Calderón Seminar: Section 2

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Outline

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Let M be a compact connected smooth n-manifold with nonempty smooth boundary, and let g be a smooth Riemannian metric. For an open set $\emptyset \neq \Gamma \subset \partial M$,

$$H^{1/2}_{\mathrm{co}}(\Gamma) = \{f \in H^{1/2}(\partial M) : \operatorname{supp} f \subset \Gamma\}.$$

For $f \in H^{1/2}(\partial M)$, let $u^g_f \in H^1(M)$ be the unique weak solution of

$$-\Delta_g u^g_f = 0 \quad \text{in } M, \qquad u^g_f|_{\partial M} = f.$$

The local energy Dirichlet-to-Neumann map is the bounded operator

$$\Lambda_{g,\Gamma} \colon H^{1/2}_{\mathrm{co}}(\Gamma) \to (H^{1/2}_{\mathrm{co}}(\Gamma))^*,$$

$$\langle \Lambda_{g,\Gamma} f, h\rangle = \int_M \langle du^g_f, du^g_h\rangle_g \, dV_g = \int_\Gamma h \partial_{\nu_g} u^g_f \, dS_g.$$

where ν_g is the outward unit normal.

Remark: This includes the density.

Theorem ([OpenAI '26])

If two smooth Riemannian metrics g_1, g_2 on M satisfy $\Lambda_{g_1,\Gamma} = \Lambda_{g_2,\Gamma}$, then there is a smooth diffeomorphism $\Phi\colon M \to M$, smooth through the boundary, such that $\Phi|_\Gamma = \mathrm{Id}$, $g_2 = \Phi^* g_1$.

Proof:

- **Lemma 2.1**: The jets coincide at the boundary.
- **Proposition 2.2**: There are compact connected smooth extensions (M_j^e, g_j^e) of (M, g_j) and an identified connected open exterior cap E on which the extended metrics agree.
- **Proposition 7.1**: There is an isometry $\Phi\colon \mathring{M}_2^e \to \mathring{M}_1^e$ such that

$$\Phi|_E = \mathrm{Id}, \qquad I_2(p) = I_1(\Phi(p)).$$

- **Proposition 7.2**: The previous isometry extends to a smooth diffeomorphism $M_2^e \to M_1^e$, preserves the original copies of M, and its restriction satisfies $\Phi|_\Gamma = \mathrm{Id}$, $g_2 = \Phi^* g_1$.

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Same jets

Lemma (Lemma 2.1)

Let g_1, g_2 be smooth metrics on a compact connected smooth n-manifold M with nonempty boundary, where $n \geq 3$. Suppose their energy maps $f \mapsto \partial_{\nu_{g_i}} u_f^{g_i} \, dS_{g_i}$ agree for inputs and observations in a nonempty relatively open set $\Gamma \subset \partial M$. In boundary-normal coordinates for the respective metrics, based on the same coordinates on Γ, their complete boundary Taylor jets agree.

This is [Lee–Uhlmann '89].

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Proposition (Proposition 2.2)

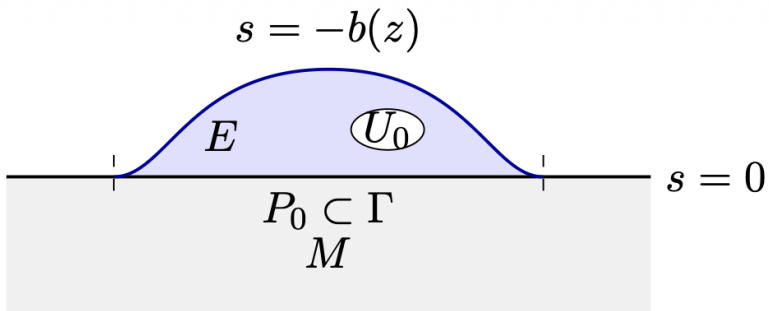
There are compact connected smooth extensions (M_j^e, g_j^e) of (M, g_j) and an identified connected open exterior cap E on which the extended metrics agree. The cap is attached through a nonempty open set $P_0 \Subset \Gamma$. The extended zero-Dirichlet Green kernels satisfy

$$G_1(x, y) = G_2(x, y), \quad x, y \in E, \quad x \neq y.$$

For every $F \in C_c^\infty(E)$, the two extended zero-Dirichlet solutions with source F agree on E.

Step 1: Extension

Take a boundary coordinate patch $P \Subset \Gamma$, with coordinates $z \mapsto q(z)$. For each $j = 1, 2$, consider a normal collar



$$C_j \colon P \times [0, \varepsilon) \to M,$$

$$C_j(z, s) = \exp^{g_j}_{q(z)}(-s\nu_{g_j}(q(z))).$$

We can take the same $\varepsilon > 0$ for both collars (by shrinking).
Let $b \geq 0$ be smooth with $\operatorname{supp} b \Subset P$, and such that $P_0 := \{b > 0\} \neq \emptyset$ is connected. Extend each collar region to $\{(z, s) : z \in P,\ -b(z) \leq s < \varepsilon\}$. The common added open region is

$$E = \{(z, s) : z \in P_0,\ -b(z) < s < 0\}$$

It is connected, since it is the image of the diffeomorphism

$$P_0 \times (0, 1) \to E, \quad (z, \tau) \mapsto (z, -\tau b(z)).$$

Step 2: Metric extension

- We have that $C_j^* g_j = ds^2 + h_j(z, s)$.
- Extend h_1 to E smoothly. By shrinking E, we can assume that the extension $\tilde{h}_1$ is still positive definite.
- Extend g_2 by

$$\begin{cases} ds^2 + h_2(z, s), & \text{if } s \geq 0, \\ ds^2 + \tilde{h}_1(z, s) & \text{if } s < 0. \end{cases}$$

- By Lemma 2.1, this extension is smooth.
- In this way, the extended metrics, which we still denote by g_j, coincide in E.

Step 3: Dirichlet solutions with source on E

- Let $F \in C_c^\infty(E)$ be a real-valued function. We can consider its smooth extension by zero to M_j^e. In $H_0^1(M_j^e)$, let

$$\mathcal{J}_j(w) = \frac{1}{2}\int_{M_j^e} |dw|^2_{g_j} dV_{g_j} - \int_E F w dV_{g_j}.$$

- By Poincaré inequality, this is coercive. Thus, $\mathcal{J}_j$ has a unique minimizer w_j, characterized by

$$-\Delta_{g_j^e} w_j = F, \quad w_j|_{\partial M_j^e} = 0.$$

- Elliptic regularity gives $w_j \in C^\infty(M_j^e)$.
- Let $f_j = w_j|_{\partial M}$, and observe that $\operatorname{supp} f_j \subset \operatorname{supp} b$. Thus, $f_j \in C_c^\infty(\Gamma)$. Since $w_j|_M$ is g_j-harmonic, uniqueness of the Dirichlet problem in M implies that $w_j|_M = u_j^{f_j}$.

To compare the two minima (w_1 and w_2), we would like to keep w_1 unchanged on E and replace its restriction to M by $u_2^{f_1}$. First, we prove that this is an element of $H_0^1(M_2^e)$.

- Let

$$T\colon M_1^e \to M_2^e, \quad T = \begin{cases} \mathrm{Id}_M, & \text{in } M \\ \mathrm{Id}_E, & \text{in } E. \end{cases}$$

- Note that the definitions agree at $s = 0$. So, T is an homeomorphism.
- In coordinates, we have

$$T(z,s) = \begin{cases} C_2^{-1} \circ C_1(z,s) & \text{if } s \geq 0, \\ (z,s) & \text{if } s < 0. \end{cases}$$

- Each branch is smooth with locally bounded derivatives, and the branches agree on $s = 0$. Thus, it is a continuous piecewise smooth map which is locally Lipschitz across that hypersurface. Since the same argument applies to its inverse, compactness supplies the needed global bounds, implying that T is bi-Lipschitz. Then, $T\colon H_0^1(M_1^e) \to H_0^1(M_2^e)$.

- Let $r = u_2^{f_1} - u_1^{f_1} \in H_0^1(M)$.
- Extend r by zero to obtain to $\tilde{r} \in H_0^1(M_2^e)$
- The function that we want is $v_2 = w_1 \circ T^{-1} + \tilde{r}$. Indeed,

$$v_2 = \begin{cases} u_2^{f_1} & \text{in } M, \\ w_1 & \text{in } E. \end{cases}$$

- Since the DN maps coincide, and $w_1|_M = u_1^{f_1}$, we obtain $\mathcal{J}_2(v_2) = \mathcal{J}_1(w_1)$.
- Since w_2 minimizes $\mathcal{J}_2$, we then obtain $\mathcal{J}_2(w_2) \leq \mathcal{J}_1(w_1)$.
- Repeating the same arguments with the reversed indices, gives the reverse inequality, showing that $\mathcal{J}_2(w_2) = \mathcal{J}_2(v_2)$.
- By uniqueness of minimizers, we obtain $w_2 = v_2$. Thus, $w_1 = w_2$ on E.

Step 4: Green kernels

- We have
$$w_j(x) = \int_{M_j^e} G_j(x,y) F(y) dV_{g_j^e}(y)$$
- For sources $F \in C_c^\infty(E)$, this becomes
$$w_j(x) = \int_E G_j(x,y) F(y) dV_{g_j}(y)$$
- Since $w_1 = w_2$ in E, we have
$$\int_E (G_1(x,y) - G_2(x,y)) F(y) dV_{g_j}(y) = 0.$$

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Fix a nonempty open source set $U_0 \Subset E$. For $f \in C_c^\infty(U_0)$, let u_f^j be the zero-Dirichlet solution of $L_{g_j} u_f^j = f$ on M_j^e. Define its **evaluation at a point** by

$$I_j(p)(f) = u_f^j(p). \tag{1}$$

Equality of evaluations means equality for every such source f. The last part of Proposition 2.2 gives $I_1 = I_2$ on E. Note that the functions u_f^j are harmonic away from $\overline{U_0}$.

Lemma (Lemma 2.3)

Let (N, g) be a compact connected smooth manifold with nonempty boundary. Let $U \Subset \mathring{N}$ be nonempty and open, and let $R = L_{g,D}^{-1}$. Then:

(i) *The functionals $I(p) : f \mapsto Rf(p), f \in C_c^\infty(U)$, separate distinct interior points. They are nonzero in the interior and zero on the boundary. For each fixed f, they depend continuously on $p \in N$.*

(ii) *If $p \in \mathring{N} \setminus \overline{U}$, the second jets of Rf at p fill exactly the hyperplane defined by $L_g u(p) = 0$. In particular they provide harmonic coordinates near p and any finite spanning family of harmonic Hessians in those coordinates.*

Item (i) Continuity

- Let $f \in C_c^\infty(U)$. We can extend it smoothly by zero to the whole manifold.
- By elliptic regularity, we obtain that Rf is smooth up to the boundary of N.
- Thus, $I(p)(f) = Rf(p)$ is continuous in p.
- The zero-Dirichlet condition implies that I is zero on ∂N.

Item (i) Non-zero

$I(p)$ is a non-zero functional: we will find some $f \in C_c^\infty(U)$ such that $I(p)(f) \neq 0$.

- Let $f \in C_c^\infty(U)$ be a smooth nonzero nonnegative bump function.
- Since $-\Delta_g Rf = f \geq 0$, the weak minimum principle implies that $Rf \geq 0$.
- If Rf vanishes at an interior point, then it attains its minumum there.
- Since $\Delta_g Rf \leq 0$, the strong maximum principle then gives that Rf is constant, thus is identically zero (since it vanishes at the boundary).
- This contradicts the definition of f.

Item (i): point separation

- By contradiction, let us assume that $Rf(p) = Rf(q)$ for all $f \in C_c^\infty(U)$.
- Write $T = \delta_p - \delta_q$, and define w by

$$\langle w, \varphi \rangle = \langle T, R\varphi \rangle, \qquad \varphi \in C_c^\infty(\mathring{N}).$$

- One has $L_g w = \delta_p - \delta_q$. Then w is harmonic in $\mathring{N} \setminus \{p, q\}$. Furthermore, it vanishes in $U \setminus \{p, q\}$
- Unique continuation then gives $\operatorname{supp} w \subset \{p, q\}$
- Let B be a ball containing p and not containing q. Then, $w|_B$ is a nonzero distribution supported at p. We can write $w|_B = \sum_{|\alpha| \leq k} c_\alpha \partial^\alpha \delta_p$.
- The order of $L_g w|_B$ is $k + 2$, while the order of δ_p is zero.
- This contradiction shows that I separate points.

Item (ii): jets

We will show that the set of second jets of Rf, with $f \in C_c^\infty(U)$, is exactly the set of jets that satisfies $L_g u(p) = 0$.

- Write $j_p^2 u = (u(p), \partial_i u(p), \partial_i \partial_j u(p))$, and let J_p^2 denote the vector space of these collections.
- Since R is linear, the set of all jets obtained by sources $S := \{j_p^2(Rf) : f \in C_c^\infty(U)\}$ is a subspace of J_p^2.
- Define
$$\ell \colon S \to \mathbb{R}, \quad j_p^2 u \mapsto L_g u(p).$$
- Since $p \in \mathring{N} \setminus \overline{U}$, we obtain $S \subset \ker \ell$.
- We will show that $\dim S = \dim \ker \ell$.

- Consider $T \in (J^2_p)^*$ such that $T(j^2_p(Rf)) = 0$ for every $f \in C^\infty_c(U)$.
- **Claim:** $T = c\ell$, for some constant c.
- Note that $T = \sum_{|\alpha|\leq 2} a_\alpha \partial^\alpha \delta_p$. Define w as before:

$$\langle w, \varphi \rangle = \langle T, R\varphi \rangle, \qquad \varphi \in C^\infty_c(\mathring{N}).$$

- We have $L_g w = T$. As before, we obtain $\operatorname{supp} w \subset \{p\}$. Thus, $w = \sum_{|\alpha|\leq k} c_\alpha \partial^\alpha \delta_p$.
- If $w \neq 0$, we obtain a contradiction as in the proof of item (i) unless $k \leq 0$. Thus, $w = \sum_{|\alpha|\leq k} c_\alpha \partial^\alpha \delta_p$
- Hence, $w = c\delta_p$, that is, $T = L_g w = cL_g \delta_p = c\ell$.
- In particular, $S^\circ = \{T \in (J^2_p)^* : T|_S = 0\}$ is spanned by ℓ. Then

$$\dim S = \dim J^2_p - \dim S^\circ = \dim J^2_p - 1 = \dim \ker \ell.$$

Item (ii): harmonic coordinates

"The second jets of Rf at p provide harmonic coordinates"

- Fix $p \in \mathring{N} \setminus \overline{U}$. In local coordinates $y = (y^1, \dots, y^n)$ near p, write $\Delta_g = g^{ij}\partial_{y^i}\partial_{y^j} + b^k\partial_{y^k}$.
- For each $a = 1, \dots, n$, prescribe the following second jet at p:

$$u^a(p) = 0, \quad \partial_{y^i} u^a(p) = \delta_i^a, \quad \partial_{y^i}\partial_{y^j} u^a(p) = -\frac{b^a(p)}{n} g_{ij}(p).$$

- Note that $L_g u^a(p) = 0$. Thus, by the previous argument, there exists $f_a \in C_c^\infty(U)$ such that $u^a = Rf_a$ has this prescribed jet. Now define $x = (u^1, \dots, u^n)$.
- Since its Jacobian at p is the identity matrix, we can apply the inverse function theorem to obtain a neighborhood W of p on which $x\colon W \to x(W) \subset \mathbb{R}^n$ is a smooth coordinate chart.
- Shrink W so that $W \cap \overline{U} = \emptyset$. In this way, we obtain that $L_g u^a = f_a = 0$ on W, i.e., $x = (Rf_1, \dots, Rf_n)$ is a harmonic coordinate chart near p.

Item (ii) harmonic Hessians

"the second jets of Rf at p [...] provide [...] any finite spanning family of harmonic Hessians in those coordinates."
We show that we can prescribe any finite spanning family of admissible Hessian matrices at p and realize them by source solutions.

- Let $x = (x^1, \dots, x^n)$ be the harmonic coordinates already constructed on a neighborhood W of p, with $W \cap \overline{U} = \emptyset$. Write $\partial_i = \partial/\partial x^i$.
- Note that $0 = \Delta_g x^a = b^a$. Thus, in these coordinates, $\Delta_g = g^{ij}\partial_i\partial_j$.
- Write $\mathcal{H}_p = \{H \in \mathrm{Sym}^2(\mathbb{R}^n) : g^{ij}(p)H_{ij} = 0\}$, and let d be its dimension.
- Let $H^1, \dots, H^d$ be an linear independent set of $\mathcal{H}_p$. For each α, prescribe the second jet

$$v^\alpha(p) = 0, \quad \partial_i v^\alpha(p) = 0, \quad \partial_i\partial_j v^\alpha(p) = H^\alpha_{ij}.$$

- Since $L_g v^\alpha(p) = -g^{ij}(p)H^\alpha_{ij} = 0$, the first part of the proof of item (ii) gives $h_\alpha \in C^\infty_c(U)$ such that $v^\alpha = Rh_\alpha$ has this prescribed jet, and its Hessian at p is H^α.

Thank you!