. Topological Markov chains and Gurevich entropy

Let G be an oriented graph and V its set of vertices. We define Γ_G as the set of two-sided infinite paths in G , that is,

$$\Gamma_G := \{(v_n)_{n \in \mathbb{Z}} \mid \text{for all } n \in \mathbb{Z}, v_n \rightarrow v_{n+1} \text{ in } G\} \subset V^{\mathbb{Z}}.$$

The map σ is the shift on Γ_G . The *topological Markov chain* on the graph G is the dynamical system (Γ_G, σ) .

The set V is endowed with the discrete topology, and Γ_G is endowed with the induced topology of $V^{\mathbb{Z}}$. The space Γ_G is not compact unless G is finite.

The topological Markov chain (Γ_G, σ) is transitive if and only if the graph G is strongly connected. It is topologically mixing if and only if the graph G is strongly connected and aperiodic.

If G is a finite graph, Γ_G is compact and the topological entropy $h_{\text{top}}(\Gamma_G, \sigma)$ is well defined (see, e.g., [Denker et al. 1976] for the definition of the topological entropy). If G is a countable graph, the *Gurevich entropy* [1969] of the graph G (or of the topological Markov chain Γ_G) is given by

$$h(G) := \sup\{h_{\text{top}}(\Gamma_H, \sigma) \mid H \subset G, H \text{ finite}\}.$$

This entropy can also be computed in a combinatorial way, as the exponential growth of the number of paths with fixed endpoints.

Proposition 5 [Gurevich 1970]. *Let G be a strongly connected oriented graph. Then for all vertices u, v ,*

$$h(G) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log p_{uv}^G(n) = -\log R(G).$$

Moreover, the variational principle is still valid for topological Markov chains.

Theorem 6 [Gurevich 1969]. *Let G be an oriented graph. Then*

$$h(G) = \sup\{h_\mu(\Gamma_G) \mid \mu \text{ } \sigma\text{-invariant probability measure}\}.$$

In this variational principle, the supremum is not necessarily reached. The next theorem gives a necessary and sufficient condition for the existence of a measure of maximal entropy (that is, a probability measure μ such that $h(G) = h_\mu(\Gamma_G)$) when the graph is strongly connected.

Theorem 7 [Gurevich 1970]. *Let G be a strongly connected oriented graph of finite positive entropy. Then the topological Markov chain on G admits a measure of maximal entropy if and only if the graph G is positive recurrent. Moreover, such a measure is unique if it exists.*

3. Construction of graphs of given entropy and given period that are either positive recurrent or transient

Lemma 8. *Let $\beta \in (1, +\infty)$. There exist a sequence of nonnegative integers $(a(n))_{n \geq 1}$ and positive constants c, M such that*

- $a(1) = 1,$
- $\sum_{n \geq 1} a(n)(1/\beta^n) = 1,$
- for all $n \geq 2, c \cdot \beta^{n^2-n} \leq a(n^2) \leq c \cdot \beta^{n^2-n} + M,$
- for all $n \geq 1, 0 \leq a(n) \leq M$ if n is not a square.

These properties imply that the radius of convergence of $\sum_{n \geq 1} a(n)z^n$ is $L = 1/\beta$ and that $\sum_{n \geq 1} na(n)L^n < +\infty$.

Proof. First we look for a constant $c > 0$ such that

$$(1) \quad \frac{1}{\beta} + c \sum_{n \geq 2} \beta^{n^2-n} \frac{1}{\beta^{n^2}} = 1.$$

We have

$$\sum_{n \geq 2} \beta^{n^2-n} \frac{1}{\beta^{n^2}} = \sum_{n \geq 2} \beta^{-n} = \frac{1}{\beta(\beta - 1)}.$$

Thus,

$$(1) \iff \frac{1}{\beta} + \frac{c}{\beta(\beta-1)} = 1 \iff c = (\beta-1)^2.$$

Since $\beta > 1$, the constant $c := (\beta-1)^2$ is positive. We define the sequence $(b(n))_{n \geq 1}$ by

- $b(1) := 1$,
- $b(n^2) := \lfloor c\beta^{n^2-n} \rfloor$ for all $n \geq 2$,
- $b(n) := 0$ for all $n \geq 2$ such that n is not a square.

Then

$$\sum_{n \geq 1} b(n) \frac{1}{\beta^n} \leq \frac{1}{\beta} + c \sum_{n \geq 2} \beta^{n^2-n} \frac{1}{\beta^{n^2}} = 1.$$

We set $\delta := 1 - \sum_{n \geq 1} b(n)(1/\beta^n) \in [0, 1)$ and $k := \lfloor \beta^2 \delta \rfloor$. Then $k \leq \beta^2 \delta < k+1 < k+\beta$, which implies that $0 \leq \delta - k/\beta^2 < 1/\beta$. We write the β -expansion of $\delta - k/\beta^2$ (see, e.g., [Dajani and Kraaikamp 2002, p. 51] for the definition): there exist integers $d(n) \in \{0, \dots, \lfloor \beta \rfloor\}$ such that $\delta - k/\beta^2 = \sum_{n \geq 1} d(n)(1/\beta^n)$. Moreover, $d(1) = 0$ because $\delta - k/\beta^2 < 1/\beta$. Thus, we can write

$$\delta = \sum_{n \geq 2} d'(n) \frac{1}{\beta^n}$$

where $d'(2) := d(2) + k$ and $d'(n) := d(n)$ for all $n \geq 3$.

We set $a(1) := b(1)$ and $a(n) := b(n) + d'(n)$ for all $n \geq 2$. Let $M := \beta + k$. We then have

- $a(1) = 1$,
- $\sum_{n \geq 1} a(n)(1/\beta^n) = 1$,
- for all $n \geq 2$, $c \cdot \beta^{n^2-n} \leq a(n^2) \leq c \cdot \beta^{n^2-n} + \beta \leq c \cdot \beta^{n^2-n} + M$,
- $0 \leq a(2) \leq \beta + k = M$,
- for all $n \geq 3$, $0 \leq a(n) \leq \beta \leq M$ if n is not a square.

The radius of convergence L of $\sum_{n \geq 1} a(n)z^n$ satisfies

$$-\log L = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log a(n) = \lim_{n \rightarrow +\infty} \frac{1}{n^2} \log a(n^2) = \log \beta$$

because $a(n^2) \sim c\beta^{n^2-n}$.

Thus, $L = 1/\beta$. Moreover,

$$\sum_{n \geq 1} na(n) \frac{1}{\beta^n} \leq M \sum_{n \geq 1} n \frac{1}{\beta^n} + c \sum_{n \geq 1} n^2 \beta^{n^2-n} \frac{1}{\beta^{n^2}} = M \sum_{n \geq 1} \frac{n}{\beta^n} + c \sum_{n \geq 1} \frac{n^2}{\beta^n} < +\infty. \quad \square$$

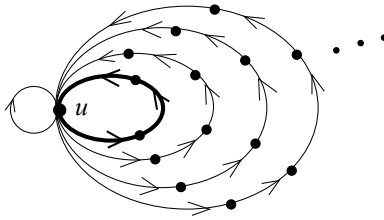


Figure 1. The graphs $G(\beta)$ and $G'(\beta)$; the bold loop belongs to $G(\beta)$ and not to $G'(\beta)$, otherwise the two graphs coincide.

Lemma 9 [Ruelle 2003, Lemma 2.4]. *Let G be a strongly connected oriented graph and u a vertex.*

- (i) $R < L_{uu}$ if and only if $\sum_{n \geq 1} f_{uu}^G(n)L_{uu}^n > 1$.
- (ii) If G is recurrent, then R is the unique positive number x such that

$$\sum_{n \geq 1} f_{uu}^G(n)x^n = 1.$$

Proof. For (i) and (ii), use Table 1 and the fact that $F(x) = \sum_{n \geq 1} f_{uu}^G(n)x^n$ is increasing for $x \in [0, +\infty)$. □

Proposition 10. *Let $\beta \in (1, +\infty)$. There exist aperiodic strongly connected graphs $G'(\beta) \subset G(\beta)$ such that $h(G(\beta)) = h(G'(\beta)) = \log \beta$, $G(\beta)$ is positive recurrent, and $G'(\beta)$ is transient.*

Remark. Salama [1988, Theorem 3.9] proved the part of this proposition concerning positive recurrent graphs.

Proof. This is a variant of the proof of [Ruelle 2003, Example 2.9].

Let u be a vertex, and let $(a(n))_{n \geq 1}$ be the sequence given by Lemma 8 for β . The graph $G(\beta)$ is composed of $a(n)$ loops of length n based at the vertex u for all $n \geq 1$ (see Figure 1). More precisely, define the set of vertices of $G(\beta)$ as

$$V := \{u\} \cup \bigcup_{n=1}^{+\infty} \{v_k^{n,i} \mid i \in \llbracket 1, a(n) \rrbracket, k \in \llbracket 1, n-1 \rrbracket\},$$

where the vertices $v_k^{n,i}$ above are distinct. Let $v_0^{n,i} = v_n^{n,i} = u$ for all $i \in \llbracket 1, a(n) \rrbracket$. There is an arrow $v_k^{n,i} \rightarrow v_{k+1}^{n,i}$ for all $k \in \llbracket 0, n-1 \rrbracket, i \in \llbracket 1, a(n) \rrbracket$, and $n \geq 2$; there is an arrow $u \rightarrow u$; and there is no other arrow in $G(\beta)$. The graph $G(\beta)$ is strongly connected, and $f_{uu}^{G(\beta)}(n) = a(n)$ for all $n \geq 1$.

By Lemma 8, the sequence $(a(n))_{n \geq 1}$ is defined such that $L = 1/\beta$ and

$$(2) \quad \sum_{n \geq 1} a(n)L^n = 1,$$

where $L = L_{uu}(G(\beta))$ is the radius of convergence of the series $\sum a(n)z^n$. If $G(\beta)$ is transient, then $R(G(\beta)) = L_{uu}(G(\beta))$ by Proposition 4. But (2) contradicts the definition of transient (see the first line of Table 1). Thus, $G(\beta)$ is recurrent, and $R(G(\beta)) = L$ by (2) and Lemma 9(ii). Moreover,

$$\sum_{n \geq 1} na(n)L^n < +\infty$$

by Lemma 8, and thus the graph $G(\beta)$ is positive recurrent (see Table 1). By Proposition 5, $h(G(\beta)) = -\log R(G(\beta)) = \log \beta$.

The graph $G'(\beta)$ is obtained from $G(\beta)$ by deleting a loop starting at u of length n_0 for some $n_0 \geq 2$ such that $a(n_0) \geq 1$ (such an integer n_0 exists because $L < +\infty$). Obviously one has $L_{uu}(G'(\beta)) = L$ and

$$\sum_{n \geq 1} f_{uu}^{G'(\beta)}(n)L^n = 1 - L^{n_0} < 1.$$

Since $R(G'(\beta)) \leq L_{uu}(G'(\beta))$, this implies that $G'(\beta)$ is transient. Moreover, $R(G'(\beta)) = L_{uu}(G'(\beta))$ by Proposition 4, so $R(G'(\beta)) = R(G(\beta))$, and hence $h(G'(\beta)) = h(G(\beta))$ by Proposition 5. Finally, both $G(\beta)$ and $G'(\beta)$ are of period 1 because of the arrow $u \rightarrow u$. □

Corollary 11. *Let p be a positive integer and $h \in (0, +\infty)$. There exist strongly connected graphs G, G' of period p such that $h(G) = h(G') = h$, G is positive recurrent, and G' is transient.*

Proof. For G , we start from the graph $G(\beta)$ given by Proposition 10 with $\beta = e^{hp}$. Let V denote the set of vertices of $G(\beta)$. The set of vertices of G is $V \times \llbracket 1, p \rrbracket$, and the arrows in G are

- $(v, i) \rightarrow (v, i + 1)$ if $v \in V$ and $i \in \llbracket 1, p - 1 \rrbracket$,
- $(v, p) \rightarrow (w, 1)$ if $v, w \in V$ and $v \rightarrow w$ is an arrow in $G(\beta)$.

According to the properties of $G(\beta)$, G is strongly connected, of period p , and positive recurrent. Moreover, $h(G) = (1/p)h(G(\beta)) = (1/p) \log \beta = h$.

For G' , we do the same starting with $G'(\beta)$. □

According to Theorem 7, the graphs of Corollary 11 satisfy that the topological Markov chain on G admits a measure of maximal entropy whereas the topological Markov chain on G' admits no measure of maximal entropy; both are transitive, of Gurevich entropy h , and supported by a graph of period p .

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SYLVIE RUETTE
 LABORATOIRE DE MATHÉMATIQUES D’ORSAY, UMR 8628
 UNIVERSITÉ PARIS-SACLAY
 ORSAY
 FRANCE
sylvie.ruette@universite-paris-saclay.fr

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Department of Mathematics
University of California
Los Angeles, CA 90095-1555
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Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Matthias Aschenbrenner
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
matthias@math.ucla.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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Princeton NJ 08544-1000
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