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Recurrence for vertex-reinforced random walks on $\mathbb{Z}$ with weak reinforcements*

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Abstract

We prove that any vertex-reinforced random walk on the integer lattice with non-decreasing reinforcement sequence w satisfying $w(k) = o(k^\alpha)$ for some $\alpha < 1/2$ is recurrent. This improves on previous results of Volkov [9] and Schapira [6].

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1 Introduction

In this paper, we consider a one-dimensional vertex-reinforced random walk (VRRW) with non-decreasing weight sequence $w : \mathbb{N} \to (0, \infty)$, that is a stochastic process $X = (X_n)_{n\geq 0}$ on $\mathbb{Z}$, starting from $X_0 = 0$, with transition probabilities:

$$\mathbf{P}\{X_{n+1} = X_n \pm 1 \,|\, \mathcal{F}_n\} = \frac{w(Z_n(X_n \pm 1))}{w(Z_n(X_n + 1)) + w(Z_n(X_n - 1))}$$

where $\mathcal{F}_n \stackrel{\text{def}}{=} \sigma(X_1, \dots, X_n)$ is the natural filtration of the process and $Z_n(x) \stackrel{\text{def}}{=} \#\{0 \leq k \leq n, X_k = x\}$ is the local time of X on site x at time n. This process was first introduced by Pemantle in [3] and then studied in the linear case $w(k) = k + 1$ by Pemantle and Volkov in [5]. They proved the surprising fact that the walk visits only finitely many sites. This result was subsequently improved by Tarrès [7, 8] who showed that the walk eventually gets stuck on exactly 5 consecutive sites almost surely. When the reinforcement sequence grows faster than linearly, the walk still gets stuck on a finite set but whose cardinality may be smaller than 5, see [1, 9] for details. On the other hand, Volkov [9] proved that for sub-linearly growing weight sequences of order n^α with $\alpha < 1$, the walk necessarily visits infinitely many sites almost-surely. Later, Schapira [6] improved this result showing that, when $\alpha < 1/2$, the VRRW is either transient or recurrent. The main result of this paper is to show that the walk is, indeed, recurrent.

Theorem 1.1. *Assume that the weight sequence is non-decreasing and satisfies $w(k) = o(k^\alpha)$ for some $\alpha < 1/2$. Then X is recurrent* i.e. *it visits every site infinitely often almost-surely.*

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Let us mention that, simultaneously with the writing of this paper, a similar result was independently obtained by Chen and Kozma [2] who proved recurrence for the VRRW with weights of order n^α, $\alpha < 1/2$, using a clever martingale argument combined with previous local time estimates from Schapira [6]. The argument in this paper, while also making use of a martingale, is self-contained and does not rely upon previous results of Volkov [9] or Schapira [6]. In particular, we do not require any assumption on the regular variation of the weight function w.

2 A martingale

Obviously, multiplying the weight function by a positive constant does not change the process X. Thus, we now assume without loss of generality that $w(0) = 1$. We define the two-sided sequence $(a_x)_{x\in\mathbb{Z}}$ by

$$a_x \stackrel{\text{def}}{=} \begin{cases} 1 - \frac{1}{(x+2)^{1+\varepsilon}} & \text{for } x \geq 0 \\ \frac{1}{2} & \text{for } x < 0 \end{cases}$$

where $\varepsilon > 0$ will be chosen later during the proof of the theorem. Define also

$$A_k \stackrel{\text{def}}{=} \prod_{x=-k}^{\infty} a_x \in (0,1).$$

We construct from X two processes $(M_n)_{n\geq 0}$ and $(\Delta_n(z),\, z < X_n)_{n\geq 0}$ in the following way:

1. Initially set $M_0 \stackrel{\text{def}}{=} 0$ and $\Delta_0(z) \stackrel{\text{def}}{=} 1$ for all $z < 0 = X_0$.
2. By induction, M_n and $(\Delta_n(z),\, z < X_n)$ having been constructed,

   - if $X_n = x$ and $X_{n+1} = x - 1$, then

   $$\begin{aligned} M_{n+1} &\stackrel{\text{def}}{=} M_n - a_x \Delta_n(x-1) \\ \Delta_{n+1}(z) &\stackrel{\text{def}}{=} \Delta_n(z) \quad \text{for } z < x-1, \end{aligned}$$

   - if $X_n = x$ and $X_{n+1} = x + 1$, then

   $$\begin{aligned} M_{n+1} &\stackrel{\text{def}}{=} M_n + a_x \Delta_n(x-1) \tfrac{w(Z_n(x-1))}{w(Z_n(x+1))} \\ \Delta_{n+1}(z) &\stackrel{\text{def}}{=} \begin{cases} \Delta_n(z) & \text{for } z < x, \\ a_x \Delta_n(x-1) \frac{w(Z_n(x-1))}{w(Z_n(x+1))} & \text{for } z = x. \end{cases} \end{aligned}$$

Note that the quantities Δ have a simple interpretation: for any n and $z < X_n$, the value $\Delta_n(z)$ is positive and corresponds to the increments of M_n the last time before n that the walk X jumped from site z to site $z+1$ (with the convention $\Delta_n(z) = 1$ for negative z if no such jumps occurred yet). By extension, we also define $\Delta_n \stackrel{\text{def}}{=} \Delta_n(X_n)$ at the current position as the "would be" increment of M_n if X makes its next jumps to the right (at time $n+1$) *i.e.*

$$\Delta_n \stackrel{\text{def}}{=} a_{X_n} \Delta_n(X_n - 1) \frac{w(Z_n(X_n - 1))}{w(Z_n(X_n + 1))}.$$

We will also use the notation τ_y to denote the hitting time of site y,

$$\tau_y \stackrel{\text{def}}{=} \inf\{n \geq 0,\, X_n = y\} \in [0, \infty].$$

Proposition 2.1. *The process M is an $\mathcal{F}_n$-martingale and, for $n \geq 0$, we have*

$$M_n = \sum_{i=0}^{n-1} \mathbf{1}_{\{X_{i+1}=X_i+1\}} \left(1 - a_{X_{i+1}} \mathbf{1}_{\{\exists j \in (i,n],\, X_j = X_i\}}\right) \Delta_i + \frac{1}{2} \inf_{i \leq n} X_i \tag{2.1}$$

In particular, for $y = 1, 2, \ldots$, the process $M_{n \wedge \tau_{-y}}$ is bounded below by $-y/2$, hence it converges a.s.

Proof. Since $\Delta_n(\cdot)$ and $Z_n(\cdot)$ are $\mathcal{F}_n$-measurable, by definition of M,

$$\begin{aligned}
&\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] \\
&= \mathbb{E}\left[M_n + a_{X_n} \Delta_n(X_n - 1) \left(\frac{w(Z_n(X_n-1))}{w(Z_n(X_n+1))} \mathbf{1}_{\{X_{n+1}=X_n+1\}} - \mathbf{1}_{\{X_{n+1}=X_n-1\}} \right) \Big|\, \mathcal{F}_n \right] \\
&= M_n + a_{X_n} \Delta_n(X_n - 1) \left(\frac{w(Z_n(X_n-1))}{w(Z_n(X_n+1))} \mathbf{P}\{X_{n+1} = X_n + 1 \,|\, \mathcal{F}_n\} - \mathbf{P}\{X_{n+1} = X_n - 1 \,|\, \mathcal{F}_n\} \right) \\
&= M_n
\end{aligned}$$

thus M is indeed a martingale. Furthermore, by construction, at each time i when the process X crosses an edge $\{x, x+1\}$ from left to right, the process M increases by $\Delta_i = \Delta_{i+1}(x) > 0$. If at some later time, say $j > i$, X crosses this edge again (and thus in the other direction), the martingale decreases by $a_{x+1}\Delta_j(x) = a_{x+1}\Delta_i$. Moreover, by convention $\Delta_0(z) = 1$ and $a_z = \frac{1}{2}$ for $z < 0$ so that M decreases by $\frac{1}{2}$ each time it crosses a new edge of the negative half line for the first time. Putting these facts together, we get exactly (2.1). Finally, since $a_z < 1$ for any $z \in \mathbb{Z}$, each term in the sum (2.1) is positive, hence $M_{n \wedge \tau_{-y}}$ is bounded below by $\frac{1}{2} \inf_{i \leq n \wedge \tau_{-y}} X_i \geq -y/2$. □

Proposition 2.2. *Let $y > 0$. For $n \leq \tau_{-y}$, we have*

$$\Delta_n(z) \geq \frac{A_y}{w(Z_n(z)) w(Z_n(z+1))} \quad \textit{for any } -y \leq z \leq X_n. \tag{2.2}$$

Proof. We prove by induction on n that for $n \leq \tau_{-y}$,

$$\Delta_n(z) \geq \frac{\prod_{i=-y}^{z} a_i}{w(Z_n(z)) w(Z_n(z+1))} \text{ for any } -y \leq z \leq X_n. \tag{2.3}$$

Recalling that $w(k) \geq 1$ and $a_k \leq 1$ for any k, it is straightforward that (2.3) holds for $n = 0$. Now, assume the result for n and consider the two cases:

- If $X_{n+1} = X_n - 1$. Then for any $-y \leq z \leq X_{n+1}$, we have $\Delta_{n+1}(z) = \Delta_n(z)$ whereas $w(Z_{n+1}(z)) \geq w(Z_n(z))$. Thus (2.3) holds for $n+1$.
- If $X_{n+1} = X_n + 1$. Again, we have $\Delta_{n+1}(z) = \Delta_n(z)$ for any $-y \leq z \leq X_n$. It remains to check that $\Delta_{n+1}(X_{n+1})$ satisfies the inequality:

$$\begin{aligned}
\Delta_{n+1}(X_{n+1}) &= \Delta_{n+1} = a_{X_{n+1}} \Delta_{n+1}(X_n) \frac{w(Z_{n+1}(X_n))}{w(Z_{n+1}(X_{n+1}+1))} \\
&\geq a_{X_{n+1}} \frac{\prod_{i=-y}^{X_n} a_i}{w(Z_n(X_n)) w(Z_n(X_n+1))} \frac{w(Z_{n+1}(X_n))}{w(Z_{n+1}(X_{n+1}+1))} \\
&\geq \frac{\prod_{i=-y}^{X_{n+1}} a_i}{w(Z_{n+1}(X_{n+1})) w(Z_{n+1}(X_{n+1}+1))}.
\end{aligned}$$

□

We can now recover, with our assumptions on w, Volkov's result [9] stating that the walk does not get stuck on any finite interval.

Proposition 2.3. *For any $y > 0$, we have*

$$\limsup_n X_n = +\infty \quad \text{on the event } \{\tau_{-y} = \infty\}.$$

Proof. On $\{\tau_{-y} = \infty\}$, the combination of (2.1) and Proposition 2.2 give

$$M_n \geq \sum_{i=0}^{n-1} \mathbf{1}_{\{X_{i+1}=X_i+1\}} \left(1 - a_{X_{i+1}} \mathbf{1}_{\{\exists j>i,\, X_j=X_i\}}\right) \frac{A_y}{w(Z_i(X_i))w(Z_i(X_i+1))} - \frac{y-1}{2}. \quad (2.4)$$

Denoting by $e_n = (s_n, s_n+1)$ the edge which has been most visited at time n, we deduce that on the event $\{\tau_{-y} = \infty\}$,

$$M_n \geq Z_n(e_n)\,(1 - a_{s_n+1})\, \frac{A_y}{w(Z_n(s_n))w(Z_n(s_n+1))} - \frac{y-1}{2},$$

where $Z_n(e_n)$ denotes the number of times the edge e_n has been crossed from left to right before time n. Using that $\max(Z_n(s_n), Z_n(s_n+1)) \leq 2Z_n(e_n)$ and that $w(k) = o(\sqrt{k})$ and that $M_{n\wedge\tau_{-y}}$ converges, we conclude that on $\{\tau_{-y} = \infty\}$, either $Z_n(e_n)$ remains bounded or a_{s_n+1} takes values arbitrarily close to 1. In any case, this means that X goes arbitrarily far to the right hence $\limsup_n X_n = +\infty$. □

3 Proof of theorem 1.1

Fix $y > 0$ and consider the event $\mathcal{E}_y = \{\inf_n X_n = -y+1\}$. Pick $v > 0$ and define N_z to be the number of jumps of X from site z to site $z+1$ before time τ_v (according to the previous proposition τ_v is finite on $\mathcal{E}_y$ so all the N_z are finite). From (2.4), grouping together the contributions to M of each edge $(z, z+1)$, we get, on $\mathcal{E}_y$,

$$\begin{aligned}
M_{\tau_v} &\geq A_y \sum_{z=-y+1}^{v-1} \frac{1 + (N_z - 1)(1 - a_v)}{w(Z_{\tau_v}(z))w(Z_{\tau_v}(z+1))} - \frac{y-1}{2} \\
&\geq A_y \sum_{z=-y+1}^{v-1} \frac{1 + (N_z - 1)(1 - a_v)}{w(N_{z-1} + N_z)w(N_z + N_{z+1})} - \frac{y-1}{2} \\
&\geq A_y \sum_{z=-y+1}^{v-1} \frac{\frac{1}{2} + N_z(1 - a_v)}{w(N_{z-1} + N_z)w(N_z + N_{z+1})} - \frac{y-1}{2} \\
&\geq CA_y \sum_{z=-y+1}^{v-1} \frac{\frac{1}{2} + \frac{N_z}{(v+2)^{1+\varepsilon}}}{(N_{z-1} + N_z)^\alpha (N_z + N_{z+1})^\alpha} - \frac{y-1}{2}
\end{aligned}$$

where $C > 0$ and $\alpha < 1/2$ only depend on the weight function w. Finally, lemma 4.1 below states that if we choose $\varepsilon > 0$ small enough, the sum above becomes arbitrarily large almost surely as v tends to infinity. On the other hand, we also know that M converges on this event so necessarily $\mathbf{P}\{\mathcal{E}_y\} = 0$. Since this result holds for any $y > 0$, we get $\inf X_n = -\infty$ a.s. By symmetry, $\sup X_n = +\infty$ a.s. which implies that the walk visits every site of the integer lattice infinitely often almost surely.

4 An analytic lemma

Lemma 4.1. *For any $0 < \alpha < \frac{1}{2}$, there exists $\varepsilon > 0$ such that*

$$\limsup_{K\to\infty} \inf_{(b_0,\ldots,b_K)\in[1,\infty)^{K+1}} \sum_{i=0}^{K} \frac{\frac{1}{2} + \frac{b_i}{(K+2)^{1+\varepsilon}}}{(b_{i-1} + b_i)^\alpha (b_i + b_{i+1})^\alpha} = \infty \quad (4.1)$$

(with the convention $b_{-1} = b_{K+1} = 0$).

Proof. The idea is to group the b_i's into packets with respect to their value. Consider a reordering of the b_i's:

$$\tilde{b}_0 \geq \tilde{b}_2 \geq \ldots \geq \tilde{b}_K.$$

Fix a positive integer l and group these numbers into $l+1$ packets

$$\underbrace{\tilde{b}_0, \ldots, \tilde{b}_{K_1}}_{\text{packet } \mathcal{P}_1}, \underbrace{\tilde{b}_{K_1+1}, \ldots, \tilde{b}_{K_2}}_{\text{packet } \mathcal{P}_2}, \ldots, \underbrace{\tilde{b}_{K_{l-1}}, \ldots, \tilde{b}_{K_l}}_{\text{packet } \mathcal{P}_l}, \underbrace{\tilde{b}_{K_l+1}, \ldots, \tilde{b}_K}_{\text{packet } \mathcal{P}_{l+1}}.$$

We can choose the K_i's growing geometrically such that the sizes of the packets satisfy

$$\#\mathcal{P}_1 \geq \frac{K}{4^l} \quad \text{and} \quad \#\mathcal{P}_i \geq 3(\#\mathcal{P}_1 + \ldots + \#\mathcal{P}_{i-1}). \tag{4.2}$$

We now regroup each term of the sum (4.1) according to which packet the central b_i (the one appearing in the numerator) belongs. Assume by contradiction that the sum (4.1) is bounded, say by A.

We first consider only the terms corresponding to packet $\mathcal{P}_1$. Since there are at least $\frac{K}{4^l}$ terms, we obtain the inequality

$$A \geq \frac{K}{4^l} \frac{\frac{1}{2} + \frac{\tilde{b}_{K_1}}{(K+2)^{1+\varepsilon}}}{(2\tilde{b}_0)^\alpha (2\tilde{b}_0)^\alpha} \geq C \frac{\tilde{b}_{K_1}}{K^\varepsilon \tilde{b}_0^{2\alpha}}$$

where the constant C (which may change from one line to the next) does not depend on K or on the sequence (b_i). We now deal with packets $k = 2, \ldots, l$. Thanks to (4.2) and since every denominator in (4.1) involves only two b_j other than the one appearing in the numerator, there are least one third of the terms belonging to packet $\mathcal{P}_k$ that do not contain any b_j from a packet with smaller index (*i.e.* with larger value). So, there is at least $\frac{K}{4^l}$ such terms for which we can get a lower bound the same way we did for packet $\mathcal{P}_1$. We deduce that, summing over the terms corresponding to packet $\mathcal{P}_k$,

$$\tilde{b}_{K_k} \leq C K^\varepsilon \tilde{b}_{K_{k-1}}^{2\alpha} \quad \text{for } i = 1, \ldots, l, \tag{4.3}$$

with the convention $K_0 = 0$ and where C again does not depend on K or (b_i). Finally, we obtain a lower bound for the sum of the terms belonging to the last packet $\mathcal{P}_{l+1}$ by taking $\frac{1}{2}$ as the lower bound for the numerator, and considering only the terms for which no b_i's from any other packet appear in the denominator (again, there are at least $\frac{K}{4^l}$ such terms). This give the inequality

$$K \leq C \tilde{b}_{K_l}^{2\alpha}. \tag{4.4}$$

Combining (4.3) and (4.4), we get by induction that for some constant C depending on l,

$$K \leq C K^{\varepsilon(2\alpha + (2\alpha)^2 + \ldots + (2\alpha)^l)} \tilde{b}_0^{(2\alpha)^{l+1}} \leq C K^{\frac{\varepsilon}{1-2\alpha}} \tilde{b}_0^{(2\alpha)^{l+1}}.$$

For ε small enough such that $\frac{\varepsilon}{1-2\alpha} \leq \frac{1}{2}$ we obtain

$$\tilde{b}_0 \geq \frac{1}{C} K^{\frac{1}{2(2\alpha)^{l+1}}}.$$

Recalling that the sum (4.1) contains the term corresponding to $\tilde{b}_0$ but is also, by assumption, bounded above by A, we find

$$A \geq \frac{\frac{1}{2} + \frac{\tilde{b}_0}{(K+2)^{1+\varepsilon}}}{(2\tilde{b}_0)^\alpha (2\tilde{b}_0)^\alpha} \geq C K^{\frac{1-2\alpha}{2(2\alpha)^{l+1}} - 1 - \varepsilon}.$$

Finally, we choose l large enough such that $\frac{1-2\alpha}{2(2\alpha)^{l+1}} - 1 - \varepsilon > 0$ and we get a contradiction by letting K tends to infinity. □

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