# LIMITING BEHAVIOR OF A DIFFUSION IN AN ASYMPTOTICALLY STABLE ENVIRONMENT

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Abstract. Let $\mathbb{V}$ be a two sided random walk and let $X$ denote a real valued diffusion process with generator $\frac{1}{2}e^{\mathbb{V}[x]}\frac{d}{dx}\left(e^{-\mathbb{V}[x]}\frac{d}{dx}\right)$. This process is the continuous equivalent of the one dimensional random walk in random environment with potential $\mathbb{V}$. Hu and Shi (1997) described the Lévy classes of $X$ in the case where $\mathbb{V}$ behaves approximately like a Brownian motion. In this paper, based on some fine results on the fluctuations of random walks and stable processes, we obtain an accurate image of the almost sure limiting behavior of $X$ when $\mathbb{V}$ behaves asymptotically like a stable process. These results also apply for the corresponding random walk in random environment.

Résumé: Étant donnée une marche aléatoire $\mathbb{V}$, on considère une diffusion aléatoire réelle $X$ de générateur $\frac{1}{2}e^{\mathbb{V}[x]}\frac{d}{dx}\left(e^{-\mathbb{V}[x]}\frac{d}{dx}\right)$. Ce processus est l'équivalent continu de la marche aléatoire en milieu aléatoire au plus proche voisin en dimension 1. Hu et Shi (1997) ont déterminé les classes de Lévy de $X$ lorsque $\mathbb{V}$ se comporte approximativement comme un mouvement Brownien. Dans cet article, une étude fine des fluctuations du potentiel $\mathbb{V}$ nous permet d'obtenir des résultats précis sur le comportement limite presque-sûre de la diffusion lorsque $\mathbb{V}$ est dans le domaine d'attraction d'une loi stable. Ces résultats se transposent également au cas discret de la marche aléatoire en milieu aléatoire.

**Key words.** Random environment, stable process,iterated logarithm law.
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## 1. Introduction

Let $(\mathbb{V}_x, x \in \mathbb{R})$ be a càdlàg, real-valued locally bounded stochastic process on some probability space $(\Omega, \mathbb{P})$ with $\mathbb{V}_0 = 0$ almost surely. Let also $(X_t)_{t \geqslant 0}$ be the coordinate process on the space of continuous functions $C([0,\infty))$ equipped with the topology of uniform convergence on compact set and the associated $\sigma$-field. For each realization of $\mathbb{V}$, let $P_{\mathbb{V}}$ be a probability on

$C([0,\infty))$ such that $X$ is a diffusion process with $X_0=0$ and generator

$$\frac{1}{2}e^{\mathbb{V}_x}\frac{d}{dx}\left(e^{-\mathbb{V}_x}\frac{d}{dx}\right).$$

It is well known in [13] that $X$ may be constructed from a standard Brownian motion through a change of scale and a change of time. We consider the annealed probability $\mathbf{P}$ on $\mathbf{\Omega}=\Omega\times C([0,\infty))$ defined as the semi-direct product $\mathbf{P}=\mathbb{P}\times P_{\mathbb{V}}$. $X$ under $\mathbf{P}$ is called a diffusion in the random potential $\mathbb{V}$. This process was first studied by Schumacher [19] and Brox [6] who proved that, when $\mathbb{V}$ is a Brownian motion, $X_t/\log^2 t$ converges in law as $t$ goes to infinity to some non-degenerate distribution on $\mathbb{R}$. Extension of this result when $\mathbb{V}$ is a stable process may be found in [7, 15, 19]. In this paper, we are concerned with the case where $\mathbb{V}$ is a two sided random walk. More precisely, $(\mathbb{V}_x\ ,\ x\in\mathbb{R})$ satisfies:

$$\begin{cases}\mathbb{V}\text{ is identically } 0 \text{ on } (-1,1),\\ \mathbb{V}\text{ is flat on } (n,n+1)\text{ for all } n\in\mathbb{Z},\\ \mathbb{V}\text{ is right continuous on } [0,\infty)\text{ and left continuous on } (-\infty,0],\\ (\mathbb{V}_{n+1}-\mathbb{V}_n)_{n\in\mathbb{Z}}\text{ is a sequence of i.i.d. variables under } \mathbf{P}.\end{cases}$$

Our goal is to describe the almost sure asymptotics of $X_t$, $\sup_{s\leqslant t}X_s$ and $\sup_{s\leqslant t}|X_s|$. This has been done by Hu and Shi [12] in the case where $\mathbb{V}$ behaves roughly like a Brownian motion. We will instead consider the more general setting where a typical step of the random walk is in the domain of attraction of a stable law. Precisely, we make the following assumption which is similar to that of Kawazu, Tamura and Tanaka [15].

**Assumption 1.** *There exists a positive sequence* $(a_n)_{n\geqslant 0}$ *such that*

$$\frac{\mathbb{V}_n}{a_n}\underset{n\to\infty}{\overset{law}{\longrightarrow}}\mathbb{S}$$

*where* $\mathbb{S}$ *is a random variable whose law is strictly stable with index* $\alpha\in(0,2]$ *and whose density is everywhere positive on* $\mathbb{R}$.

This implies of course that $\mathbb{V}_{-n}/a_n$ converges in law toward $-\mathbb{S}$. It is known that the norming sequence $(a_n)$ is regularly varying with index $1/\alpha$ and we can without loss of generality assume that $(a_n)$ is strictly increasing with $a_1=1$. We will denote by $a(\cdot)$ a continuous, strictly increasing interpolation of $(a_n)$ and $a^{-1}(\cdot)$ will stand for its inverse. It is to be noted that $a(\cdot)$ and $a^{-1}(\cdot)$ are respectively regularly varying with index $1/\alpha$ and $\alpha$. Let $p$ denote the positivity parameter of $\mathbb{S}$ and $q$ its negativity parameter, namely:

$$p=\mathbf{P}(\mathbb{S}>0)=1-\mathbf{P}(\mathbb{S}<0)=1-q.$$

The assumption that $\mathbb{S}$ has a positive density in the whole of $\mathbb{R}$ implies that $p,q\in(0,1)$. More precisely, for $\alpha>1$ it is known in [22] that $1-1/\alpha\leqslant p,q\leqslant 1/\alpha$. In any case, we have

$$0<\alpha p,\alpha q\leqslant 1.$$

Note also that the Fourier transform of $\mathbb{S}$ is well known to be

$$\mathbf{E}\left(e^{i\lambda\mathbb{S}}\right) = e^{-\gamma|\lambda|^\alpha\left(1-i\frac{\lambda}{|\lambda|}\tan\left(\pi\alpha\left(p-\frac{1}{2}\right)\right)\right)} \tag{1.1}$$

where $\gamma$ is some strictly positive constant. Let us now extend $\mathbb{S}$ into a two sided strictly stable process $(\mathbb{S}_x\ ,\ x \in \mathbb{R})$ such that $\mathbb{S}_1$ has same law as $\mathbb{S}$. By two sided, we mean that the processes $(\mathbb{S}_t\ ,\ t \geqslant 0)$ and $(-\mathbb{S}_{-t}\ ,\ t \geqslant 0)$ are independent, both càdlàg, and have the same law. Notice in particular that, when $\alpha = 1$, $\mathbb{S}$ is a symmetric Cauchy process with drift, whereas for $\alpha = 2$ we have $p = 1/2$ and $\mathbb{S}$ is a Brownian motion. Furthermore, the extremal cases $\alpha p = 1$ (resp. $\alpha q = 1$) can only happen when $\alpha > 1$ and are equivalent to the assumption that $\mathbb{S}$ has no positive jumps (resp. no negative jumps). When $\mathbb{S}$ has no positive jumps, it is known that the Fourier transform can be extended such that

$$\mathbf{E}\left(e^{\lambda\mathbb{S}_1}\right) = e^{\gamma'\lambda^\alpha} \text{for all } \lambda \geqslant 0 \tag{1.2}$$

where $\gamma'$ is a positive constant that we will assume to be 1 (we can reduce to this case by changing the norming sequence $(a_n)$). Similarly, when $\mathbb{S}$ has no negative jumps, we will assume $\mathbf{E}\left(\exp(-\lambda\mathbb{S}_1)\right) = \exp(\lambda^\alpha)$ for all $\lambda \geqslant 0$. Let $E_\alpha$ denote the Mittag-Leffler function with parameter $\alpha$:

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)} \text{ for } x \in \mathbb{R}.$$

Define $-\rho_1(\alpha)$ to be the first negative root of $E_\alpha$ and $-\rho_2(\alpha)$ to be the first negative root of $\alpha x E''_\alpha(x) + (\alpha - 1)E'_\alpha(x)$. The first result of this paper is a law of the iterated logarithm for the limsup of the diffusion $X$ in the random environment $\mathbb{V}$.

**Theorem 1.** *We have, almost surely,*

$$\limsup_{t\to\infty} \frac{X_t}{a^{-1}(\log t)\log\log\log t} = \frac{1}{K^\#}$$

*where* $K^\# \in (0,\infty)$ *is a constant that only depends on the limit law* $\mathbb{S}$ *and is given by*

$$K^\# = -\lim_{t\to\infty} \frac{1}{t}\log \mathbf{P}\Big(\sup_{0\,\leqslant\, u\,\leqslant\, v\,\leqslant\, t} (\mathbb{S}_v - \mathbb{S}_u) \leqslant 1\Big).$$

*Furthermore, when* $\mathbb{S}$ *is completely asymmetric, the value of* $K^\#$ *is given by*

$$K^\# = \begin{cases} \rho_1(\alpha) & \text{when } \mathbb{S} \text{ has no positive jumps,} \\ \rho_2(\alpha) & \text{when } \mathbb{S} \text{ has no negative jumps.} \end{cases}$$

Note that $X_t$ and $\sup_{s\,\leqslant\, t} X_s$ have the same running maximum, hence Theorem 1 also holds with $\sup_{s\,\leqslant\, t} X_s$ in place of $X_t$. A symmetry argument yields

$$\limsup_{t\to\infty} \frac{-\inf_{s\,\leqslant\, t} X_t}{a^{-1}(\log t)\log\log\log t} = \frac{1}{\widetilde{K}^\#} \text{ a.s.}$$

where $\widetilde{K}^{\#} = -\lim_{t\to\infty} \log \mathbf{P}\left(\sup_{0\,\leqslant\, u\,\leqslant\, v\,\leqslant\, t} (\mathbb{S}_{-v} - \mathbb{S}_{-u}) \leqslant 1\right)/t$. Hence,

$$\limsup_{t\to\infty} \frac{\sup_{s\,\leqslant\, t} |X_t|}{a^{-1}(\log t) \log\log\log t} = \frac{1}{\widetilde{K}^{\#} \wedge K^{\#}} \text{ a.s.}$$

In the case where $\alpha = 2$, we have $E_\alpha(-x) = \cos(\sqrt{x})$ for all $x \geqslant 0$, therefore $\widetilde{K}^{\#} = K^{\#} = \pi^2/4$, and we recover the law of the iterated logarithm of Theorem 1.6 of [12].

Let $\mathbf{T}_n$ denote the $n^{\text{th}}$ strictly descending ladder time of the random walk $\mathbb{V}$, formally,

$$\begin{cases} \mathbf{T}_0 = 0, \\ \mathbf{T}_{n+1} = \min\left(k > \mathbf{T}_n\ ,\ \mathbb{V}_k < \mathbb{V}_{\mathbf{T}_n}\right). \end{cases}$$

Since $\mathbb{V}$ is oscillatory, $\mathbf{T}_n$ is finite for all $n$. Theorem 4 of [18] states that $\mathbf{T}_1$ is in the domain of attraction of a positive stable law with index $q$. Moreover, $\mathbf{T}_1$ is in the domain of normal attraction of this distribution if and only if

$$\sum_{n=1}^{\infty} \frac{\mathbf{P}(\mathbb{V}_n < 0) - q}{n} < \infty. \tag{1.3}$$

Let $(b_n)$ denote a (strictly increasing) sequence of norming constants for $\mathbf{T}_1$ and $b(\cdot)$ will stand for a continuous, strictly increasing interpolation of this sequence. The function $b^{-1}(\cdot)$ is therefore regularly varying with index $q$. The next theorem characterizes the liminf behavior of $\sup_{s\,\leqslant\, t} X_s$.

**Theorem 2.** *For any positive, non-decreasing function $f$ define*

$$J(f) = \int^{\infty} \frac{b^{-1}\left(a^{-1}(\log t)/f(t)\right) dt}{b^{-1}\left(a^{-1}(\log t)\right) t \log t}.$$

*We have, almost surely,*

$$\liminf_{t\to\infty} \frac{f(t)}{a^{-1}(\log t)} \sup_{s\,\leqslant\, t} X_s = \begin{cases} 0 \\ \infty \end{cases} \iff J(f) \begin{cases} = \infty \\ < \infty. \end{cases}$$

*In particular, with probability* 1,

$$\liminf_{t\to\infty} \frac{(\log\log t)^{\beta}}{a^{-1}(\log t)} \sup_{s\,\leqslant\, t} X_s = \begin{cases} 0, & \text{if } \beta < 1/q, \\ \infty, & \text{if } \beta > 1/q. \end{cases} \tag{1.4}$$

Notice that (1.3) holds whenever $\mathbb{V}_1$ is strictly stable or when $\mathbf{E}\left(\mathbb{V}_1^2\right) < \infty$ (according to Theorem 1 of [10], p 575). In those cases, $\mathbb{V}_1$ is also in the domain of normal attraction of $\mathbb{S}$ so that we can both choose $a(x) = x^{1/\alpha}$ and $b(x) = x^{1/q}$ and the last theorem is simplified:

$$\liminf_{t\to\infty} \frac{f(t)}{(\log t)^{\alpha}} \sup_{s\,\leqslant\, t} X_s = \begin{cases} 0 \\ \infty \end{cases} \iff \int^{\infty} \frac{dt}{f^q(t) t \log t} \begin{cases} = \infty \\ < \infty. \end{cases}$$

In particular, the liminf for the critical case $\beta = 1/q$ in (1.4) is infinite.

We are also interested in the asymptotic behavior of the bilateral supremum $\sup_{s\,\leqslant\, t} |X_s|$. We already mentioned that the limsup behavior of this process may be deduced from Theorem 1. Although we were not able to deal with the general case (as it seems that many different behaviors may occur in the completely asymmetric case, depending on the distribution tail of $\mathbb{V}_1$), we can still obtain, when the limiting process has jumps of both signs, the following integral test:

**Theorem 3.** *When the limiting stable process $\mathbb{S}$ has jumps of both signs, we have, for any non-decreasing positive function $f$, almost surely,*

$$\liminf_{t\to\infty} \frac{f(t)}{a^{-1}(\log t)} \sup_{s\leqslant t} |X_s| = \begin{cases} 0 \\ \infty \end{cases} \iff \int^{\infty} \frac{dt}{tf(t)^2 \log t} \begin{cases} = \infty \\ < \infty. \end{cases}$$

*In particular, with probability* 1,

$$\liminf_{t\to\infty} \frac{(\log\log t)^{\beta}}{a^{-1}(\log t)} \sup_{s\leqslant t} |X_s| = \begin{cases} 0, & \text{if } \beta \leqslant 1/2, \\ \infty, & \text{if } \beta > 1/2. \end{cases}$$

Notice that in this case, the limiting behavior does not depend on the symmetry parameter and notice also that this behavior is quite different from the Brownian case (Theorem 1.7 of [12]). This may be informally explained from the facts that when the limiting process has jumps of both signs, typical valleys for the diffusion are much deeper than in the Brownian case.

Although we are mainly concerned with the almost-sure behavior of $X$, our approach also allows us to prove a convergence in law for the supremum process.

**Theorem 4.** *There exists a non-degenerate random variable $\Xi$ depending only on the limiting process $\mathbb{S}$ such that under the annealed probability $\mathbf{P}$,*

$$\frac{\sup_{s\leqslant t} X_s}{a^{-1}(\log t)} \xrightarrow[t\to\infty]{law} \Xi.$$

*Moreover, when $\mathbb{S}$ has no positive jumps the law of $\Xi$ is characterized by its Laplace transform,*

$$\mathbf{E}\left(e^{-q\Xi}\right) = \Gamma(\alpha+1)\frac{E'_{\alpha}(q)}{E_{\alpha}(q)} \quad \text{for } q \geqslant 0,$$

*and in the case where $\mathbb{S}$ has no negative jumps, we have*

$$\mathbf{E}\left(e^{-q\Xi}\right) = (\alpha-1)\frac{E'_{\alpha}(q)}{\alpha q E''_{\alpha}(q) + (\alpha-1)E'_{\alpha}(q)} \quad \text{for } q \geqslant 0.$$

This paper is organized as follows: in Section 2, we prove sharp results on the fluctuations of the potential $\mathbb{V}$ as well as on the limiting stable process $\mathbb{S}$. These estimates, which may be of independent interest, ultimately play an important role in the proof of the main theorems. In Section 3, we reduce the study of the hitting times of $X$ to the study of some functionals of the potential process $\mathbb{V}$. This step is similar to [12], namely, we make use of Laplace's method and the reader may refer to [20] for an overview of the key ideas. The proofs of the main theorems are given in Section 4. We shall eventually discuss these results in the last section, in particular, we show that Theorems $1-4$ still hold when $\mathbb{V}$ is a strictly stable process. We also explain how similar results can be obtained for a random walk in a random environment with an asymptotically stable potential.

## 2. Fluctuations of $\mathbb{V}$ and $\mathbb{S}$

In this section we prove several results about fluctuations of the random walk $\mathbb{V}$. Some of these estimates will be obtained via the study of the limiting process $\mathbb{S}$. In the first subsection, we recall elementary properties of the stable process $\mathbb{S}$ as well as a result of functional convergence of the random walk toward the limiting stable process. In the following, for any process $Z$, we will use indifferently the notation $Z_x$ or $Z(x)$.

2.1. **Preliminaries and functional convergence in $\mathbb{D}$.** We introduce the space $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ of càdlàg functions $Z:\mathbb{R}_+\to\mathbb{R}$ equipped with the Skorohod topology. Let $\theta$ stand for the shift operator, that is, for any $Z\in\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ and any $x_0\geqslant 0$, we have

$$((\theta_{x_0}Z)_x, x\geqslant 0)=(Z_{x+x_0}-Z_{x_0}, x\geqslant 0)\,. \tag{2.1}$$

Since our processes are double-sided, we will also need the space $\mathbb{D}(\mathbb{R},\mathbb{R})$ of functions $f:\mathbb{R}\to\mathbb{R}$ which are right continuous with left limits on $[0,\infty)$ and left continuous with right limits on $(-\infty,0]$ considered jointly with the associated Skorohod topology. Recall that $\mathbb{S}$ and $\mathbb{V}$ have paths on $\mathbb{D}(\mathbb{R},\mathbb{R})$. We will be interested in the following functionals: for any $a\in\mathbb{R}$ and for any $Z\in\mathbb{D}(\mathbb{R},\mathbb{R})$ we define (we give two notations for each definition):

$$\begin{aligned}
\overline{Z}_a &= F_a^{(1)}(Z) &=& \begin{cases}\sup_{y\in[0,a]} Z_y, & \text{for } a\geqslant 0,\\ \sup_{y\in[a,0]} Z_y, & \text{for } a<0,\end{cases}\\
\underline{Z}_a &= F_a^{(2)}(Z) &=& \begin{cases}\inf_{y\in[0,a]} Z_y, & \text{for } a\geqslant 0,\\ \inf_{y\in[a,0]} Z_y, & \text{for } a<0,\end{cases}\\
Z^*_a &= F_a^{(3)}(Z) &=& \begin{cases}\sup_{y\in[0,a]} |Z_y|, & \text{for } a\geqslant 0,\\ \sup_{y\in[a,0]} |Z_y|, & \text{for } a<0,\end{cases}\\
Z^R_a &= F_a^{(4)}(Z) &=& Z_a-\underline{Z}_a,\\
Z^{\#}_a &= F_a^{(5)}(Z) &=& \begin{cases}\sup_{0\leqslant y\leqslant a} Z^R_y, & \text{for } a\geqslant 0,\\ \sup_{a\leqslant y\leqslant 0} Z^R_y, & \text{for } a<0,\end{cases}\\
\sigma_Z(a) &= F_a^{(6)}(Z) &=& \begin{cases}\inf\left(x\geqslant 0\ ,\ Z_x\geqslant a\right), & \text{for } a\geqslant 0,\\ \inf\left(x\geqslant 0\ ,\ Z_x\leqslant a\right), & \text{for } a<0,\end{cases}\\
\widetilde{\sigma}_Z(a) &= F_a^{(7)}(Z) &=& \begin{cases}\inf\left(x\geqslant 0\ ,\ Z_{-x}\geqslant a\right), & \text{for } a\geqslant 0,\\ \inf\left(x\geqslant 0\ ,\ Z_{-x}\leqslant a\right), & \text{for } a<0,\end{cases}\\
U_Z(a) &= F_a^{(8)}(Z) &=& a-\underline{Z}(\sigma_Z(a)),\ \text{for } a\geqslant 0,\\
\widetilde{U}_Z(a) &= F_a^{(9)}(Z) &=& a-\underline{Z}\left(\widetilde{\sigma}_Z(a)\right),\ \text{for } a\geqslant 0,\\
\widetilde{G}_Z(a) &= F_a^{(10)}(Z) &=& \widetilde{U}_Z(\overline{Z}_a)\vee Z^{\#}_a,\ \text{for } a\geqslant 0.
\end{aligned}$$

Let $\mathcal{D}_i(a)$ for $i \in \{1, \cdots, 10\}$ denote the set of discontinuity points in $\mathbb{D}(\mathbb{R}, \mathbb{R})$ of $F_a^{(i)}$ and for $v \geqslant 1$ let $\mathbb{V}^{(v)} = (\mathbb{V}_{vx}/a(v)\ ,\ x \in \mathbb{R})$. From a theorem of Skorohod [21], Assumption 1 implies that $(\mathbb{V}^{(v)}\ ,\ v \geqslant 1)$ converges in law in the Skorohod space towards $\mathbb{S}$ as $v \to \infty$. It remains to check that the previously defined functionals have nice continuous properties (with respect to $\mathbb{S}$) in order to obtain results such as $F_a^{(i)}(\mathbb{V}^{(v)}) \to F_a^{(i)}(\mathbb{S})$ in law as $v \to \infty$.

For $Z \in \mathbb{D}(\mathbb{R}, \mathbb{R})$ and $a \in \mathbb{R}$, we will say that

$$Z \text{ is oscillating at } a^- \text{ if } \quad \forall \varepsilon > 0, \quad \inf_{(a-\varepsilon,a)} Z < Z_{a-} < \sup_{(a-\varepsilon,a)} Z,$$

$$Z \text{ is oscillating at } a^+ \text{ if } \quad \forall \varepsilon > 0, \quad \inf_{(a,a+\varepsilon)} Z < Z_{a+} < \sup_{(a,a+\varepsilon)} Z.$$

The following lemma collects some easy results about the sample path of $\mathbb{S}$.

**Lemma 2.1.** *We have*

(1) $\sup_{[0,\infty)} \mathbb{S} = \sup_{(-\infty,0]} \mathbb{S} = \infty$ *almost surely.*
(2) *With probability* 1, *any path of* $\mathbb{S}$ *is such that if* $\mathbb{S}$ *is discontinuous at a point* $x$, *then* $\mathbb{S}$ *is oscillating at* $x^-$ *and* $x^+$.
(3) *For any fixed* $a \in \mathbb{R}$, $\mathbb{S}$ *is almost surely continuous at* $a$ *and oscillating at* $a^-$ *and* $a^+$.

*Proof.* (1) and (2) come from Lemma 3.1 of [15], p531. As for (3), it is well known that $\mathbb{S}$ is almost surely continuous at any given point and the fact that it is oscillating follows from the assumption that $|\mathbb{S}|$ is not a subordinator. □

Note that (2) implies that, almost surely, $\mathbb{S}$ is continuous at all its local extrema. (2) also implies that, with probability 1, $\mathbb{S}$ attains its bound on any compact interval. These facts enable us to prove the following:

**Proposition 2.2.** *For any fixed* $a \in \mathbb{R}$ *and* $i \in \{1, \cdots, 10\}$, *we have*

$$\mathbf{P}\left(\mathbb{S} \in \mathcal{D}_i(a)\right) = 0.$$

*Proof.* Let $a$ be fixed. The functionals $F_a^{(i)}, i \in \{1, 2, 3, 4, 5\}$ are continuous at all $Z \in \mathbb{D}(\mathbb{R}, \mathbb{R})$ such that $Z$ is continuous at point $a$ (refer to Proposition 2.11 on p305 of [14] for further details) and the result follows from (3) of the previous lemma. It is also easily checked from the definition of the Skorohod topology that the functionals $F_a^{(i)}, i \in \{6, 8\}$ are continuous at all $Z$ which have the following properties:

(a) $\sigma_Z(a) < \infty$,
(b) $Z$ is oscillating a $\sigma_Z(a)+$,
(c) $Z$ attains its bounds on any compact interval.

Using again the previous lemma, we see that (a) and (c) hold for almost any path of $\mathbb{S}$. Notice that, from the Markov property, part (3) of the lemma is unchanged when $a$ is replaced by an arbitrary stopping time. Hence, $(b)$ is also true for almost any path of $\mathbb{S}$. The proof for $F_a^{(i)}, i \in \{7, 9\}$ is of course similar. Finally, the result for $F_a^{(10)}$ may easily be deduced from previous ones using the independence of $(\mathbb{S}_x\ ,\ x\geqslant 0)$ and $(\mathbb{S}_{-x}\ ,\ x\geqslant 0)$. $\square$

We will also use the fact that the random variables $F_a^{(i)}$ have continuous cumulative functions (except for the degenerated case $a=0$).

**Proposition 2.3.** *For all $a\neq 0$ and $b\in\mathbb{R}$ and $i\in\{1,\cdots,10\}$, we have*

$$\mathbf{P}\left(F_a^{(i)}(\mathbb{S})=b\right)=0.$$

We skip the proof as this may be easily checked from the facts that $\mathbb{S}$ has a continuous density and the assumption that it is not a subordinator.

Finally, throughout the rest of this paper, the notation $\mathbf{C}_i$ will always denote a finite strictly positive constant depending only on our choice of $\mathbf{P}$. In the case of a constant depending on some other parameters, these parameters will appear in the subscript. We will also repeatedly use the following lemma easily deduced from the Uniform Convergence Theorem for regularly varying functions ([4], p22) combined with monotonicity property.

**Lemma 2.4.** *Let $f:[1,\infty)\mapsto\mathbb{R}_+$ be a strictly positive non-decreasing function which is regularly varying at infinity with index $\beta\geqslant 0$. Then, for any $\varepsilon>0$ there exist $\mathbf{C}_{1,\varepsilon,f},\mathbf{C}_{2,\varepsilon,f}$ such that for any $1\leqslant x\leqslant y$,*

$$\mathbf{C}_{1,\varepsilon,f}\left(\frac{x}{y}\right)^{\beta+\varepsilon}\leqslant\frac{f(x)}{f(y)}\leqslant\mathbf{C}_{2,\varepsilon,f}\left(\frac{x}{y}\right)^{\beta-\varepsilon}.$$

2.2. **Supremum of the reflected process.** We now give some bounds and asymptotics about $\mathbb{V}^{\#}$. These estimates which may look quite technical will play a central role in the proof of Theorem 1. This subsection is devoted to the proofs of the following three propositions.

**Proposition 2.5.** *We have*

$$\lim_{\substack{x\to\infty\\ v/a^{-1}(x)\to\infty}}\frac{a^{-1}(x)}{v}\log\mathbf{P}\left(\mathbb{V}_v^{\#}\leqslant x\right)=-K^{\#}$$

*where $K^{\#}=-\lim_{v\to\infty}\frac{1}{v}\log\mathbf{P}\left(\mathbb{S}_v^{\#}\leqslant 1\right)$ is strictly positive and finite.*

**Proposition 2.6.** *For all $0<b<1$, there exists $\mathbf{C}_{3,b}>0$ such that for all $x$ large enough (depending on $b$) and all $v>0$,*

$$\mathbf{C}_{3,b}\mathbf{P}\left(\mathbb{V}_v^{\#}\leqslant x\right)\leqslant\mathbf{P}\left(\mathbb{V}_v^{\#}\leqslant x,\overline{\mathbb{V}}_v\leqslant bx\right)\leqslant\mathbf{P}\left(\mathbb{V}_v^{\#}\leqslant x\right).$$

**Proposition 2.7.** *There exists $\mathbf{C}_4>0$ such that for all $x$ large enough and all $v_1,v_2>0$,*

$$\mathbf{C}_4\mathbf{P}\left(\mathbb{V}_{v_1}^{\#}\leqslant x\right)\mathbf{P}\left(\mathbb{V}_{v_2}^{\#}\leqslant x\right)\leqslant\mathbf{P}\left(\mathbb{V}_{v_1+v_2}^{\#}\leqslant x\right).$$

Notice that using Proposition 2.6 we deduce that Proposition 2.5 is unchanged if we replace $\mathbf{P}(\mathbb{V}_v^{\#} \leqslant x)$ by $\mathbf{P}(\mathbb{V}_v^{\#} \leqslant x \ , \overline{\mathbb{V}}_v \leqslant bx)$ for all $b > 0$. The proof of the first proposition relies on the following lemma:

**Lemma 2.8.** *There exists a constant $K^{\#} \in (0, \infty)$ such that, for any $a, c > 0$ and any $b \geqslant 0$,*

$$\lim_{t\to\infty} \frac{a^\alpha}{t} \log \mathbf{P}\left(\mathbb{S}_t^{\#} \leqslant a \ , \underline{\mathbb{S}}_t \leqslant -b \ , \mathbb{S}_t - \underline{\mathbb{S}}_t \leqslant c\right) = -K^{\#}.$$

*In particular $K^{\#} = -\lim_{v\to\infty} \frac{1}{v} \log(\mathbf{P}(\mathbb{S}_v^{\#} \leqslant 1))$.*

*Proof.* Using the scaling property, we only need to prove the lemma in the case $a = 1$. For the sake of clarity, let

$$\mathcal{E}_1 = \left\{\mathbb{S}_t^{\#} \leqslant 1 \ , \underline{\mathbb{S}}_t \leqslant -b \ , \mathbb{S}_t - \underline{\mathbb{S}}_t \leqslant c\right\},$$

and let $f(t) = \log \mathbf{P}(\mathbb{S}_t^{\#} \leqslant 1)$. Using the Markov property of the stable process $\mathbb{S}$, we deduce that $f(t+s) \leqslant f(t) + f(s)$ for any $s, t \geqslant 0$. Since $f$ is subadditive, elementary analysis shows that the limit $K^{\#} = -\lim_{t\to\infty} f(t)/t$ exists and furthermore $K^{\#} \in (0, \infty]$. In order to prove that $K^{\#} < \infty$, note that $\{\mathbb{S}_t^{\#} \leqslant 1\} \supset \{\mathbb{S}_t^{*} \leqslant 1/2\}$ which implies $f(t)/t \geqslant \log \mathbf{P}\left(\mathbb{S}_t^{*} \leqslant 1/2\right)/t$. Using Proposition 3 of [1], p220, the r.h.s. of this last inequality converges to some finite constant when $t$ converges to infinity. Therefore $K^{\#}$ must be finite. So we have obtained

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbf{P}\left(\mathcal{E}_1\right) \leqslant \lim_{t\to\infty} \frac{1}{t} f\left(t\right) \leqslant -K^{\#}.$$

It remains to prove the lower bound. Let $0 < \varepsilon < \min\left(c, 1\right)$ and let $t > 1$. Define

$$\mathcal{E}_2 = \left\{\mathbb{S}_{t-1}^{\#} \leqslant 1 - \varepsilon\right\},$$
$$\mathcal{E}_3 = \left\{(\theta_{t-1}\mathbb{S})_1^{\#} \leqslant \varepsilon \ , \underline{(\theta_{t-1}\mathbb{S})}_1 \leqslant -b - 1\right\}.$$

We have $\mathcal{E}_1 \supset \mathcal{E}_2 \cap \mathcal{E}_3$. Since $\mathbb{S}$ has independent increments, $\mathcal{E}_2$ and $\mathcal{E}_3$ are independent. Therefore $\mathbf{P}\left(\mathcal{E}_1\right) \geqslant \mathbf{P}\left(\mathcal{E}_2\right) \mathbf{P}\left(\mathcal{E}_3\right)$. Furthermore, using scaling, $\mathbf{P}(\mathcal{E}_2) = f\left((t-1)/(1-\varepsilon)^\alpha\right)$. Hence

$$\frac{1}{t} \log \mathbf{P}\left(\mathcal{E}_1\right) \geqslant \frac{\log \mathbf{P}\left(\mathcal{E}_3\right)}{t} + \frac{1}{t} f\left(\frac{t}{(1-\varepsilon)^\alpha}\right), \tag{2.2}$$

and $\mathbf{P}(\mathcal{E}_3) = \mathbf{P}(\mathbb{S}_1^{\#} \leqslant \varepsilon \ , \underline{\mathbb{S}}_1 \leqslant -b - 1)$ does not depend on $t$ and is not zero (this is easy to check since $\mathbb{S}$ is not a subordinator). Taking the limit in (2.2) we conclude that

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbf{P}\left(\mathcal{E}_1\right) \geqslant \lim_{t\to\infty} \frac{1}{t} f\left(\frac{t}{(1-\varepsilon)^\alpha}\right) = \frac{-K^{\#}}{(1-\varepsilon)^\alpha}.$$

□

*Proof of Proposition 2.5.* Let us choose $\varepsilon > 0$. The previous lemma and the scaling property of $\mathbb{S}^{\#}$ give

$$K^{\#} = -\lim_{y\to\infty} \frac{1}{y^{\alpha}} \log \mathbf{P}\left(\mathbb{S}_1^{\#} < \frac{1}{y}\right).$$

Hence, we can choose $y_0 > 0$ such that $\log \mathbf{P}(\mathbb{S}_1^{\#} \leqslant 1/y_0) \leqslant -(K^{\#} - \varepsilon)y_0^{\alpha}$. Combining results of Proposition 2.2 and 2.3 for the functional $F^{(3)}$, we get

$$\lim_{k\to\infty} \log \mathbf{P}\left(\frac{1}{a(k)}\mathbb{V}_k^{\#} \leqslant \frac{1}{y_0}\right) = \log \mathbf{P}\left(\mathbb{S}_1^{\#} \leqslant \frac{1}{y_0}\right) \leqslant -\left(K^{\#} - \varepsilon\right) y_0^{\alpha}.$$

Therefore, for all $k$ large enough,

$$\log \mathbf{P}\left(\frac{1}{a(k)}\mathbb{V}_k^{\#} \leqslant \frac{1}{y_0}\right) \leqslant -\left(K^{\#} - 2\varepsilon\right) y_0^{\alpha}. \tag{2.3}$$

Let us choose $k = [a^{-1}(xy_0)]+1$, thus (2.3) holds whenever $x$ is large enough. Notice that

$$\left\{\mathbb{V}_v^{\#} \leqslant x\right\} \subset \bigcap_{n=0}^{[v/k]-1} \left\{(\theta_{nk}\mathbb{V})_k^{\#} \leqslant x\right\},$$

hence, using the independence and stationarity of the increments of the random walk at integer times, we obtain

$$\mathbf{P}\left(\mathbb{V}_v^{\#} \leqslant x\right) \leqslant \left(\mathbf{P}\left(\mathbb{V}_k^{\#} \leqslant x\right)\right)^{\left[\frac{v}{k}\right]}. \tag{2.4}$$

Since $a(\cdot)$ is non-decreasing, our choice of $k$ implies $x/a(k) \leqslant 1/y_0$, therefore

$$\mathbf{P}\left(\mathbb{V}_k^{\#} \leqslant x\right) \leqslant \mathbf{P}\Big(\frac{\mathbb{V}_k^{\#}}{a(k)} \leqslant \frac{1}{y_0}\Big).$$

Combining this inequality with (2.3) and (2.4) yields

$$\log \mathbf{P}\left(\mathbb{V}_v^{\#} \leqslant x\right) \leqslant -\left[\frac{v}{k}\right] y_0^{\alpha}\left(K^{\#} - 2\varepsilon\right).$$

It is easy to check from the regular variation of $a^{-1}(\cdot)$ with index $\alpha$ that $[v/k]y_0^{\alpha} \sim v/a^{-1}(x)$ when $x$ and $v/a^{-1}(x)$ both go to infinity, hence

$$\limsup \frac{a^{-1}(x)}{v} \log \mathbf{P}\left(\mathbb{V}_v^{\#} \leqslant x\right) \leqslant -K^{\#}.$$

The proof of the lower bound is quite similar yet slightly more technical. Using Lemma 2.8 and the scaling property, we can find $y_0 > 0$ such that

$$\log \mathbf{P}\left(\mathbb{S}_1^{\#} \leqslant \frac{1-\varepsilon}{y_0}, \underline{\mathbb{S}}_1 \leqslant -\frac{2\varepsilon}{y_0}, \mathbb{S}_1 - \underline{\mathbb{S}}_1 \leqslant \frac{\varepsilon}{y_0}\right) \geqslant -\frac{K^{\#}y_0^{\alpha}}{(1-2\varepsilon)^{\alpha}}. \tag{2.5}$$

Let us set

$$\mathcal{E}_4(k) = \left\{\frac{\mathbb{V}_k^{\#}}{a(k)} \leqslant \frac{1-\varepsilon}{y_0}, \frac{\underline{\mathbb{V}}_k}{a(k)} \leqslant -\frac{2\varepsilon}{y_0}, \frac{\mathbb{V}_k - \underline{\mathbb{V}}_k}{a(k)} \leqslant \frac{\varepsilon}{y_0}\right\}.$$

Using Proposition 2.2 and 2.3, we check that

$$\lim_{k\to\infty} \mathbf{P}\left(\mathcal{E}_4\left(k\right)\right) = \mathbf{P}\left(\mathbb{S}_1^{\#} \leqslant \frac{1-\varepsilon}{y_0}\ ,\ \underline{\mathbb{S}}_1 \leqslant\ -\frac{2\varepsilon}{y_0}\ ,\ \mathbb{S}_1 - \underline{\mathbb{S}}_1 \leqslant \frac{\varepsilon}{y_0}\right).$$

Hence for all $k$ large enough, it follows from (2.5) that

$$\log \mathbf{P}\left(\mathcal{E}_4\left(k\right)\right)\ \geqslant \frac{-K^{\#} y_0^{\alpha}}{(1-3\varepsilon)^{\alpha}}. \tag{2.6}$$

We now choose $k = [a^{-1}\left(xy_0\right)]$. Notice that $1/y_0 \leqslant x/a(k) \leqslant 2/y_0$ for all $x$ large enough, thus

$$\mathcal{E}_4(k) \subset \left\{\mathbb{V}_k^{\#} \leqslant (1-\varepsilon)x\ ,\ \underline{\mathbb{V}}_k \leqslant\ -\varepsilon x\ ,\ \mathbb{V}_k - \underline{\mathbb{V}}_k \leqslant \varepsilon x\right\}.$$

One may check by induction that

$$\begin{aligned}\left\{\mathbb{V}_v^{\#} \leqslant x\right\}\ \ \supset\ \ \bigcap_{n=0}^{[v/k]} \Big\{(\theta_{nk}\mathbb{V})_k^{\#} \leqslant (1-\varepsilon)x\ ,\ \underline{(\theta_{nk}\mathbb{V})}_k \leqslant\ -\varepsilon x,&\\ (\theta_{nk}\mathbb{V})_k - \underline{(\theta_{nk}\mathbb{V})}_k \leqslant \varepsilon x\Big\},&\end{aligned}$$

hence, using independence and stationarity of the increments of $\mathbb{V}$ at integer times, we get

$$\begin{aligned}\mathbf{P}\left(\mathbb{V}_v^{\#} \leqslant x\right) \quad &\geqslant \quad \mathbf{P}\Big(\mathbb{V}_k^{\#} \leqslant (1-\varepsilon)x\ ,\ \underline{\mathbb{V}}_k \leqslant\ -\varepsilon x\ ,\ \mathbb{V}_k - \underline{\mathbb{V}}_k \leqslant \varepsilon x\Big)^{\left[\frac{v}{k}\right]+1}\\ &\geqslant \quad \mathbf{P}\left(\mathcal{E}_4(k)\right)^{\left[\frac{v}{k}\right]+1}.\end{aligned}$$

Combining this inequality with (2.6), this shows that for all $x$ large enough,

$$\log \mathbf{P}\left(\mathbb{V}_v^{\#} \leqslant x\right)\ \geqslant \frac{-K^{\#}}{(1-3\varepsilon)^{\alpha}}\left(\left[\frac{v}{k}\right]+1\right) y_0^{\alpha}.$$

Notice that $([v/k]+1)y_0^{\alpha} \sim v/a^{-1}(x)$ as $x$ and $v/a^{-1}\left(x\right)$ go to infinity simultaneously, which completes the proof. □

*Proof of Proposition 2.6.* The upper bound is trivial. Let $0 < b < 1$, define $v_1 = [a^{-1}(x)]$ and set $c = (b-1)x$,

$$\begin{aligned}\left\{\mathbb{V}_v^{\#} \leqslant x, \overline{\mathbb{V}}_v \leqslant bx\right\}\ \ &\supset\ \ \left\{\mathbb{V}_v^{\#} \leqslant x, \overline{\mathbb{V}}_v \leqslant bx, \sigma_{\mathbb{V}}(c) \leqslant v_1\right\}\\ &\supset\ \ \left\{\mathbb{V}_{\sigma_{\mathbb{V}}(c)}^{\#} \leqslant bx, \sigma_{\mathbb{V}}(c) \leqslant v_1\right\} \cap \left\{\left(\theta_{\sigma_{\mathbb{V}}(c)}\mathbb{V}\right)_v^{\#}\ \leqslant x\right\},\end{aligned}$$

thus

$$\begin{aligned}\mathbf{P}\left(\mathbb{V}_v^{\#} \leqslant x, \overline{\mathbb{V}}_v \leqslant bx\right)\quad &\geqslant\quad \mathbf{P}\left(\mathbb{V}_{\sigma_{\mathbb{V}}(c)}^{\#} \leqslant bx, \sigma_{\mathbb{V}}(c) \leqslant v_1\right)\mathbf{P}\left(\mathbb{V}_v^{\#} \leqslant x\right)\\ &\geqslant\quad \mathbf{P}\left(\mathbb{V}_{v_1}^{\#} \leqslant bx, \underline{\mathbb{V}}_{v_1} \leqslant c\right)\mathbf{P}\left(\mathbb{V}_v^{\#} \leqslant x\right).\end{aligned}$$

Just like in the previous proof, we see that $\mathbf{P}(\mathbb{V}_{v_1}^{\#} \leqslant bx, \underline{\mathbb{V}}_{v_1} \leqslant c)$ converges when $x$ goes to infinity toward $\mathbf{P}(\mathbb{S}_1^{\#} \leqslant b, \underline{\mathbb{S}}_1 \leqslant b-1)$ and this quantity is strictly positive since $|\mathbb{S}|$ is not a subordinator. □

*Proof of Proposition 2.7.* Notice that

$$\begin{aligned}\left\{\mathbb{V}^{\#}_{v_1+v_2}\leqslant x\right\} &\supset \left\{\mathbb{V}^{\#}_{[v_1]+[v_2]+2}\leqslant x\right\}\\ &\supset \left\{\mathbb{V}_1\leqslant 0\ ,\ \mathbb{V}_2-\mathbb{V}_1\leqslant 0\right\}\\ &\quad\cap\left\{(\theta_2\mathbb{V})^{\#}_{[v_1]}\leqslant x\ ,\ (\theta_2\mathbb{V})_{[v_1]}-\underline{(\theta_2\mathbb{V})}_{[v_1]}\leqslant\frac{x}{2}\right\}\\ &\quad\cap\left\{(\theta_{2+[v_1]}\mathbb{V})^{\#}_{[v_2]}\leqslant x\ ,\ \overline{(\theta_{2+[v_1]}\mathbb{V})}_{[v_2]}\leqslant\frac{x}{2}\right\}.\end{aligned}$$

Using the independence and stationarity of the increments of $\mathbb{V}$ at integer time and setting $\mathbf{C}_5=\mathbf{P}(\mathbb{V}_1\leqslant 0)>0$, we see that $\mathbf{P}(\mathbb{V}^{\#}_{v_1+v_2}\leqslant x)$ is larger than

$$\mathbf{C}_5^2\mathbf{P}\left(\mathbb{V}^{\#}_{[v_1]}\leqslant x\ ,\ \mathbb{V}_{[v_1]}-\underline{\mathbb{V}}_{[v_1]}\leqslant\frac{x}{2}\right)\mathbf{P}\left(\mathbb{V}^{\#}_{[v_2]}\leqslant x\ ,\ \overline{\mathbb{V}}_{[v_2]}\leqslant\frac{x}{2}\right).$$

Time reversal of the random walk $\mathbb{V}$ shows that

$$\mathbf{P}\left(\mathbb{V}^{\#}_{[v_1]}\leqslant x,\mathbb{V}_{[v_1]}-\underline{\mathbb{V}}_{[v_1]}\leqslant x/2\right)=\mathbf{P}\left(\mathbb{V}^{\#}_{[v_1]}\leqslant x,\overline{\mathbb{V}}_{[v_1]}\leqslant x/2\right),$$

hence, using Proposition 2.6,

$$\begin{aligned}\mathbf{P}(\mathbb{V}^{\#}_{v_1+v_2}\leqslant x) &\geqslant (\mathbf{C}_{3,\frac{1}{2}}\mathbf{C}_5)^2\mathbf{P}\left(\mathbb{V}^{\#}_{[v_1]}\leqslant x\right)\mathbf{P}\left(\mathbb{V}^{\#}_{[v_2]}\leqslant x\right)\\ &\geqslant (\mathbf{C}_{3,\frac{1}{2}}\mathbf{C}_5)^2\mathbf{P}\left(\mathbb{V}^{\#}_{v_1}\leqslant x\right)\mathbf{P}\left(\mathbb{V}^{\#}_{v_2}\leqslant x\right).\end{aligned}$$

□

2.3. **The case where $\mathbb{S}$ is a completely asymmetric stable process.** One may wish to calculate the value of the constant $K^{\#}$ that appears in the last section. Unfortunately, we do not know its value in general. However, the completely asymmetric case is a particularly nice setting where calculations may be carried out to their full extend. We now assume throughout this section that the stable process $(\mathbb{S}_x\ ,\ x\geqslant 0)$ either has no positive jumps hence the exponential moments of $\mathbb{S}$ are finite and (1.2) holds (recall that we assume $\gamma'=1$) or $\mathbb{S}$ has no negative jumps and $\mathbf{E}\left(\exp(-\lambda\mathbb{S}_t)\right)=\exp(t\lambda^{\alpha})$ for all $t,\lambda\geqslant 0$. For $a,b>0$, define the stopping times:

$$\begin{aligned}\tau_b &= \inf(t\geqslant 0\ ,\ \mathbb{S}_t\geqslant b)=\sigma_{\mathbb{S}}(b),\\ \tau_b^{\#} &= \inf(t\geqslant 0\ ,\ \mathbb{S}^{\#}_t\geqslant b)=\sigma_{\mathbb{S}^{\#}}(b),\\ \tau^*_{a,b} &= \inf(t\geqslant 0\ ,\ \mathbb{S}_t \text{ not in } (-a,b)).\end{aligned}$$

Recall that $E_\alpha$ stands for the Mittag-Leffler function with parameter $\alpha$.

**Proposition 2.9.** *When $\mathbb{S}$ has no positive jumps, we have*

$$\mathbf{E}\left(e^{-q\tau_1^{\#}}\right)=\frac{1}{E_\alpha(q)},$$

*and when $\mathbb{S}$ has no negative jumps, we have*

$$\mathbf{E}\left(e^{-q\tau_1^{\#}}\right)=E_\alpha(q)-\frac{\alpha q(E'_\alpha(q))^2}{\alpha qE''_\alpha(q)+(\alpha-1)E'_\alpha(q)}.$$

This proposition is a particular case of Proposition 2 of [17], p191. Still, we give here a simpler proof when $\mathbb{S}$ is stable using the solution of the two sided exit problem given by Bertoin [2].

*Proof.* We suppose that $\mathbb{S}$ has no negative jumps. Let $\eta(q)$ be an exponential random time of parameter $q$ independent of $\mathbb{S}$. Let also $a, b$ be strictly positive real numbers such that $a+b=1$. We may without loss of generality assume any path of $\mathbb{S}$ attains its bounds on any compact interval and is continuous at all local extrema (because this happens with probability 1 according to Lemma 2.1). Thus, on the one hand, the event $\{\tau_1^{\#} > \eta(q)\}$ contains

$$\left\{\tau^*_{a,b} > \eta(q)\right\} \cup \left(\left\{\tau^*_{a,b} \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau^*_{a,b}} \leqslant\ -a\right\} \cap \left\{(\theta_{\tau^*_{a,b}}\mathbb{S})^{\#}_{\eta(q)-\tau^*_{a,b}} < 1\right\}\right).$$

Using the strong Markov property of $\mathbb{S}$, the lack of memory, and the independence of the exponential time, it follows that $\mathbf{P}(\tau_1^{\#} > \eta(q))$ is larger than

$$\mathbf{P}\left(\tau^*_{a,b} > \eta(q)\right) + \mathbf{P}\left(\tau^*_{a,b} \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau^*_{a,b}} \leqslant\ -a\right)\mathbf{P}\left(\tau_1^{\#} > \eta(q)\right),$$

therefore

$$\mathbf{P}\left(\tau_1^{\#} > \eta(q)\right) \geqslant \frac{\mathbf{P}\left(\tau^*_{a,b} > \eta(q)\right)}{1-\mathbf{P}\left(\tau^*_{a,b} \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau^*_{a,b}} \leqslant\ -a\right)}. \tag{2.7}$$

On the other hand, one may check that the event $\{\tau_1^{\#} > \eta(q)\}$ is a subset of

$$\left\{\tau^*_{a,b} > \eta(q)\right\} \cup \left(\left\{\tau^*_{a,b} \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau^*_{a,b}} \leqslant\ -a\right\} \cap \left\{(\theta_{\tau^*_{a,b}}\mathbb{S})^{\#}_{\eta(q)-\tau^*_{a,b}} < b\right\}\right),$$

and similarly we deduce

$$\mathbf{P}\left(\tau_b^{\#} > \eta(q)\right) \leqslant \frac{\mathbf{P}\left(\tau^*_{a,b} > \eta(q)\right)}{1-\mathbf{P}\left(\tau^*_{a,b} \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau^*_{a,b}} \leqslant\ -a\right)}. \tag{2.8}$$

Obviously $\tau_b^{\#}$ converges to $\tau_1^{\#}$ almost surely as $b$ converges to 1. Combining this observation with (2.7) and (2.8), we find

$$\mathbf{P}\left(\tau_1^{\#} > \eta(q)\right) = \lim_{b\nearrow 1} \frac{\mathbf{P}\left(\tau^*_{1-b,b} > \eta(q)\right)}{1-\mathbf{P}\left(\tau^*_{1-b,b} \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau^*_{1-b,b}} \leqslant b-1\right)}. \tag{2.9}$$

The probabilities of the r.h.s. of this equation have been calculated by Bertoin [2]:

$$\mathbf{P}\left(\tau^*_{1-b,b} > \eta(q)\right) = 1 - E_\alpha(b^\alpha) + \tfrac{b^{\alpha-1}E'_\alpha(qb^\alpha)}{E'_\alpha(q)}\left(E_\alpha(q)-1\right), \tag{2.10}$$

$$\mathbf{P}\left(\tau^*_{1-b,b} \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau^*_{1-b,b}} \leqslant b-1\right) = \tfrac{b^{\alpha-1}E'_\alpha(qb^\alpha)}{E'_\alpha(q)}. \tag{2.11}$$

Taylor expansions of $E_\alpha$ and $E'_\alpha$ near point $q$ enables us to calculate the limit in (2.9) in term of $E_\alpha$ and its first and second derivatives. After a few lines of elementary calculus, we get

$$\mathbf{P}\left(\tau_1^\# > \eta(q)\right) = 1 - E_\alpha(q) + \frac{\alpha q (E'_\alpha(q))^2}{\alpha q E''_\alpha(q) + (\alpha-1)E'_\alpha(q)}.$$

We complete the proof using the well known relation $\mathbf{E}(\exp(-q\tau_1^\#)) = 1 - \mathbf{P}(\tau_1^\# > \eta(q))$. The proof in the case where $\mathbb{S}$ has no positive jumps is similar (and the calculation of the limit is even easier). We omit it. □

**Corollary 2.10.** *Recall that $-\rho_1(\alpha)$ is the first negative root of $E_\alpha$ and $-\rho_2(\alpha)$ is the first negative root of $\alpha x E''_\alpha(x) + (\alpha-1)E'_\alpha(x)$. The constant of Proposition 2.5 is given by*

$$K^\# = \begin{cases} \rho_1(\alpha) & \text{when } \mathbb{S} \text{ has no positive jumps,} \\ \rho_2(\alpha) & \text{when } \mathbb{S} \text{ has no negative jumps.} \end{cases}$$

*Proof.* Recall that $K^\# = -\lim_{t\to\infty} \mathbf{P}(\mathbb{S}_t^\# \leqslant 1)/t$. Using the same argument as in Corollary 1 of [2], we see that $K^\# = \rho_1(\alpha)$ when $\mathbb{S}$ has no positive jumps. Similarly, when $\mathbb{S}$ has no negative jumps $-K^\#$ is equal to the first negative pole of

$$g(x) = \frac{\alpha x (E'_\alpha(x))^2}{\alpha x E''_\alpha(x) + (\alpha-1)E'_\alpha(x)} = E_\alpha(x) - \mathbf{E}\left(e^{-x\tau_1^\#}\right).$$

Let $-x_0$ be the first negative root of $E'_\alpha$. Since $E'_\alpha(0) > 0$, this implies that $E_\alpha$ is strictly increasing on $[-x_0, 0]$. Notice also that $x \mapsto -\mathbf{E}(\exp(-x\tau_1^\#))$ is increasing on $(-K^\#, 0]$, thus $g(x)$ is strictly increasing on $(-(K^\# \wedge x_0), 0]$. Since $g(-x_0) = g(0) = 0$ (this holds even when $-x_0$ is a zero of multiple order) we deduce from the monotonicity of $g$ that $K^\# < x_0$ and this shows that the first negative pole of $g$ is indeed $-\rho_2(\alpha)$. □

We conclude this subsection by calculating the Laplace transform of $\tau_1^\# \wedge \tau_b$. This will be useful for the determination of the limiting law in the proof of Theorem 4.

**Corollary 2.11.** *For $0 < b \leqslant 1$, when $\mathbb{S}$ has no positive jumps*

$$\mathbf{E}\left(e^{-q\tau_1^\# \wedge \tau_b}\right) = \frac{E_\alpha(q(1-b)^\alpha)}{E_\alpha(q)},$$

*and when $\mathbb{S}$ has no negative jumps*

$$\mathbf{E}\left(e^{-q\tau_1^\# \wedge \tau_b}\right) = E_\alpha(qb^\alpha) - b^{\alpha-1}\frac{\alpha q E'_\alpha(qb^\alpha)E'_\alpha(q)}{\alpha q E''_\alpha(q) + (\alpha-1)E'_\alpha(q)}.$$

*Proof.* Let $\eta(q)$ still denote an exponential time with parameter $q$ independent of $\mathbb{S}$. Suppose that $\mathbb{S}$ has no negative jumps, using the Markov property and the lack of memory of the exponential law, we get

$$\begin{aligned}\mathbf{P}\left(\tau_1^{\#}\wedge\tau_b>\eta(q)\right) &= \mathbf{P}\left(\tau^*_{1-b,b}\leqslant\eta(q),\mathbb{S}_{\tau^*_{1-b,b}}\leqslant b-1\right)\mathbf{P}\left(\tau_1^{\#}>\eta(q)\right)\\ &\quad+\mathbf{P}\left(\tau^*_{1-b,b}>\eta(q)\right).\end{aligned}$$

The r.h.s. of the last equality may be calculated explicitly using again (2.10), (2.11), and Proposition 2.9. Hence, after simplification,

$$\mathbf{P}\left(\tau_1^{\#}\wedge\tau_b>\eta(q)\right)=1-E_\alpha(qb^\alpha)+b^{\alpha-1}\frac{\alpha qE'_\alpha(qb^\alpha)E'_\alpha(q)}{\alpha qE''_\alpha(q)+(\alpha-1)E'_\alpha(q)}.$$

The no positive jumps case may be treated the same way. □

2.4. **The exit problem for the random walk $\mathbb{V}$.** Let us define for $x,y>0$ the following events:

$$\begin{aligned}\Lambda(x,y) &= \left\{(\mathbb{V}_s)_{s\geqslant 0}\text{ hits }(y,\infty)\text{ before it hits }(-\infty,-x)\right\},\\ \Lambda'(x,y) &= \left\{(\mathbb{V}_s)_{s\geqslant 0}\text{ hits }[y,\infty)\text{ before it hits }(-\infty,-x]\right\},\\ \widetilde{\Lambda}'(x,y) &= \left\{(\mathbb{V}_{-s})_{s\geqslant 0}\text{ hits }(-\infty,-y]\text{ before it hits }[x,\infty)\right\}.\end{aligned}$$

We are interested in the behavior of the probabilities of these events for large $x,y$. In the case of a fixed $x$, when $y$ goes to infinity, this study was done by Bertoin and Doney [3]. Here, we need to study this quantities when both $x$ and $y$ go to infinity with the ratio $y/x$ also going to infinity. We already defined the sequence $(\mathbf{T}_n)_{n\geqslant 0}$ of strictly descending ladder times, we now consider the associated ladder heights:

$$\mathbf{H}_n=-\mathbb{V}_{\mathbf{T}_n}\quad\text{for }n\geqslant 0.$$

We will also need the sequence $(\mathbf{M}_n)_{n\geqslant 1}$:

$$\mathbf{M}_n=\max\left(\mathbb{V}_k+\mathbf{H}_{n-1}\ ,\ \mathbf{T}_{n-1}\leqslant k<\mathbf{T}_n\right).$$

Note that the sequence $(\mathbf{T}_{n+1}-\mathbf{T}_n,\mathbf{H}_{n+1}-\mathbf{H}_n,\mathbf{M}_n)_{n\geqslant 1}$ is independent and identically distributed. We know that $\mathbf{T}_1$ is in the domain of attraction of a positive stable law of index $q$ with norming constants $(b_n)$. Now, Corollary 3 of [8] states that $\mathbf{P}(\mathbf{M}_1>x)$ is regularly varying with index $-\alpha q$. More precisely, it gives

$$\mathbf{P}(\mathbf{M}_1>x)\underset{x\to\infty}{\sim}\frac{\mathbf{C}_6}{b^{-1}\left(a^{-1}(x)\right)}. \tag{2.12}$$

In particular, this shows that $\mathbf{M}_1$ is in the domain of attraction of a positive stable law when $\alpha q<1$ and that $\mathbf{M}_1$ is relatively stable when $\alpha q=1$ (relatively stable means that $\frac{1}{a(b(n))}\sum_{k\leqslant n}\mathbf{M}_k$ converges in probability to some strictly positive constant).

For $\mathbf{H}_1$, using Theorem 9 of [18], we see that $\mathbf{H}_1$ is in the domain of attraction of a positive stable law with index $\alpha q$ when $\alpha q<1$ and that $\mathbf{H}_1$ is relatively stable in the case $\alpha q=1$. Furthermore, the lemma of [8], p358 shows that we can choose $a(b(n))$ as norming constant for $\mathbf{H}_1$ in any of those two cases, thus

$$\frac{\mathbf{H}_n}{a(b(n))} \text{ converges to } \begin{cases} \text{some constant } \mathbf{C}_7, \text{ in probability when } \alpha q = 1, \\ \text{a positive stable law of index } \alpha q \text{ otherwise.} \end{cases}$$

When $\alpha q < 1$, this shows that (2.12) holds with $\mathbf{H}_1$ in place of $\mathbf{M}_1$ (for a different value of $\mathbf{C}_6$). Unfortunately, in the case $\alpha q = 1$, the relative stability of $\mathbf{H}_1$ does not imply the regular variation of $\mathbf{P}(\mathbf{H}_1 > x)$ (look at the counter example in [18], p 576). However, we can still prove a smooth behavior for the associated renewal function

$$\mathbf{R}(x) = \sum_{n=0}^{\infty} \mathbf{P}(\mathbf{H}_n \leqslant x).$$

**Lemma 2.12.** *There exists a constant* $\mathbf{C}_8 > 0$ *such that*

$$\mathbf{R}(x) \underset{x\to\infty}{\sim} \mathbf{C}_8 b^{-1}\left(a^{-1}(x)\right).$$

*Proof.* When $\alpha q < 1$ we mentioned that $\mathbf{P}(\mathbf{H}_1 > x) \sim \mathbf{C}_9 / b^{-1}\left(a^{-1}(x)\right)$ where $\mathbf{C}_9$ is some strictly positive constant. In this case, the asymptotic behavior of $\mathbf{R}$ follows from the Tauberian Theorem as in the lemma on p446 of [10]. We now consider the case $\alpha q = 1$. Let $L(\lambda) = \mathbf{E}\left(e^{-\lambda \mathbf{H}_1}\right)$ stand for the Laplace transform of $\mathbf{H}_1$. We know that

$$\frac{\mathbf{H}_n}{a(b(n))} \underset{n\to\infty}{\overset{\text{Prob.}}{\longrightarrow}} \mathbf{C}_7.$$

Therefore, for any $\lambda \geqslant 0$ and when $n$ ranges trough the set of integers, we have

$$\left(L\left(\frac{\lambda}{a(b(n))}\right)\right)^n \underset{n\to\infty}{\longrightarrow} e^{-\mathbf{C}_7 \lambda}. \tag{2.13}$$

Since $L$ is continuous at 0 with $L(0) = 1$, setting $\lambda = 1$ and taking the logarithm in (2.13) give

$$n\left(1 - L\left(\frac{1}{a(b(n))}\right)\right) \underset{n\to\infty}{\longrightarrow} \mathbf{C}_7. \tag{2.14}$$

Using the monotonicity of $L$ and $a(b(\cdot))$, it is easy to check that (2.14) still holds when $n$ now ranges trough the set of real numbers, thus

$$1 - L\left(\frac{1}{x}\right) \underset{x\to\infty}{\sim} \frac{\mathbf{C}_7}{b^{-1}(a^{-1}(x))}. \tag{2.15}$$

Let us now define $\widehat{\mathbf{R}}(y) = \int_0^\infty e^{-yx}\mathbf{R}(dx)$. The well-known relation $\widehat{\mathbf{R}}(y) = 1/(1 - L(y))$ combined with (2.15) shows that $\widehat{\mathbf{R}}$ is regularly varying near 0, hence we can use Karamata's Tauberian/Abelian Theorem to conclude the proof. □

**Proposition 2.13.** *There exists* $\mathbf{C}_{10}$ *such that when* $x \to \infty$ *and* $y/x \to \infty$,

$$\mathbf{P}(\Lambda(x,y)) \sim \mathbf{C}_{10} \frac{b^{-1}\left(a^{-1}(x)\right)}{b^{-1}(a^{-1}(x+y))}.$$

*This result also holds for* $\mathbf{P}(\Lambda'(x,y))$ *and* $\mathbf{P}(\widetilde{\Lambda}'(x,y))$.

*Proof.* The two processes $(\mathbb{V}_s)_{s\geqslant 0}$ and $(-\mathbb{V}_{-s})_{s\geqslant 0}$ have the same law, hence $\mathbf{P}(\Lambda'(x,y)) = \mathbf{P}(\widetilde{\Lambda}'(x,y))$. We also have the trivial inclusion $\Lambda(x-1,y) \subset \Lambda'(x,y) \subset \Lambda(x,y-1)$, so we only need to prove the proposition for $\Lambda(x,y)$. The first part of the proof is borrowed from Bertoin and Doney [3], p2157. The probability $\mathbf{P}(\Lambda(x,y))$ is equal to

$$\mathbf{P}(\mathbf{M}_1 > y) \quad + \quad \sum_{k=1}^{\infty} \mathbf{P}\Big(\mathbf{M}_1 \leqslant y + \mathbf{H}_0, \cdots, \mathbf{M}_k \leqslant y + \mathbf{H}_{k-1}, \mathbf{H}_k \leqslant x, \mathbf{M}_{k+1} > y + \mathbf{H}_k\Big), \tag{2.16}$$

thus

$$\begin{aligned}\mathbf{P}(\Lambda(x,y)) &\leqslant \mathbf{P}(\mathbf{M}_1 > y) + \sum_{k=1}^{\infty} \mathbf{P}(\mathbf{H}_k \leqslant x, \mathbf{M}_{k+1} > y + \mathbf{H}_k) \\ &\leqslant \mathbf{P}(\mathbf{M}_1 > y) + \sum_{k=1}^{\infty} \mathbf{P}(\mathbf{H}_k \leqslant x, \mathbf{M}_{k+1} > y) \\ &\leqslant \mathbf{P}(\mathbf{M}_1 > y)\,\mathbf{R}(x).\end{aligned}$$

Using (2.12), Lemma 2.12, and the equivalence $\mathbf{P}(\mathbf{M}_1 > y) \sim \mathbf{P}(\mathbf{M}_1 > x+y)$ when $x$ and $y/x$ go to infinity, we obtain the upper bound with $\mathbf{C}_{10} = \mathbf{C}_6\mathbf{C}_8$. We now prove the result pertaining to the lower bound. Let $k_0 \in \mathbb{N}^*$. From (2.16), we see that $\mathbf{P}(\Lambda(x,y))$ is bigger than

$$\begin{aligned}&\mathbf{P}(\mathbf{M}_1 > y) + \sum_{k=1}^{\infty} \mathbf{P}(\mathbf{M}_1 \leqslant y, \cdots, \mathbf{M}_k \leqslant y, \mathbf{H}_k \leqslant x, \mathbf{M}_{k+1} > x + y) \\ &\qquad \geqslant \mathbf{P}(\mathbf{M}_1 > x + y)\Big(1 + \sum_{k=1}^{k_0} \mathbf{P}(\mathbf{M}_1 \leqslant y, \cdots, \mathbf{M}_k \leqslant y, \mathbf{H}_k \leqslant x)\Big),\end{aligned}$$

hence

$$\mathbf{P}(\Lambda(x,y)) \geqslant \mathbf{P}(\mathbf{M}_1 > x+y)\Big(\mathbf{R}(x) - \mathbf{R}_{k_0}(x) - \mathbf{W}_{k_0}(y)\Big), \tag{2.17}$$

with

$$\begin{aligned}\mathbf{R}_{k_0}(x) &= \sum_{k=k_0+1}^{\infty} \mathbf{P}(\mathbf{H}_k \leqslant x), \\ \mathbf{W}_{k_0}(y) &= \sum_{k=1}^{k_0} \mathbf{P}(\mathbf{M}_1 > y \text{ or } \cdots \text{ or } \mathbf{M}_k > y).\end{aligned}$$

On the one hand, using (2.12) and Lemma 2.12, for $y$ large enough,

$$\mathbf{W}_{k_0}(y) \leqslant \sum_{k=1}^{K} k_0 \mathbf{P}(\mathbf{M}_1 > y) \leqslant k_0^2 \mathbf{P}(\mathbf{M}_1 > y) \leqslant \frac{\mathbf{C}_{11}k_0^2}{\mathbf{R}(y)}.$$

On the other hand,

$$\begin{aligned}\mathbf{R}_{k_0}(x) &= \sum_{k=0}^{\infty}\mathbf{P}\left(\mathbf{H}_{k_0+1}+\left(\mathbf{H}_{k+k_0+1}-\mathbf{H}_{k_0+1}\leqslant x\right)\right)\\ &\leqslant \sum_{k=0}^{\infty}\mathbf{P}\left(\mathbf{H}_{k+k_0+1}-\mathbf{H}_{k_0+1}\leqslant x\right)\mathbf{P}\left(\mathbf{H}_{k_0+1}\leqslant x\right)\\ &\leqslant \mathbf{R}(x)\,\mathbf{P}\left(\mathbf{H}_{k_0}\leqslant x\right).\end{aligned}$$

Combining these two bounds with (2.17) yields, for all $x, y$ large enough,

$$\mathbf{P}\left(\Lambda(x,y)\right) \geqslant \mathbf{P}\left(\mathbf{M}_1 > x+y\right)\mathbf{R}(x)\left(1-\mathbf{P}\left(\mathbf{H}_{k_0}\leqslant x\right)-\frac{\mathbf{C}_{11}k_0^2}{\left(\mathbf{R}(y)\right)^2}\right).$$

It only remains to show that for a good choice of $k_0 = k_0(x,y)$, we have

$$\mathbf{P}\left(\mathbf{H}_{k_0}\leqslant x\right)+\frac{\mathbf{C}_{11}k_0^2}{\left(\mathbf{R}(y)\right)^2}\underset{x,\frac{y}{x}\to\infty}{\longrightarrow}0.$$

Let $k_0 = \left[b^{-1}\left(a^{-1}\left(x\log\left(y/x\right)\right)\right)\right]$. Note that $k_0$ is such that $k_0\to\infty$, when $x$ and $y/x$ go to infinity simultaneously, and we know that

$$\frac{\mathbf{H}_{k_0}}{a\left(b\left(k_0\right)\right)}\underset{k_0\to\infty}{\overset{\text{law}}{\longrightarrow}} J_\infty$$

where $J_\infty$ is either a positive stable law ($\alpha q < 1$) or a strictly positive constant ($\alpha q = 1$). In either cases $\mathbf{P}\left(J_\infty = 0\right) = 0$. Since $x/a(b(k_0))\to 0$ when $x$ and $y/x$ go to infinity simultaneously, we deduce that

$$\mathbf{P}\left(\mathbf{H}_{k_0}\leqslant x\right)=\mathbf{P}\left(\frac{\mathbf{H}_{k_0}}{a\left(b\left(k_0\right)\right)}\leqslant\frac{x}{a\left(b\left(k_0\right)\right)}\right)\underset{x,\frac{y}{x}\to\infty}{\longrightarrow}0.$$

Finally, using Lemmas 2.4 and 2.12, we also check that

$$\frac{\mathbf{C}_{11}k_0^2}{\left(\mathbf{R}(y)\right)^2}\underset{x,\frac{y}{x}\to\infty}{\sim}\frac{\mathbf{C}_{11}}{\mathbf{C}_8^2}\left(\frac{\mathbf{R}\left(x\log\frac{y}{x}\right)}{\mathbf{R}(y)}\right)^2\underset{x,\frac{y}{x}\to\infty}{\longrightarrow}0.$$

□

2.5. **Other estimates.** We conclude the section about the fluctuations of $\mathbb{V}$ by collecting several results on the functionals $\overline{\mathbb{V}}$ and $\underline{\mathbb{V}}$. We start with a reflection principle for $\mathbb{V}$.

**Lemma 2.14.** *There exists $\mathbf{C}_{12}$ such that for all $v, x > 0$,*

$$\mathbf{P}\left(\overline{\mathbb{V}}_v\geqslant x\right)\leqslant\mathbf{C}_{12}\mathbf{P}\left(\mathbb{V}_v\geqslant x\right),$$

*similarly*

$$\mathbf{P}\left(\underline{\mathbb{V}}_v\leqslant -x\right)\leqslant\mathbf{C}_{12}\mathbf{P}\left(\mathbb{V}_v\leqslant -x\right).$$

*Proof.* We only need to prove the first inequality (the second inequality can be obtained in the same way, with a possibly enlarged value for $\mathbf{C}_{12}$).

$$\begin{aligned}\mathbf{P}\left(\overline{\mathbb{V}}_v \geqslant x\right) &= \mathbf{P}\left(\sigma_{\mathbb{V}}(x) \leqslant [v]\right)\\ &\leqslant \mathbf{P}\left(\sigma_{\mathbb{V}}(x) \leqslant [v]\ ,\ \mathbb{V}_{[v]} < x\right) + \mathbf{P}\left(\mathbb{V}_v \geqslant x\right)\\ &\leqslant \sum_{k=1}^{[v]} \mathbf{P}\left(\sigma_{\mathbb{V}}(x) = k\ ,\ \mathbb{V}_{[v]} < x\right) + \mathbf{P}\left(\mathbb{V}_v \geqslant x\right).\end{aligned}$$

From the Markov property, we check that $\mathbf{P}(\sigma_{\mathbb{V}}(x) = k, \mathbb{V}_{[v]} < x)$ is equal to

$$\begin{aligned}&\mathbf{P}\left(\sigma_{\mathbb{V}}(x) = k\right) \int_{y \geqslant x} \mathbf{P}\left(\mathbb{V}_{[v]-k} < x - y\right) \mathbf{P}\left(\mathbb{V}_{\sigma_{\mathbb{V}}(x)} = dy | \sigma_{\mathbb{V}}(x) = k\right)\\ &\quad \leqslant \mathbf{P}\left(\sigma_{\mathbb{V}}(x) = k\right) \mathbf{P}\left(\mathbb{V}_{[v]-k} < 0\right).\end{aligned}$$

Our assumption on $\mathbb{V}$ implies that $\lim_{n\to\infty} \mathbf{P}\left(\mathbb{V}_n < 0\right) = \mathbf{P}\left(\mathbb{S} < 0\right) = q < 1$. Thus, there exists $\mathbf{C}_{13} > 0$ such that $\sup_n \mathbf{P}\left(\mathbb{V}_n < 0\right) = \mathbf{C}_{13} < 1$. Therefore

$$\begin{aligned}\mathbf{P}\left(\overline{\mathbb{V}}_v \geqslant x\right) &\leqslant \mathbf{C}_{13} \sum_{k=1}^{[v]} \mathbf{P}\left(\sigma_{\mathbb{V}}(x) = k\right) + \mathbf{P}\left(\mathbb{V}_v \geqslant x\right)\\ &\leqslant \mathbf{C}_{13} \mathbf{P}\left(\sigma_{\mathbb{V}}(x) \leqslant v\right) + \mathbf{P}\left(\mathbb{V}_v \geqslant x\right)\\ &\leqslant \frac{1}{1 - \mathbf{C}_{13}} \mathbf{P}\left(\mathbb{V}_v \geqslant x\right).\end{aligned}$$

□

We now estimate the large deviations of $\mathbf{P}\left(\mathbb{V}_v > x\right)$. Using the characterization of the domains of attraction to a stable law (see Chapter IX, Section 8 of [10]), Assumption 1 implies

$$a^{-1}(x)\mathbf{P}\left(\mathbb{V}_1 > x\right) \underset{x\to\infty}{\longrightarrow} \begin{cases} \mathbf{C}_{14} > 0 & \text{if } \mathbb{S} \text{ has positive jumps,}\\ 0 & \text{otherwise.}\end{cases} \tag{2.18}$$

Similarly,

$$a^{-1}(x)\mathbf{P}\left(\mathbb{V}_1 < -x\right) \underset{x\to\infty}{\longrightarrow} \begin{cases} \mathbf{C}_{15} > 0 & \text{if } \mathbb{S} \text{ has negative jumps,}\\ 0 & \text{otherwise.}\end{cases} \tag{2.19}$$

**Proposition 2.15.** *There exists $\mathbf{C}_{16} > 0$ such that for all $v \geqslant 1$ and all $x \geqslant 1$,*

$$\mathbf{P}\left(\mathbb{V}_v > x\right) \leqslant \mathbf{C}_{16} \frac{v}{a^{-1}(x)}. \tag{2.20}$$

*Moreover, if $\mathbb{S}$ has positive jumps,*

$$\mathbf{P}\left(\mathbb{V}_v > x\right) \underset{\substack{v \to \infty \\ \frac{a^{-1}(x)}{v} \to \infty}}{\sim} v\mathbf{P}\left(\mathbb{V}_1 > x\right) \underset{\substack{v \to \infty \\ \frac{a^{-1}(x)}{v} \to \infty}}{\sim} \mathbf{C}_{14} \frac{v}{a^{-1}(x)}. \tag{2.21}$$

*There is of course a similar result for $\mathbf{P}\left(\mathbb{V}_v < -x\right)$.*

*Proof.* Result (2.21) is already known and is stated in [5], yet we could not find a proof of this result in English. A weaker result is proved by Heyde [11] but a slight modification of his argument will enable us to prove the proposition. Let us choose $1/2 < \delta < 1$ and set $z = (x/a(v))^{\delta} a(v)$. Define for $k \geqslant 1$,

$$\zeta_{k,z} = \begin{cases} \mathbb{V}_k - \mathbb{V}_{k-1} & \text{if } |\mathbb{V}_k - \mathbb{V}_{k-1}| \leqslant z, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\varepsilon > 0$ and define

$$\begin{aligned} \mathcal{E}_5 &= \Big\{ \mathbb{V}_k - \mathbb{V}_{k-1} > (1-\varepsilon)x \text{ for at least one } k \text{ in } \{1,\ldots,[v]\} \Big\}, \\ \mathcal{E}_6 &= \Big\{ \mathbb{V}_k - \mathbb{V}_{k-1} > z \text{ for at least two } k\text{'s in } \{1,\ldots,[v]\} \Big\}, \\ \mathcal{E}_7 &= \Big\{ \zeta_{1,z} + \ldots + \zeta_{[v],z} > \varepsilon x \Big\}. \end{aligned}$$

We see that $\{\mathbb{V}_v > x\} \subset \mathcal{E}_5 \cap \mathcal{E}_6 \cap \mathcal{E}_7$, hence

$$\mathbf{P}(\mathbb{V}_v > x) \leqslant \mathbf{P}(\mathcal{E}_5) + \mathbf{P}(\mathcal{E}_6) + \mathbf{P}(\mathcal{E}_7). \tag{2.22}$$

We deal with each term on the r.h.s. of (2.22) separately. Let us choose $C > \mathbf{C}_{14}$ if $\mathbb{S}$ has positive jumps and set $C = 1$ otherwise. We now assume that $v$ and $a^{-1}(x)/v$ are very large. According to (2.18) and using the regular variation of $a^{-1}(\cdot)$, we get

$$\mathbf{P}(\mathcal{E}_5) \leqslant v\mathbf{P}(\mathbb{V}_1 > (1-\varepsilon)x) \leqslant \frac{C}{(1-\varepsilon)^{\alpha}} \frac{v}{a^{-1}(x)}. \tag{2.23}$$

We now deal with $\mathbf{P}(\mathcal{E}_6)$. Let $\eta > 0$. Lemma 2.4 gives for all $v$ and $a^{-1}(x)/v$ large enough,

$$\begin{aligned} \frac{va^{-1}(x)}{(a^{-1}(z))^2} &= \frac{a^{-1}\left(a(v)\frac{x}{a(v)}\right)}{a^{-1}(a(v))} \left( \frac{a^{-1}(a(v))}{a^{-1}\left(a(v)\left(\frac{x}{a(v)}\right)^{\delta}\right)} \right)^2 \\ &\leqslant \left(\frac{x}{a(v)}\right)^{\alpha+\eta} \left(\frac{a(v)}{x}\right)^{2\delta(\alpha-\eta)}. \end{aligned}$$

Since $\delta > 1/2$, we can assume $\eta$ small enough such that $2\delta(\alpha-\eta)-(\alpha+\eta) > \eta$. Thus, we have

$$\frac{va^{-1}(x)}{(a^{-1}(z))^2} \leqslant \left(\frac{a(v)}{x}\right)^{\eta}, \tag{2.24}$$

therefore, using (2.18) and (2.24), we get

$$\mathbf{P}(\mathcal{E}_6) \leqslant v^2 \mathbf{P}(\mathbb{V}_1 > z)^2 \leqslant C\frac{v^2}{(a^{-1}(z))^2} \leqslant C\frac{v}{a^{-1}(x)} \left(\frac{a(v)}{x}\right)^{\eta}. \tag{2.25}$$

Turning our attention to $\mathbf{P}(\mathcal{E}_7)$, we deduce from Tchebychev's inequality that

$$\mathbf{P}(\mathcal{E}_7) \leqslant \frac{1}{\varepsilon^2 x^2}\mathbf{E}\left((\zeta_{1,z} + \ldots + \zeta_{[v],z})^2\right) \leqslant \frac{v}{\varepsilon^2 x^2}\mathbf{E}\left(\zeta_{1,z}^2\right) + \frac{v^2}{\varepsilon^2 x^2}\mathbf{E}(\zeta_{1,z})^2. \tag{2.26}$$

Let $f(z) = \mathbf{E}\left((\zeta_{1,z})^2\right) = \int_{-z}^{z} y^2 \mathbf{P}(\mathbb{V}_1 \in dy)$. This function is non-decreasing and non-zero for $z$ large enough. It is also known from the characterization of the domains of attraction (c.f. (8.14) of [10], p304) that the norming constants $(a_n)$ are such that $nf(a_n)/a_n^2 \to \mathbf{C}_{17} > 0$, hence $f(z) \sim \mathbf{C}_{17} z^2/a^{-1}(z)$ as $z$ goes to infinity ($f$ is regularly varying with index $2-\alpha$). Therefore, for $v$ and $a^{-1}(x)/v$ large enough, we have

$$\frac{v}{\varepsilon^2 x^2}\mathbf{E}\left((\zeta_{1,z})^2\right) = \frac{vf(z)}{\varepsilon x^2} \leqslant \mathbf{C}_{18,\varepsilon}\frac{v}{a^{-1}(x)}\frac{f(z)}{f(x)} \leqslant \mathbf{C}_{18,\varepsilon}\frac{v}{a^{-1}(x)}. \tag{2.27}$$

We can sharpen this estimate when $\alpha < 2$. Indeed, in this case, $f$ is regularly varying with index $2-\alpha > 0$. Thus, using Lemma 2.4 and setting $\eta' = (1-\delta)(2-\alpha)/2$,

$$\frac{f(z)}{f(x)} \leqslant \left(\frac{z}{x}\right)^{(2-\alpha)/2} = \left(\frac{a(v)(x/a(v))^{\delta}}{x}\right)^{(2-\alpha)/2} = \left(\frac{a(v)}{x}\right)^{\eta'}.$$

When $\alpha < 2$, we therefore obtain

$$\frac{v}{\varepsilon^2 x^2}\mathbf{E}\left((\zeta_{1,z}\right) \leqslant \mathbf{C}_{18,\varepsilon}\frac{v}{a^{-1}(x)}\left(\frac{a(v)}{x}\right)^{\eta'}. \tag{2.28}$$

Let $g(z) = \mathbf{E}(\zeta_{1,z}) = \int_{-z}^{z} y\mathbf{P}(\mathbb{V}_1 \in dy)$. Since $\mathbb{V}_1$ is in the domain of attraction of a stable law, it is known that the centering constants $c(n)$ such that $\mathbb{V}_n/a(n) - c(n)$ converge to a stable law may be chosen to be $c(n) = ng(a(n))/a(n)$ (see [10], p305), but the main assumption of this paper states that the sequence $c(n)$ may also be chosen to be identically 0. This implies in particular that the sequence $ng(a(n))/a(n)$ is bounded. So we deduce that there exists $\mathbf{C}_{19} > 0$ such that

$$|g(z)| \leqslant \mathbf{C}_{19}\frac{z}{a^{-1}(z)} \text{ for all } z \geqslant 1.$$

Using this inequality, we get for $v$ and $a^{-1}(x)/v$ large enough,

$$\begin{aligned}
\frac{v^2}{\varepsilon^2 x^2}\mathbf{E}(\zeta_{1,z})^2 &\leqslant \mathbf{C}_{20,\varepsilon}\frac{v^2 z^2}{x^2(a^{-1}(z))^2} \\
&= \mathbf{C}_{20,\varepsilon}\frac{v}{a^{-1}(x)}\frac{va^{-1}(x)}{(a^{-1}(z))^2}\left(\frac{z}{x}\right)^2 \\
&\leqslant \mathbf{C}_{20,\varepsilon}\frac{v}{a^{-1}(x)}\frac{va^{-1}(x)}{(a^{-1}(z))^2} \\
&\leqslant \mathbf{C}_{20,\varepsilon}\frac{v}{a^{-1}(x)}\left(\frac{a(v)}{x}\right)^{\eta},
\end{aligned} \tag{2.29}$$

where we used (2.24) for the last inequality. Putting the pieces together, (2.22)-(2.23)-(2.25)-(2.26)-(2.27) and (2.29) yield (2.20). Moreover, when $\mathbb{S}$ has positive jumps, we have $\alpha < 2$, hence we can use (2.28) instead of (2.27) and we deduce that

$$\limsup_{\substack{v\to\infty\\ \frac{a^{-1}(x)}{v}\to\infty}} \frac{a^{-1}(x)\mathbf{P}\left(\mathbb{V}_v > x\right)}{v} \leqslant \mathbf{C}_{14}.$$

It remains to prove that the lower bound holds. Assume that $\mathbb{S}$ has positive jumps and notice that the event $\{\mathbb{V}_v > x\}$ contains

$$\bigcap_{k=0}^{[v]-1} \Big\{ \mathbb{V}^*_k \leqslant \varepsilon x \ , \ \mathbb{V}_{k+1} - \mathbb{V}_k > (1+2\varepsilon)x \ , \ (\theta_{k+1}\mathbb{V})^*_{[v]-k-1} \leqslant \varepsilon x \Big\}.$$

Moreover, the events of the last formula are disjoints. The independence and the stationarity of the increments of the random walk $\mathbb{V}$ yield

$$\begin{aligned} \mathbf{P}\left(\mathbb{V}_v > x\right) &\geqslant \sum_{k=0}^{[v]-1} \mathbf{P}\Big(\mathbb{V}^*_k \leqslant \varepsilon x\Big)\mathbf{P}\Big(\mathbb{V}_1 > (1+2\varepsilon)x\Big)\mathbf{P}\Big(\mathbb{V}^*_{[v]-k-1} \leqslant \varepsilon x\Big) \\ &\geqslant [v]\mathbf{P}\Big(\mathbb{V}^*_v \leqslant \varepsilon x\Big)^2 \mathbf{P}\Big(\mathbb{V}_1 > (1+2\varepsilon)x\Big). \end{aligned}$$

From (2.18) and the regular variation of $a^{-1}(\cdot)$ we see that

$$[v]\mathbf{P}\left(\mathbb{V}_1 > (1+2\varepsilon)x\right) \sim \frac{\mathbf{C}_{14} v}{(1+2\varepsilon)^{\alpha} a^{-1}(x)}$$

as $v$ and $a^{-1}(x)/v$ both go to infinity. We also know from the results of Section 2.1 that $\mathbb{V}^*_v/a(v)$ converges in law towards $\mathbb{S}^*_1$, therefore

$$\lim_{\substack{v\to\infty\\ \frac{a^{-1}(x)}{v}\to\infty}} \mathbf{P}\Big(\mathbb{V}^*_v \leqslant \varepsilon x\Big) = \lim_{\substack{v\to\infty\\ \frac{a^{-1}(x)}{v}\to\infty}} \mathbf{P}\left(\frac{\mathbb{V}^*_v}{a(v)} \leqslant \varepsilon \frac{x}{a(v)}\right) = 1.$$

We conclude that

$$\liminf_{\substack{v\to\infty\\ \frac{a^{-1}(x)}{v}\to\infty}} \frac{a^{-1}(x)\mathbf{P}\left(\mathbb{V}_v > x\right)}{v} \geqslant \frac{\mathbf{C}_{14}}{(1+2\varepsilon)^{\alpha}}.$$

□

**Corollary 2.16.** *By possibly extending the value of $\mathbf{C}_{16}$, the equation (2.20) also holds with $\overline{\mathbb{V}}_v$, $-\underline{\mathbb{V}}_v$, $\mathbb{V}^{\#}_v$ and $\mathbb{V}^*_v$ in place of $\mathbb{V}_v$.*

*Proof.* The results for $\overline{\mathbb{V}}_v$ and $-\underline{\mathbb{V}}_v$ are straightforward using Lemma 2.14. As for $\mathbb{V}^*$ and $\mathbb{V}^{\#}$, simply notice that $\{\mathbb{V}^{\#}_v \geqslant 2x\} \subset \{\mathbb{V}^*_v \geqslant x\} \subset \{\overline{\mathbb{V}}_v \geqslant x\} \cup \{-\underline{\mathbb{V}}_v \geqslant x\}$. □

**Corollary 2.17.** *For any $0 < \delta < \alpha$, we have*

$$\lim_{v\to\infty} \mathbf{E}\left(\left(\frac{\overline{\mathbb{V}}_v}{a(v)}\right)^{\delta}\right) = \mathbf{E}\left(\left(\overline{\mathbb{S}}_1\right)^{\delta}\right) \ \textit{and} \ \lim_{v\to\infty} \mathbf{E}\left(\left|\frac{\underline{\mathbb{V}}_v}{a(v)}\right|^{\delta}\right) = \mathbf{E}\left(\left(-\underline{\mathbb{S}}_1\right)^{\delta}\right).$$

*Proof.* It follows from the last corollary and the regular variation of $a^{-1}(\cdot)$ with index $\alpha$ that for any $0 < \delta < \alpha$,

$$\sup_{v \geqslant 1} \mathbf{E}\left(\left(\frac{\overline{\mathbb{V}}_v}{a(v)}\right)^{\delta}\right) < \infty.$$

The family $\left((\overline{\mathbb{V}}_v/a(v))^{\delta}, v \geqslant 1\right)$ is therefore uniformly integrable for all $0 < \delta < \alpha$. We also know that $\overline{\mathbb{V}}_v/a(v)$ converges in law toward $\overline{\mathbb{S}}_1$ as $v$ goes to infinity. These two facts combined yield the first assertion. The proof of the second part of the corollary is similar. □

**Proposition 2.18.** *For all $0 < \delta < q$ (recall that $q$ is the negativity parameter of $\mathbb{S}$) there exists $\mathbf{C}_{21,\delta}$ such that, for all $v, x \geqslant 1$,*

$$\mathbf{P}\left(-\underline{\mathbb{V}}_v \leqslant x\right) \leqslant \mathbf{C}_{21,\delta}\left(\frac{a^{-1}(x)}{v}\right)^{\delta}.$$

*We have a similar result for $\mathbf{P}\left(\overline{\mathbb{V}}_v \leqslant x\right)$ when changing the condition $\delta < q$ by $\delta < p$.*

*Proof.* We only prove the result for $\underline{\mathbb{V}}_v$. By possibly extending the value of $\mathbf{C}_{21,\delta}$, it suffices to prove the inequality for $x$ and $v/a^{-1}(x)$ large enough. Let us choose $\delta'$ such that $\delta < \delta' < q < 1$ and notice that for any $y > 0$,

$$\begin{aligned}\{-\underline{\mathbb{V}}_v \leqslant x\} &\subset \Lambda(x,y) \cup (\{-\underline{\mathbb{V}}_v \leqslant x\} \cap \Lambda(x,y)^c) \\ &\subset \Lambda(x,y) \cup \left\{\mathbb{V}_v^{\#} \leqslant x + y\right\},\end{aligned}$$

thus

$$\mathbf{P}\left(-\underline{\mathbb{V}}_v \leqslant x\right) \leqslant \mathbf{P}\left(\Lambda\left(x,y\right)\right) + \mathbf{P}\left(\mathbb{V}_v^{\#} \leqslant x + y\right). \tag{2.30}$$

On the one hand, for $x$ and $y/x$ large enough, using Proposition 2.13 and Lemma 2.4, we get

$$\begin{aligned}\mathbf{P}\left(\Lambda\left(x,y\right)\right) &\leqslant \mathbf{C}_{22}\frac{b^{-1}\left(a^{-1}(x)\right)}{b^{-1}\left(a^{-1}(x+y)\right)} \\ &\leqslant \mathbf{C}_{23,\delta'}\left(\frac{a^{-1}(x)}{a^{-1}(x+y)}\right)^{\delta'}.\end{aligned} \tag{2.31}$$

On the other hand, for $x+y$ and $v/a^{-1}(x+y)$ large enough, using Proposition 2.5, we obtain

$$\mathbf{P}\left(\mathbb{V}_v^{\#} \leqslant x + y\right) \leqslant \exp\left(-\frac{K^{\#}}{2}\frac{v}{a^{-1}(x+y)}\right). \tag{2.32}$$

Let us choose $y = a\left(\frac{K^{\#}v}{2\log(v/a^{-1}(x))}\right) - x$. It is easy to check that (2.31) and (2.32) hold whenever $x$ and $v/a^{-1}(x)$ are large enough, thus, (2.30) yields

$$\begin{aligned}\mathbf{P}\left(-\underline{\mathbb{V}}_v \leqslant x\right) &\leqslant \mathbf{C}_{23,\delta'}\left(\frac{2}{K^{\#}}\right)^{\delta'}\left(\frac{a^{-1}(x)}{v}\left(\log\frac{v}{a^{-1}(x)}\right)\right)^{\delta'} + \frac{a^{-1}(x)}{v} \\ &\leqslant \mathbf{C}_{24,\delta'}\left(\frac{a^{-1}(x)}{v}\right)^{\delta}.\end{aligned}$$

□

## 3. Behavior of $X$

In this section, we now study the diffusion $X$ in the random potential $\mathbb{V}$. We will see that the behavior of this process depends strongly on the environment. In order to do so, we will adapt the ideas of Hu and Shi [12] to our setting, in particular, we will show that Lemma 4.1 and Lemma 4.2 of [12] still hold with a slight modification.

Recall the well known representation of $X$ (c.f. [6, 12, 13]) which states that we can construct $X$ from a Brownian motion through a (random) change of scale and a (random) change of time, hence we will assume that $X$ has the form:

$$X_t = \mathbb{A}^{-1}\left(B_{\mathbb{T}^{-1}(t)}\right) \tag{3.1}$$

where $B$ is a standard Brownian motion independent of $\mathbb{V}$ and where $\mathbb{A}^{-1}$ and $\mathbb{T}^{-1}$ are the respective inverses of

$$\begin{aligned}\mathbb{A}(x) &= \int_0^x e^{\mathbb{V}_y}dy \qquad \text{for } x \in \mathbb{R}, \\ \mathbb{T}(t) &= \int_0^t e^{-2\mathbb{V}_{\mathbb{A}^{-1}(B_s)}}ds \qquad \text{for } t \geqslant 0.\end{aligned}$$

Note that our assumption on $\mathbb{V}$ implies with probability 1 that $\mathbb{A}$ is an increasing homeomorphism on $\mathbb{R}$ and that $\mathbb{T}$ is an increasing homeomorphism on $\mathbb{R}_+$, thus $\mathbb{A}^{-1}$ and $\mathbb{T}^{-1}$ are well defined. Let $v > 0$ and recall the definition of $\sigma$ given in Section 2.1. Using (3.1), we have

$$\sigma_X(v) = \mathbb{T}\left(\sigma_B\left(\mathbb{A}(v)\right)\right).$$

Let $(L(t,x), t \geqslant 0, x \in \mathbb{R})$ stand for the bi-continuous version of the local time process of $B$. The last equality may be rewritten:

$$\begin{aligned}\sigma_X(v) &= \int_0^{\sigma_B(\mathbb{A}(v))} e^{-2\mathbb{V}_{\mathbb{A}^{-1}(B_s)}}ds \\ &= \int_{-\infty}^{\mathbb{A}(v)} e^{-2\mathbb{V}_{\mathbb{A}^{-1}(x)}}L(\sigma_B(\mathbb{A}(v)), x)dx \\ &= \int_{-\infty}^{v} e^{-\mathbb{V}_y}L(\sigma_B(\mathbb{A}(v)), \mathbb{A}(y))dy\end{aligned}$$

where we have used the change of variable $x = \mathbb{A}(y)$. Let us now define $I_1$ and $I_2$,

$$I_1(v) = \int_0^v e^{-\mathbb{V}_y} L(\sigma_B(\mathbb{A}(v)), \mathbb{A}(y))dy, \tag{3.2}$$

$$I_2(v) = \int_0^\infty e^{-\mathbb{V}_{-y}} L(\sigma_B(\mathbb{A}(v)), \mathbb{A}(-y))dy. \tag{3.3}$$

Using the definition of $\sigma_X$, we get

$$\left\{\overline{X}_t \geqslant v\right\} = \{I_1(v) + I_2(v) \leqslant t\}. \tag{3.4}$$

The next two propositions show the connection between $\mathbb{V}$ and $X$. These estimates will enable us to reduce the study of the limiting behavior of $X$ to the study of some functionals of the potential $\mathbb{V}$. The streamline of the proofs is the same as that of Lemmas 4.1 and 4.2 of [12] and one should refer to the proof of these two lemmas for further details.

**Proposition 3.1.** *There exists* $\mathbf{C}_{25}$ *such that for all* $v$ *large enough*

$$\mathbb{V}^{\#}_{v-\frac{1}{2}} - (\log v)^4 \leqslant \log I_1(v) \leqslant \mathbb{V}^{\#}_v + (\log v)^4 \quad \text{on } \mathcal{E}_8(v),$$

*where* $\mathcal{E}_8(v)$ *is a measurable set such that*

$$\mathbf{P}\left(\mathcal{E}_8(v)^c\right) \leqslant \mathbf{C}_{25} e^{-(\log v)^2}.$$

**Proposition 3.2.** *There exists* $\mathbf{C}_{26}$ *such that for all* $v$ *large enough*

$$\log I_2(v) \leqslant \widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + (\log v)^4\right) \quad \text{on } \mathcal{E}_9(v),$$

$$\log I_2(v) \geqslant \widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}_{v-\frac{1}{2}} - (\log v)^4\right) \quad \text{on } \mathcal{E}_9(v) \cap \left\{\overline{\mathbb{V}}_{v-\frac{1}{2}} > (\log v)^4\right\},$$

*where* $\widetilde{U}$ *was defined in Section 2.1 and where* $\mathcal{E}_9(v)$ *is a measurable set such that*

$$\mathbf{P}\left(\mathcal{E}_9(v)^c\right) \leqslant \mathbf{C}_{26} e^{-(\log v)^2}.$$

*Proof of Proposition 3.1.* For $v > 0$, let $\mathcal{R}^2$ be defined as:

$$\mathcal{R}^2(t) = \frac{L\left(\sigma_B(\mathbb{A}(v)), \mathbb{A}(v) - t\mathbb{A}(v)\right)}{\mathbb{A}(v)} \qquad \text{for } 0 \leqslant t \leqslant 1.$$

Let $\mathcal{R}$ be the positive root of $\mathcal{R}^2$. Just as in [12], p1498, we see that, using Ray-Knight Theorem and the scaling property of the Brownian motion, for any fixed $v$ the process $(\mathcal{R}(t), 0 \leqslant t \leqslant 1)$ has the law of a two dimensional Bessel process starting from 0. Moreover, $\mathcal{R}$ is independent of $\mathbb{V}$. We can now rewrite (3.2) as

$$I_1(v) = \mathbb{A}(v) \int_0^v e^{-\mathbb{V}_s} \mathcal{R}^2\left(\frac{\mathbb{A}(v) - \mathbb{A}(s)}{\mathbb{A}(v)}\right) ds.$$

Let us define

$$\mathcal{E}_{10} = \left\{\sup_{0<t\leqslant 1} \frac{\mathcal{R}(t)}{\sqrt{t\log(8/t)}} \leqslant \sqrt{v}\right\}.$$

Using Lemma 6.1 p1497 of [12], we get $\mathbf{P}(\mathcal{E}_{10}^c) \leqslant \mathbf{C}_{27}e^{-v/2}$. On $\mathcal{E}_{10}$, we have

$$I_1(v) \leqslant v \int_0^v e^{-\mathbb{V}_s} (\mathbb{A}(v) - \mathbb{A}(s)) \log\left(\frac{8\mathbb{A}(v)}{\mathbb{A}(v) - \mathbb{A}(s)}\right) ds,$$

and for all $s \leqslant v$

$$e^{-\mathbb{V}_s}(\mathbb{A}(v) - \mathbb{A}(s)) = \int_s^v e^{\mathbb{V}_y - \mathbb{V}_s} dy \leqslant v e^{\mathbb{V}_v^\#}.$$

This implies

$$I_1(v) \leqslant v^2 e^{\mathbb{V}_v^\#} \int_0^v \log\left(\frac{8\mathbb{A}(v)}{\mathbb{A}(v) - \mathbb{A}(s)}\right) ds. \tag{3.5}$$

We also have

$$\mathbb{A}(v) = \int_0^v e^{\mathbb{V}_s} ds \leqslant v e^{\overline{\mathbb{V}}_v} \quad \text{and} \quad \mathbb{A}(v) - \mathbb{A}(s) = \int_s^v e^{\mathbb{V}_y} dy \geqslant (v-s)e^{\underline{\mathbb{V}}_v},$$

thus

$$\begin{aligned}\int_0^v \log\left(\frac{8\mathbb{A}(v)}{\mathbb{A}(v) - \mathbb{A}(s)}\right) ds &\leqslant v\left(\overline{\mathbb{V}}_v - \underline{\mathbb{V}}_v\right) + \int_0^v \log\left(\frac{8v}{v-s}\right) ds \\ &\leqslant v\left(\overline{\mathbb{V}}_v - \underline{\mathbb{V}}_v + 1 + \log(8)\right).\end{aligned}$$

Combining this with (3.5) yields $\log(I_1(v)) \leqslant \mathbb{V}_v^\# + \log\left(\overline{\mathbb{V}}_v - \underline{\mathbb{V}}_v\right) + 4\log(v)$ for all $v$ large enough. We now define $\mathcal{E}_{11} = \left\{\log\left(\overline{\mathbb{V}}_v - \underline{\mathbb{V}}_v\right) \leqslant \log^3(v)\right\}$. On $\mathcal{E}_{10} \cap \mathcal{E}_{11}$, for all $v$ large enough, we get the upper bound,

$$\log(I_1(v)) \leqslant \mathbb{V}_v^\# + \log^4(v).$$

Notice that $\{\overline{\mathbb{V}}_v - \underline{\mathbb{V}}_v > a\} \subset \{\mathbb{V}_v^* > a/2\}$. Thus, using Corollary 2.16 and the regular variation of $a^{-1}(\cdot)$, it is easily checked that $\mathbf{P}(\mathcal{E}_{11}^c) \leqslant \exp(-\log^2(v))$ for any $v$ large enough. We now prove the existence of the lower bound. For the sake of clarity, we will use the notation $l = \log(v)$ and $\delta = \exp(-l^2)$. For $v > 1/2$, there exist two integers $0 \leqslant k^- \leqslant k^+ \leqslant v - \frac{1}{2}$ such that $\mathbb{V}_{v-\frac{1}{2}}^\# = \mathbb{V}_{k^+} - \mathbb{V}_{k^-}$. Let us define the sets:

$$\begin{aligned}\mathcal{E}_{12} &= \left\{\inf_{k^- \leqslant s \leqslant k^- + \frac{1}{2}} \mathcal{R}\left(\frac{\mathbb{A}(v) - \mathbb{A}(s)}{\mathbb{A}(v)}\right) > \delta\sqrt{\frac{\mathbb{A}(v) - \mathbb{A}(k^-)}{\mathbb{A}(v)}}\right\}, \\ \mathcal{E}_{13} &= \left\{\mathbb{V}_{v-\frac{1}{2}}^\# \geqslant 3l^2\right\}.\end{aligned}$$

Using again Lemma 6.1 p1497 of [12] combined with the independence of $\mathcal{R}$ and $\mathbb{V}$, we get

$$\mathbf{P}\left((\mathcal{E}_{12} \cap \mathcal{E}_{13})^c\right) \leqslant \mathbf{P}(\mathcal{E}_{13}^c) + 2\delta + 2\mathbf{E}\left(e^{-\frac{\delta^2}{2}J(v)}\mathbf{1}_{\mathcal{E}_{13}}\right), \tag{3.6}$$

where $J$ is given by

$$J(v) = \frac{\mathbb{A}(v) - \mathbb{A}(k^-)}{\mathbb{A}\left(k^- + \frac{1}{2}\right) - \mathbb{A}(k^-)}.$$

On the one hand, we have

$$\mathbb{A}(v)-\mathbb{A}(k^-)=\int_{k^-}^{v}e^{\mathbb{V}_s}ds\geqslant\int_{k^+}^{k^++\frac{1}{2}}e^{\mathbb{V}_s}ds=\frac{1}{2}e^{\mathbb{V}_{k^+}}.$$

On the other hand, since $k^-$ is an integer and $\mathbb{V}$ is flat on $[k^-,k^-+1)$, we also have

$$\mathbb{A}\left(k^-+\frac{1}{2}\right)-\mathbb{A}(k^-)=\int_{k^-}^{k^-+\frac{1}{2}}e^{\mathbb{V}_s}ds=\frac{1}{2}e^{\mathbb{V}_{k^-}}.$$

This implies $J(v)\geqslant\exp(\mathbb{V}^{\#}_{v-1/2})$. Using this inequality combined with (3.6), we get

$$\mathbf{P}\left((\mathcal{E}_{12}\cap\mathcal{E}_{13})^c\right)\leqslant\mathbf{P}\left(\mathcal{E}_{13}^c\right)+2\delta+2\exp(-\delta^2\exp(3l^2)/2).$$

Hence, we have $\mathbf{P}\left((\mathcal{E}_{12}\cap\mathcal{E}_{13})^c\right)\leqslant\mathbf{P}\left(\mathcal{E}_{13}^c\right)+3\exp(-l^2)$ for all $v$ large enough. Using Proposition 2.5, it is easily seen that $\mathbf{P}\left(\mathcal{E}_{13}^c\right)\leqslant e^{-l^2}$ for all large enough $v$'s. Let us finally set $\mathcal{E}_8=\mathcal{E}_{10}\cap\mathcal{E}_{11}\cap\mathcal{E}_{12}\cap\mathcal{E}_{13}$. We have proved that there exists $\mathbf{C}_{25}>0$ such that $\mathbf{P}\left(\mathcal{E}_8^c\right)\leqslant\mathbf{C}_{25}\exp(-l^2)$. Notice that

$$\begin{aligned}I_1(v)&=\mathbb{A}(v)\int_0^v e^{-\mathbb{V}_s}\mathcal{R}^2\left(\frac{\mathbb{A}(v)-\mathbb{A}(s)}{\mathbb{A}(v)}\right)ds\\&\geqslant\mathbb{A}(v)e^{-\mathbb{V}_{k^-}}\int_{k^-}^{k^-+\frac{1}{4}}\mathcal{R}^2\left(\frac{\mathbb{A}(v)-\mathbb{A}(s)}{\mathbb{A}(v)}\right)ds,\end{aligned}$$

therefore, on $\mathcal{E}_8$,

$$I_1(v)\geqslant\delta^2e^{-\mathbb{V}_{k^-}}\int_{k^-}^{k^-+\frac{1}{4}}\left(\mathbb{A}(v)-\mathbb{A}(s)\right)ds,$$

but for all $s$ such that $k^-\leqslant s\leqslant k^-+\frac{1}{4}$ we also have

$$\mathbb{A}(v)-\mathbb{A}(s)\geqslant\mathbb{A}(v)-\mathbb{A}\left(k^-+\frac{1}{4}\right)=\int_{k^-+\frac{1}{4}}^{v}e^{\mathbb{V}_y}dy\geqslant\int_{k^++\frac{1}{4}}^{k^++\frac{1}{2}}e^{\mathbb{V}_y}dy=\frac{1}{4}e^{\mathbb{V}_{k^+}},$$

hence

$$\int_{k^-}^{k^-+\frac{1}{4}}\left(\mathbb{A}(v)-\mathbb{A}(s)\right)ds\geqslant\frac{1}{16}e^{\mathbb{V}_{k^+}}.$$

We finally get

$$I_1(v)\geqslant\frac{\delta^2}{16}e^{\mathbb{V}^{\#}_{v-\frac{1}{2}}}\quad\text{on }\mathcal{E}_8.$$

We conclude the proof of the proposition by taking the logarithm. □

*Proof of Proposition 3.2.* For $v>0$, we define the process $\mathcal{Z}$ by

$$\mathcal{Z}(t)=\frac{L\left(\sigma_B\left(\mathbb{A}(v)\right),-t\mathbb{A}(v)\right)}{\mathbb{A}(v)}\qquad\text{for }t\geqslant 0.$$

Using Ray-Knight Theorem and the scaling property of the Brownian motion, we see that for any fixed $v$ the process $\mathcal{Z}$ has the law of a squared Bessel process of dimension 0 such that $\mathcal{Z}(0)$ has an exponential distribution with mean 2. Moreover, $\mathcal{Z}$ is independent of $\mathbb{V}$. We can now rewrite (3.3):

$$I_2(v) = \mathbb{A}(v) \int_0^{\infty} e^{-\mathbb{V}_{-s}} \mathcal{Z}\left(\frac{-\mathbb{A}(-s)}{\mathbb{A}(v)}\right) ds.$$

We know that 0 is an absorbing state for $\mathcal{Z}$. Let $\zeta = \inf(s \geqslant 0\ ,\ \mathcal{Z}_s = 0)$ be the absorption time of $\mathcal{Z}$ and let us also define

$$\zeta(v) = \inf\left(s \geqslant 0\ ,\ \mathcal{Z}\left(\frac{-\mathbb{A}(-s)}{\mathbb{A}(v)}\right) = 0\right).$$

We can now write

$$I_2(v) = \mathbb{A}(v) \int_0^{\zeta(v)} e^{-\mathbb{V}_{-s}} \mathcal{Z}\left(\frac{-\mathbb{A}(-s)}{\mathbb{A}(v)}\right) ds.$$

We keep the notation $l = \log(v)$, note that $\mathbb{A}(v) = \int_0^v e^{\mathbb{V}_s} ds \leqslant \exp(\overline{\mathbb{V}}_v + l)$, therefore

$$\begin{aligned} I_2(v) &\leqslant e^{\overline{\mathbb{V}}_v + l} \zeta(v) \sup_{0 \leqslant s \leqslant \zeta(v)} \left(e^{-\mathbb{V}_{-s}}\right) \sup_{s \geqslant 0} \mathcal{Z}(s) \\ &\leqslant \zeta(v) \sup_{s \geqslant 0} \mathcal{Z}(s) e^{l + \overline{\mathbb{V}}(v) - \underline{\mathbb{V}}(-\zeta(v))}. \end{aligned}$$

Let us define $\mathcal{E}_{14} = \{\sup_{s \geqslant 0} \mathcal{Z}(s) \leqslant \exp(l^2)\}$. Using Lemma 7.1, p1501 of [12], we get $\mathbf{P}(\mathcal{E}_{14}^c) \leqslant 4\exp(-l^2)$. Thus, on $\mathcal{E}_{14}$, we have

$$I_2(v) \leqslant \zeta(v) e^{2l^2 + \overline{\mathbb{V}}(v) - \underline{\mathbb{V}}(-\zeta(v))}. \tag{3.7}$$

Let $\mathcal{E}_{15} = \left\{\zeta(v) \leqslant \widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + l^4\right) + \frac{1}{2}\right\}$ and notice that for all $a \geqslant 0$,

$$\{\zeta(v) > a\} = \left\{\frac{-\mathbb{A}(-a)}{\mathbb{A}(v)} < \zeta\right\}.$$

Therefore

$$\mathbf{P}\left(\mathcal{E}_{15}^c\right) = \mathbf{P}\left(\frac{-\mathbb{A}\left(-\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + l^4\right) - \frac{1}{2}\right)}{\mathbb{A}(v)} < \zeta\right),$$

but

$$-\mathbb{A}\left(-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4) - \frac{1}{2}\right) \geqslant \int_{-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4) - \frac{1}{2}}^{-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4)} e^{\mathbb{V}_s} ds \geqslant \frac{1}{2} e^{\overline{\mathbb{V}}_v + l^4},$$

and we have already seen that $\mathbb{A}_v \leqslant \exp(\overline{\mathbb{V}}_v + l)$. Combining this two inequalities yields for all $v$ large enough,

$$\frac{-\mathbb{A}\left(-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4) - \frac{1}{2}\right)}{\mathbb{A}(v)} \geqslant e^{l^3},$$

hence

$$\mathbf{P}\left(\mathcal{E}_{15}^c\right) \leqslant \mathbf{P}\left(\zeta > e^{l^3}\right) \leqslant e^{-l^3},$$

where we have used Lemma 7.1 on p1501 of [12] for the last inequality. On $\mathcal{E}_{14} \cap \mathcal{E}_{15}$, for $v$ large enough, we deduce from (3.7) that

$$I_2(v) \leqslant \zeta(v) e^{2l^2 + \overline{\mathbb{V}}(v) - \underline{\mathbb{V}}(-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4) + \frac{1}{2})}.$$

But $\overline{\mathbb{V}}(v) - \underline{\mathbb{V}}(-\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + l^4\right) + \frac{1}{2}) = \widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4) - l^4$ (recall that $\mathbb{V}$ is flat on $(-n-1, -n]$ , $n \in N$). Therefore, on $\mathcal{E}_{14} \cap \mathcal{E}_{15}$,

$$I_2(v) \leqslant \zeta(v) e^{-l^3 + \widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4)}.$$

Let $\mathcal{E}_{16} = \{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4) + \frac{1}{2} \leqslant \exp(l^3)\}$. On $\mathcal{E}_{17} = \mathcal{E}_{14} \cap \mathcal{E}_{15} \cap \mathcal{E}_{16}$, we have $\zeta(v) \leqslant \exp(l^3)$. Hence, on $\mathcal{E}_{17}$ and for all $v$ large enough,

$$\log\left(I_2(v)\right) \leqslant \widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + \log^4 v\right).$$

This gives the upper bound on $\mathcal{E}_{17}$. Let us check that

$$\mathbf{P}\left(\mathcal{E}_{16}^c\right) \leqslant \mathbf{C}_{28} \exp(-l^2). \tag{3.8}$$

We have $\mathbf{P}(\mathcal{E}_{16}^c) \leqslant \mathbf{P}\left(\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4) > \exp(l^3)/2\right)$, thus

$$\mathbf{P}\left(\mathcal{E}_{16}^c\right) \leqslant \mathbf{P}\left(\overline{\mathbb{V}}(-\frac{1}{2}e^{l^3}) \leqslant 2\overline{\mathbb{V}}(v)\right) + \mathbf{P}\left(\overline{\mathbb{V}}(-\frac{1}{2}e^{l^3}) \leqslant 2l^4\right).$$

We also have

$$\mathbf{P}\left(\overline{\mathbb{V}}(-\frac{1}{2}e^{l^3}) \leqslant 2\overline{\mathbb{V}}(v)\right) \leqslant \mathbf{P}\left(\overline{\mathbb{V}}(-\frac{1}{2}e^{l^3}) \leqslant e^{l^{5/2}}\right) + \mathbf{P}\left(\overline{\mathbb{V}}(v) > \frac{1}{2}e^{l^{5/2}}\right).$$

Using Corollary 2.16 and the regular variation of $a^{-1}(\cdot)$, for all $v$ large enough,

$$\mathbf{P}\left(\overline{\mathbb{V}}(v) > \frac{1}{2}e^{l^{5/2}}\right) \leqslant e^{-l^2}.$$

Recall that $(\mathbb{V}(x), x \geqslant 0)$ and $(-\mathbb{V}(-x), x \geqslant 0)$ have the same law, thus Proposition 2.18 yields

$$\mathbf{P}\left(\overline{\mathbb{V}}(-\frac{1}{2}e^{l^3}) \leqslant 2l^4\right) \leqslant \mathbf{P}\left(\overline{\mathbb{V}}(-\frac{1}{2}e^{l^3}) \leqslant e^{l^{5/2}}\right) \leqslant e^{-l^2}.$$

These inequalities give $\mathbf{P}\left(\mathcal{E}_{16}^c\right) \leqslant 3e^{-l^2}$, hence $\mathbf{P}\left(\mathcal{E}_{17}^c\right) \leqslant 8e^{-l^2}$. We now prove the lower bound. Notice that

$$\mathbb{A}(v) \geqslant \int_{\sigma_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac{1}{2}))}^{\sigma_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac{1}{2}))+\frac{1}{2}} e^{\mathbb{V}_s} ds = \frac{1}{2} e^{\overline{\mathbb{V}}(v-\frac{1}{2})}, \tag{3.9}$$

and for all $x \leqslant \widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_{v-\frac{1}{2}} - l^4) \leqslant \widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)$,

$$-\mathbb{A}(-x) = \int_{-x}^{0} e^{\mathbb{V}(s)} ds \leqslant e^{\overline{\mathbb{V}}(v-\frac{1}{2}) - l^4} \widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v). \tag{3.10}$$

Thus, for all $x \leqslant \widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac{1}{2}) - l^4)$ we have $-\mathbb{A}_{-x}/\mathbb{A}_v \leqslant \exp(-l^4)\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)$. Let $\mathcal{E}_{18} = \{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v) \leqslant \exp(l^3)\}$. As for the estimate of $\mathbf{P}\left(\mathcal{E}_{16}^c\right)$, it is easily checked that for all $v$ large enough, $\mathbf{P}\left(\mathcal{E}_{18}^c\right) \leqslant 3\exp(-l^2)$. Moreover, on the set $\mathcal{E}_{18}$, combining (3.9) and (3.10), we have $-\mathbb{A}(-x)/\mathbb{A}(v) \leqslant e^{-\frac{1}{2}l^4}$ for all $0 \leqslant x \leqslant \widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac{1}{2}) - l^4)$. Let us now define

$$\mathcal{E}_{19} = \left\{ \inf_{0 \leqslant s \leqslant e^{-\frac{1}{2}l^4}} \mathcal{Z}(s) \geqslant e^{-l^2} \right\}.$$

Using Lemma 7.1 on p1501 of [12], we see that $\mathbf{P}(\mathcal{E}_{19}^c) \leqslant 2e^{-l^2}$. Recall that

$$\begin{aligned} I_2(v) &= \mathbb{A}(v) \int_0^\infty e^{-\mathbb{V}_{-s}} \mathcal{Z}\left(\frac{-\mathbb{A}(-s)}{\mathbb{A}(v)}\right) ds \\ &\geqslant \mathbb{A}(v) \int_0^{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac{1}{2})-l^4)} e^{-\mathbb{V}_{-s}} \mathcal{Z}\left(\frac{-\mathbb{A}(-s)}{\mathbb{A}(v)}\right) ds, \end{aligned}$$

therefore, on $\mathcal{E}_{20} = \mathcal{E}_{18} \cap \mathcal{E}_{19}$,

$$I_2(v) \geqslant \widetilde{\sigma}_{\mathbb{V}}\Big(\overline{\mathbb{V}}\big(v - \frac{1}{2}\big) - l^4\Big)\mathbb{A}(v) e^{-\underline{\mathbb{V}}\left(-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac{1}{2})-l^4)\right)-l^2}.$$

Using (3.9) again, on $\mathcal{E}_{20}$,

$$\begin{aligned} I_2(v) &\geqslant \frac{1}{2}\widetilde{\sigma}_{\mathbb{V}}\Big(\overline{\mathbb{V}}\big(v - \frac{1}{2}\big) - l^4\Big) e^{\overline{\mathbb{V}}(v-\frac{1}{2}) - \underline{\mathbb{V}}\left(-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac{1}{2})-l^4)\right)-l^2} \\ &\geqslant \frac{1}{2}\widetilde{\sigma}_{\mathbb{V}}\Big(\overline{\mathbb{V}}\big(v - \frac{1}{2}\big) - l^4\Big) e^{\widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}(v-\frac{1}{2})-l^4\right)+l^4-l^2}. \end{aligned}$$

Notice that on $\{\overline{\mathbb{V}}\big(v-1/2\big) > l^4\}$, we have $\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-1/2)-l^4) \geqslant 1$ (because $\mathbb{V}$ is identically 0 on $(-1,0]$). This implies that on $\mathcal{E}_{20} \cap \{\overline{\mathbb{V}}(v-1/2) > l^4\}$,

$$I_2(v) \geqslant e^{\widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}\left(v-\frac{1}{2}\right)-l^4\right)},$$

which yields the lower bound by taking the logarithm. Finally, let $\mathcal{E}_9 = \mathcal{E}_{20} \cap \mathcal{E}_{17}$, we have

$$\mathbf{P}\left(\mathcal{E}_9^c\right) \leqslant \mathbf{P}\left(\mathcal{E}_{17}^c\right) + \mathbf{P}\left(\mathcal{E}_{20}^c\right) \leqslant 13e^{-(\log v)^2}$$

for all large enough $v$'s and the upper bound holds on $\mathcal{E}_9$ as well as the lower bound on $\mathcal{E}_9 \cap \{\overline{\mathbb{V}}\big(v-1/2\big) > l^4\}$. □

## 4. Proof of the main theorems

4.1. **Proof of Theorem 1.** We first state two lemmas before we give the proof of the theorem.

**Lemma 4.1.** *For any $c_0 > 0$, we have*

$$\limsup_{t\to\infty} \frac{\log \mathbf{P}\left(\overline{X}_t \geqslant c_0 a^{-1} (\log t) \log\log\log t\right)}{\log\log\log t} \leqslant -c_0 K^{\#},$$

*where $K^{\#}$ was defined in Proposition 2.5.*

*Proof.* Let $v = c_0 a^{-1}(\log t)\log\log\log t$, using (3.4) and Proposition 3.1 we get for all $t$ large enough,

$$\begin{aligned}\mathbf{P}\left(\overline{X}_t \geqslant v\right) &\leqslant \mathbf{P}\left(I_1(v) \leqslant t\right)\\ &\leqslant \mathbf{P}\left(\mathbb{V}^{\#}_{v-\frac{1}{2}} \leqslant \log t + (\log v)^4\right) + \mathbf{C}_{25}\exp\left(-(\log v)^2\right).\end{aligned}$$

Using Proposition 2.5, for any $\varepsilon > 0$ and for all $t$ large enough (depending on $\varepsilon$), we obtain

$$\begin{aligned}\mathbf{P}\left(\mathbb{V}^{\#}_{v-\frac{1}{2}} \leqslant \log t + (\log v)^4\right) &\leqslant \exp\left(-(K^{\#}-\varepsilon)\frac{v-1/2}{a^{-1}\left(\log t + (\log v)^4\right)}\right)\\ &\leqslant \exp\left(-c_0(K^{\#}-2\varepsilon)\log\log\log t\right)\end{aligned}$$

where we used the regular variation of $a^{-1}(\cdot)$ to check that $a^{-1}(\log t + (\log v)^4) \sim a^{-1}(\log t)$. Therefore, for all $t$ large enough,

$$\begin{aligned}\mathbf{P}\left(\overline{X}_t \geqslant v\right) &\leqslant \exp\left(-c_0(K^{\#}-2\varepsilon)\log\log\log t\right) + \exp\left(-(\log v)^2\right)\\ &\leqslant 2\exp\left(-c_0(K^{\#}-2\varepsilon)\log\log\log t\right).\end{aligned}$$

□

**Lemma 4.2.** *For any $c_0 > 0$ and for all $t$ large enough (depending on $c_0$) we have*

$$\begin{aligned}\left\{\overline{X}_t \geqslant v\right\} &\supset \left\{\mathbb{V}^{\#}_v \leqslant \log t - \sqrt{\log t}\ ,\ \overline{\mathbb{V}}_v \leqslant \frac{\log t}{5}\right\}\\ &\quad\cap\left\{\widetilde{U}_{\mathbb{V}}\left(\frac{\log t}{4}\right) \leqslant \frac{\log t}{2}\right\}\cap\mathcal{E}_{21}(v)\end{aligned}$$

*where $v = c_0 a^{-1}(\log t)\log\log\log t$ and where $\mathcal{E}_{21}(v)$ is a measurable set such that*

$$\mathbf{P}\left(\mathcal{E}_{21}^c(v)\right) \leqslant \mathbf{C}_{29}e^{-(\log v)^2}.$$

*Proof.* Using (3.4) combined with Proposition 3.1 and 3.2, for $t$ sufficiently large,

$$\begin{aligned}\left\{\overline{X}_t \geqslant v\right\} &= \left\{I_1(v) + I_2(v) \leqslant t\right\}\\ &\supset \left\{e^{\mathbb{V}^{\#}_v + (\log v)^4} + e^{\widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + (\log v)^4\right)} \leqslant t\right\}\cap\mathcal{E}_{21}(v)\end{aligned}$$

with $\mathcal{E}_{21}(v) = \mathcal{E}_8(v)\cap\mathcal{E}_9(v)$, thus $\mathbf{P}\left(\mathcal{E}_{21}^c(v)\right) \leqslant \mathbf{C}_{29}e^{-(\log v)^2}$. Notice also that

$$\left\{\mathbb{V}^{\#}_v \leqslant \log t - \sqrt{\log t}\right\} \subset \left\{\mathbb{V}^{\#}_v + \log^4 v \leqslant \log\frac{t}{2}\right\}.$$

Hence, $\left\{\overline{X}_t \geqslant v\right\}$ contains

$$\left\{\mathbb{V}^{\#}_v \leqslant \log t - \sqrt{\log t}\right\}\cap\left\{\widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + (\log v)^4\right) \leqslant \log\left(\frac{t}{2}\right)\right\}\cap\mathcal{E}_{21}(v). \quad (4.1)$$

We also have $\left\{\overline{\mathbb{V}}_v \leqslant \frac{\log t}{5}\right\} \subset \left\{\overline{\mathbb{V}}_v + (\log v)^4 \leqslant \frac{\log t}{4}\right\}$, therefore

$$\left\{\overline{\mathbb{V}}_v \leqslant \frac{\log t}{5}\,,\, \widetilde{U}_{\mathbb{V}}\left(\frac{\log t}{4}\right) \leqslant \frac{\log t}{2}\right\} \subset \left\{\widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + (\log v)^4\right) \leqslant \frac{\log t}{2}\right\}.$$

This inclusion combined with (4.1) completes the proof. □

*Proof of Theorem 1.* As we already mentioned in the introduction, $X$ and $\overline{X}$ have the same upper function so we only need to prove the theorem for $\overline{X}$. Let us choose $K$ such that $K < K^{\#}$ and $\varepsilon > 0$. Define the sequence $t_i = \exp(\exp(\varepsilon i))$. We also use the notation $f(x) = a^{-1}(\log x) \log\log\log x$. Using regular variation of $a(\cdot)$ we easily check that $f(t_i)/f(t_{i+1})$ converges to $\exp(-\alpha\varepsilon)$. Thus, for all $i$ large enough

$$\mathbf{P}\left(\overline{X}_{t_{i+1}} \geqslant \frac{f(t_i)}{K}\right) \leqslant \mathbf{P}\left(\overline{X}_{t_{i+1}} \geqslant \frac{f(t_{i+1})}{e^{2\varepsilon}K}\right).$$

Using Lemma 4.1, we get

$$\limsup_{i\to\infty} \frac{1}{\log(\varepsilon(i+1))} \log\left(\mathbf{P}\left(\overline{X}_{t_{i+1}} \geqslant \frac{f(t_i)}{K}\right)\right) \leqslant -\frac{K^{\#}}{e^{2\varepsilon}K}.$$

Since $K < K^{\#}$, we can choose $\varepsilon$ small enough such that $K^{\#}/(K\exp(2\varepsilon)) < 1$ and we deduce from the last inequality that the sum $\sum \mathbf{P}(\overline{X}_{t_{i+1}} \geqslant f(t_i)/K)$ converges. Using Borel-Cantelli's Lemma, with probability 1, for all $i$ large enough $\overline{X}_{t_{i+1}} \leqslant f(t_i)/K$. For $t \in [t_i, t_{i+1}]$, using monotonicity of $f$ and $\overline{X}$,

$$\overline{X}_t \leqslant \overline{X}_{t_{i+1}} \leqslant \frac{f(t_i)}{K} \leqslant \frac{f(t)}{K}.$$

This holds for all $K < K^{\#}$. Hence, we proved that

$$\limsup_{t\to\infty} \frac{\overline{X}_t}{f(t)} \leqslant \frac{1}{K^{\#}} \text{ a.s.}$$

We now prove the lower bound. Choose $K > K^{\#}$ and change the sequence $(t_i)$ for $t_i = \exp(\exp i)$. From Lemma 4.2, for $i$ large enough,

$$\left\{\overline{X}_{t_i} \geqslant \frac{f(t_i)}{K}\right\} \supset \mathcal{E}_{21}(f(t_i)/K) \cap \mathcal{E}_{22}(i)$$

where $\mathcal{E}_{21}$ was defined in Lemma 4.2 and where $\mathcal{E}_{22}(i) = \mathcal{E}_{23}(i) \cap \mathcal{E}_{24}(i) \cap \mathcal{E}_{25}(i)$ with

$$\begin{aligned}
\mathcal{E}_{23}(i) &= \left\{\widetilde{U}_{\mathbb{V}}\left(e^i/4\right) \leqslant e^i/2\right\},\\
\mathcal{E}_{24}(i) &= \left\{\mathbb{V}^{\#}_{f(t_i)/K} \leqslant e^i - e^{i/2}\right\},\\
\mathcal{E}_{25}(i) &= \left\{\overline{\mathbb{V}}_{f(t_i)/K} \leqslant e^i/5\right\}.
\end{aligned}$$

Moreover, $\sum \mathbf{P}(\mathcal{E}^c_{21}(f(t_i)/K)) < \infty$. So it only remains to prove that the events $\mathcal{E}_{22}(i)$ happen infinitely often almost surely. It follows from the results of Section 2.1 that $\lim_{i\to\infty} \mathbf{P}(\mathcal{E}_{23}(i)) = \mathbf{P}(\widetilde{U}_{\mathbb{S}}(1/4) \leqslant 1/2)$ and it is clear that this quantity is not 0. Since $\mathcal{E}_{24}(i) \cap \mathcal{E}_{25}(i)$ and $\mathcal{E}_{23}(i)$ are independent events

$\mathbf{P}(\mathcal{E}_{22}(i)) \geqslant \mathbf{C}_{30}\mathbf{P}(\mathcal{E}_{24}(i) \cap \mathcal{E}_{25}(i))$ for all $i$ large enough. Thus, we deduce from Proposition 2.6 that for all large enough $i$'s, we have

$$\mathbf{C}_{31}\mathbf{P}\left(\mathcal{E}_{24}(i)\right) \leqslant \mathbf{P}\left(\mathcal{E}_{22}(i)\right) \leqslant \mathbf{P}\left(\mathcal{E}_{24}(i)\right). \tag{4.2}$$

We now use Proposition 2.5 to check that

$$\log\left(\mathbf{P}\left(\mathcal{E}_{24}(i)\right)\right) \underset{i\to\infty}{\sim} -\frac{K^{\#}}{K}\frac{f(t_i)}{a^{-1}\left(e^i - e^{i/2}\right)} \underset{i\to\infty}{\sim} -\frac{K^{\#}}{K}\log i, \tag{4.3}$$

where we used the regular variation of $a(\cdot)$ for the last equivalence. In particular, combining this with (4.2) and the fact that $K^{\#}/K < 1$, we see that $\sum_i \mathbf{P}(\mathcal{E}_{22}(i)) = \infty$. We now estimate $\mathbf{P}(\mathcal{E}_{22}(i) \cap \mathcal{E}_{22}(j))$ for $i$ large enough and for $j > i$.

$$\begin{aligned}\mathcal{E}_{22}(i) \cap \mathcal{E}_{22}(j) \quad &\subset \quad \mathcal{E}_{24}(i) \cap \mathcal{E}_{24}(j)\\ &\subset \quad \mathcal{E}_{24}(i) \cap \left\{\left(\theta_{f(t_i)/K}\mathbb{V}\right)^{\#}_{f(t_j)/K - f(t_i)/K} \leqslant e^j - e^{j/2}\right\}.\end{aligned}$$

Hence, from the independence and the stationarity of the increments of $\mathbb{V}$ (at integer times), combined with Proposition 2.7, for all $i$ large enough (i.e. all $j$ large enough), we get

$$\begin{aligned}\mathbf{P}\left(\mathcal{E}_{22}(i) \cap \mathcal{E}_{22}(j)\right) \quad &\leqslant \quad \mathbf{P}\left(\mathcal{E}_{24}(i)\right)\mathbf{P}\left(\mathbb{V}^{\#}_{f(t_j)/K - f(t_i)/K} \leqslant e^j - e^{j/2}\right)\\ &\leqslant \quad \mathbf{C}_{32}\frac{\mathbf{P}\left(\mathcal{E}_{24}(i)\right)\mathbf{P}\left(\mathcal{E}_{24}(j)\right)}{\mathbf{P}\left(\mathbb{V}^{\#}_{f(t_i)/K} \leqslant e^j - e^{j/2}\right)}.\end{aligned}$$

Using Lemma 2.4, one may check after a few lines of calculus that for all $i$ sufficiently large, $\exp(j) - \exp(j/2) \geqslant a^{-1}(f(t_i)/K)$ whenever $j - i \geqslant \log i$, thus

$$\mathbf{P}\left(\mathbb{V}^{\#}_{f(t_i)/K} \leqslant e^j - e^{j/2}\right) \geqslant \mathbf{P}\left(\frac{\mathbb{V}^{\#}_{f(t_i)/K}}{a\left(f(t_i)/K\right)} \leqslant 1\right).$$

Since the r.h.s. of the last inequality converges to $\mathbf{P}(\mathbb{S}^{\#}_1 \leqslant 1) \neq 0$ as $i$ goes to infinity, we deduce that for all $i$ large enough and all $j - i \geqslant \log i$,

$$\mathbf{P}\left(\mathbb{V}^{\#}_{f(t_i)/K} \leqslant e^j - e^{j/2}\right) \geqslant \mathbf{C}_{33} > 0.$$

Finally, for all $i$ large enough and for all $j \geqslant i$,

$$\mathbf{P}\left(\mathcal{E}_{22}(i) \cap \mathcal{E}_{22}(j)\right) \leqslant \begin{cases}\mathbf{P}\left(\mathcal{E}_{22}(i)\right), & \text{if } 0 \leqslant j - i < \log i,\\ \mathbf{C}_{34}\mathbf{P}\left(\mathcal{E}_{22}(i)\right)\mathbf{P}\left(\mathcal{E}_{24}(j)\right), & \text{if } j - i \geqslant \log i.\end{cases} \tag{4.4}$$

Combining (4.2),(4.3) and (4.4), we see that

$$\liminf_{n\to\infty} \sum_{i,j\,\leqslant\, n} \mathbf{P}\left(\mathcal{E}_{22}(i) \cap \mathcal{E}_{22}(j)\right) \Big/ \Big(\sum_{i\,\leqslant\, n} \mathbf{P}\left(\mathcal{E}_{22}(i)\right)\Big)^2 \leqslant \mathbf{C}_{35}.$$

The Borel-Cantelli Lemma of [16] yields $\mathbf{P}\left(\mathcal{E}_{22}(i) \text{ i.o.}\right) > 1/\mathbf{C}_{35}$. We now use a classical 0-1 argument (compare with [12], p1511 for details) to conclude that $\mathbf{P}\left(\mathcal{E}_{22}(i) \text{ i.o.}\right) = 1$. Hence, with probability 1,

$$\limsup_{t\to\infty} \frac{\overline{X}_t}{f(t)} \geqslant \frac{1}{K^{\#}}.$$

Moreover, the value of $K^{\#}$ when the process $\mathbb{V}$ is completely asymmetric was calculated in Corollary 2.10. □

## 4.2. **Proof of Theorem 2.**

**Lemma 4.3.** *Let $\rho > 0$, for all $t$ large enough (depending on $\rho$) and all $1 \leqslant \lambda \leqslant (\log\log t)^{\rho}$, we have*

$$\mathbf{P}\left(\overline{X_t} < \frac{a^{-1}(\log t)}{\lambda}\right) \leqslant \mathbf{C}_{36}\frac{b^{-1}(a^{-1}(\log t)/\lambda)}{b^{-1}(a^{-1}(\log t))}.$$

*Proof.* We use the notation $v = a^{-1}(\log t)/\lambda$, the bounds on $\lambda$ give

$$\frac{a^{-1}(\log t)}{(\log\log t)^{\rho}} \leqslant v \leqslant a^{-1}(\log t).$$

We assume that $t$ is very large, hence $v$ is also large. From (3.4) combined with Proposition 3.1 and Proposition 3.2, we deduce that

$$\begin{aligned}
\mathbf{P}\left(\overline{X}_t < v\right) &\leqslant \mathbf{P}\left(I_1(v) \geqslant \frac{t}{2}\right) + \mathbf{P}\left(I_2(v) \geqslant \frac{t}{2}\right)\\
&\leqslant \mathbf{P}\left(\mathbb{V}_v^{\#} \geqslant \log\frac{t}{2} - (\log v)^4\right)\\
&\quad + \mathbf{P}\left(\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v + (\log v)^4) \geqslant \log\frac{t}{2}\right) + \mathbf{C}_{37}e^{-(\log v)^2}.
\end{aligned}$$

Remind that $b^{-1}(\cdot)$ is regularly varying with index $q < 1$. Therefore, using Corollary 2.16 and Lemma 2.4,

$$\begin{aligned}
\mathbf{P}\left(\mathbb{V}_v^{\#} \geqslant \log\frac{t}{2} - (\log v)^4\right) &\leqslant \mathbf{P}\left(\mathbb{V}_v^{\#} \geqslant \frac{1}{2}\log t\right)\\
&\leqslant \mathbf{C}_{38}\frac{v}{a^{-1}(\log t)}\\
&\leqslant \mathbf{C}_{39}\frac{b^{-1}(v)}{b^{-1}(a^{-1}(\log t))}.
\end{aligned}$$

It is also easy to check from the bounds on $v$ and the regular variations of $a^{-1}(\cdot)$ and $b^{-1}(\cdot)$ that

$$e^{-(\log v)^2} \leqslant \frac{b^{-1}(v)}{b^{-1}\left(a^{-1}(\log t)\right)}.$$

We still have to prove a similar bound for $\mathbf{P}(\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v + (\log v)^4) \geqslant \log(t/2))$. Notice that for any $y > x > 0$, $\{\widetilde{U}_{\mathbb{V}}(x) \geqslant y\} = \widetilde{\Lambda}'(x, y - x)$. Hence, using Proposition 2.13 and the independence of $(\mathbb{V}_x)_{x\geqslant 0}$ and $(\mathbb{V}_{-x})_{x\geqslant 0}$, we get

$$\begin{aligned}
\mathbf{P}&\left(\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v + (\log v)^4) \geqslant \log\frac{t}{2}\right) \\
&\leqslant \mathbf{C}_{40}\mathbf{E}\left(\frac{b^{-1}(a^{-1}(\overline{\mathbb{V}}_v + (\log v)^4))}{b^{-1}(a^{-1}(\log\frac{t}{2}))}\right) \\
&\leqslant \mathbf{C}_{40}\frac{b^{-1}(v)}{b^{-1}(a^{-1}(\log\frac{t}{2}))}\mathbf{E}\left(\frac{b^{-1}(a^{-1}(\overline{\mathbb{V}}_v + (\log v)^4))}{b^{-1}(a^{-1}(a(v)))}\right). \qquad (4.5)
\end{aligned}$$

Let us pick $\varepsilon > 0$. We now use Lemma 2.4 for the regularly varying function $b^{-1}(a^{-1}(\cdot))$ to check that (4.5) is smaller than

$$\mathbf{C}_{41,\varepsilon}\frac{b^{-1}(v)}{b^{-1}(a^{-1}(\log\frac{t}{2}))}\mathbf{E}\left(\left(\frac{\overline{\mathbb{V}}_v + (\log v)^4}{a(v)}\right)^{\alpha q+\varepsilon} + 1\right).$$

Finally, since $q < 1$, we can choose $\varepsilon$ small enough such that $\alpha q + \varepsilon < \alpha$, therefore Corollary 2.17 implies

$$\mathbf{E}\left(\left(\frac{\overline{\mathbb{V}}_v + (\log v)^4}{a(v)}\right)^{\alpha q+\varepsilon}\right) \leqslant \mathbf{E}\left(\left(\frac{\overline{\mathbb{V}}_v}{a(v)} + 1\right)^{\alpha q+\varepsilon}\right) \leqslant \mathbf{C}_{42,\varepsilon}.$$

We conclude the proof noticing that $b^{-1}(a^{-1}(\log\frac{t}{2})) \sim b^{-1}(a^{-1}(\log t))$. □

**Lemma 4.4.** *Let $\rho > 0$, for all $t$ large enough (depending on $\rho$) and for all $1 \leqslant \lambda \leqslant (\log\log t)^\rho$, we have*

$$\left\{\overline{X}_t < \frac{a^{-1}(\log t)}{\lambda}\right\} \supset \left\{\widetilde{U}_{\mathbb{V}}(a(v)) \geqslant \log t\right\} \cap \left\{\overline{\mathbb{V}}_{v/2} \geqslant 2a(v)\right\} \cap \mathcal{E}_9(v),$$

*with $v = a^{-1}(\log t)/\lambda$, and where $\mathcal{E}_9(v)$ was defined in Proposition 3.2 and satisfies*

$$\mathbf{P}(\mathcal{E}_9(v)^c) \leqslant \mathbf{C}_{26}e^{-(\log v)^2}.$$

*Proof.* Recall that Relation (3.4) gives $\{\overline{X}_t < v\} = \{I_1(v) + I_2(v) > t\}$ and notice that $I_1(v) > 0$ for all $v > 0$, thus, $\{\overline{X}_t < v\} \supset \{I_2(v) \geqslant t\}$. We now use Proposition 3.2 to see that for all $t$ large enough (i.e. $v$ large enough), the event $\{\overline{X}_t < v\}$ contains

$$\begin{aligned}
&\left\{\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_{v-\frac{1}{2}} - (\log v)^4) \geqslant \log t\right\} \cap \left\{\overline{\mathbb{V}}_{v-\frac{1}{2}} > (\log v)^4\right\} \cap \mathcal{E}_9(v) \\
&\supset \left\{\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_{v/2} - a(v)) \geqslant \log t\right\} \cap \left\{\overline{\mathbb{V}}_{v/2} \geqslant 2a(v)\right\} \cap \mathcal{E}_9(v) \\
&\supset \left\{\widetilde{U}_{\mathbb{V}}(a(v)) \geqslant \log t\right\} \cap \left\{\overline{\mathbb{V}}_{v/2} \geqslant 2a(v)\right\} \cap \mathcal{E}_9(v),
\end{aligned}$$

where we used the fact that $x \mapsto \widetilde{U}_{\mathbb{V}}(x)$ is a non-decreasing function and the trivial inequalities $\overline{\mathbb{V}}_{v/2} \leqslant \mathbb{V}_{v-1/2}$ and $(\log v)^4 \leqslant a(v)$ which hold for all large enough $v$'s. □

*Proof of Theorem 2.* For any positive non-decreasing function $f$, recall that

$$J(f)=\int^{\infty}\frac{b^{-1}(a^{-1}(\log t)/f(t))dt}{b^{-1}(a^{-1}(\log t))t\log t}$$

(we do not specify the lower bound for the integral since we are only concerned with the convergence of $J(f)$ at infinity).

Let us first prove the theorem when $J(f)<\infty$. Since $f$ is non-decreasing, it is clear that $f(t)\to\infty$ as $t\to\infty$. Thus, $f(t)\geqslant e^{2\alpha}$ for all $t$ large enough. Let $f_0(t)=(\log\log t)^{2/q}$, we have $J(f_0)<\infty$. It is known that we may assume without loss of generality that

$$f(t)\leqslant f_0(t)=(\log\log t)^{2/q}\quad\text{for all large } t \tag{4.6}$$

(compare with the argument given in the beginning of the proof of Theorem 1 in [9]). Let us set $t_i=\exp(\exp i)$. Since $a^{-1}(\cdot)$ is regularly varying with index $\alpha$, for all $i$ large enough, we have

$$a^{-1}(\log t_{i+1})\leqslant e^{2\alpha}a^{-1}(\log t_i). \tag{4.7}$$

Hence, Lemma 4.3 yields, $i$ still being very large,

$$\begin{aligned}\mathbf{P}\left(\overline{X}_{t_i}<\frac{a^{-1}(\log t_{i+1})}{f(t_i)}\right) &\leqslant \mathbf{P}\left(\overline{X}_{t_i}<\frac{a^{-1}(\log t_i)}{e^{-2\alpha}f(t_i)}\right)\\ &\leqslant \mathbf{C}_{36}\frac{b^{-1}(e^{2\alpha}a^{-1}(\log t_i)/f(t_i))}{b^{-1}(a^{-1}(\log t_i))}\\ &\leqslant \mathbf{C}_{43}\frac{b^{-1}(a^{-1}(\log t_{i-1})/f(t_i))}{b^{-1}(a^{-1}(\log t_i))}\\ &\leqslant \mathbf{C}_{43}\int_{t_{i-1}}^{t_i}\frac{b^{-1}(a^{-1}(\log t)/f(t))dt}{b^{-1}(a^{-1}(\log t))t\log t},\end{aligned}$$

where we used again (4.7) and the regular variation of $b^{-1}$ for the third inequality and the monotonicity of $a^{-1}$,$b^{-1}$ and $f$ for the last inequality. Since $J(f)<\infty$, we conclude that $\sum_i\mathbf{P}(\overline{X}_{t_i}<a^{-1}(\log t_{i+1})/f(t_i))<\infty$ and Borel-Cantelli's Lemma implies that, almost surely,

$$\overline{X}_{t_i}\geqslant\frac{a^{-1}(\log t_{i+1})}{f(t_i)}\quad\text{for all } i \text{ large enough.}$$

But for $t_i\leqslant t\leqslant t_{i+1}$, we have $a^{-1}(\log t_{i+1})/f(t_i)>a^{-1}(\log t)/f(t)$ and $\overline{X}_t\geqslant\overline{X}_{t_i}$, therefore, with probability 1,

$$\liminf_{t\to\infty}\frac{f(t)}{a^{-1}(\log t)}\overline{X}_t\geqslant 1\text{ a.s.} \tag{4.8}$$

Changing $f$ for $Cf$ for any $C>0$ does not alter the convergence of $J(f)$. Thus, the lim inf in (4.8) is in fact infinite.

We now prove the second part of the theorem. Let $f$ be a positive, non-decreasing function such that $J(f)=\infty$. Again, we may without loss of generality assume that (4.6) holds (compare with the argument given in the proof of Theorem 3 in [9]). Moreover, notice that the theorem is straightforward for any bounded function $f$ provided that we prove the theorem for at least one function $h$ going to infinity with $J(h) = \infty$ (we may choose for example $h(t) = (\log\log t)^{1/(2q)}$). Thus, we can also assume that $f(t) \to \infty$ as $t \to \infty$. We use the notation $v_i = a^{-1}(\log t_i)/f(t_i)$. Our assumptions on $f$ yield the following estimates:

$$\frac{a^{-1}(\log t_i)}{(\log\log t_i)^{2/q}} \leqslant v_i \leqslant a^{-1}(\log t_i) \text{ for } i \text{ large enough}, \tag{4.9}$$

and

$$\lim_{i\to\infty} v_i = \infty, \qquad \lim_{i\to\infty} \frac{v_i}{a^{-1}(\log t_i)} = 0. \tag{4.10}$$

From now on, we assume that $i$ is very large. Using Lemma 4.4, we get

$$\left\{\overline{X}_{t_i} \leqslant \frac{a^{-1}(\log t_i)}{f(t_i)}\right\} \supset \mathcal{E}_9(v_i) \cap \mathcal{E}_{26}(i),$$

where $\mathcal{E}_{26}(i) = \mathcal{E}_{27}(i) \cap \mathcal{E}_{28}(i)$ with

$$\begin{aligned} \mathcal{E}_{27}(i) &= \{\widetilde{U}_{\mathbb{V}}(a(v_i)) \geqslant \log(t_i)\}, \\ \mathcal{E}_{28}(i) &= \{\overline{\mathbb{V}}_{v_i/2} \geqslant 2a(v_i)\}. \end{aligned}$$

Since $\mathbf{P}(\mathcal{E}_9(v_i)^c) \leqslant \mathbf{C}_{26} \exp(-\log^2 v_i)$, it is easy to check from (4.9) that $\sum_i \mathbf{P}(\mathcal{E}_9(v_i)^c) < \infty$. So it only remains to prove that $\mathbf{P}(\mathcal{E}_{26}(i) \text{ i.o.}) = 1$. Since $v_i \to \infty$ as $i \to \infty$, results of Section 2.1 imply that

$$\lim_{i\to\infty} \mathbf{P}(\mathcal{E}_{28}(i)) = \mathbf{P}(\overline{\mathbb{S}}_{1/2} \geqslant 2) > 0.$$

Therefore, the independence of $\mathcal{E}_{27}(i)$ and $\mathcal{E}_{28}(i)$ yields

$$\mathbf{C}_{43}\mathbf{P}\left(\mathcal{E}_{27}(i)\right) \leqslant \mathbf{P}\left(\mathcal{E}_{26}(i)\right) \leqslant \mathbf{P}\left(\mathcal{E}_{27}(i)\right). \tag{4.11}$$

Recall that $\{\widetilde{U}_{\mathbb{V}}(a(v_i)) \geqslant \log(t_i)\} = \widetilde{\Lambda}'(a(v_i), \log(t_i) - a(v_i))$. Keeping in mind (4.10), we can estimate $\mathbf{P}\left(\mathcal{E}_{27}(i)\right)$ using Proposition 2.13:

$$\mathbf{C}_{44}\frac{b^{-1}(v_i)}{b^{-1}(a^{-1}(\log t_i))} \leqslant \mathbf{P}\left(\mathcal{E}_{27}(i)\right) \leqslant \mathbf{C}_{45}\frac{b^{-1}(v_i)}{b^{-1}(a^{-1}(\log t_i))}. \tag{4.12}$$

Combining the inequalities (4.11) and (4.12), the assumption that $J(f) = \infty$ implies

$$\sum_i \mathbf{P}(\mathcal{E}_{26}(i)) = \infty.$$

We now estimate $\mathbf{P}(\mathcal{E}_{26}(i) \cap \mathcal{E}_{26}(j))$. Let $g(i) = \log(t_i) - a(v_i)$. It is easy to check from (4.10) that g is ultimately increasing. Let us pick $j > i$. We can rewrite

$$\mathcal{E}_{27}(i) \cap \mathcal{E}_{27}(j) = \widetilde{\Lambda}'\left(a(v_i), g(i)\right) \cap \widetilde{\Lambda}'\left(a(v_i), g(j)\right).$$

There are two cases (which are not disjoint):

(1) $(\mathbb{V}_{-n})_{n\,\geqslant\,0}$ hits $(-\infty, -g(j)]$ before hitting $[a(v_i), \infty)$. Using Proposition 2.13, we can check that the probability of this case is less than $\mathbf{C}_{46}b^{-1}(v_i)/b^{-1}(a^{-1}(\log t_j))$.

(2) $(\mathbb{V}_{-n})_{n\geqslant 0}$ hits $(-\infty,-g(i)]$ before hitting $[a(v_i),\infty)$ (i.e. $\mathcal{E}_{27}(i)$ happens) and also the shifted random walk $(\mathbb{V}_{-\widetilde{\sigma}_{\mathbb{V}}(a(v_i))-n})_{n\geqslant 0}$ hits $(-\infty,-g(j)]$ before hitting $[a(v_j),+\infty)$ (the probability of this event is clearly smaller than $\mathbf{P}(\mathcal{E}_{27}(j))$). Using the Markov property for the random walk $(\mathbb{V}_{-n})_{n\geqslant 0}$ we conclude that the probability of this case is smaller than $\mathbf{P}(\mathcal{E}_{27}(i))\mathbf{P}(\mathcal{E}_{27}(j))$.

Combining (1) and (2) we deduce that $\mathbf{P}(\mathcal{E}_{27}(i)\cap\mathcal{E}_{27}(j))$ is smaller than

$$\begin{aligned}&\mathbf{P}\left(\mathcal{E}_{27}(i)\right)\mathbf{P}\left(\mathcal{E}_{27}(j)\right)+\mathbf{C}_{46}\frac{b^{-1}(v_i)}{b^{-1}(a^{-1}(\log t_j))}\\ &\qquad\leqslant \mathbf{P}\left(\mathcal{E}_{27}(i)\right)\mathbf{P}\left(\mathcal{E}_{27}(j)\right)+\frac{\mathbf{C}_{46}}{\mathbf{C}_{44}}\mathbf{P}\left(\mathcal{E}_{27}(i)\right)\frac{b^{-1}(a^{-1}(\log t_i))}{b^{-1}(a^{-1}(\log t_j))},\end{aligned}$$

where we used (4.12) for the second inequality. Finally, using Lemma 2.4 and (4.11), we conclude that for all $i$ large enough and all $j>i$,

$$\begin{aligned}\mathbf{P}\left(\mathcal{E}_{26}(i)\cap\mathcal{E}_{26}(j)\right) &\leqslant \mathbf{P}\left(\mathcal{E}_{27}(i)\cap\mathcal{E}_{27}(j)\right)\\ &\leqslant \mathbf{C}_{47}\left(\mathbf{P}\left(\mathcal{E}_{26}(i)\right)\mathbf{P}\left(\mathcal{E}_{26}(j)\right)+\mathbf{P}\left(\mathcal{E}_{26}(i)\right)e^{-\mathbf{C}_{48}(j-i)}\right),\end{aligned}$$

hence

$$\liminf_{n\to\infty}\sum_{i,j\leqslant n}\mathbf{P}\left(\mathcal{E}_{26}(i)\cap\mathcal{E}_{26}(j)\right)\Big/\Big(\sum_{i\leqslant n}\mathbf{P}\left(\mathcal{E}_{26}(i)\right)\Big)^2\leqslant\mathbf{C}_{47}.$$

Just like in the proof of Theorem 1, we apply the Borel-Cantelli Lemma of [16] and a standard 0-1 argument to conclude that $\mathbf{P}(\mathcal{E}_{26}(i)\text{ i.o.})=1$. Since this result still holds when changing $f$ for $Cf$ for any $C>0$, we have proved that, with probability 1,

$$\liminf_{t\to\infty}\frac{\overline{X}_t}{f(t)}=0.$$

□

4.3. **Proof of Theorem 3.** Just like the previous two theorems, the proof is based on the following two lemmas.

**Lemma 4.5.** *Let $\rho>0$, for all $t$ large enough and all $1\leqslant\lambda\leqslant(\log\log t)^\rho$ we have*

$$\mathbf{P}\left(X_t^*<\frac{a^{-1}(\log t)}{\lambda}\right)\leqslant\frac{\mathbf{C}_{49}}{\lambda^2}.$$

*Proof.* We use the notation $v=a^{-1}(\log t)/\lambda$. Let $Y=-X$, it is clear from a symmetry argument that $Y$ is a diffusion in the "reversed" random environment $\mathbb{W}=(\mathbb{V}_{-x},x\in\mathbb{R})$. Let us notice that

$$\begin{aligned}\mathbf{P}\left(X_t^*<v\right) &= \mathbf{P}\left(\overline{X}_t<v\ ,\ \overline{Y}_t<v\right)\\ &\leqslant \mathbf{P}\left(\overline{X}_t<v\ ,\ \overline{\mathbb{V}}_v\leqslant\overline{\mathbb{V}}_{-v}\right)+\mathbf{P}\left(\overline{Y}_t<v\ ,\ \overline{\mathbb{W}}_v\leqslant\overline{\mathbb{W}}_{-v}\right).\end{aligned}$$

Let us also note that all the assumptions we made on $\mathbb{V}$ also hold for $\mathbb{W}$. Hence, we only need to prove the upper bound for the first member on the r.h.s. of the last inequality. According to (3.4)

$$\left\{\overline{X}_t < v\right\} = \{I_1(v) + I_2(v) > t\},$$

thus,

$$\begin{aligned}
&\mathbf{P}\left(\overline{X}_t < v\ ,\ \overline{\mathbb{V}}_v \leqslant \overline{\mathbb{V}}_{-v}\right)\\
&\quad\leqslant \mathbf{P}\left(\frac{1}{4}\log t \leqslant \overline{\mathbb{V}}_v \leqslant \overline{\mathbb{V}}_{-v}\right) &(4.13)\\
&\qquad+\mathbf{P}\left(I_1(v) \geqslant \frac{t}{2}, \overline{\mathbb{V}}_v \leqslant \frac{\log t}{4}\right) &(4.14)\\
&\qquad+\mathbf{P}\left(I_2(v) \geqslant \frac{t}{2}, \overline{\mathbb{V}}_v \leqslant \overline{\mathbb{V}}_{-v}, \overline{\mathbb{V}}_v \leqslant \frac{\log t}{4}\right). &(4.15)
\end{aligned}$$

We deal with each term separately. First, using independence of $(\mathbb{V}_x)_{x\,\geqslant\,0}$ and $(\mathbb{V}_{-x})_{x\,\geqslant\,0}$ we see that (4.13) is smaller than

$$\mathbf{P}\left(\overline{\mathbb{V}}_v \geqslant \frac{1}{4}\log t\right)\mathbf{P}\left(\overline{\mathbb{V}}_{-v} \geqslant \frac{1}{4}\log t\right) \leqslant \frac{\mathbf{C}_{49}}{\lambda^2},$$

where we used Corollary 2.16 for the last inequality. We now turn our attention to (4.14). Using Proposition 3.1, we check that for $t$ large enough, this probability is smaller than

$$\mathbf{P}\left(\mathbb{V}_v^{\#} \geqslant \log\frac{t}{2} - \log^4 v\ ,\ \overline{\mathbb{V}}_v \leqslant \frac{1}{4}\log t\right) + \mathbf{C}_{25}e^{-\log^2 v}.$$

For $t$ large enough, using the Markov property,

$$\begin{aligned}
&\mathbf{P}\left(\mathbb{V}_v^{\#} \geqslant \log\frac{t}{2} - \log^4 v\ ,\ \overline{\mathbb{V}}_v \leqslant \frac{\log t}{4}\right)\\
&\quad\leqslant \mathbf{P}\left(\mathbb{V}_v^{\#} \geqslant \frac{\log t}{2}\ ,\ \overline{\mathbb{V}}_v \leqslant \frac{\log t}{4}\right)\\
&\quad\leqslant \mathbf{P}\left(\sigma_{\mathbb{V}}\Big(-\frac{\log t}{4}\Big) \leqslant v, \Big(\theta_{\sigma_{\mathbb{V}}(-\frac{\log t}{4})}\mathbb{V}\Big)_v^{\#} \geqslant \frac{\log t}{2}\right)\\
&\quad\leqslant \mathbf{P}\left(\underline{\mathbb{V}}_v \leqslant -\frac{\log t}{4}\right)\mathbf{P}\left(\mathbb{V}_v^{\#} \geqslant \frac{\log t}{2}\right)\\
&\quad\leqslant \frac{\mathbf{C}_{50}}{\lambda^2},
\end{aligned}$$

where we used again Corollary 2.16 for the last line. It is also clear from our bounds on $\lambda$ that $e^{-\log^2 v} \leqslant 1/\lambda^2$ for all $t$ large enough. This gives the desired bound for (4.14). It remains to prove the existence of a similar inequality for (4.15). We first use Proposition 3.2 to see that, for all $t$ large enough, (4.15) is smaller than

$$\mathbf{P}\Big(\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v + \log^4 v) \geqslant \log\frac{t}{2}\ ,\ \overline{\mathbb{V}}_v \leqslant \overline{\mathbb{V}}_{-v}\ ,\ \overline{\mathbb{V}}_v \leqslant \frac{1}{4}\log t\Big) + \mathbf{C}_{26}e^{-\log^2 v}.$$

We can rewrite:

$$\begin{aligned}
&\Big\{\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v+\log^4 v)\geqslant \log\frac{t}{2}\ ,\ \overline{\mathbb{V}}_v\leqslant \overline{\mathbb{V}}_{-v}\ ,\ \overline{\mathbb{V}}_v\leqslant \frac{1}{4}\log t\Big\}\\
&\quad=\Big\{\widetilde{\sigma}_{\mathbb{V}}\Big(\overline{\mathbb{V}}_v+\log^4 v-\log\frac{t}{2}\Big)<\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+\log^4 v\right)\ ,\\
&\qquad\quad \widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\leqslant v\ ,\ \overline{\mathbb{V}}_v\leqslant \frac{1}{4}\log t\Big\}\\
&\quad\subset\Big\{\widetilde{\sigma}_{\mathbb{V}}\Big(-\frac{\log t}{2}\Big)<\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+\log^4 v\right)\ ,\ \widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\leqslant v\Big\}\\
&\quad\subset\Big\{\widetilde{\sigma}_{\mathbb{V}}\Big(-\frac{\log t}{2}\Big)<\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\leqslant v\Big\}\\
&\qquad\quad\cup\Big\{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)<\widetilde{\sigma}_{\mathbb{V}}\Big(-\frac{\log t}{2}\Big)<\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+\log^4 v\right)\Big\}.
\end{aligned}$$

Notice that on the event $\{\widetilde{\sigma}_{\mathbb{V}}(-(\log t)/2)<\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\leqslant v\}$, the process $(\mathbb{V}_{-x})_{x\geqslant 0}$ hits $(-\infty,-(\log t/2)]$ before time $v$, and then hits $[0,\infty)$, again before time $v$. The Markov property with the stopping time $\widetilde{\sigma}_{\mathbb{V}}(-(\log t)/2)$ and Corollary 2.16 yield

$$\begin{aligned}
&\mathbf{P}\left(\widetilde{\sigma}_{\mathbb{V}}\left(-\frac{\log t}{2}\right)<\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\leqslant v\right)\\
&\qquad\leqslant \mathbf{P}\left(\underline{\mathbb{V}}_{-v}\leqslant -\frac{\log t}{2}\right)\mathbf{P}\left(\overline{\mathbb{V}}_{-v}\geqslant\frac{\log t}{2}\right)\leqslant\frac{\mathbf{C}_{51}}{\lambda^2}.
\end{aligned}$$

It is also easy to check from the Markov property of $(\mathbb{V}_{-x})_{x\geqslant 0}$ applied to the stopping time $\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)$ that the probability of the event $\{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)<\widetilde{\sigma}_{\mathbb{V}}(-(\log t)/2)<\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+\log^4 v\right)\}$ is smaller than the probability that the random walk $(\mathbb{V}_{-x})_{x\geqslant 0}$ hits $(-\infty,-(\log t)/2]$ before it hits $[\log^4 v,\infty)$. Using the estimate for the exit problem (Proposition 2.13) and the regular variation of $b^{-1}(a^{-1}(\cdot))$, for $t$ large enough, we obtain

$$\begin{aligned}
&\mathbf{P}\left(\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)<\widetilde{\sigma}_{\mathbb{V}}\left(-\frac{\log t}{2}\right)<\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+\log^4 v\right)\right)\\
&\qquad\leqslant \mathbf{C}_{52}\frac{b^{-1}\left(a^{-1}\left((\log v)^4\right)\right)}{b^{-1}\left(a^{-1}\left(\frac{\log t}{2}\right)\right)}\leqslant\frac{1}{\lambda^2},
\end{aligned}$$

so we conclude that (4.15) is smaller than $\mathbf{C}_{53}/\lambda^2$. □

**Lemma 4.6.** *Let $\rho>0$, for all $t$ large enough and all $1\leqslant\lambda\leqslant(\log\log t)^\rho$, we have*

$$\left\{X^*_t<\frac{a^{-1}(\log t)}{\lambda}\right\}\supset\left\{\mathbb{V}_{v-\frac{1}{2}}\geqslant 2\log t\ ,\ \mathbb{V}_{-v+\frac{1}{2}}\geqslant 2\log t\right\}\cap\mathcal{E}_{29}(v),$$

*with the notation $v=a^{-1}(\log t)/\lambda$ and where $\mathbf{P}\left(\mathcal{E}_{29}(v)^c\right)$ is a measurable set such that*

$$\mathbf{P}\left(\mathcal{E}_{29}(v)^c\right)\leqslant\mathbf{C}_{54}e^{-\log^2 v}.$$

*Proof.* Recall that $X$ is given by the formula

$$X_t = \mathbb{A}^{-1}\left(B_{\mathbb{T}^{-1}(t)}\right),$$

where $B$ is a Brownian motion independent of $\mathbb{V}$. Let $\widetilde{B} = -B$ and let $\widetilde{L}$ denote the bi-continuous version of the local time process of $\widetilde{B}$. Recall also that $\mathbb{W}$ stands for the reversed process $(\mathbb{V}_{-x}, x \in \mathbb{R})$. In the beginning of Section 3, we proved that

$$\sigma_X(v) = I_1(v) + I_2(v) \quad \text{for all } v > 0. \tag{4.16}$$

It is easily checked, using similar arguments, that

$$\sigma_X(-v) = \widetilde{I}_1(v) + \widetilde{I}_2(v) \quad \text{for all } v > 0, \tag{4.17}$$

with

$$\begin{aligned} \widetilde{I}_1(v) &= \int_0^v e^{-\mathbb{W}_y} \widetilde{L}(\sigma_{\widetilde{B}}(\widetilde{\mathbb{A}}(v)), \widetilde{\mathbb{A}}(y))dy, \\ \widetilde{I}_2(v) &= \int_0^\infty e^{-\mathbb{W}_{-y}} \widetilde{L}(\sigma_{\widetilde{B}}(\widetilde{\mathbb{A}}(v)), \widetilde{\mathbb{A}}(-y))dy, \end{aligned}$$

and where

$$\widetilde{\mathbb{A}}(x) = \int_0^x e^{\mathbb{W}_y} dy.$$

Thus, $\widetilde{I}_1$ and $\widetilde{I}_2$ are given by the same formulas as $I_1$ and $I_2$ by simply changing the process $B$ for $\widetilde{B}$ and changing $\mathbb{V}$ for $\mathbb{W}$. Notice that $\widetilde{B}$ is again a Brownian motion independent of $\mathbb{W}$ and that $\mathbb{W}$ fulfills all the assumptions we made on $\mathbb{V}$. Therefore, Propositions 3.1 and 3.2 also hold for $\widetilde{I}_1$ and $\widetilde{I}_2$ with $\mathbb{W}$ instead of $\mathbb{V}$ (with different events and different values of the constants). In particular, we deduce that for all $v$ large enough

$$\log \widetilde{I}_1(v) \geqslant \mathbb{V}^{\#}_{-v+1/2} - (\log v)^4 \quad \text{on } \mathcal{E}_{30}(v), \tag{4.18}$$

where $\mathcal{E}_{30}(v)$ is a measurable set such that $\mathbf{P}(\mathcal{E}_{30}^c(v)) \leqslant \mathbf{C}_{55} \exp(-\log^2 v)$. We also know from Proposition 3.1 that

$$\log I_1(v) \geqslant \mathbb{V}^{\#}_{v-1/2} - (\log v)^4 \quad \text{on } \mathcal{E}_8(v). \tag{4.19}$$

Let $\mathcal{E}_{29}(v) = \mathcal{E}_8(v) \cap \mathcal{E}_{30}(v)$, then $\mathbf{P}\left(\mathcal{E}_{29}^c(v)\right) \leqslant \mathbf{C}_{54} \exp\left(-\log^2 v\right)$. Combining (4.16),(4.17),(4.18) and (4.19), we get

$$\begin{aligned} \left\{X_t^* < v\right\} &= \left\{\sigma_X(v) > t\right\} \cap \left\{\sigma_X(-v) > t\right\} \\ &\supset \left\{I_1(v) > t\right\} \cap \left\{\widetilde{I}_1(v) > t\right\} \\ &\supset \left\{\mathbb{V}^{\#}_{v-1/2} > \log t + \log^4 v\right\} \\ &\quad \cap \left\{\mathbb{V}^{\#}_{-v+1/2} > \log t + \log^4 v\right\} \cap \mathcal{E}_{29}(v) \\ &\supset \left\{\mathbb{V}^{\#}_{v-1/2} \geqslant 2\log t\right\} \cap \left\{\mathbb{V}^{\#}_{-v+1/2} \geqslant 2\log t\right\} \cap \mathcal{E}_{29}(v) \\ &\supset \left\{\mathbb{V}_{v-1/2} \geqslant 2\log t\right\} \cap \left\{\mathbb{V}_{-v+1/2} \geqslant 2\log t\right\} \cap \mathcal{E}_{29}(v). \end{aligned}$$

□

*Proof of Theorem 3.* This theorem is an easy consequence (using similar technics as in the proof of Theorem 2) of the last two lemmas and of Proposition 2.15 (when the limiting process has jumps of both signs). We feel free to omit it. □

## 4.4. **Proof of Theorem 4.**

**Proposition 4.7.** *We have*

$$\frac{1}{a(v)}\left(\log\sigma_X(v) - \mathbb{V}_v^{\#}\vee\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\right) \underset{v\to\infty}{\overset{Prob.}{\longrightarrow}} 0.$$

The proof of this Proposition is very similar to that of Proposition 11.1 of [12] using the estimates for $I_1$ and $I_2$ obtained in Propositions 3.1 and 3.2, we therefore skip the details.

*Proof of Theorem 4.* Let $\lambda > 0$ and let $v$ be a large number,

$$\begin{aligned}\mathbf{P}\left(\frac{\overline{X}_v}{a^{-1}(\log v)}\geqslant\lambda\right) &= \mathbf{P}\left(\log\sigma_X(\lambda a^{-1}(\log v))\leqslant\log v\right)\\ &= \mathbf{P}\left(\frac{\log\sigma_X(x)}{c(x)}\leqslant\frac{1}{\lambda^{1/\alpha}}\right),\end{aligned}$$

with the change of variable $x = \lambda a^{-1}(\log v)$ and where

$$c(x) = \lambda^{1/\alpha}a(x/\lambda) \underset{x\to\infty}{\sim} a(x). \tag{4.20}$$

Results of Section 2.1 ensure that $(\mathbb{V}_x^{\#}\vee\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_x))/a(x)$ converges in law as $x\to\infty$ towards $\mathbb{S}_1^{\#}\vee\widetilde{U}_{\mathbb{S}}(\overline{\mathbb{S}}_1)$ whose cumulative function is continuous. Hence, it follows from Proposition 4.7 and from (4.20) that

$$\lim_{v\to\infty}\mathbf{P}\left(\frac{\overline{X}_v}{a^{-1}(\log v)}\geqslant\lambda\right) = \mathbf{P}\left(\mathbb{S}_1^{\#}\vee\widetilde{U}_{\mathbb{S}}(\overline{\mathbb{S}}_1)\leqslant\frac{1}{\lambda^{1/\alpha}}\right).$$

We have proved the convergence in law of $\overline{X}_v/a^{-1}(\log v)$ towards the non-degenerate random variable $\Xi = (\mathbb{S}_1^{\#}\vee\widetilde{U}_{\mathbb{S}}(\overline{\mathbb{S}}_1))^{-\alpha}$. Let us calculate the Laplace transform of this law when $\mathbb{S}$ is completely asymmetric. Recall the notation $\tau_x^{\#}$ and $\tau_x$ defined in Section 2.3. Let also $r_1$ be the stopping time:

$$r_1 = \inf\left(x\geqslant 0\ ,\ (\mathbb{S}_{-t})_{t\,\geqslant\,0}\text{ hits }(-\infty,-(1-x))\text{ before it hits }(x,\infty)\right).$$

From the scaling property of $\mathbb{S}$,

$$\begin{aligned}\mathbf{P}\left((\mathbb{S}_1^{\#}\vee\widetilde{U}_{\mathbb{S}}(\overline{\mathbb{S}}_1))^{-\alpha}\leqslant\lambda\right) &= \mathbf{P}\left(\mathbb{S}_\lambda^{\#}\vee\widetilde{U}_{\mathbb{S}}(\overline{\mathbb{S}}_\lambda)\geqslant 1\right)\\ &= \mathbf{P}\left(\tau_1^{\#}\wedge\tau_{r_1}\leqslant\lambda\right),\end{aligned}$$

therefore $\Xi$ and $\tau_1^{\#}\wedge\tau_{r_1}$ have the same law. Let us first assume that $\mathbb{S}$ has no positive jumps and recall that $(-\mathbb{S}_{-t}\ ,\ t\geqslant 0)$ and $(\mathbb{S}_t\ ,\ t\geqslant 0)$ have the same law. It follows from the well known solution of the exit problem for a completely asymmetric Levy process via its scale function $W$ (c.f. [1] , p194) that

$$\begin{aligned} \mathbf{P}(r_1 > x) &= \mathbf{P}\left((\mathbb{S}_{-t})_{t \geqslant 0} \text{ hits } (x,\infty) \text{ before it hits } (-\infty, -(1-x))\right) \\ &= 1 - \mathbf{P}\left((\mathbb{S}_{t})_{t \geqslant 0} \text{ hits } (1-x,\infty) \text{ before it hits } (-\infty, -x)\right) \\ &= 1 - \frac{W(x)}{W(1)}, \end{aligned}$$

and it is known that in our case $W(x) = x^{\alpha-1}/\Gamma(\alpha)$. Hence, the density of $r_1$ is

$$\mathbf{P}(r_1 \in dx) = \frac{\alpha-1}{x^{2-\alpha}}dx \qquad \text{for } x \in (0,1).$$

Using Corollary 2.11 and the independence of $(\mathbb{S}_t)_{t \geqslant 0}$ and $(\mathbb{S}_{-t})_{t \geqslant 0}$, we have, for $q \geqslant 0$,

$$\begin{aligned} \mathbf{E}\left(e^{-q\tau_1^{\#} \wedge \tau_{r_1}}\right) &= \int_0^1 \mathbf{E}\left(e^{-q\tau_1^{\#} \wedge \tau_x}\right) \frac{\alpha-1}{x^{2-\alpha}}dx \\ &= \frac{\alpha-1}{E_\alpha(q)} \int_0^1 \frac{E_\alpha(q(1-x)^\alpha)}{x^{2-\alpha}}dx \\ &= \frac{\alpha-1}{E_\alpha(q)} \sum_{n=0}^{\infty} \frac{q^n}{\Gamma(1+\alpha n)} \int_0^1 \frac{(1-x)^{\alpha n}}{x^{2-\alpha}}dx, \end{aligned}$$

but

$$\frac{1}{\Gamma(1+\alpha n)} \int_0^1 \frac{(1-x)^{\alpha n}}{x^{2-\alpha}}dx = \frac{\Gamma(\alpha-1)}{\Gamma(\alpha(n+1))},$$

hence

$$\begin{aligned} \mathbf{E}\left(e^{-q\tau_1^{\#} \wedge \tau_{r_1}}\right) &= \frac{\Gamma(\alpha)}{E_\alpha(q)} \sum_{n=0}^{\infty} \frac{q^n}{\Gamma(\alpha(n+1))} \\ &= \Gamma(\alpha+1)\frac{E'_\alpha(q)}{E_\alpha(q)}. \end{aligned}$$

We now assume that $\mathbb{S}$ has no negative jumps. Just like in the previous case, we can calculate the density of $r_1$ from the scale function and we find $\mathbf{P}(r_1 \in dx) = (\alpha-1)/(1-x)^{2-\alpha}$ for $x \in (0,1)$. Thus, using Corollary 2.11 we get

$$\begin{aligned} \mathbf{E}\left(e^{-q\tau_1^{\#} \wedge \tau_{r_1}}\right) &= \int_0^1 \mathbf{E}\left(e^{-q\tau_1^{\#} \wedge \tau_x}\right) \frac{\alpha-1}{x^{2-\alpha}}dx \\ &= (\alpha-1) \int_0^1 \frac{E_\alpha(qx^\alpha)}{(1-x)^{2-\alpha}}dx \\ &\quad - \frac{E'_\alpha(q)\alpha(\alpha-1)q}{\alpha q E''_\alpha(q) + (\alpha-1)E'_\alpha(q)} \int_0^1 \frac{x^{\alpha-1}E'_\alpha(qx^\alpha)}{(1-x)^{2-\alpha}}dx. \end{aligned}$$

We already calculated the first integral:

$$\int_0^1 \frac{E_\alpha(qx^\alpha)}{(1-x)^{2-\alpha}}dx = \int_0^1 \frac{E_\alpha(q(1-y)^\alpha)}{y^{2-\alpha}}dy = \frac{\Gamma(\alpha+1)}{\alpha-1}E'_\alpha(q).$$

As for the second integral,

$$\int_0^1 \frac{x^{\alpha-1}E'_\alpha(qx^\alpha)}{(1-x)^{2-\alpha}}dx = \sum_{n=0}^\infty \frac{(n+1)q^n}{\Gamma(\alpha(n+1)+1)}\int_0^1 \frac{x^{\alpha(n+1)-1}}{(1-x)^{2-\alpha}}dx,$$

and it is known that

$$\int_0^1 \frac{x^{\alpha(n+1)-1}}{(1-x)^{2-\alpha}}dx = \frac{\Gamma(\alpha(n+1))\Gamma(\alpha-1)}{\Gamma(\alpha(n+2)-1)},$$

hence

$$\begin{aligned}
&\int_0^1 \frac{x^{\alpha-1}E'_\alpha(qx^\alpha)}{(1-x)^{2-\alpha}}dx \\
&= \frac{\Gamma(\alpha-1)}{\alpha}\sum_{n=0}^\infty \frac{q^n}{\Gamma(\alpha(n+2)-1)} \\
&= \Gamma(\alpha-1)\sum_{n=0}^\infty \frac{(n+2)(\alpha(n+2)-1)q^n}{\Gamma(\alpha(n+2)+1)} \\
&= \Gamma(\alpha-1)\left(\alpha\sum_{n=0}^\infty \frac{(n+1)(n+2)q^n}{\Gamma(\alpha(n+2)+1)} + (\alpha-1)\sum_{n=0}^\infty \frac{(n+2)q^n}{\Gamma(\alpha(n+2)+1)}\right) \\
&= \frac{\Gamma(\alpha-1)}{q}\left(q\alpha E''_\alpha(q) + (\alpha-1)E'_\alpha(q) - \frac{\alpha-1}{\Gamma(\alpha+1)}\right).
\end{aligned}$$

Putting the pieces together, we conclude that

$$\mathbf{E}\left(e^{-q\tau_1^{\#}\wedge\tau_{r_1}}\right) = \frac{(\alpha-1)E'_\alpha(q)}{\alpha q E''_\alpha(q) + (\alpha-1)E'_\alpha(q)}.$$

□

## 5. Comments

5.1. **The case where $\mathbb{V}$ is a stable process.** In the whole paper, we assumed $\mathbb{V}$ to be a random walk in the domain of attraction of a stable process $\mathbb{S}$. Let us now assume that $\mathbb{V}$ itself is a strictly stable process (such that $|\mathbb{V}|$ is not a subordinator) and let us explain why Theorems $1-4$ still hold in this case. It is clear that all the results dealing with the fluctuations of $\mathbb{V}$ remain unchanged (in fact, they even take a nicer form since we can now choose $a(x)=x^\alpha$ and $b(x)=x^q$). Notice also that we did not use the fact that $\mathbb{V}$ was a random walk in the proofs of the theorems in Section 4. Indeed, the only time we really used the assumption that $\mathbb{V}$ was flat on the intervals $(n,n+1)$ , $n\in\mathbb{Z}$ was in the proofs of Propositions 3.1 and 3.2 (we needed to make sure that $\mathbb{V}$ spends "enough" time around its local extremas). Looking closely at those two proofs, we see that they will still hold if we can show that there exists a measurable event $\mathcal{E}_{31}(v)$ such that:

(a) there exists $\mathbf{C}_{56}$ such that $\mathbf{P}\left(\mathcal{E}_{31}(v)^c\right) \leqslant \mathbf{C}_{56}\exp(-\log^2 v)$.

(b) On $\mathcal{E}_{31}(v)$, any path of $\mathbb{V}$ is such that for all $x \in [-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + \log^4 v), v]$, we have $|\mathbb{V}_y - \mathbb{V}_x| \leqslant 1$ either for all $y$ in $[x, x + \exp(-\log^3 v)]$ or for all $y$ in the interval $[x - \exp(-\log^3 v), x]$.

Let us quickly explain how we can construct this event. Define the sequence of random variables $(\gamma_n)_{n\in\mathbb{Z}}$:

$$\begin{cases} \gamma_0 = 0, \\ \gamma_{n+1} = \inf(t > \gamma_n \ , \ |\mathbb{V}_t - \mathbb{V}_{\gamma_n}| \geqslant \frac{1}{2}) \text{ for } n \geqslant 0, \\ \gamma_{-n-1} = \inf(t < \gamma_{-n} \ , \ |\mathbb{V}_t - \mathbb{V}_{\gamma_{-n}}| \geqslant \frac{1}{2}) \text{ for } n \geqslant 0. \end{cases}$$

Let us set

$$\begin{aligned} \mathcal{E}_{32}(v) &= \left\{\gamma_{i+1} - \gamma_i > 2e^{-\log^3 v} \text{ for all } -e^{\frac{1}{2}\log^3 v} \leqslant i \leqslant e^{\frac{1}{2}\log^3 v}\right\}, \\ \mathcal{E}_{33}(v) &= \left\{\gamma_{-[e^{\frac{1}{2}\log^3 v}]} > e^{\log^{5/2} v} \ , \ \gamma_{[e^{\frac{1}{2}\log^3 v}]} > e^{\log^{5/2} v}\right\}, \\ \mathcal{E}_{34}(v) &= \left\{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + \log^4 v) \leqslant e^{\log^{5/2} v}\right\}, \\ \mathcal{E}_{31}(v) &= \mathcal{E}_{32}(v) \cap \mathcal{E}_{33}(v) \cap \mathcal{E}_{34}(v). \end{aligned}$$

It is clear that (b) holds for $\mathcal{E}_{31}$. We now assume that $v$ is very large. We have

$$\mathbf{P}\left(\mathcal{E}_{32}(v)^c\right) \leqslant 2e^{\frac{1}{2}\log^3 v}\mathbf{P}\left(\gamma_1 \leqslant 2e^{-\log^3 v}\right) \leqslant \mathbf{C}_{57}e^{-\frac{1}{2}\log^3 v},$$

where we used the relation $\mathbf{P}(\gamma_1 \leqslant x) = \mathbf{P}(\mathbb{V}^*_x \geqslant \frac{1}{2})$ and Corollary 2.16 for the last inequality. Using Cramer's large deviation theorem, it is easy to check that $\mathbf{P}\left(\mathcal{E}_{33}(v)^c\right) \leqslant e^{-v}$ (in fact, we can obtain a much better bound). We also have $\mathbf{P}(\mathcal{E}_{34}(v)^c) \leqslant 3e^{-\log^2 v}$ (compare with the proof of the inequality (3.8) for details). Thus, (a) holds.

5.2. **Non-symmetric environments.** In the whole paper, in order to avoid even more complicated notations, we assumed that the processes $(\mathbb{V}_x \ , \ x \geqslant 0)$ and $(-\mathbb{V}_{-x} \ , \ x \leqslant 0)$ have the same law. However it is easy to see that this assumption can be relaxed. Indeed, we may swap Assumption 1 for the following assumption.

**Assumption 2.** *$(\mathbb{V}_n)_{n\geqslant 0}$ and $(\mathbb{V}_{-n})_{n\geqslant 0}$ are independent random walks and there exists a positive sequence $(a_n)_{n\geqslant 0}$ such that*

$$\frac{\mathbb{V}_n}{a_n} \xrightarrow[n\to\infty]{law} \mathbb{S}^1 \text{ and } \frac{-\mathbb{V}_{-n}}{a_n} \xrightarrow[n\to\infty]{law} \mathbb{S}^2,$$

*where $\mathbb{S}^1$ and $\mathbb{S}^2$ are random variables whose laws are strictly stable with respective parameters $(\alpha, p_1)$ and $(\alpha, p_2)$ and whose densities are everywhere positive on $\mathbb{R}$.*

It is crucial to assume that the norming sequence $(a_n)$ may be chosen to be the same for both random walk (in order to keep the results of functional convergence of Section 2.1) but the positivity parameters $p_1$ and $p_2$ need not be the same. Theorem 1-4 must be adapted in consequences. For example, Theorem 1 now takes the form:

**Theorem 5.** *Under the annealed probability* $\mathbf{P}$*, almost surely,*

$$\limsup_{t\to\infty} \frac{X_t}{a^{-1}(\log t)\log\log\log t} = \frac{1}{K^{\#,1}},$$

*where* $K^{\#,1}$ *depends only on* $\mathbb{S}^1$ *and is given by*

$$K^{\#,1} = -\lim_{t\to\infty} \frac{1}{t} \log \mathbf{P}\left( \sup_{0 \leqslant u \leqslant v \leqslant t} \left(\mathbb{S}^1_v - \mathbb{S}^1_u\right) \leqslant 1\right).$$

*Furthermore, when* $\mathbb{S}^1$ *is completely asymmetric,* $K^{\#,1}$ *is given by*

$$K^{\#,1} = \begin{cases} \rho_1(\alpha) & \text{when } \mathbb{S}^1 \text{ has no positive jumps,} \\ \rho_2(\alpha) & \text{when } \mathbb{S}^1 \text{ has no negative jumps.} \end{cases}$$

Let now $(\mathbf{T}_n)$ stand for the sequence of strictly ascending ladder indices of the random walk $(\mathbb{V}_{-x})_{x\geqslant 0}$:

$$\begin{cases} \mathbf{T}_0 = 0, \\ \mathbf{T}_{n+1} = \min\left(k > \mathbf{T}_n \ , \ \mathbb{V}_{-k} > \mathbb{V}_{-\mathbf{T}_n}\right). \end{cases}$$

Hence, $\mathbf{T}_1$ is in the domain of attraction of a positive stable law with index $p_2$ and we choose $b(\cdot)$ to be a continuous positive increasing function such that $(b(n))_{n\geqslant 1}$ is a norming sequence for $\mathbf{T}_1$. Theorem 2 now takes the form:

**Theorem 6.** *For any positive, non-decreasing function* $f$ *define*

$$K(f) = \int^{\infty} \frac{b^{-1}\left(a^{-1}(\log t)/f(t)\right) dt}{b^{-1}\left(a^{-1}(\log t)\right) t \log t}.$$

*We have, almost surely,*

$$\liminf_{t\to\infty} \frac{f(t)}{a^{-1}(\log t)} \sup_{s\leqslant t} X_s = \begin{cases} 0 \\ \infty \end{cases} \iff K(f) \begin{cases} = \infty \\ < \infty. \end{cases}$$

*In particular, with probability* 1*,*

$$\liminf_{t\to\infty} \frac{(\log\log t)^{\beta}}{a^{-1}(\log t)} \sup_{s\leqslant t} X_s = \begin{cases} 0, & \text{if } \beta < 1/p_2, \\ \infty, & \text{if } \beta > 1/p_2. \end{cases} \tag{5.1}$$

Theorems 3 and 4 must be adapted similarly. Notice that like in Theorem 4, we can again calculate the Laplace transform of the limiting law when $\mathbb{S}^1$ and $\mathbb{S}^2$ have both completely asymmetric laws.

5.3. **Random walk in random environment.** Let us recall the connection between the diffusion in random potential and the model of Sinai's random walk in random environment. Let $\omega = (\omega_i)_{i\in\mathbb{Z}}$ be an i.i.d. family of random variables in $(0,1)$ and define for each realization of this family a Markov chain $(Z_n)_{n\geqslant 0}$ by $Z_0 = 0$ and

$$\mathbf{P}\left(Z_{n+1} = Z_n + e \mid Z_n = x, (\omega_i)_{i\in\mathbb{Z}}\right) = \begin{cases} \omega_x & \text{if } e = 1, \\ 1-\omega_x & \text{if } e = -1. \end{cases}$$

$(Z_n)$ is a random walk in the random environment $\omega$. We now define the associated two-sided random walk $(\mathbb{V}_n)_{n\in\mathbb{Z}}$ by $\mathbb{V}_0 = 0$ and $\mathbb{V}_{n+1} - \mathbb{V}_n =$

$\log((1-\omega_n)/\omega_n)$ for all $n \in \mathbb{Z}$. Let $X$ still denote the random diffusion in the random potential $\mathbb{V}$. The following result from Schumacher [19] relates the two processes $X$ and $Z$.

**Proposition 5.1.** *Define the sequence* $(\mu_n)_{n \geqslant 0}$ *by*

$$\begin{cases} \mu_0 = 0, \\ \mu_{n+1} = \inf\left(t > \mu_n\ ,\ |X_{\mu_{n+1}} - X_{\mu_n}| = 1\right). \end{cases}$$

*Under the annealed probability* $\mathbf{P}$*, the sequence* $(\mu_{n+1} - \mu_n)_{n \geqslant 0}$ *is i.i.d. and* $\mu_1$ *is distributed as the first hitting time of* 1 *of a reflected standard Brownian motion. Moreover, for each realization of the environment* $\omega$*, the processes* $(X_{\mu_n})_{n \geqslant 0}$ *and* $(Z_n)_{n \geqslant 0}$ *have the same law.*

Using this proposition, we can easily adapt Theorem 1-4 for the random walk in random environment $Z$ in the case where $\mathbb{V}_1 = \log((1-\omega_0)/\omega_0)$ satisfies Assumption 1 (see Section 10 of [12] for details). For example, Theorem 3 for $Z$ takes the form:

**Theorem 7.** *When* $\mathbb{S}$ *has jumps of both signs, we have, with probability* 1*, for any non-decreasing positive sequence* $(c_n)_{n \geqslant 0}$*,*

$$\liminf_{n\to\infty} \frac{c_n}{a^{-1}(\log n)} \sup_{k \leqslant n} |Z_k| = \begin{cases} 0 \\ \infty \end{cases} \iff \sum_{n \geqslant 2} \frac{1}{(c_n)^2 n \log n} \begin{cases} = \infty \\ < \infty. \end{cases}$$

*In particular, almost surely,*

$$\liminf_{n\to\infty} \frac{(\log\log n)^\beta}{a^{-1}(\log n)} \sup_{k \leqslant n} |Z_k| = \begin{cases} 0, & \text{if } \beta \leqslant 1/2, \\ \infty, & \text{if } \beta > 1/2. \end{cases}$$

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