# OpenIA's proof of Calderón: Section 7

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# Outline

1. Review
2. Proposition 7.1
   - Step 1: Maximal match
   - Step 2: Green kernels
   - Step 3: Construction of local charts
   - Step 4: Spanning Hessians
   - Step 5: Metric extension
   - Step 6: Contradiction
   - Step 7: Surjectivity
3. Proposition 7.2
   - Step 1: Extension to the boundary
   - Step 2: Smoothness up to the boundary
   - Step 3: Preservation of the interior and the boundary

# Review

**Proposition 2.2:** There are compact connected $(M_j^e, g_j^e)$ of $(M, g_j)$ and an identified connected open exterior cap $E$ such that

- the extended metrics agree on $E$.
- $G_1(x, y) = G_2(x, y)$ for $x \neq y \in E$.
- For every $F \in C_c^\infty(E)$, the two extended zero-Dirichlet solutions with source $F$ agree on $E$.

**Lemma 2.3:** Let $(N, g)$ be a compact connected smooth manifold with nonempty boundary. Let $U \Subset \mathring{N}$ be nonempty and open, and let $R = L_{g,D}^{-1}$. Then:

(i) The functionals $I(p) : f \mapsto Rf(p), f \in C_c^\infty(U)$, separate distinct interior points. They are nonzero in the interior and zero on the boundary. For each fixed $f$, they depend continuously on $p \in N$.

(ii) If $p \in \mathring{N} \setminus \overline{U}$, the second jets of $Rf$ at $p$ fill exactly the hyperplane defined by $L_g u(p) = 0$. In particular they provide harmonic coordinates near $p$ and any finite spanning family of harmonic Hessians in those coordinates.

The key element will be

**Theorem 3.1:** Let $n \geq 3$, $(N_1, g_1)$ and $(N_2, g_2)$ be smooth compact connected Riemannian manifolds with nonempty smooth boundary, with Dirichlet Green functions $G_1, G_2$. Identify two interior harmonic coordinate neighborhoods with one neighborhood $\Omega \subset \mathbb{R}^n$ of 0. Suppose an open $W \subset \Omega$ contains, near 0, one side of a smooth hypersurface through 0, and suppose

- $g_1 = g_2$ in $W$,
- $G_1(x, y) = G_2(x, y)$ for $x, y \in W$ $(x \neq y)$.

Suppose further that there are finitely many smooth pairs $u_j^a$ harmonic for $g_j$ on $\Omega$, agreeing on $W$, whose first-side coordinate Hessians at 0 span

$$\{H \in \mathrm{Sym}^2(\mathbb{R}^n) : g_1^{ij}(0) H_{ij} = 0\}.$$

Then the two coordinate metrics agree on a neighborhood of 0.
The proof is in Section 6, but its proof requires results from Section 3, 4 and 5.

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1. Review
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   - Step 1: Maximal match
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   - Step 4: Spanning Hessians
   - Step 5: Metric extension
   - Step 6: Contradiction
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   - Step 1: Extension to the boundary
   - Step 2: Smoothness up to the boundary
   - Step 3: Preservation of the interior and the boundary

An **admissible match** is a smooth local isometry

$$\Phi\colon W \to \mathring{M}_1^e, \quad W \subset \mathring{M}_2^e,$$

where $W$ is connected, open, and contains $E$, such that

$$\Phi|_E = \mathrm{Id}, \quad I_2(p) = I_1(\Phi(p)) \text{ for } p \in W.$$

Thus $\Phi^* g_1 = g_2$ on $W$.
**Remark:** $\mathrm{Id}_E$ is an admissible match by Proposition 2.2.

## Proposition (Proposition 7.1)

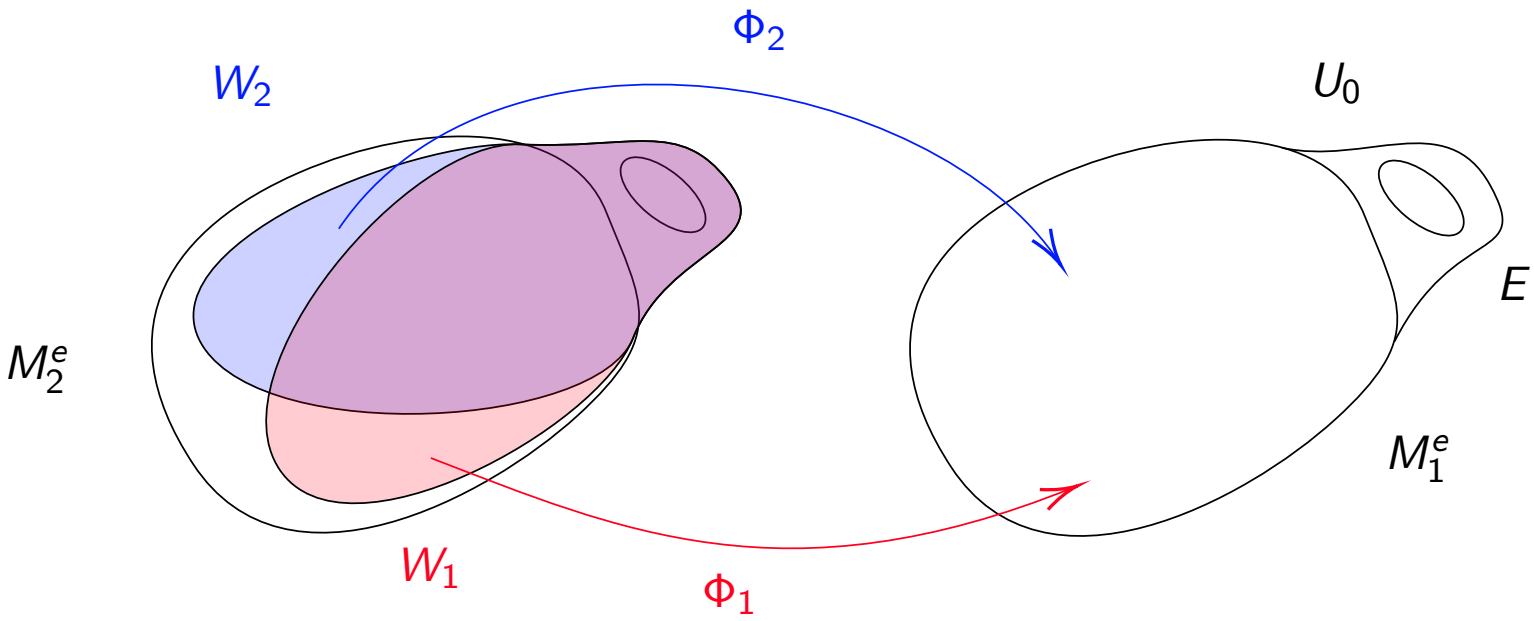
There is an isometry $\Phi\colon \mathring{M}_2^e \to \mathring{M}_1^e$ such that

$$\Phi|_E = \mathrm{Id}, \qquad I_2(p) = I_1(\Phi(p))$$

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1. Review
2. Proposition 7.1
   - Step 1: Maximal match
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   - Step 6: Contradiction
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   - Step 1: Extension to the boundary
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Let us consider first two admissible match $\Phi_j \colon W_j \to \mathring{M}_1^e$.




Define the "union map"

$$\Phi_{12} \colon W_1 \cup W_2 \to \mathring{M}_1^e, \quad \Phi_{12}(p) = \begin{cases} \Phi_1(p), & \text{for } p \in W_1, \\ \Phi_2(p), & \text{for } p \in W_2. \end{cases}$$

Thus, we can apply this construction to obtain a maximal admissible match, which we denote by $\Phi$.

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1. Review
2. Proposition 7.1
   - Step 1: Maximal match
   - Step 2: Green kernels
   - Step 3: Construction of local charts
   - Step 4: Spanning Hessians
   - Step 5: Metric extension
   - Step 6: Contradiction
   - Step 7: Surjectivity
3. Proposition 7.2
   - Step 1: Extension to the boundary
   - Step 2: Smoothness up to the boundary
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**Claim:** for different points $p, q \in W$, we have $G_2(p,q) = G_1(\Phi(p), \Phi(q))$.

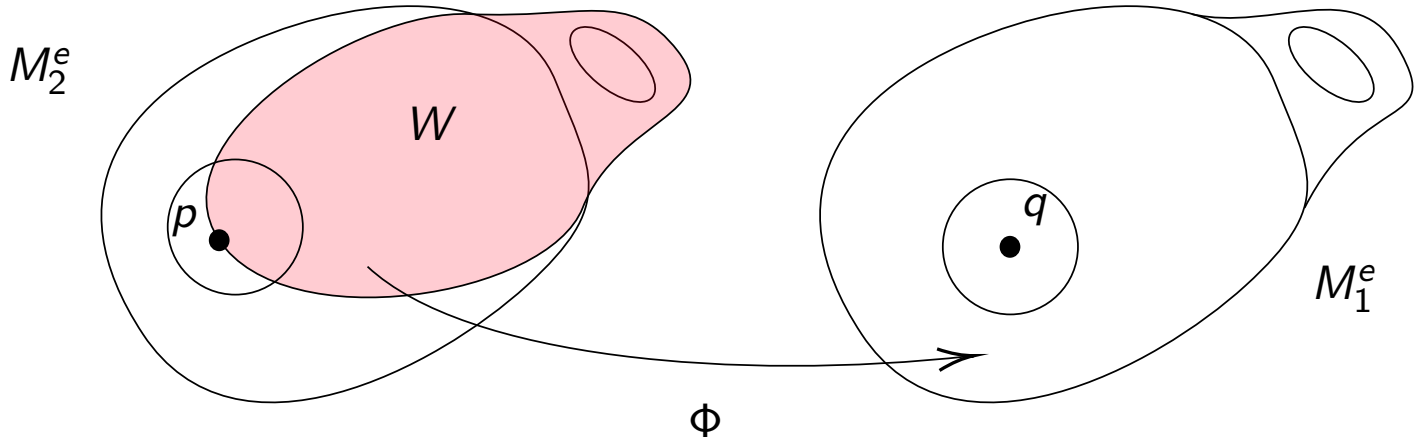
- By admisibility, $I_2(p) = I_1(\Phi(p))$.
- Thus, for $f \in C_c^\infty(U_0)$, we have $u_2^f = u_1^f \circ \Phi$.
- So,
$$\int_{U_0} (G_2(p,q) - G_1(\Phi(p), q)) f(q) dV(q) = 0.$$
- Since $U_0 \subset E$, we have that $\Phi(q) = q$ for $q \in U_0$.
- Then, $G_2(p,q) = G_1(\Phi(p), \Phi(q))$ for $p \in W$, $q \in U_0$, as distributions.
- Since $\Phi$ is a local isometry, $\Phi^* g_1 = g_2$. Hence, for $p$ fixed and $q \in W \setminus \{p\}$ we have
$$\Delta_{g_2}(G_2(p,q) - G_1(\Phi(p), \Phi(q))) = 0.$$
- Now apply unique continuation.

**Claim:** $W = \mathring{M}_2^e$. Assume by contradiction that $W \neq \mathring{M}_2^e$. In the next steps we will try to extend $W$ and $\Phi$.

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2. Proposition 7.1
   - Step 1: Maximal match
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   - Step 4: Spanning Hessians
   - Step 5: Metric extension
   - Step 6: Contradiction
   - Step 7: Surjectivity
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   - Step 2: Smoothness up to the boundary
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- $\partial W \neq \emptyset$, otherwise $\overline{W} = W$ is a nonempty clopen in $M_2^e$ which is connected. Take $p \in \partial W$.
- Let $\{p_k\}_k \subset W$ converging to $p$. Since $M_1^e$ is compact, up to taking a subsequence, we obtain a limit point $q$ of $\{\Phi(p_k)\}$.
- Continuity of the evaluations (Lemma 2.3 item (i)) gives $I_2(p) = I_1(q)$.
- Any other subsequence of $\{\Phi(p_k)\}$ must converge to $q$ too. Indeed, if it converges $\tilde{q} \neq q$, we obtain $I_1(q) = I_1(\tilde{q})$. Observe that since $p$ is in the interior of $M_2^e$, then $I_2(p) \neq 0$ by Lemma 2.3 item (i). Thus, $I_1(q) = I_1(\tilde{q}) \neq 0$ , so $q \in \mathring{M}_1^e$. Since $I_1$ separate points in the interior (Lemma 2.3 item (i)), we conclude that $q = \tilde{q}$.
- This shows that **for every open neighborhood $V_q$ of $q$, there exists a neighborhood $V_p$ of $p$ such that $\Phi(W \cap V_p) \subset V_q$.** Indeed, if this is false for some neighborhood $V_q$, we could chose points $r_k \in W$ with $r_k \to p$ but $\Phi(r_k) \notin V_q$. However, our previous arguments show that $\Phi(r_k) \to q$, a contradiction.



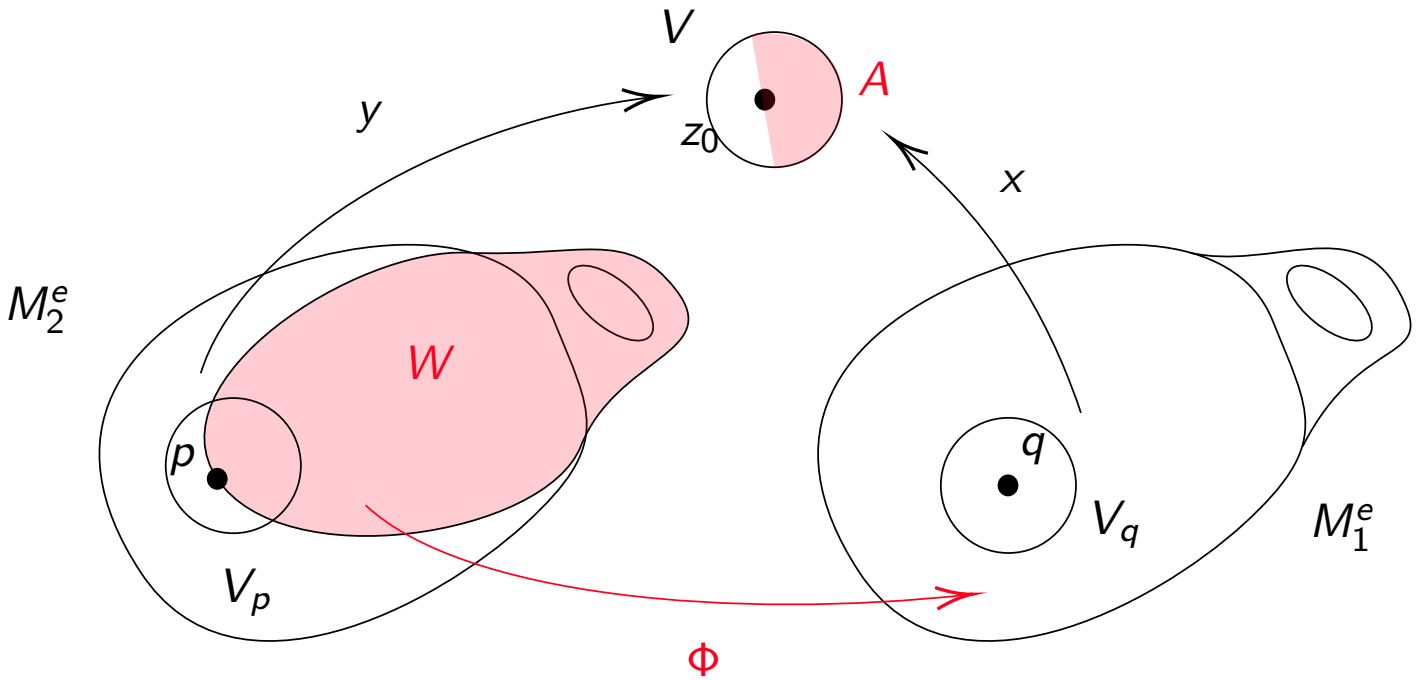

**Claim:** $p, q \notin U_0$.

- For $p$, this follows from the fact that $p \in \partial W$ and $U_0 \Subset E \subset W$.
- If $q \in U_0$, then $q \in E$, and since $\Phi|_E = \mathrm{Id}$, we obtain $I_2(q) = I_1(\Phi(q)) = I_1(q)$.
- We know that $I_2(p) = I_1(q)$.
- Separation (Lemma 2.3 item (i)) then implies $p = q \in E$, a contradiction.

- Since $p \notin \overline{U_0}$, we can choose $f_1, \dots, f_n \in C_c^\infty(U_0)$ so that $y = (u^2_{f_1}, \dots, u^2_{f_n})$ has independent differentials at $p$. (Recall: here $-\Delta_{g_2} u^2_f = f$ in $\mathring{M}^2_2$ with Dirichlet condition).
- The IFT says that $y$ restricts to a diffeomorphism from a neighborhood of $p$ onto an open subset of $\mathbb{R}^n$, and we obtain a coordinate chart around $p$.
- To obtain a chart around $q$, let $x = (u^1_{f_1}, \dots, u^1_{f_n})$.
- Since $\{p_k\} \subset W$, and in $W$ we have $\Phi^* g_1 = g_2$, then

$$(\langle du^2_{f_a}, du^2_{f_b} \rangle_{g_2}(p_k))_{a,b} = (\langle du^1_{f_a}, du^1_{f_b} \rangle_{g_1}(\Phi(p_k)))_{a,b}.$$

- Since the differentials are linearly independent, the left-hand side, and hence the right-hand side is positive definitive.
- Thus, $dx(q)$ is invertible, and then $x$ gives a chart around $q$.
- Observe that since $I_2(p) = I_1(q)$, then $y(p) = x(q)$.




- We can shrink the image of the intersection of the coordinate neighborhoods in $\mathbb{R}^n$, so that the closure of the domains of $x$ and $y$ do not intersect $\overline{U_0}$.
- We obtain that for $r$ in a neighborhood of $p$ intersected with $W$, we have $y(r) = x(\Phi(r))$ (note that we need to intersect with $W$, since there is where $\Phi$ is defined).
- So far we have coordinates charts $y\colon V_p \to V$ and $x\colon V_q \to V$. Let $A = y(W \cap V_p)$, and write $z_0 = x(q) = y(p)$.

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   - Step 2: Smoothness up to the boundary
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- Lemma 2.3 item (ii) allows us to find finitely many sources $h_1, \dots, h_d$ supported in $U_0$ such that the Hessians of the source solutions (with respect to $g_1$) at $z_0$ span the hyperplane annihilated by $g_1^{-1}(z_0)$.
- After shrinking $V$ if necessary, we can assume that the Hessian are independent in the whole $V$.
- For $z \in A$, the coordinate representations of $g_1$ and $g_2$ agree. Indeed, since $\Phi^* g_1 = g_2$,

$$(x^{-1})^* g_1 = (\Phi \circ y^{-1})^* g_1 = (y^{-1})^* (\Phi^* g_1) = (y^{-1})^* g_2.$$

- The coordinate representations of $u_f^1$ and $u_f^2$ agree for any $f \in C_c^\infty(U_0)$. Indeed, by admissibility

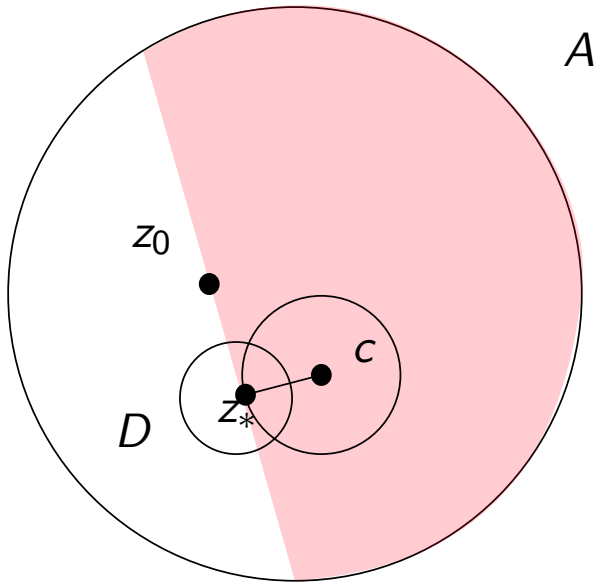
$$u_f^1(x^{-1}(z)) = u_f^1(\Phi(y^{-1}(z))) = u_f^2(y^{-1}(z)).$$

- In particular, the representations of $u_{h_\ell}^2$ and $u_{h_\ell}^1$ coincide in $A$, and the first ones are $g_2$-harmonic and then their Hessian also spawn.

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   - Step 6: Contradiction
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- Let $c \in A$ be such that if $R := \operatorname{dist}(c, V \setminus A) > 0$, then $B(c, R) \Subset V$.
- The distance $R$ is attained at some point $z_* \in \partial B(c, R) \cap (V \setminus A)$.
- By the previous steps, in $B(c, R)$ the metrics, the finite harmonic pairs (they are harmonic since we are away from $U_0$), and the Green kernels in both variables agree. The Hessian spanning condition holds for $z_*$ by our choice of $V$.




- Thus, Theorem 3.1 gives equality of the coordinate metrics in a neighborhood of $z_*$. Inside such neighborhood, take a ball $D$ that intersects $B(c, R)$.

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1. Review
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   - Step 1: Maximal match
   - Step 2: Green kernels
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   - Step 4: Spanning Hessians
   - Step 5: Metric extension
   - Step 6: Contradiction
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- Let $f \in C_c^\infty(U_0)$. Then, $u_f^1(x^{-1}(z)) - u_f^2(y^{-1}(z))$ is harmonic in $D$, and vanishes in $D \cap B(c, R)$.
- Define $\Psi := x^{-1} \circ y \colon y^{-1}(D) \to x^{-1}(D)$. We have $u_f^2 x^{-1} = u_f^1 \circ \Psi$ in $y^{-1}(D)$, for any $f \in C_c^\infty(U_0)$.
- Equality of metrics makes $\Psi$ an isometry.
- Since $I_1 \circ \Psi = I_2 = I_1 \circ \Phi$ in $y^{-1}(D)$ and $I_1$ separate interior points (Lemma 2.3 item (i)), $\Psi \equiv \Phi$ there.
- Thus,

$$\widetilde{\Phi} \colon W \cup y^{-1}(D) \to \mathring{M}_1^e, \quad \widetilde{\Phi} = \begin{cases} \Phi & \text{in } W, \\ \Psi & \text{in } y^{-1}(D), \end{cases}$$

  is an admissible match, with connected domain (since $y^{-1}(D)$ and $W$ are connected and they overlap)
- Since $y^{-1}(z_*) \in y^{-1}(D) \setminus W$, we obtain that $\widetilde{\Phi}$ is an extension of $\Phi$, contradicting its maximality. Therefore, $W = \mathring{M}_2^e$.

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   - Step 1: Maximal match
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   - Step 4: Spanning Hessians
   - Step 5: Metric extension
   - Step 6: Contradiction
   - Step 7: Surjectivity
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   - Step 1: Extension to the boundary
   - Step 2: Smoothness up to the boundary
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So far we have an admissible match $\Phi\colon \mathring{M}_2^e \to \mathring{M}_1^e$. We still need to show that it is a diffeomorphism. To prove this last part, it is enough to show that $\Phi$ is surjective, since an injective surjective local diffeomorphism has a smooth inverse. We will use the conectedness of $\mathring{M}_1^e$.

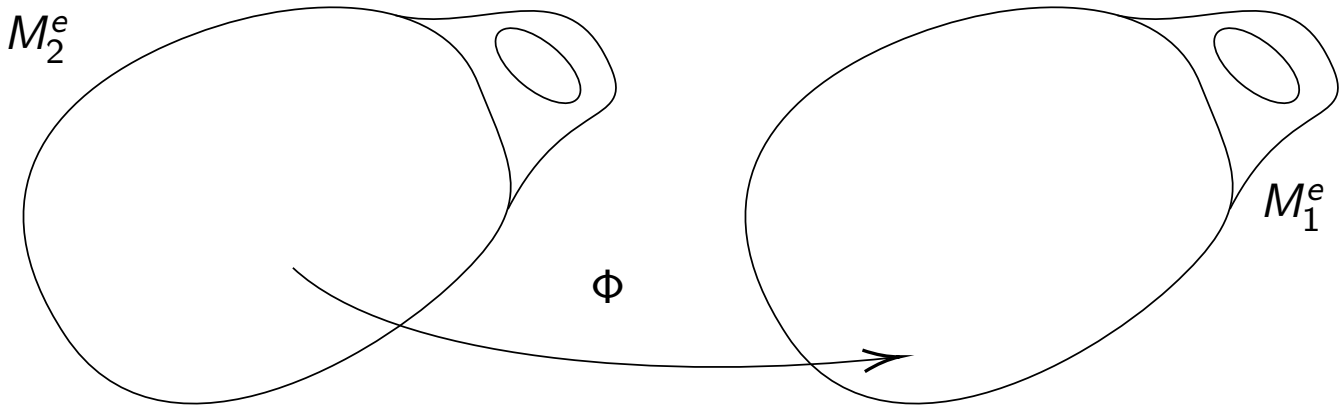
- In first place, $\Phi(\mathring{M}_1^e)$ is open since $\Phi$ is a local diffeomorphism.
- To show that it is closed, suppose that $\Phi(p_k) \to q \in \mathring{M}_1^e$.
- By compactness, a subsequence of $\{p_{k_j}\}_j$ converges in $M_2^e$ to some point $p \in M_2^e$.
- By continuity of evaluations (Lemma 2.3 item (i)), $I_2(p) = I_1(q)$.
- If $p \in \partial M_2^e$, then $0 = I_2(p) = I_1(q)$, but since $q$ is an interior point, Lemma 2.3 item (i) shows that $I_1(q) \neq 0$.
- Thus, $p \in \mathring{M}_2^e$. So, $\Phi(p_{k_j}) \to \Phi(p)$, showing that $\Phi(p) = q$. Therefore, the image of $\Phi$ is closed.
- Since $M_1^e$ is connected and $\Phi(\mathring{M}_2^e) \neq \emptyset$ (since $E \subset \Phi(\mathring{M}_2^e)$), $M_1^e = \Phi(\mathring{M}_2^e)$, showing that $\Phi$ is surjective, hence a diffeomorphism.

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1. Review
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   - Step 1: Maximal match
   - Step 2: Green kernels
   - Step 3: Construction of local charts
   - Step 4: Spanning Hessians
   - Step 5: Metric extension
   - Step 6: Contradiction
   - Step 7: Surjectivity
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   - Step 1: Extension to the boundary
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   - Step 3: Preservation of the interior and the boundary

## Proposition (Proposition 7.2)

The isometry of Proposition 7.1 $\Phi\colon \mathring{M}_2^e \to \mathring{M}_1^e$ extends to a smooth diffeomorphism $M_2^e \to M_1^e$, preserves the original copies of $M$, and its restriction satisfies $\Phi|_\Gamma = \mathrm{Id}$, $g_2 = \Phi^* g_1$.



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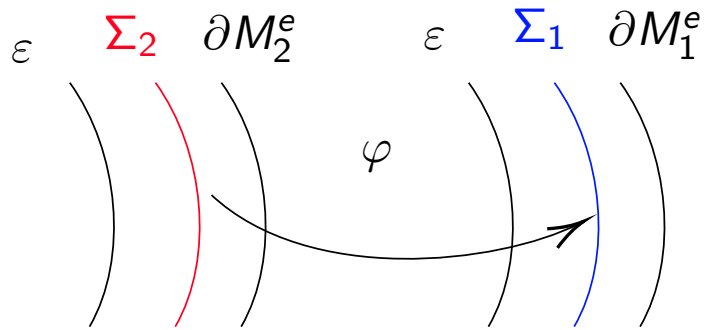

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   - Step 1: Maximal match
   - Step 2: Green kernels
   - Step 3: Construction of local charts
   - Step 4: Spanning Hessians
   - Step 5: Metric extension
   - Step 6: Contradiction
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3. Proposition 7.2
   - Step 1: Extension to the boundary
   - Step 2: Smoothness up to the boundary
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- Since $\Phi\colon \mathring{M}_2^e \to \mathring{M}_1^e$ is an isometry, we have that $d_{g_2}(x,y) = d_{g_1}(\Phi(x),\Phi(y))$ for $x, y \in \mathring{M}_2^e$.
- To extend $\Phi$ to $\partial M_2^e$, given $p \in \partial M_2^e$, let $\{p_k\} \subset \mathring{M}_2^e$ with $p_j \to p$.
- Since $\Phi$ preserves distances, then $\{\Phi(p_k)\}$ is a Cauchy sequence. The completeness of $(M_1^e, d_{g_1})$ shows that $\lim \Phi(p_k)$ exists.
- Define $\Phi(p) := \lim \Phi(p_k)$. Is not difficult to see that the definition does not depend on the choice of the sequence.
- A similar construction applied to $\Phi^{-1}$, allows to obtain an extension $\Phi\colon M_2^e \to M_1^e$ which is still a homeomorphism and preserves distances.
- Since the interiors correspond bijectively, we also have $\Phi(\partial M_2^e) = \partial M_1^e$.

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1. Review
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   - Step 1: Maximal match
   - Step 2: Green kernels
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   - Step 5: Metric extension
   - Step 6: Contradiction
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   - Step 1: Extension to the boundary
   - Step 2: Smoothness up to the boundary
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- Define
$$s_j \colon M_j^e \to \mathbb{R}, \quad s_j(x) = d_{g_j}(x, \partial M_j^e).$$
- Since $\Phi$ preserves distances, we obtain $s_1 \circ \Phi = s_2$.
- Let $U_j := \{x \in M_j^e : s_j(x) < \varepsilon\}$, and consider normal collars:
$$C_j \colon \partial M_j^e \times [0, \varepsilon) \to U_j, \quad C_j(q, t) = \exp_q^{g_j}(t \nu_{g_j}(q)).$$
- Let $0 < s_0 < \varepsilon$, $\Sigma_j := \{x \in M_j^e : s_j(x) = s_0\}$.
- Since $\Phi$ intertwines the $s_j$'s we have that $\varphi := \Phi|_{\Sigma_2} \colon \Sigma_2 \to \Sigma_1$ is a smooth diffeomorphism.



- Define the smooth diffeomorphisms

$$r_j : \partial M_j^e \to \Sigma_j, \quad r_j(q) = C_j(q, s_0).$$

$$F_j : \Sigma_j \times [0, \varepsilon) \to U_j, \quad F_j(z, t) = C_j(r_j^{-1}(z), t).$$

- Observe that $\partial_t F_j(z,t) = \nabla^{g_j} s_j(F_j(z,t))$.
- Since $\Phi$ intertwines the $s_j$'s, we have $D\Phi_x(\nabla^{g_2} s_2(x)) = \nabla^{g_1} s_1(\Phi(x))$.
- A computation and uniqueness for ODEs shows that $\Phi(F_2(z,t)) = F_1(\varphi(z), t)$ for $t \in (0, \varepsilon)$ a priori, but we can extend up to $t = 0$.
- Since $F_j$ and $\varphi$ are smooth, the same holds for $\Phi$. The same formula shows that $\Phi^{-1}$ is smooth up to the boundary.

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- Since $\Phi|_E = \mathrm{Id}_E$, then $\Phi$ is the identity in $\overline{E}$. Thus, it maps $\mathring{M}_2$ to $\mathring{M}_1$. By continuity, it maps the closures to the closures, that is, it is a diffeomorphism of $M$.
- Let $f \in C_c^\infty(\Gamma)$, and denote by $v_f^j$ its harmonic extension to $(M, g_j)$.
- Since $\Phi$ is the identity in $\overline{E}$, it fixes pointwise $P_0 = \{b > 0\}$. Thus, $v_f^2$ and $v_f^1 \circ \Phi$ are $g_2$ harmonic in $M$, and its difference is zero in $P_0$.
- By hypothesis, its energy maps coincide in $\Gamma$, and by (the proof of) Lemma 2.1, the densities $dS_{g_1}$ and $dS_{g_2}$ are equal in $P_0$.
- Then, its difference have zero normal derivative in $P_0$.
- Boundary unique continuation gives vanishing of $v_f^2 - v_f^1 \circ \Phi$ on a nonempty open set in $M$.
- By interior unique continuation, we conclude that the function is zero in $M$.
- Therefore, $f(q) = f(\Phi(q))$ for $q \in \Gamma$. This holds for any $f \in C_c^\infty(\Gamma)$.
- Then, given $q \neq \Phi(q)$, we can take $f$ supported in a small neighborhood of $q$, such that $f(q) = 1$. The previous equality then forces a contradiction.
- Since $q$ is arbitrary, we conclude that $\Phi$ fixes $\Gamma$ pointwise.

# Thank you!