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TRANSITIVE TOPOLOGICAL MARKOV CHAINS OF GIVEN ENTROPY AND PERIOD WITH OR WITHOUT MEASURE OF MAXIMAL ENTROPY

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We show that, for every positive real number h and every positive integer p, there exist oriented graphs G, G' (with countably many vertices) that are strongly connected, of period p, of Gurevich entropy h, and such that G is positive recurrent (thus the topological Markov chain on G admits a measure of maximal entropy) and G' is transient (thus the topological Markov chain on G' admits no measure of maximal entropy).

1. Vere-Jones classification of graphs

In this paper, all the graphs are oriented and have a finite or countable set of vertices, and if u, v are two vertices, there is at most one arrow $u \to v$. A *path* of length n in the graph G is a sequence of vertices $(u_0, u_1, \ldots, u_n)$ such that $u_i \to u_{i+1}$ in G for all $i \in [\![0, n-1]\!]$. This path is called a *loop* if $u_0 = u_n$.

Definition 1. Let G be an oriented graph, and let u, v be two vertices in G. We define the following quantities:

- $p^G_{uv}(n)$ is the number of paths $(u_0, u_1, \ldots, u_n)$ such that $u_0 = u$ and $u_n = v$; $R_{uv}(G)$ is the radius of convergence of the series $\sum p^G_{uv}(n) z^n$.
- $f^G_{uv}(n)$ is the number of paths $(u_0, u_1, \ldots, u_n)$ such that $u_0 = u$, $u_n = v$, and $u_i \neq v$ for all $0 < i < n$; $L_{uv}(G)$ is the radius of convergence of the series $\sum f^G_{uv}(n) z^n$.

Definition 2. Let G be an oriented graph and V its set of vertices. The graph G is *strongly connected* if, for all $u, v \in V$, there exists a path from u to v in G. The *period* of a strongly connected graph G is the greatest common divisor of $(p^G_{uu}(n))_{u \in V,\, n \geq 0}$. The graph G is *aperiodic* if its period is 1.

Proposition 3 [Vere-Jones 1962]. *Let G be an oriented graph. If G is strongly connected, $R_{uv}(G)$ does not depend on u and v; it is denoted by $R(G)$.*

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	transient	null recurrent	positive recurrent
$\sum_{n>0} f^G_{uu}(n)R^n$	<1	1	1
$\sum_{n>0} n f^G_{uu}(n)R^n$	$\le +\infty$	$+\infty$	$<+\infty$
$\sum_{n\ge 0} p^G_{uv}(n)R^n$	$<+\infty$	$+\infty$	$+\infty$
$\lim_{n\to+\infty} p^G_{uv}(n)R^n$	0	0	$\lambda_{uv}>0$
	$R=L_{uu}$	$R=L_{uu}$	$R\le L_{uu}$

Table 1. Properties of the series associated to a transient, null recurrent or positive recurrent graph G (G is strongly connected); these properties do not depend on the vertices u, v.

If there is no confusion, $R(G)$ and $L_{uv}(G)$ will be written R and L_{uv}.

Vere-Jones [1962] gives a classification of strongly connected graphs as transient, null recurrent, or positive recurrent. These definitions are lines 1 and 2 in Table 1. The other lines of Table 1 state properties of the series $\sum p^G_{uv}(n)z^n$, which give alternative definitions (lines 3 and 4 are in [Vere-Jones 1962], and the last line is Proposition 4).

Proposition 4 [Salama 1992]. *Let G be a strongly connected oriented graph. If G is transient or null recurrent, then $R = L_{uu}$ for all vertices u. Equivalently, if there exists a vertex u such that $R < L_{uu}$, then G is positive recurrent.*

2. Topological Markov chains and Gurevich entropy

Let G be an oriented graph and V its set of vertices. We define Γ_G as the set of two-sided infinite paths in G, that is,

$$\Gamma_G := \{(v_n)_{n\in\mathbb{Z}} \mid \text{for all } n\in\mathbb{Z},\ v_n \to v_{n+1} \text{ in } G\} \subset V^{\mathbb{Z}}.$$

The map σ is the shift on Γ_G. The *topological Markov chain* on the graph G is the dynamical system (Γ_G, σ).

The set V is endowed with the discrete topology, and Γ_G is endowed with the induced topology of $V^{\mathbb{Z}}$. The space Γ_G is not compact unless G is finite.

The topological Markov chain (Γ_G, σ) is transitive if and only if the graph G is strongly connected. It is topologically mixing if and only if the graph G is strongly connected and aperiodic.

If G is a finite graph, Γ_G is compact and the topological entropy $h_{\text{top}}(\Gamma_G, \sigma)$ is well defined (see, e.g., [Denker et al. 1976] for the definition of the topological entropy). If G is a countable graph, the *Gurevich entropy* [1969] of the graph G (or of the topological Markov chain Γ_G) is given by

$$h(G) := \sup\{h_{\text{top}}(\Gamma_H, \sigma) \mid H \subset G,\ H \text{ finite}\}.$$

This entropy can also be computed in a combinatorial way, as the exponential growth of the number of paths with fixed endpoints.

Proposition 5 [Gurevich 1970]. *Let G be a strongly connected oriented graph. Then for all vertices u, v,*

$$h(G) = \lim_{n\to+\infty} \frac{1}{n} \log p_{uv}^G(n) = -\log R(G).$$

Moreover, the variational principle is still valid for topological Markov chains.

Theorem 6 [Gurevich 1969]. *Let G be an oriented graph. Then*

$$h(G) = \sup\{h_\mu(\Gamma_G) \mid \mu \ \sigma\text{-invariant probability measure}\}.$$

In this variational principle, the supremum is not necessarily reached. The next theorem gives a necessary and sufficient condition for the existence of a measure of maximal entropy (that is, a probability measure μ such that $h(G) = h_\mu(\Gamma_G)$) when the graph is strongly connected.

Theorem 7 [Gurevich 1970]. *Let G be a strongly connected oriented graph of finite positive entropy. Then the topological Markov chain on G admits a measure of maximal entropy if and only if the graph G is positive recurrent. Moreover, such a measure is unique if it exists.*

3. Construction of graphs of given entropy and given period that are either positive recurrent or transient

Lemma 8. *Let $\beta \in (1, +\infty)$. There exist a sequence of nonnegative integers $(a(n))_{n\geq 1}$ and positive constants c, M such that*

- $a(1) = 1$,
- $\sum_{n\geq 1} a(n)(1/\beta^n) = 1$,
- *for all* $n \geq 2$, $c \cdot \beta^{n^2-n} \leq a(n^2) \leq c \cdot \beta^{n^2-n} + M$,
- *for all* $n \geq 1$, $0 \leq a(n) \leq M$ *if n is not a square.*

These properties imply that the radius of convergence of $\sum_{n\geq 1} a(n) z^n$ is $L = 1/\beta$ and that $\sum_{n\geq 1} n a(n) L^n < +\infty$.

Proof. First we look for a constant $c > 0$ such that

$$\frac{1}{\beta} + c \sum_{n\geq 2} \beta^{n^2-n} \frac{1}{\beta^{n^2}} = 1. \tag{1}$$

We have

$$\sum_{n\geq 2} \beta^{n^2-n} \frac{1}{\beta^{n^2}} = \sum_{n\geq 2} \beta^{-n} = \frac{1}{\beta(\beta-1)}.$$

Thus,

$$(1) \iff \frac{1}{\beta}+\frac{c}{\beta(\beta-1)}=1 \iff c=(\beta-1)^2.$$

Since $\beta>1$, the constant $c:=(\beta-1)^2$ is positive. We define the sequence $(b(n))_{n\geq 1}$ by

- $b(1):=1$,
- $b(n^2):=\lfloor c\beta^{n^2-n}\rfloor$ for all $n\geq 2$,
- $b(n):=0$ for all $n\geq 2$ such that n is not a square.

Then

$$\sum_{n\geq 1} b(n)\frac{1}{\beta^n}\leq \frac{1}{\beta}+c\sum_{n\geq 2}\beta^{n^2-n}\frac{1}{\beta^{n^2}}=1.$$

We set $\delta:=1-\sum_{n\geq 1} b(n)(1/\beta^n)\in[0,1)$ and $k:=\lfloor\beta^2\delta\rfloor$. Then $k\leq\beta^2\delta<k+1<k+\beta$, which implies that $0\leq\delta-k/\beta^2<1/\beta$. We write the β-expansion of $\delta-k/\beta^2$ (see, e.g., [Dajani and Kraaikamp 2002, p. 51] for the definition): there exist integers $d(n)\in\{0,\ldots,\lfloor\beta\rfloor\}$ such that $\delta-k/\beta^2=\sum_{n\geq 1} d(n)(1/\beta^n)$. Moreover, $d(1)=0$ because $\delta-k/\beta^2<1/\beta$. Thus, we can write

$$\delta=\sum_{n\geq 2} d'(n)\frac{1}{\beta^n}$$

where $d'(2):=d(2)+k$ and $d'(n):=d(n)$ for all $n\geq 3$.

We set $a(1):=b(1)$ and $a(n):=b(n)+d'(n)$ for all $n\geq 2$. Let $M:=\beta+k$. We then have

- $a(1)=1$,
- $\sum_{n\geq 1} a(n)(1/\beta^n)=1$,
- for all $n\geq 2$, $c\cdot\beta^{n^2-n}\leq a(n^2)\leq c\cdot\beta^{n^2-n}+\beta\leq c\cdot\beta^{n^2-n}+M$,
- $0\leq a(2)\leq\beta+k=M$,
- for all $n\geq 3$, $0\leq a(n)\leq\beta\leq M$ if n is not a square.

The radius of convergence L of $\sum_{n\geq 1} a(n)z^n$ satisfies

$$-\log L=\limsup_{n\to+\infty}\frac{1}{n}\log a(n)=\lim_{n\to+\infty}\frac{1}{n^2}\log a(n^2)=\log\beta \qquad \text{because } a(n^2)\sim c\beta^{n^2-n}.$$

Thus, $L=1/\beta$. Moreover,

$$\sum_{n\geq 1} na(n)\frac{1}{\beta^n}\leq M\sum_{n\geq 1} n\frac{1}{\beta^n}+c\sum_{n\geq 1} n^2\beta^{n^2-n}\frac{1}{\beta^{n^2}}=M\sum_{n\geq 1}\frac{n}{\beta^n}+c\sum_{n\geq 1}\frac{n^2}{\beta^n}<+\infty. \qquad \square$$

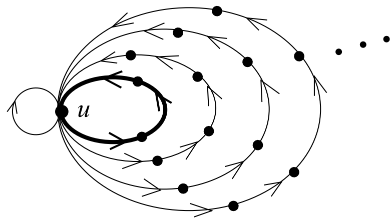

Figure 1. The graphs $G(\beta)$ and $G'(\beta)$; the bold loop belongs to $G(\beta)$ and not to $G'(\beta)$, otherwise the two graphs coincide.

Lemma 9 [Ruette 2003, Lemma 2.4]. *Let G be a strongly connected oriented graph and u a vertex.*

(i) $R < L_{uu}$ *if and only if* $\sum_{n\geq 1} f_{uu}^G(n) L_{uu}^n > 1$.

(ii) *If G is recurrent, then R is the unique positive number x such that*

$$\sum_{n\geq 1} f_{uu}^G(n) x^n = 1.$$

Proof. For (i) and (ii), use Table 1 and the fact that $F(x) = \sum_{n\geq 1} f_{uu}^G(n) x^n$ is increasing for $x \in [0, +\infty)$. □

Proposition 10. *Let $\beta \in (1, +\infty)$. There exist aperiodic strongly connected graphs $G'(\beta) \subset G(\beta)$ such that $h(G(\beta)) = h(G'(\beta)) = \log \beta$, $G(\beta)$ is positive recurrent, and $G'(\beta)$ is transient.*

Remark. Salama [1988, Theorem 3.9] proved the part of this proposition concerning positive recurrent graphs.

Proof. This is a variant of the proof of [Ruette 2003, Example 2.9].

Let u be a vertex, and let $(a(n))_{n\geq 1}$ be the sequence given by Lemma 8 for β. The graph $G(\beta)$ is composed of $a(n)$ loops of length n based at the vertex u for all $n \geq 1$ (see Figure 1). More precisely, define the set of vertices of $G(\beta)$ as

$$V := \{u\} \cup \bigcup_{n=1}^{+\infty} \{v_k^{n,i} \mid i \in [\![1, a(n)]\!],\ k \in [\![1, n-1]\!]\},$$

where the vertices $v_k^{n,i}$ above are distinct. Let $v_0^{n,i} = v_n^{n,i} = u$ for all $i \in [\![1, a(n)]\!]$. There is an arrow $v_k^{n,i} \to v_{k+1}^{n,i}$ for all $k \in [\![0, n-1]\!]$, $i \in [\![1, a(n)]\!]$, and $n \geq 2$; there is an arrow $u \to u$; and there is no other arrow in $G(\beta)$. The graph $G(\beta)$ is strongly connected, and $f_{uu}^{G(\beta)}(n) = a(n)$ for all $n \geq 1$.

By Lemma 8, the sequence $(a(n))_{n\geq 1}$ is defined such that $L = 1/\beta$ and

$$\sum_{n\geq 1} a(n) L^n = 1, \tag{2}$$

where $L = L_{uu}(G(\beta))$ is the radius of convergence of the series $\sum a(n)z^n$. If $G(\beta)$ is transient, then $R(G(\beta)) = L_{uu}(G(\beta))$ by Proposition 4. But (2) contradicts the definition of transient (see the first line of Table 1). Thus, $G(\beta)$ is recurrent, and $R(G(\beta)) = L$ by (2) and Lemma 9(ii). Moreover,

$$\sum_{n\geq 1} na(n)L^n < +\infty$$

by Lemma 8, and thus the graph $G(\beta)$ is positive recurrent (see Table 1). By Proposition 5, $h(G(\beta)) = -\log R(G(\beta)) = \log \beta$.

The graph $G'(\beta)$ is obtained from $G(\beta)$ by deleting a loop starting at u of length n_0 for some $n_0 \geq 2$ such that $a(n_0) \geq 1$ (such an integer n_0 exists because $L < +\infty$). Obviously one has $L_{uu}(G'(\beta)) = L$ and

$$\sum_{n\geq 1} f_{uu}^{G'(\beta)}(n)L^n = 1 - L^{n_0} < 1.$$

Since $R(G'(\beta)) \leq L_{uu}(G'(\beta))$, this implies that $G'(\beta)$ is transient. Moreover, $R(G'(\beta)) = L_{uu}(G'(\beta))$ by Proposition 4, so $R(G'(\beta)) = R(G(\beta))$, and hence $h(G'(\beta)) = h(G(\beta))$ by Proposition 5. Finally, both $G(\beta)$ and $G'(\beta)$ are of period 1 because of the arrow $u \to u$. □

Corollary 11. *Let p be a positive integer and $h \in (0, +\infty)$. There exist strongly connected graphs G, G' of period p such that $h(G) = h(G') = h$, G is positive recurrent, and G' is transient.*

Proof. For G, we start from the graph $G(\beta)$ given by Proposition 10 with $\beta = e^{hp}$. Let V denote the set of vertices of $G(\beta)$. The set of vertices of G is $V \times [\![1, p]\!]$, and the arrows in G are

- $(v, i) \to (v, i+1)$ if $v \in V$ and $i \in [\![1, p-1]\!]$,
- $(v, p) \to (w, 1)$ if $v, w \in V$ and $v \to w$ is an arrow in $G(\beta)$.

According to the properties of $G(\beta)$, G is strongly connected, of period p, and positive recurrent. Moreover, $h(G) = (1/p)h(G(\beta)) = (1/p)\log\beta = h$.

For G', we do the same starting with $G'(\beta)$. □

According to Theorem 7, the graphs of Corollary 11 satisfy that the topological Markov chain on G admits a measure of maximal entropy whereas the topological Markov chain on G' admits no measure of maximal entropy; both are transitive, of Gurevich entropy h, and supported by a graph of period p.

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