Exercice 1. Let $\xi$ be a Gaussian $\mathcal{N}(0, 1)$ random variable. Let $x > 0$.

(i) Prove that $\frac{1}{(2\pi)^{1/2}} (\frac{1}{x} - \frac{1}{x^3}) e^{-x^2/2} \leq \mathbb{P}(\xi > x) \leq \frac{1}{(2\pi)^{1/2}} \frac{1}{x} e^{-x^2/2}$.

(ii) Prove that $\mathbb{P}(\xi > x) \leq e^{-x^2/2}$.

Exercice 2. Let $\xi$ be a Gaussian $\mathcal{N}(0, 1)$ random variable.

(i) Compute $\mathbb{E}(\xi^4)$ and $\mathbb{E}(|\xi|)$.

(ii) Compute $\mathbb{E}(e^{a\xi})$, $\mathbb{E}(\xi e^{a\xi})$ and $\mathbb{E}(e^{a\xi^2})$, with $a \in \mathbb{R}$.

(iii) Let $b \geq 0$. Let $\eta$ be a Gaussian $\mathcal{N}(0, 1)$ random variable, independent of $\xi$. Prove that $\mathbb{E}(e^{b\xi^2}) = \mathbb{E}(e^{\lambda \xi \eta})$, where $\lambda := (2b)^{1/2}$.

Exercice 3. Let $\xi, \xi_1, \xi_2, \cdots$ be real-valued random variables. Assume that for each $n$, $\xi_n$ is Gaussian $\mathcal{N}(\mu_n, \sigma_n^2)$, with $\mu_n \in \mathbb{R}$ and $\sigma_n \geq 0$.

(i) Show that, if $\mu_n$ and $\sigma_n^2$ both converge to finite limits $\mu$ and $\sigma^2$, then $\xi_n$ converges in law to a Gaussian $\mathcal{N}(\mu, \sigma^2)$.

(ii) Conversely, assume that $\xi_n$ converges in law to a random variable $\xi$. Show that $\mu_n$ and $\sigma_n^2$ both converge to finite limits and that $\xi$ is Gaussian.

Exercice 4. Let $\xi, \xi_1, \xi_2, \cdots$ be random variables. Assume that for any $n$, $\xi_n$ is Gaussian $\mathcal{N}(\mu_n, \sigma_n^2)$, where $\mu_n \in \mathbb{R}$ and $\sigma_n \geq 0$, and that $\xi_n \rightarrow \xi$ in probability. Prove that $\xi_n$ converges in $L^p$, for all $p \in [1, \infty)$.

Exercice 5. Let $(\xi^n)_{n \in \mathbb{N}}$ be a sequence of Gaussian vectors in $\mathbb{R}^N$. Show that $\xi^n$ converge in distribution if and only if the sequences of their mean vectors and covariance matrices both converge and that, in this case, the limit is a Gaussian vector.

Exercice 6.

(i) Let $\Sigma$ denote an $n \times n$ positive semi-definite symmetric matrix. Show that there exists a centered (i.e. with 0 mean) Gaussian vector with $\Sigma$ as its covariance matrix.

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1 We will see that $\mathbb{P}(\xi > x) \leq \frac{1}{2} e^{-x^2/2}$. 
(ii) Let $\xi$ be a centered Gaussian vector with covariance matrix $\Sigma$. Let $e_1, \ldots, e_n$ be a basis of $\mathbb{R}^n$ in which $\Sigma$ is diagonal: $\Sigma e_i = \lambda_i e_i$ with

$$\lambda_1 > \lambda_2 > \ldots > \lambda_r > 0 = \lambda_{r+1} = \ldots = \lambda_n.$$ 

Show that

$$X = \sum_{i=1}^r Y_i e_i$$

where $(Y_1, \ldots, Y_r)$ are independent Gaussian random variables with $\text{Var}(Y_i) = \lambda_i$.

(iii) Let $\xi$ be a centered Gaussian vector with covariance matrix $\Sigma$. Assume that $\Sigma$ is definite positive. Show that the density $p_\xi$ of $\xi$ is given by the formula

$$p_\xi(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2} \langle x, \Sigma^{-1} x \rangle\right) \text{ for all } x \in \mathbb{R}^n.$$ 

Exercice 7. Let $(\xi, \eta, \theta)$ be an $\mathbb{R}^3$-valued Gaussian random vector. Assume $\mathbb{E}(\xi) = \mathbb{E}(\eta) = \mathbb{E}(\xi \eta) = 0$, $\sigma^2_\xi := \mathbb{E}(\xi^2) > 0$ and $\sigma^2_\eta := \mathbb{E}(\eta^2) > 0$.

(i) Prove that $\mathbb{E}(\theta | \xi, \eta) = \mathbb{E}(\theta | \xi) + \mathbb{E}(\theta | \eta) - \mathbb{E}(\theta)$.

(ii) Prove that $\mathbb{E}(\xi | \xi \eta) = 0$.

(iii) Prove that $\mathbb{E}(\theta | \xi \eta) = \mathbb{E}(\theta)$.

Exercice 8. Let $(\xi_{k,n}, k \geq 0, n \geq 0)$ be a collection of i.i.d. Gaussian $\mathcal{N}(0, 1)$ random variables. For all $n \geq 0$, we define the process $(X_n(t), t \in [0, 1])$ with $t \mapsto X_n(t)$ being affine on each of the intervals $[\frac{i}{2^n}, \frac{i+1}{2^n}], 0 \leq i \leq 2^n - 1$, in the following way $X_0(0) := 0, X_0(1) := \xi_{0,0}$, and by induction, for $n \geq 1$,

$$X_n(\frac{2i}{2^n}) := X_{n-1}(\frac{2i}{2^n}), \quad 0 \leq i \leq 2^{n-1},$$

$$X_n(\frac{2j + 1}{2^n}) := X_{n-1}(\frac{2j + 1}{2^n}) + \xi_{2j+1,n}(\frac{\ell}{2^{(n+1)/2}}), \quad 0 \leq j \leq 2^{n-1} - 1.$$ 

Prove that for all $n \geq 0$, $(X_n(\frac{k}{2^n}), 0 \leq k \leq 2^n)$ is a centered Gaussian vector such that $\mathbb{E}[X_n(\frac{k}{2^n})X_n(\frac{\ell}{2^n})] = \frac{k}{2^n} \wedge \frac{\ell}{2^n}$, for $0 \leq k, \ell \leq 2^n$.

Exercice 9. Let $(B^m_t, t \in [0, 1])$, for $m \geq 0$, be a sequence of independent Brownian motions defined on $[0, 1]$. Let

$$B_t := B^1_{t-[t]} + \sum_{0 \leq m < [t]} B^m_1, \quad t \geq 0.$$ 

Prove that $(B_t, t \geq 0)$ is Brownian motion.

Exercice 10. Let $I = [a, b]$ with $a < b$. 

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1. Show that any monotone function $f : I \to \mathbb{R}$ has bounded variations.

2. Show that any Lipschitz function $f : I \to \mathbb{R}$ has bounded variations.

3. Show that the set of bounded variation functions on $I$ is a vector space.

4. Show that $f : I \to \mathbb{R}$ has bounded variations if and only if it can be written $f = g - h$ where $g$ and $h$ are two non-decreasing functions.

Exercice 11. Let $T := \inf\{t \geq 0 : B_t = 1\}$ (with $\inf \emptyset := \infty$). Prove that $^2 \mathbb{P}(T < \infty) \geq \frac{1}{2}$.

Exercice 12. (i) Prove that $(-B_t, t \geq 0)$ is Brownian motion.

(ii) (Scaling) Prove that for any $a > 0$, $(\frac{1}{a^{1/2}} B_{at}, t \geq 0)$ is Brownian motion.

Exercice 13. (i) Let $\xi := \int_0^1 B_t \, dt$. Determine the law of $\xi$.

(ii) Let $\eta := \int_0^2 B_t \, dt$. Determine $\mathbb{E}(B_1 | \eta)$.

(iii) Prove that $B_7 - B_2$ is independent of $\sigma(B_s, s \in [0, 1])$.

(iv) Let $\mathcal{F}_1 := \sigma(B_s, s \in [0, 1])$. Determine $\mathbb{E}(B_5 | \mathcal{F}_1)$ and $\mathbb{E}(B_5^2 | \mathcal{F}_1)$.

Exercice 14. (i) Prove or disprove: for all $t > 0$, $\int_0^t B_s^2 \, ds$ has the same distribution as $t^2 \int_0^1 B_s^2 \, ds$.

(ii) Prove or disprove: the processes $(\int_0^t B_s^2 \, ds, t \geq 0)$ and $(t^2 \int_0^1 B_s^2 \, ds, t \geq 0)$ have the same distribution.

Exercice 15. Let $T$ be a random variable having the exponential law of parameter 1, independent of $B$. Determine the law of $B_T$.

Exercice 16. (i) Prove that $\int_0^1 \frac{B_s}{s} \, ds$ is a.s. well defined.

(ii) Let $\beta_t := B_t - \int_0^t \frac{B_s}{s} \, ds$. Prove that $(\beta_t, t \geq 0)$ is Brownian motion.

Exercice 17. Prove that $\int_0^\infty |B_s| \, ds = \infty$ a.s.

Exercice 18. Let $T := \inf\{t \geq 0 : |B_t| = 1\}$ (with $\inf \emptyset := \infty$).

(i) Prove that $T < \infty$ a.s.

(ii) Prove that $T$ and $1_{\{B_T=1\}}$ are independent.

Exercice 19. Let $B := (B_t, t \in [0, 1])$ be Brownian motion defined on $[0, 1]$. For all $t \in [0, 1]$, let

\[
\mathcal{F}_t := \sigma(B_s, s \in [0, t]), \quad \mathcal{G}_t := \mathcal{F}_t \vee \sigma(B_1) = \sigma(\{C; C \in \mathcal{F}_t \text{ or } C \in \sigma(B_1)\}).
\]

\(^2\)Later on, we will see that $T < \infty$ a.s.
(i) Let $0 \leq s < t \leq 1$. Prove that

$$
\mathbb{E}[(B_t - B_s) \mid \mathcal{G}_s] = \frac{t-s}{1-s} (B_1 - B_s).
$$

(ii) Consider the process $\beta := (\beta_t, t \in [0, 1])$ defined by

$$
\beta_t := B_t - \int_0^t \frac{B_1 - B_s}{1-s} \, ds, \quad t \in [0, 1].
$$

Prove that for $0 \leq s < t \leq 1$, $\mathbb{E}(\beta_t \mid \mathcal{G}_s) = \beta_s$ a.s.

**Exercice 20.** Let $\mathcal{F}_1 := \sigma(B_s, s \in [0, 1])$, and let $a \in \mathbb{R}$. Let $\mathbb{Q}$ be the probability measure on $\mathcal{F}_1$ defined by $\mathbb{Q}(A) := \mathbb{E}(e^{aB_1 - \frac{a^2}{2}} 1_A), A \in \mathcal{F}_1$. Define $\gamma_t := B_t - at, t \in [0, 1]$. Prove that $(\gamma_t, t \in [0, 1])$ is Brownian motion under $\mathbb{Q}$. 


Exercice 1. Let $\mathcal{A}_1 \subset \mathcal{F}, \ldots, \mathcal{A}_n \subset \mathcal{F}$ be $\pi$-systems, satisfying $\Omega \in \mathcal{A}_i, \forall i$. Assume
\[\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdots \mathbb{P}(A_n), \quad \forall A_i \in \mathcal{A}_i.\]
Then $\sigma(\mathcal{A}_1), \ldots, \sigma(\mathcal{A}_n)$ are independent.

Exercice 2. (i) (Time reversal) Fix $a > 0$. Prove that $(B_{a-t} - B_{a-t}, t \in [0, a])$ is Brownian motion on $[0, a]$.

(ii) (Time inversion) Prove that $X := (X_t, t \geq 0)$ defined by $X_t := tB_1$ (for $t > 0$) and $X_0 := 0$ is Brownian motion.

Exercice 3. Prove that there exists a constant $a > 0$ (that does not depend on $\omega$) such that $\inf_{t \in [0, 2]} B_t$ has the same distribution as $a \inf_{t \in [0, 1]} B_t$.

Exercice 4. (Brownian bridge) Let $b_t = B_t - tB_1$, $t \in [0, 1]$. It is a centered Gaussian process with a.s. continuous trajectories and with covariance $(s \wedge t) - st$. We call $b$ a Brownian bridge.

(i) The process $(b_t, t \in [0, 1])$ is independent of the random variable $B_1$.

(ii) If $b$ is a Brownian bridge, so is $(b_{1-t}, t \in [0, 1])$.

(iii) If $b$ is a Brownian bridge, then $B_t = (1 + t)b_{t/(1+t)}$, $t \geq 0$, is Brownian motion. Note that $b_t = (1 - t)B_t/(1-t)$.

Exercice 5. Prove that when $t \to \infty$, $(\int_0^t e^{B_s} \, ds)^{1/2} \to e^{\frac{1}{2}}$ in law, where $N$ is a Gaussian $\mathcal{N}(0, 1)$ random variable.

Exercice 6. (i) Prove that $0 < \sup_{t \geq 0} (|B_t| - t) < \infty$ a.s. and that $0 < \sup_{t \geq 0} \frac{|B_t|}{1+t} < \infty$ a.s.

(ii) Prove that $\sup_{t \geq 0} (|B_t| - t)$ and $(\sup_{t \geq 0} |B_t|)^2$ have the same distribution.

Hint: Use the scaling property.

(iii) Prove that for any $p > 0$, $\mathbb{E}\{[\sup_{t \geq 0} (|B_t| - t)]^p\} < \infty$.

(iv) Prove that there exists a constant $C < \infty$ such that for any non-negative random variable $T$ (not necessarily a stopping time!), $\mathbb{E}(|B_T|) \leq C [\mathbb{E}(T)]^{1/2}$. 
Hint: Write, for any \( a > 0 \), \( |B_T| = (|B_T| - aT) + aT \), and prove that \( \mathbb{E}(|B_T| - aT) \leq \frac{1}{a} \mathbb{E}[\sup_{t \geq 0}(|B_t| - t)] \).

**Exercice 7.** Let \( S_t := \sup_{s \in [0, t]} B_s \), \( t \geq 0 \). Prove that \( S_2 - S_1 \) is distributed as \( \max\{\|N\| - \|\bar{N}\|, 0\} \), where \( N \) and \( \bar{N} \) are independant Gaussian \( \mathcal{N}(0, 1) \) random variables.

**Exercice 8.** Let \( d_1 := \inf \{ t \geq 1 : B_t = 0 \} \).

(i) Is \( d_1 \) a stopping time?

(ii) Determine the law of \( d_1 \).

(iii) Recover from (ii) that \( L := \sup \{ t \leq 1 : B_t = 0 \} \) follows the arcsin law.

**Exercice 9.** Define \( T_1 := \inf \{ t > 0 : B_t = 1 \} \) and \( \tau := \inf \{ t \geq T_1 : B_t = 0 \} \).

(i) Is \( \tau \) a stopping time?

(ii) Determine the law of \( \tau \).

**Exercice 10.** (i) Study convergence in probability of \( \frac{\log(1 + B_t^2)}{\log t} \) (quand \( t \to \infty \)).

(ii) Study a.s. convergence of \( \frac{\log(1 + B_t^2)}{\log t} \).

**Exercice 11.** (i) Prove that \( \int_0^\infty \sin^2(B_t) \, dt = \infty \) a.s.

(ii) More generally, prove that if \( f : \mathbb{R} \to \mathbb{R} \) is continuous which is not identically 0, then \( \int_0^\infty f^2(B_t) \, dt = \infty \) a.s.

**Exercice 12.** (i) Let \([a, b]\) and \([c, d]\) be disjoint intervals of \( \mathbb{R}_+ \). Prove that \( \sup_{t \in [a, b]} B_s \neq \sup_{t \in [c, d]} B_s \) a.s.

(ii) Prove that a.s., each local maximum of \( B \) is a strict local maximum.

(iii) Prove that a.s., the set of times at which \( B \) realises local maxima is countable and dense in \( \mathbb{R}_+ \).

**Exercice 13.** (i) Let \( a > 0 \) and let \( T_a := \inf \{ t \geq 0 : B_t = a \} \). Recall that \( \mathbb{E}[e^{-\lambda T_a}] = e^{-a(2\lambda)^{1/2}}, \forall \lambda \geq 0 \). Prove that \( \mathbb{P}(T_a \leq t) \leq \exp(-\frac{a^2}{2t}) \), for all \( t > 0 \).

(ii) Prove that if \( \xi \) is a Gaussian \( \mathcal{N}(0, 1) \) random variable, then \( \mathbb{P}(\xi \geq x) \leq \frac{1}{2}e^{-x^2/2}, \forall x > 0 \).

**Exercice 14.** (i) Prove that for all \( t > 0 \) and all \( \varepsilon > 0 \), \( \mathbb{P}\{\sup_{s \in [0, t]} |B_s| \leq \varepsilon\} > 0 \).

(ii) Prove that there exists \( c \in (0, \infty) \) such that \( \mathbb{P}\{\sup_{s \in [0, 1]} |B_s| \leq \varepsilon\} \geq e^{-c/\varepsilon^2}, \forall \varepsilon \in (0, 1] \).

(iii) Prove that for all \( t > 0 \) and all \( x > 0 \), \( \mathbb{P}\{\sup_{s \in [0, t]} |B_s| \geq x\} > 0 \).