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Transcendence of values of logarithms of E-functions

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Abstract
Let f be an E-function (in Siegel's sense) not of the form $e^{\beta z}$, $\beta \in \overline{\mathbb{Q}}$, and let log denote any fixed determination of the complex logarithm. We first prove that there exists a finite set $S(f)$ such that for all $\xi \in \overline{\mathbb{Q}} \setminus S(f)$, $\log(f(\xi))$ is a transcendental number. We then quantify this result when f is an E-function in the strict sense with rational coefficients, by proving an irrationality measure of $\ln(f(\xi))$ when $\xi \in \mathbb{Q} \setminus S(f)$ and $f(\xi) > 0$. This measure implies that $\ln(f(\xi))$ is not an ultra-Liouville number, as defined by Marques and Moreira. The proof of our first result, which is in fact more general, uses in particular a recent theorem of Delaygue. The proof of the second result, which is independent of the first one, is a consequence of a new linear independence measure for values of linearly independent E-functions in the strict sense with rational coefficients, where emphasis is put on other parameters than on the height, contrary to the case in Shidlovskii's classical measure for instance.

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1 Introduction

In this paper, we pursue our study of the properties of the values of E-functions at algebraic points, and more specifically of the logarithms of these values. We recall the definition of E-functions, due to Siegel [18]. As usual, we embed $\overline{\mathbb{Q}}$ in $\mathbb{C}$. A power series $F(z) = \sum_{n=0}^{\infty} a_n z^n/n! \in \overline{\mathbb{Q}}[[z]]$ is said to be an E-function *in Siegel's sense* if

(i) $F(z)$ is solution of a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

(ii) For all $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, all Galois conjugates of a_n have modulus less than $n!^{\varepsilon}$.

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(iii) There exists a sequence of positive integers d_n such that $d_n a_m$ are algebraic integers for all $m \leq n$ and such that for all $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, $d_n \leq n!^{\varepsilon}$.

An *a priori* smaller class of E-functions, called *strict* E-functions, have also been considered in the literature. They are defined as follows: (i) still holds but in (ii) and (iii) we now assume that stronger bounds hold, *i.e.* we assume the existence of $C > 0$ and $D > 0$ such that, for all $n \geq 0$, all Galois conjugates of a_n have modulus less than C^{n+1}, and $d_n \leq D^{n+1}$. It is believed that an E-function in Siegel's sense is automatically a strict E-function; see [2, p. 715] for a discussion. Unless otherwise specified, E-functions below will be understood in Siegel's sense. Note that if $a_n \in \mathbb{Q}$, in (i), there exists such a differential equation with coefficients in $\mathbb{Q}(z)$, and the normalized one of minimal order also has coefficients in $\mathbb{Q}(z)$. An E-function is either a polynomial or a transcendental function.

Let f and g be two E-functions. If f is transcendental and g is a polynomial, then $f(\overline{\mathbb{Q}}) \cap g(\overline{\mathbb{Q}})$ is finite by [1]. If f and g are polynomials, two cases occur: if one of them is a constant, $f(\overline{\mathbb{Q}}) \cap g(\overline{\mathbb{Q}})$ is finite, while if none is a constant, $f(\overline{\mathbb{Q}}) \cap g(\overline{\mathbb{Q}})$ is infinite. Our first result completes the picture; it shows that a transcendental E-function is determined by the set of values it takes at algebraic numbers.

Theorem 1 *Let f, g be two transcendental E-functions such that $f(z)$ is not of the form $g(\beta z)$, $\beta \in \overline{\mathbb{Q}}$. Then $\{(\xi, \eta) \in \overline{\mathbb{Q}}^2 : f(\xi) = g(\eta)\}$ is a finite set. Equivalently, $f(\overline{\mathbb{Q}}) \cap g(\overline{\mathbb{Q}})$ is a finite set.*

The equivalence is a consequence of Proposition 2 proved in Sect. 2. The assumption in Theorem 1 is obviously also necessary to obtain finiteness when f and g are transcendental.

The finite sets in Theorem 1 can be determined explicitly in theory when f and g are known, by which we mean that we know differential equations with coefficients in $\overline{\mathbb{Q}}[z]$ satisfied by f and g, and enough of their Taylor coefficients to compute any other Taylor coefficient of f and g from these equations. We explain this in more details in the important particular case considered in Corollary 1.

Applying Theorem 1 to $f(z)$ transcendental and $g(z) = e^z$, we deduce that the set $\{(\xi, \eta) \in \overline{\mathbb{Q}}^2 : f(\xi) = e^{\eta}\}$ is finite when $f(z) \neq e^{\beta z}$ for all $\beta \in \overline{\mathbb{Q}}$. As $\{(\xi, \eta) \in \overline{\mathbb{Q}}^2 : f(\xi) = e^{\eta}\}$ is also finite if $f \in \overline{\mathbb{Q}}[z]$ by the Hermite-Lindemann Theorem, we obtain the following result.

Corollary 1 *Let f be an E-function not of the form $e^{\beta z}$, $\beta \in \overline{\mathbb{Q}}$, and let* $\log$ *denote any fixed determination of the complex logarithm. There exists a finite set $S(f)$ such that for all $\xi \in \overline{\mathbb{Q}} \setminus S(f)$, $\log(f(\xi))$ is a transcendental number.*

As the proof shows, given $\xi \in \overline{\mathbb{Q}}$, there exists an algebraic determination of the logarithm of $f(\xi)$ if, and only if, either $f(\xi) = 1$ or the E-function $f(z) - \exp(z/\varrho)$ vanishes at ξ for some ϱ in the finite set $\mathfrak{S}(f)$ considered in Sect. 2.1. This provides an algorithm to determine $S(f)$. Indeed, one can use the algorithm described in [1] to determine the finite set of $\xi \in \overline{\mathbb{Q}}$ such that $f(\xi) = 1$; moreover, we shall explain at the beginning of Sect. 2.1 how to determine a finite set $\mathfrak{S}$ of algebraic numbers that contains $\mathfrak{S}(f)$, from which again the algorithm in [1] enables to decide, for any

ϱ in $\mathfrak{S}$, whether or not $f(z) - \exp(z/\varrho)$ vanishes at some $\xi \in \overline{\mathbb{Q}}$. We may make more verifications than really necessary, but this is not important as this procedure effectively determines $S(f)$.

This corollary applies to any E-function with a minimal differential equation of order ≥ 2, for example to Bessel's function $J_0(z) := \sum_{n=0}^{\infty}(-1)^n(z/2)^{2n}/n!^2$ whose minimal equation is $zy''(z) + y'(z) + zy(z) = 0$. But the property $\{(\xi,\eta) \in \overline{\mathbb{Q}}^2 : J_0(\xi) = e^{\eta}\} = \{(0,0)\}$ is not a new result as it is a consequence of the much more general [17, p. 219, Theorem 4]: for any $\xi, \eta \in \overline{\mathbb{Q}}^*$, the numbers $J_0(\xi)$, $J_0'(\xi)$ and e^{η} are algebraically independent over $\overline{\mathbb{Q}}$. Other examples of a similar flavor involving generalized hypergeometric series ${}_pF_q[z^{q-p+1}]$ with rational parameters satisfying certain arithmetic conditions can be deduced from the very general algebraic independence result in [7, p. 300, Corollary 4.6]. However, these conditions do not exhaust all such series with rational parameters. To the best of our knowledge, Corollary 1 is new for $f(z) := {}_pF_q[z^{q-p+1}]$ with $1 \leq p \leq q$, $1/2$ as a lower parameter and no upper parameter equal to $1/2 \mod \mathbb{Z}$, because neither assumption A nor assumption B on page 280 of [7] is satisfied, for instance ${}_1F_1[1/3; 1/2; z]$.

Corollary 1 can be quantified in the rational and strict cases. Our method to prove Theorem 2 is independent though of that of Theorem 1 (based on a recent result of Delaygue [10]), as it uses a new linear independence measure for values of E-functions (Theorem 3). When x is a positive real number, we denote by $\ln(x)$ its Napierian logarithm.

Theorem 2 *Let $f \in \mathbb{Q}[[z]]$ be a strict E-function, and $\xi \in \mathbb{Q}^*$ be such that $f(\xi) > 0$ and $\ln(f(\xi)) \notin \mathbb{Q}$. Then there exist $c, d > 0$ such that, for all $(a,b) \in \mathbb{Z} \times \mathbb{N}^*$,*

$$\left|\ln(f(\xi)) - \frac{a}{b}\right| \geq \frac{1}{\exp(cb^d)}. \tag{1.1}$$

In particular, this result implies that $\ln(f(\xi))$ is not an ultra-Liouville number (as defined in [15]).

Theorem 2 applies to any $\xi \in \mathbb{Q}^*$ such that $J_0(\xi) > 0$ because $\ln(J_0(\xi))$ is then a transcendental number by the above mentioned result; the irrationality measure for $\ln(J_0(\xi))$ is new to our knowledge. Note that Theorem 2 can also be applied to any $f \in \mathbb{Q}[z] \setminus \{0\}$, but in this case the lower bound can be much improved because it is known that for any $\alpha \in \mathbb{Q}_{>0}$, $\alpha \neq 1$, the number $\ln(\alpha)$ is an irrational number and not a Liouville number; see [13, p. 150, Satz 5]. If $f(\xi) < 0$, the same result holds for $-f(\xi)$ instead of $f(\xi)$.

The constants c, d in (1.1) depend on f and ξ; they are effective but we did not try to compute them (this could be done in principle) because it is likely that the lower bound in (1.1) is not optimal and could be replaced by c/b^d for some other effective constants $c, d > 0$, proving that $\ln(f(\xi))$ is not a Liouville number. We explain in Sect. 4.2 why it does not seem that this improvement could be obtained with our method, which relies on the following observation:

$$\ln(f(\xi)) \text{ is close to } a/b \text{ if, and only if, } f(\xi) - \exp(a/b) \text{ is small.} \tag{1.2}$$

To deduce Theorem 2, it is enough to bound $|f(\xi)-\exp(a/b)|$ from below. With this aim in mind, we let $f_1 = f$ and consider strict E-functions f_2, …, f_m, with $f_m(z) = \exp(\frac{az}{b\xi})$, such that $(f_1, \dots, f_m)$ are linearly independent over $\mathbb{Q}(z)$, and solution of a first-order differential system. Then a general linear independence measure due to Shidlovskii [17, p. 357, Theorem 1] yields

$$\Big|\sum_{j=1}^{m} a_j f_j(\xi)\Big| > \frac{1}{H^c} \tag{1.3}$$

for any $c > m-1$, any $H \geq H_0$ (where H_0 depends on c), and any integers $a_1, \dots, a_m$, not all zero, such that $\max|a_k| \leq H$. This measure holds more generally for E-functions in Siegel's sense. However there are two problems in applying this result to our setting.

First, lower bounds like Eq. (1.3) are optimized for large values of H whereas in our setting, we have $(a_1, \dots, a_m) = (1, 0, \dots, 0, -1)$: the best choice of H is to take $H = H_0$, so we need to have H_0^c as small as possible. Taking small values of c (very close to $m-1$) may result in larger values of H_0 and, more importantly, of H_0^c. For proving Theorem 2 it is thus necessary to have a very precise control upon H_0. Indeed the second and most important problem for us in such linear independence measures is that the constant H_0 depends on f_1, …, f_m and ξ. In our context, this would result on a lower bound $\left|\ln(f(\xi)) - \frac{a}{b}\right| > H_0^{-c}$ where H_0^{-c} depends on a and b, since the function f_m depends on a and b. Therefore we need to control very precisely how c and H_0 depend on the E-functions and the point ξ under consideration. Usual effectivity considerations, namely proving that H_0 can be effectively computed in terms of c and the f_j, are not sufficient.

These two problems are tackled by the following result. Assume $Y := {}^t(f_1, \dots, f_m)$ is a vector of strict E-functions in $\mathbb{Q}[[z]]$, solution of a differential system $Y' = AY$ with $A \in M_m(\mathbb{Q}(z))$; assume moreover that $f_1, \dots, f_m$ are $\mathbb{Q}(z)$-linearly independent. Let $T \in \mathbb{Z}[z] \setminus \{0\}$ be a common denominator of minimal degree of the entries of $A := (A_{i,j})_{1\leq i,j\leq m}$. We denote by B the maximum modulus of the coefficients of the polynomial $T(z)$ and of all the polynomials $T(z)A_{i,j}(z)$.

Since the $f_i = \sum_{k=0}^{\infty} \frac{\varphi_{k,i}}{k!} z^k$, $i = 1, \dots, m$, are E-functions in the strict sense, there exists a constant $C > 0$ such $|\varphi_{k,i}| \leq C^{k+1}$ for all $k \geq 0$, $i \in \{1, \dots, m\}$, and there exists a constant $D > 0$ such the common denominator $d_{k,i}$ of $\varphi_{0,i}, \dots, \varphi_{k,i}$ satisfies $d_{k,i} \leq D^{k+1}$ for all $k \geq 0$, $i \in \{1, \dots, m\}$.

Let $\xi \in \mathbb{Q}$ be such that $\xi T(\xi) \neq 0$. We denote by κ any real number such that $0 < \kappa \leq \max_{1\leq j\leq m} |f_j(\xi)|$ and for a given rational number r, we set $H(r) := \max(|u|, |v|)$ where $r = u/v$ with u, v coprime.

We denote by $n_0 \geq 1$ the notorious *Shidlovskii constant*, and postpone to Sect. 3.1 a crucial discussion about its effect in our context.

Theorem 3 *In the above conditions, there exists an effective constant c, which depends on m, p, q, $T(z)$, κ and polynomially on $H(\xi)$, B, C and D, such that for any integer $H \geq \max(3, n_0^{n_0})$ and any vector $(a_1, \dots, a_m) \in \mathbb{Z}^m \setminus \{0\}$ such that* $\max|a_k| \leq H$

we have

$$\left|\sum_{j=1}^{m} a_j f_j(\xi)\right| > \frac{1}{H^c}.$$

In more precise terms, there exists a polynomial $C \in \mathbb{R}[W, X, Y, Z]$, which can be computed effectively in terms of $m, p, q, T(z), \kappa$, such that one may choose $c = C(H(\xi), B, C, D)$. For E-functions in the strict sense, Shidlovskii [16, §2, Theorem 1] (reproduced in [17, p. 421, Theorem 2]) was able to refine his lower bound (1.3) to $1/H^{m-1+\gamma m^{7/2}/\log\log(H)^{1/2}}$ where γ depends on all the above parameters except m, and $\log\log(H) \geq \gamma^2 \max(m^2, \ln(n_0))$. However, Shidlovskii did not provide an explicit expression for γ and an upper bound for n_0, nor did he explain the *functional* dependencies on the parameters $H(\xi)$, B, C and D which are crucial for us because $\exp(az/b\xi)$ is among the E-functions f_i, so that B, C and D depend on a and b and this impacts on the lower bound obtained in Theorem 2. To obtain Theorem 3 and in comparison with Shidlovskii's lower bound, we slightly increase the exponent of H with respect to m (which is an accessory parameter for us), but at the same time this enables us to take lower values of H; this turns out to be in our favor in our application because as said above we consider H as a constant and not a large parameter.

Theorem 3 can also be used to provide an irrationality measure of a real irrational zero ζ of any non-polynomial E-function f with rational coefficients. We assume further that ζ is a simple zero of f. Then, we have

$$\zeta - \frac{a}{b} = \frac{1}{f'(\eta)}\Big(f(\zeta) - f\big(\frac{a}{b}\big)\Big) = -\frac{f(a/b)}{f'(\eta)}$$

for some real number η in the interval with endpoints ζ and $a/b \in \mathbb{Q}$. Since ζ is an irrational simple zero of f, neither $f(a/b)$ nor $f'(\eta)$ is equal to 0 when a/b is close to ζ, and a/b is not a singularity of the minimal inhomogeneous linear differential equation satisfied by f. Theorem 3 with $\xi := a/b$ then implies that $|f(a/b)| > \exp(-c_0 b^{d_0})$ for some effective $c_0, d_0 > 0$. Therefore for all $(a, b) \in \mathbb{Z} \times \mathbb{N}^*$,

$$\left|\zeta - \frac{a}{b}\right| \geq \frac{1}{\exp(cb^d)}$$

for some effective $c, d > 0$. This irrationality measure is not as strong as the one obtained by Galochkin who found a lower bound $c_\varepsilon \exp(-b^\varepsilon)$ for all $\varepsilon > 0$, but $c_\varepsilon > 0$ is ineffective; see [17, p. 436].

The structure of this paper is the following. In Sect. 2 we prove Theorem 1 using Delaygue's analogue of the Lindemann–Weierstrass theorem. Then we prove Theorem 3 in Sect. 3. In passing, we deduce from it a corollary, namely Proposition 3 in Sect. 3.1, that we shall use in Sect. 4 to prove Theorem 2.

2 Proof of Theorem 1

2.1 Delaygue's analogue of the Lindemann–Weierstrass theorem

Given an E-function $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$, let $\mathfrak{S}(f) \subset \overline{\mathbb{Q}}^*$ denote the set of finite singularities of the G-function $\sum_{n=0}^{\infty} a_n z^n$. As promised after Corollary 1 in the introduction, let us explain how we can find an explicit set of algebraic numbers that contains $\mathfrak{S}(f)$. The input is f, *i.e.* we have a differential equation $My = 0$, $M \in \overline{\mathbb{Q}}[z][\frac{d}{dz}]$, satisfied by f and we know enough Taylor coefficients of f to compute any other Taylor coefficient of f from the equation. Applying Fourier transform to M, we obtain a differential equation $Ly = 0$ satisfied by g, with $L \in \overline{\mathbb{Q}}[z][\frac{d}{dz}]$ (see [2]); the set $\mathfrak{S}(f)$ is a subset of the set $\mathfrak{S}$ of finite singularities of L.

We shall derive Theorem 1 from the following special case of Delaygue's analogue of the linear version of the Lindemann–Weierstrass theorem (see [10, Corollary 2.2]).

Proposition 1 *Let f_1, f_2 be E-functions and ξ_1, ξ_2 be non-zero algebraic numbers such that $f_1(\xi_1) = f_2(\xi_2)$ is transcendental. Then ξ_1/ξ_2 can be written as ϱ_1/ϱ_2 with $\varrho_1 \in \mathfrak{S}(f_1)$ and $\varrho_2 \in \mathfrak{S}(f_2)$.*

The important point for us, in the conclusion of Proposition 1, is that ξ_1/ξ_2 belongs to a finite set determined by f_1 and f_2.

Remark 1 Proposition 1 is proved in [10] for E-functions in the strict sense. However, it also holds for E-functions in Siegel's sense (and so does the general result [10, Theorem 2.1]) because only the following properties are used in the proof, and they hold for G-functions and E-functions in Siegel's sense by the results proved in [3, 12]:

(1) The point at infinity is regular or a regular singularity of any G-function, because it is solution of a Fuchsian differential operator.
(2) A G-function without finite singularity is a polynomial. Indeed such a function is entire, and has moderate growth at infinity by (1). By Liouville's theorem, it is a polynomial.
(3) Any E-function is annihilated by an E-operator, without non-zero finite singularity.
(4) Beukers' refined version of the Siegel-Shidlovskii theorem (i.e. [8, Theorem 1.3]) holds.

2.2 Application to Theorem 1

We first prove the following result.

Proposition 2 *Let f be a non-constant E-function and $\chi \in \mathbb{C}$. Then the set $\{\alpha \in \overline{\mathbb{Q}} : f(\alpha) = \chi\}$ is finite.*

Proof of Proposition 2 If $\chi \in \overline{\mathbb{Q}}$, this is a consequence of the main result in [1]. Otherwise, let us fix $\alpha_0 \in \overline{\mathbb{Q}}$ such that $f(\alpha_0) = \chi$; if there is no such α_0, the corresponding set is empty and therefore finite. For any $\alpha \in \overline{\mathbb{Q}}$ such that $f(\alpha) = \chi$, Proposition 1 implies that α/α_0 belongs to a finite set determined by f. This concludes the proof of Proposition 2.

Proof of Theorem 1 First of all, let us consider the set of pairs $(\xi, \eta) \in \overline{\mathbb{Q}}^2$ such that $f(\xi) = g(\eta)$ is algebraic. Recall from [1] that a transcendental E-function takes algebraic values at only finitely many algebraic points. Therefore each of ξ, η belongs to a finite set determined by f and g: so does the pair (ξ, η).

Now let us move to pairs $(\xi, \eta) \in \overline{\mathbb{Q}}^2$ such that $f(\xi) = g(\eta)$ is transcendental; this implies $\xi, \eta \neq 0$. Then Proposition 1 provides a finite set (determined by f and g) that contains all quotients η/ξ. For each value β of the quotient, the E-function $f(z) - g(\beta z)$ vanishes at ξ. Since this E-function is not identically zero by hypothesis, ξ belongs to a finite set determined by β. So does η, and this concludes the proof that $\{(\xi, \eta) \in \overline{\mathbb{Q}}^2 : f(\xi) = g(\eta)\}$ is a finite set. This is equivalent to the fact that $I := f(\overline{\mathbb{Q}}) \cap g(\overline{\mathbb{Q}})$ is a finite set. Indeed, if $\{(\xi, \eta) \in \overline{\mathbb{Q}}^2 : f(\xi) = g(\eta)\}$ is finite, then obviously I is finite. Conversely, if I is finite, then for any $\chi \in I$, there are only finitely many $(\xi, \eta) \in \overline{\mathbb{Q}}^2$ such that $f(\xi) = \chi = g(\eta)$ by Proposition 2. This completes the proof of Theorem 1.

3 Proof of Theorem 3

This section is mainly devoted to the proof of Theorem 3. Before that, we recall the setting and discuss the role of n_0; moreover, the important point for our application to Theorem 2 is that when the last function is $\exp(\beta z)$, the constants are controlled in terms of β: this special case is studied in Proposition 3 stated in Sect. 3.1.

The structure of proof is similar to that of Shidlovskii's measure, so we recall it in Sect. 3.2. We proceed to the proof in Sect. 3.3.

3.1 Notations and comments related to Theorem 3

Assume $Y := {}^t(f_1, \ldots, f_m)$ is a vector of strict E-functions in $\mathbb{Q}[[z]]$, solution of a differential system $Y' = AY$ with $A \in M_m(\mathbb{Q}(z))$; assume moreover that $f_1, \ldots, f_m$ are $\mathbb{Q}(z)$-linearly independent. Let $T \in \mathbb{Z}[z] \setminus \{0\}$ be a common denominator of minimal degree of the entries of $A := (A_{i,j})_{1 \le i,j \le m}$. Let $\xi \in \mathbb{Q}$ be such that $\xi T(\xi) \neq 0$. We consider any integer $H \geq 1$ and any vector $(a_1, \ldots, a_m) \in \mathbb{Z}^m \setminus \{0\}$ such that $\max |a_k| \leq H$.

As a special case $\mathbb{K} = \mathbb{Q}$ of [17, p. 357, Theorem 1], Shidlovskii proved that for any $\varepsilon > 0$, there exists an ineffective constant $c > 0$ such that, in the above situation, we have

$$\left| \sum_{j=1}^{m} a_j f_j(\xi) \right| > \frac{c}{H^{m-1+\varepsilon}}. \tag{3.1}$$

This constant c is now effective (because the integer n_0 in Shidlovskii's multiplicity estimate can be bounded, see below). However it depends on $f_1, \ldots, f_m$ in a way which is not made explicit by Shidlovskii. This is a problem to prove Theorem 2, since in our setting f_m will be $\exp(\beta z)$ and we need constants that we control explicitly in terms of β.

We shall prove a linear independence measure, namely Theorem 3 stated in the introduction, in which the dependencies of the constants on certain parameters important for us are made explicit, unlike the classical measures in the same context. With this aim in mind, we define as in [17, p. 93]:

$$p := \min_{1\le j\le m} \mathrm{ord}_{z=0} f_j(z) \quad \text{and} \quad q := \max(\deg T, \max_{i,j} \deg(TA_{i,j})). \tag{3.2}$$

Since the $f_i = \sum_{k=0}^{\infty} \frac{\varphi_{k,i}}{k!} z^k$, $i = 1, \dots, m$, are E-functions in the strict sense, there exists a constant $C > 0$ such $|\varphi_{k,i}| \le C^{k+1}$ for all $k \ge 0$, $i \in \{1, \dots, m\}$, and there exists a constant $D > 0$ such the common denominator $d_{k,i}$ of $\varphi_{0,i}, \dots, \varphi_{k,i}$ satisfies $d_{k,i} \le D^{k+1}$ for all $k \ge 0$, $i \in \{1, \dots, m\}$.

We also denote by B the maximum modulus of the coefficients of the polynomial $T(z)$ and of all the polynomials $T(z)A_{i,j}(z)$.

We denote by $n_0 \ge 1$ the constant in Shidlovskii's zero estimate. Shidlovskii's proof of the existence of n_0 is not effective (see the proof of [17, p. 93, Lemma 8] and the definition of n_0 in [17, p. 99, Eq. (83)]). However, following the works of Bertrand, Beukers, Chirskii and Yebbou [4, 6], it is now known that the integer n_0 can be bounded above using explicit quantities that depend on the matrix A of the differential system. Certain of these quantities are themselves bounded by means of the generalized local exponents at the singularities of A and at the point at infinity (for their definition, see [5, Appendix] or [9, §2.3.4]). More precisely, from the discussion in [6, p. 252] we have that

$$n_0 \le 2(q+1)m^2(\mathcal{E} + (q+1)m + 1), \tag{3.3}$$

where $\mathcal{E}$ is the maximum of all the modulus of the generalized local exponents at the infinite point and at the finite singularities of A.

At last we denote by κ any real number such that

$$0 < \kappa \le \max_{1\le j\le m} |f_j(\xi)|$$

(such a κ exists because there exists at least one j with $f_j(\xi) \ne 0$, because $\xi T(\xi) \ne 0$) and for a given rational number r, we set $H(r) := \max(|u|, |v|)$ where $r = u/v$ with u, v coprime; we also write $\mathrm{den}(r) = |v|$.

We now recall that Theorem 3 says the following: *There exists an effective constant c, which depends on m, p, q, $T(z)$, κ and polynomially on $H(\xi)$, B, C and D, such that if $H \ge \max(3, n_0^{n_0})$ then*

$$\left| \sum_{j=1}^{m} a_j f_j(\xi) \right| > \frac{1}{H^c}.$$

Using Eq. (3.3), the lower bound on H can be replaced by an explicit lower bound in terms of $m, q, \mathcal{E}$.

The important point in Theorem 3 is that c depends on f_1, …, f_m only through a given set of parameters, and also that the dependence on $H(\xi)$, B, C, D, is polynomial. The constant c is effective because the only potential source of ineffectivity of the proof, i.e. n_0, is now known to be effective. In principle it would be possible to make c completely explicit, but this would make the statement of our results much more complicated for no immediate application better than those we present here. Moreover such explicit formulas are not sharp in general.

In this theorem, and throughout this section, the polynomial dependence of c with respect to $H(\xi)$, B, C, D means that there exists a polynomial $P \in \mathbb{R}[W, X, Y, Z]$ with non-negative coefficients and degree depending only on m, p, q, $T(z)$, κ, such that one may choose $c = P(H(\xi), B, C, D)$.

To prove Theorem 2, we shall be interested in the following particular case of the previous considerations:

$$f_m(z) = \exp(\beta z) \text{ and } f_1, \dots, f_{m-1} \text{ are independent from } \beta \in \mathbb{Q}. \tag{3.4}$$

In precise terms, when we refer to (3.4) we shall assume that $Z = {}^t(f_1, \dots, f_{m-1})$ is a vector of E-functions with rational coefficients, solution of a differential system $Z' = MZ$ with $M \in M_{m-1}(\mathbb{Q}(z))$. Then $Y = {}^t(f_1, \dots, f_m)$ is solution of $Y' = AY$ where $A \in M_m(\mathbb{Q}(z))$ is blockwise diagonal, with diagonal blocks M and β. The important point is that $f_1, \dots, f_{m-1}$ and M are independent from β. In this setting we have the following special case of Theorem 3.

Proposition 3 *In the situation (3.4), in particular $f_1, \dots, f_{m-1}, \exp(\beta z)$ are assumed to be linearly independent over $\mathbb{Q}(z)$, there exists an effective constant c, which depends only on $f_1, \dots, f_{m-1}$ and polynomially on $H(\xi)$ and $H(\beta)$, such that with $H := \max(3, n_0^{n_0})$, we have*

$$\left| f_1(\xi) - \exp(\beta\xi) \right| > \frac{1}{H^c}.$$

Moreover an upper bound for H can be given in terms of $f_1, \dots, f_{m-1}$ only, independently of β and of ξ.

Proof In the setting of (3.4), recall that $Z = {}^t(f_1, \dots, f_{m-1})$ is a solution of $Z' = MZ$ with $M \in M_{m-1}(\mathbb{Q}(z))$, and $Y = {}^t(f_1, \dots, f_m)$ of $Y' = AY$. The matrix $A \in M_m(\mathbb{Q}(z))$ is blockwise diagonal, with diagonal blocks M and β. Therefore $T(z)$ is independent of β; so are m, p, q, and also $\mathcal{E}$ because the function $\exp(\beta z)$ has null generalized local exponents everywhere. Hence, Eq. (3.3) shows that n_0 can be bounded independently of β.

Moreover at least one of f_1, …, f_{m-1} does not vanish at ξ, because ξ is not a singularity of the differential system $Z' = MZ$ (indeed it is not a pole of a coefficient of M, because all these coefficients are coefficients of A); of course the functions f_1, …, f_{m-1} are not identically zero because they are linearly independent over $\mathbb{Q}(z)$. Therefore we may choose $\kappa = \min(1, \max_{1 \le j \le m-1} |f_j(\xi)|)$.

Since $A_{m,m} = \beta$, we may take $C = \max(\widetilde{C}, |\beta|)$, $D = \mathrm{den}(\beta)\widetilde{D}$ and $B \leq \max(1, |\beta|)\widetilde{B}$, where $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$ are quantities analogous to B, C, D when we consider only $f_1, \dots, f_{m-1}$. Therefore B, C and D depend linearly on $H(\beta)$. Applying Theorem 3 concludes the proof of Proposition 3.

3.2 Shidlovskii's measure (3.1): sketch of proof

We first recall the statement of [17, p. 107, Lemma 14] when the number field $\mathbb{K} = \mathbb{Q}$, viewed as a subfield of $\mathbb{C}$. Assume $Y := {}^t(f_1, \dots, f_m)$ is a vector of E-functions in Siegel's sense in $\mathbb{Q}[[z]]$ solution of a differential system $Y' = AY$ with $A \in M_m(\mathbb{Q}(z))$. Let $n \in \mathbb{N}^*$ and $\varepsilon_1 \in (0, \frac{1}{2m-1})$; the reason of this technical assumption on ε_1 will appear in Sect. 3.3.3. There exist $P_1, \dots, P_m \in \mathbb{Z}[z]$ of degree at most n and not all zero such that:

$$|b_{i,\nu}| = \mathcal{O}(n^{(1+\varepsilon_1)n}), \quad i = 1, \dots, m, \ \nu = 0, \dots, n,$$

where $b_{i,\nu}$ is the coefficient of z^ν in $P_i(z)$ and the symbol $\mathcal{O}$ is uniform in i and ν, and such that the function

$$R(z) := \sum_{i=1}^{m} P_i(z) f_i(z) = \sum_{\nu=\tau}^{\infty} \frac{\rho_\nu}{\nu!} z^\nu$$

satisfies $\mathrm{ord}_{z=0} R(z) \geq \tau$ with

$$\tau = m(n+1) - \lfloor \varepsilon_1 n \rfloor - 1,$$

and $|\rho_\nu| = \mathcal{O}(\nu^{\varepsilon_1 n} n^n)$ for $\nu \geq \tau$.

In the same setting, define

$$R_k(z) := \sum_{i=1}^{m} P_{k,i}(z) f_i(z)$$

with $P_{k,i} \in \mathbb{Z}[z]$, by letting $R_1(z) = R(z)$ and $R_{k+1}(z) = T(z) R_k'(z)$; recall that $T \in \mathbb{Z}[z] \setminus \{0\}$ is a common denominator of minimal degree of the entries of A.

From now on, we assume that $f_1, \dots, f_m$ are $\mathbb{Q}(z)$-linearly independent, and that $n \geq n_0$ where n_0 was introduced in Sect. 3.1.

Then by Lemma 10 of [17, p. 101], for any $\xi \in \mathbb{C}$ such $\xi T(\xi) \neq 0$, the linear forms $R_k(\xi)$, $k = 1, \dots, m + t_1$, include m linearly independent forms, where

$$t_1 := q \frac{(m-1)m}{2} + \lfloor \varepsilon_1 n \rfloor + p; \tag{3.5}$$

we recall from Sect. 3.1 (and [17, p. 93]) that

$$p := \min_{1 \leq i \leq m} \mathrm{ord}_{z=0} f_i(z) \quad \text{and} \quad q := \max(\deg T, \max_{i,j} \deg(T A_{i,j}))$$

with $A := (A_{i,j})_{1\le i,j\le m}$. Now Lemma 15 in [17, p. 110] says the following. Suppose that $\varepsilon_1 = \varepsilon/(6(m+1))$ for some $\varepsilon \in (0,1)$. Then for any $\xi \in \mathbb{Q}$ such that $\xi T(\xi) \neq 0$, we have

$$|R_k(\xi)| = \mathcal{O}(n^{-(m-1-\varepsilon/2)n}), \quad k \le m + t_1,$$

and

$$|P_{k,i}(\xi)| = \mathcal{O}(n^{(1+\varepsilon/2)n}), \quad k \le m + t_1,\ i = 1, \dots, m.$$

From these estimates, Shidlovskii deduces [17, p. 357, Theorem 1], *i.e.*, (3.1) for E-functions in Siegel's sense.

With Shidlovskii's original method, the constant c in (3.1) was ineffective because of the ineffectivity of the integer n_0. As we have explained before, this is no longer the case and this is crucial for us. For our purpose, we need a different version of the measure (3.1), with a control of the dependencies of the constants on the parameters. We shall make this proof as explicit as possible only for E-functions in the strict sense as in [16], otherwise it would be difficult to obtain a good control on the quantities we are interested in, as for instance the parameters C and D do not exist for E-functions in Siegel's sense. Nonetheless, we follow all the steps in Shidlovskii's proof for E-functions in Siegel's sense as presented in [17] because he gave all the details of the construction and we can then better explain the improvements we make with respect to functional dependencies. Note that this amounts to make as precise as possible the dependence of the constants $\gamma_1, \dots, \gamma_{19}$ in [16, pp. 394–398] on the parameters we are interested in, for which Shidlovskii gave no information neither in [16] nor in [17].

3.3 Proof of Theorem 3

In this section we shall prove Theorem 3 by completing the sketch of proof given in Sect. 3.2 for E-functions in the strict sense, and by making as explicit as possible the dependencies on the parameters. We split the proof in four lemmas proved in Sects. 3.3.1, 3.3.2 and 3.3.3.

In all what follows, as in Sect. 3.1, when we say that *a constant c depends polynomially on a parameter k* we mean that there exists a polynomial P with non-negative coefficients such that $|c| \le P(|k|)$. The polynomial P, including its degree, may depend on all other parameters c depends on.

We shall follow now the sketch of proof given in the previous section; at each step we shall make all bounds explicit (to be precise, we shall make the dependencies in terms of the parameters explicit).

As in Sect. 3.1 we consider a vector $Y = {}^t(f_1, \dots, f_m)$ of strict E-functions in $\mathbb{Q}[[z]]$, solution of a differential system $Y' = AY$ with $A \in M_m(\mathbb{Q}(z))$. We assume that $f_1, \dots, f_m$ are $\mathbb{Q}(z)$-linearly independent, and denote by $T \in \mathbb{Z}[z]\setminus\{0\}$ a common denominator of minimal degree of the entries of A. We fix $\xi \in \mathbb{Q}$ be such that $\xi T(\xi) \neq 0$. As in Sect. 3.2 we consider also $\varepsilon_1 > 0$ such that $\varepsilon_1 < \frac{1}{2m-1}$. We assume from now on that z is a complex variable.

3.3.1 Construction of the polynomials

Following the proof of [17, p. 107, Lemma 14] and using Siegel's lemma, we find that for any $1 \le i \le m$ and any $0 \le \nu \le n$,

$$|b_{i,\nu}| \le n!2\big(m(n+1)CD\big)^{2m/\varepsilon_1}(2CD)^{4m^2n/\varepsilon_1}$$

and for any $\nu \ge \tau$,

$$|\rho_\nu| \le n!m2^{\nu+1}\big(m(n+1)CD\big)^{2m/\varepsilon_1}(2CD)^{4m^2n/\varepsilon_1}C^{\nu+1}.$$

Since

$$R(z) = \sum_{\nu=\tau}^{\infty} \frac{\rho_\nu}{\nu!} z^\nu,$$

we deduce that, for any $z \in \mathbb{C}$,

$$|R(z)| \le n!m\big(m(n+1)CD\big)^{2m/\varepsilon_1}(2CD)^{4m^2n/\varepsilon_1}2C\sum_{\nu=\tau}^{\infty}\frac{(2C|z|)^\nu}{\nu!}.$$

As $\big(CD\big)^{2m/\varepsilon_1}(2CD)^{4m^2n/\varepsilon_1} \le (2CD)^{6m^2n/\varepsilon_1}$ and

$$\sum_{\nu=\tau}^{\infty}\frac{t^\nu}{\nu!} = \frac{1}{(\tau-1)!}\int_0^t (t-x)^{\tau-1}e^x dx \le \frac{t^{\tau-1}e^t}{(\tau-1)!} \quad \text{for any } t \ge 0,$$

this proves the following result.

Lemma 1 *We have*

$$|R(z)| \le C_1(2CD)^{6m^2n/\varepsilon_1}e^{2C|z|}\frac{n!}{\big(m(n+1)-\lfloor\varepsilon_1 n\rfloor\big)!}(2C|z|)^{m(n+1)-\lfloor\varepsilon_1 n\rfloor-2}$$

where $C_1 > 0$ depends on ε_1, m, polynomially on n, not on z and not on the Taylor coefficients of the f_j's. This constant C_1 also satisfies

$$|\rho_\nu| \le n!\,C_1\,(2CD)^{6m^2n/\varepsilon_1}\,(2C)^{\nu+1} \textit{ for any } \nu \ge \tau \tag{3.6}$$

and

$$|b_{i,\nu}| \le n!\,C_1\,(2CD)^{6m^2n/\varepsilon_1} \textit{ for any } 1 \le i \le m \textit{ and any } 0 \le \nu \le n. \tag{3.7}$$

3.3.2 Upper bounds on the linear forms

By a similar analysis of the proof of [17, p. 110, Lemma 15], using Eq. (3.6) in Lemma 1 we see that, for any $z \in \mathbb{C}$,

$$|R_k(z)| \le C_2(2CD)^{6m^2n/\varepsilon_1} \frac{n!k!(2q)^k e^{2C|z|}(1+|z|)^{(k-1)q}}{\big(m(n+1)-\lfloor\varepsilon_1 n\rfloor-k-2\big)!}(2C|z|)^{m(n+1)-\lfloor\varepsilon_1 n\rfloor-k-2}$$

for all $n \ge n_0$ and all $k \in \{1,\dots,m+t_1\}$, where $C_2 > 0$ depends on ε_1, m, p, q, polynomially on n, linearly on the k-th power of the maximum modulus of the coefficients of the polynomial $T(z)$, and not on z.

Moreover, the degree of each $P_{k,i}$ is less than $n+(k-1)q$ by [17, p. 115] and using Eq. (3.7) in Lemma 1 we have, for any $z \in \mathbb{C}$,

$$|P_{k,i}(z)| \le C_3(2CD)^{6m^2n/\varepsilon_1}(1+|z|)^{n+(k-1)q}(m+n)^k n!k!q^k, \quad k \le m+t_1,\ i=1,\dots,m$$

where C_3 depends on ε, m, p, q, polynomially on n and z, and linearly on E^k. As a polynomial in n and z, the degree of C_3 depends only on ε_1, m, p, q.

In the above upper bounds for R_k and $P_{k,i}$, we now use the fact that $k \le m+t_1 \le \lfloor\varepsilon_1 n\rfloor + C_4$, where C_4 depends only on m, p, q (by Eq. (3.5)), but not on n or z. As in [17] we take $z = \xi$ and multiply by a common denominator. After some simplifications, the situation can now be summarized by the two following lemmas (this makes explicit [17, p. 114, Lemma 16]).

Lemma 2 *For every $\xi \in \mathbb{Q}$ such that $\xi T(\xi) \neq 0$ and for all $n \ge n_0$, there exist m linearly independent linear forms*

$$L_j := \sum_{i=1}^{m} a_{j,i} f_i(\xi), \quad a_{j,i} \in \mathbb{Z}, \quad j=1,\dots,m,$$

(that depend on n) such that for any $1 \le j \le m$,

$$|L_j| \le C_5(2CD)^{6m^2n/\varepsilon_1} e^{2C|\xi|}(2q\max(\varepsilon_1, C_4))^{\varepsilon_1 n} \times (1+|\xi|)^{q\varepsilon_1 n}\mathrm{den}(\xi)^{(1+q\varepsilon_1)n}\frac{n^{n+\varepsilon_1 n}(2C|\xi|)^{mn-2\varepsilon_1 n}}{(mn-2\lfloor\varepsilon_1 n\rfloor)!}, \quad (3.8)$$

where the factor C_5 depends on ε_1, m, p, q, polynomially on $H(\xi)$ and n, and linearly on the $(m+t_1)$-th power of the maximum modulus of the coefficients of the polynomial $T(z)$.

We also observe that since $m+t_1 \le \varepsilon_1 n + C_4$ we have $C_5 \le \widetilde{c_5}^n$ where $\widetilde{c_5}$ depends on $\varepsilon_1, m, p, q, T(z)$, polynomially on $H(\xi)$ but not on n.

Lemma 3 *For any* $1 \le i, j \le m$,

$$|a_{j,i}| \le C_6 (2CD)^{6m^2 n/\varepsilon_1} (2q \max(\varepsilon_1, C_4))^{\varepsilon_1 n} \big((1+|\xi|)\mathrm{den}(\xi)\big)^{(1+q\varepsilon_1)n} n^{n+2\varepsilon_1 n}, \tag{3.9}$$

where C_6 *depends on* ε_1, m, p, q, *polynomially on* $H(\xi)$ *and* n, *and linearly in* B^{m+t_1}. *As a polynomial in* $H(\xi)$ *and* n, *the total degree of* C_6 *does not depend on* B.

3.3.3 Conclusion

We are now ready to analyze the proof of [17, p. 357, Theorem 1] in the case $\mathbb{K} = \mathbb{Q}$ in order to make Shidlovskii's measure explicit. Shidlovskii proves that for any $\varepsilon \in (0, 1)$, any integer $H \ge 1$ and any vector $(a_1, \dots, a_m) \in \mathbb{Z}^m \setminus \{0\}$ such that $\max |a_i| \le H$, we have

$$\Big|\sum_{i=1}^m a_i f_i(\xi)\Big| \ge b_4 n^{-(m-1+\varepsilon)n}\big(1 - b_5 H n^{-(1-\varepsilon)n}\big),$$

where $b_4, b_5 > 0$ are not computed but are known to be independent of H. He then chooses the smallest n such that $n \ge n_0$ and $n^{(1-\varepsilon)n} > 2b_5 H$ (such an n obviously exists) to deduce the expected linear independence measure:

$$\Big|\sum_{i=1}^m a_i f_i(\xi)\Big| \ge \frac{b_7}{H^{m-1+2m\varepsilon}}$$

for a constant $b_7 > 0$ which is again not computed but is known to be independent of H.

In the proof of [17, p. 357, Theorem 1] in the case $\mathbb{K} = \mathbb{Q}$, we have $i = h = 1$. Shidlovskii defines the determinant Δ whose entries are the coefficients (in $\mathbb{Z}$) of the m linear forms $L_1, \dots, L_{m-1}, L_0 := \sum_{i=1}^m a_i f_i(\xi)$ (which up to reordering can be assumed to be linearly independent without loss of generality when $n \ge n_0$), and the determinant $\Delta_{j,\ell}$ which is the cofactor of the entry in the j-th row and ℓ-column of Δ. (Each line corresponds to a linear form, with L_1 at the top and L_0 at the bottom of the determinant.) In [17, p. 358, Eq. (41)], each occurence of $i (= 1)$ can be deleted and we have

$$|\Delta_{m,\ell}| \cdot |L_0| \ge |f_\ell(\xi)| \cdot |\Delta| - (m-1) \max_{1 \le j \le m-1} |\Delta_{j,\ell}| \cdot \max_{1 \le j \le m-1} |L_j|. \tag{3.10}$$

This inequality holds for any $\ell \in \{1, \dots, m\}$ such that $f_\ell(\xi) \ne 0$; such an ℓ exists because $\xi T(\xi) \ne 0$. We have $|\Delta| \ge 1$ when $n \ge n_0$. Then using the bounds (3.8) and (3.9) given in Lemmas 2 and 3 respectively, for any $n \ge n_0$, we can replace the three bounds in [17, p. 359, Eq. (42)] by

$$\max_{1\le j\le m-1}|L_j| \le C_5(2CD)^{6m^2n/\varepsilon_1}e^{2C|\xi|}(2q\max(\varepsilon_1, C_4))^{\varepsilon_1 n} \times (1+|\xi|)^{q\varepsilon_1 n}\mathrm{den}(\xi)^{(1+q\varepsilon_1)n}\frac{n^{n+\varepsilon_1 n}(2C|\xi|)^{mn-2\varepsilon_1 n}}{(mn-2\lfloor\varepsilon_1 n\rfloor)!},$$

$$|\Delta_{m,\ell}| \le C_7\big((2CD)^{6m^2n/\varepsilon_1}(2q\max(\varepsilon_1, C_4))^{\varepsilon_1 n}\big((1+|\xi|)\mathrm{den}(\xi)\big)^{(1+q\varepsilon_1)n}n^{n+2\varepsilon_1 n}\big)^{m-1}$$

and

$$\max_{1\le j\le m-1}|\Delta_{j,\ell}| \le C_8 H\big((2CD)^{6m^2n/\varepsilon_1}(2q\max(\varepsilon_1, C_4))^{\varepsilon_1 n}\big((1+|\xi|)\mathrm{den}(\xi)\big)^{(1+q\varepsilon_1)n}n^{n+2\varepsilon_1 n}\big)^{m-2},$$

where $C_7, C_8 > 0$ have the same dependencies as C_6^m, where C_6 is considered in (3.9), and both can be bounded accordingly. We also recall that from the results proved in Sect. 3.3.2, we have that:

- C_4 depends only on m, p, q, but not on n, ξ.
- $C_5 \le \widetilde{c_5}^n$ where $\widetilde{c_5}$ depends on $\varepsilon_1, m, p, q, T(z)$, polynomially on $H(\xi)$ but not on n.
- $C_6 \le \widetilde{c_6}^n$ where $\widetilde{c_6}$ depends on ε_1, m, p, q, polynomially on $H(\xi)$ and on E, but is independent of n. All polynomials involved here have degrees bounded in terms of ε_1, m, p, q.

Putting together these three bounds, we then obtain the following result.

Lemma 4 *For any $n \ge n_0$, we have*

$$\max_{1\le j\le m-1}|L_j| \le C_9 e^{2C|\xi|}n^{C_{10}}C_{11}^n n^{-n(m-1-3\varepsilon_1)}$$

$$|\Delta_{m,\ell}| \le C_9 n^{C_{10}}C_{11}^n n^{n(m-1)(1+2\varepsilon_1)},$$

and

$$\max_{1\le j\le m-1}|\Delta_{j,\ell}| \le C_9 H n^{C_{10}}C_{11}^n n^{n(m-2)(1+2\varepsilon_1)},$$

where :

- *$C_9 \ge 1$ depends on ε_1, m, p, q, polynomially on $H(\xi)$, C and D, but not on n.*
- *$C_{10} \ge 0$ depends on ε_1, m, p, q, polynomially on $H(\xi)$, not on n and not on the Taylor coefficients of the f_j's.*
- *$C_{11} \ge 1$ depends on $\varepsilon_1, m, p, q, T(z)$, polynomially on $H(\xi)$, B, C and D, but not on n.*
- *The total degree of C_9 as a polynomial in $H(\xi)$, C and D, and the one of C_{11} in $H(\xi)$, B, C and D, are bounded in terms of ε_1, m, p, q.*

Now we choose ℓ such that

$$|f_\ell(\xi)| = \max_{1\le j\le m}|f_j(\xi)| \ge \kappa$$

and we deduce from (3.10) and Lemma 4 that, for $n \geq n_0$,

$$|L_0| \geq \frac{\kappa}{C_9 n^{C_{10}} C_{11}^n n^{n(m-1)(1+2\varepsilon_1)}} \left(1 - un^v w^n n^{-\delta n} H\right) \tag{3.11}$$

where

$$u := \max\left(1, (m-1)C_9^2 e^{2C|\xi|}/\kappa\right), \quad v := 2C_{10}, \quad w := C_{11}^2,$$

and $\delta := 1 - (2m-1)\varepsilon_1 \in (0, 1)$ by the assumption made on ε_1 at the beginning of Sect. 3.3. This implies

$$|L_0| \geq \frac{\kappa}{2C_9 n^{C_{10}} C_{11}^n n^{n(m-1)(1+2\varepsilon_1)}}$$

provided

$$n \geq n_0 \quad \text{and} \quad 2uH \leq w^{-n} n^{\delta n - v}. \tag{3.12}$$

We can now complete the proof of Theorem 3. Since $u \geq 1$ and we assume $H \geq \max(3, n_0^{n_0})$ in Theorem 3, we have $2uH \geq n_0^{n_0}$. Accordingly for any $n < n_0$ we have $2uH > n^n \geq w^{-n} n^{\delta n - v}$, so that the minimal value of n (denoted by N from now on) that satisfies $2uH \leq w^{-n} n^{\delta n - v}$ is automatically $\geq n_0$: it satisfies the assumptions (3.12).

We want to find an upper bound for N in terms of $2uH, w, v, \delta$. An equivalent definition of N is that it is the largest integer such that

$$(N-1)^{N-1} < (2uH)^{1/\delta} (w^{1/\delta})^{N-1} (N-1)^{v/\delta}. \tag{3.13}$$

For any $X \geq 1$ we have the following lower bounds:

$$\begin{aligned}
\frac{X^X}{(2uH)^{1/\delta} w^{X/\delta} X^{v/\delta}} &\geq \frac{X^X}{(2uH)^{1/\delta} X^{X/4} X^{v/\delta}} \quad \text{assuming that } X \geq w^{4/\delta}, \\
&= \frac{X^{3X/4}}{(2uH)^{1/\delta} X^{v/\delta}} \\
&\geq \frac{X^{X/4}}{(2uH)^{1/\delta}} \quad \text{assuming that } X \geq \frac{2v}{\delta} \\
&= \left(\frac{X^X}{(2uH)^{4/\delta}}\right)^{1/4} \\
&\geq 1 \quad \text{assuming that } X \geq \frac{8\ln(2uH)}{\delta \ln\ln((2uH)^{4/\delta})},
\end{aligned}$$

where in the last line we use the elementary fact that for any $y > e$, if $x \geq \frac{2\ln(y)}{\ln\ln(y)}$, then $x^x \geq y$. Note that $H \geq 3$ and $u \geq 1$ so that $(2uH)^{4/\delta} \geq 6^{4/\delta} > e$ because $\delta \in (0, 1)$ and thus we can use this fact with $y := (2uH)^{4/\delta}$. All assumptions in the lower bound above are satisfied for instance for any

$$X \geq 1 + w^{4/\delta} + \frac{2v}{\delta} + \frac{8\ln(2uH)}{\delta \ln\ln((2uH)^{4/\delta})}.$$

Since for $X = N - 1$ the lower bound $\frac{X^X}{(2uH)^{1/\delta} w^{X/\delta} X^{v/\delta}} \geq 1$ does not hold, we deduce that

$$N \leq 1 + w^{4/\delta} + \frac{2v}{\delta} + \frac{8\ln(2uH)}{\delta \ln\ln((2uH)^{4/\delta})} =: \Phi. \tag{3.14}$$

Hence, since (3.12) holds with $n := N$, we have

$$\begin{aligned} |L_0| &\geq \frac{\kappa}{2C_9 N^{C_{10}} C_{11}^N N^{N(m-1)(1+2\varepsilon_1)}} \\ &= \frac{\kappa}{2C_9 e^{C_{10}\ln(N)+\ln(C_{11})N+(m-1)(1+2\varepsilon_1)N\ln(N)}} \\ &\geq \frac{1}{H^c} \end{aligned}$$

because

$$N\ln(N) \leq \Phi\ln(\Phi) \leq C_{12}\ln(H) \text{ and } H \geq 3,$$

where c and C_{12} depend on $\varepsilon_1, m, p, q, \kappa, T(z)$, and polynomially on $H(\xi)$, B, C and D. The total degrees of these polynomials are bounded in terms of ε_1, m, p, q. To conclude the proof of Theorem 3, we choose $\varepsilon_1 = \frac{1}{2m}$.

4 Proof of Theorem 2

4.1 First reductions

Let $f \in \mathbb{Q}[[z]]$ be a strict E-function, and $\xi \in \mathbb{Q}^*$ be such that $f(\xi) > 0$. Considering $f(\xi z)$ instead of f, we may assume that $\xi = 1$. Recall that $f(1) \neq 1$, because $\ln(f(1)) \notin \mathbb{Q}$ is an assumption of Theorem 2. Accordingly, if $f(1)$ is algebraic then $\ln(f(1))$ is not a Liouville number by [14, p. 386, Theorem 3], and the conclusion of Theorem 2 follows at once. Therefore we may assume that $f(1)$ is transcendental and apply the following consequence of [11, Proposition 2], which is a variant of Beukers' desingularization lemma [8, Theorem 1.5].

Proposition 4 *Let g_1, ..., g_k be E-functions with rational coefficients, such that 1, g_1, ..., g_k are linearly independent over $\mathbb{Q}(z)$ and $g_1(1)$ is transcendental. Assume also that the vector ${}^t(1, g_1, \ldots, g_k)$ is solution of a differential system $Y' = AY$ with $A \in M_{k+1}(\mathbb{Q}(z))$.*

Then there exist E-functions $f_1, \ldots, f_k$ with rational coefficients, such that $1, f_1, \ldots, f_k$ are linearly independent over $\mathbb{Q}(z)$, $f_1(1) = g_1(1)$, and the vector ${}^t(1, f_1, \ldots, f_k)$ is solution of a differential system $Y' = MY$ with $M \in M_{k+1}(\mathbb{Q}[z, \frac{1}{z}])$.

Moreover, when the g_j's are strict E-functions, then the f_j's are strict E-functions as well.

To apply Proposition 4, we consider the inhomogeneous differential equation of minimal order satisfied by f. We denote its order by $\mu - 1$, with $\mu \geq 2$ because f is transcendental (since $f(1)$ is). The functions $1, f, f', \ldots, f^{(\mu-2)}$ are linearly independent over $\mathbb{Q}(z)$: Proposition 4 provides E-functions $f_1, \ldots, f_{\mu-1}$ with rational coefficients such that $f_1(1) = f(1)$ is the number we are interested in the logarithm of. Letting $f_\mu = 1$, the functions $f_1, \ldots, f_\mu$ are linearly independent over $\mathbb{Q}(z)$ and make up a vector solution of a differential system without non-zero singularity; in particular 1 is not a singularity.

Now recall that the functions $\exp(\beta z)$, $\beta \in \overline{\mathbb{Q}}$, are linearly independent over $\mathbb{Q}(z)$. Therefore at most μ of them belong to the $\mathbb{Q}(z)$-vector space spanned by $f_1, \ldots, f_\mu$. Since $\ln(f(1))$ is irrational, we may exclude these finitely many values of $\beta = a/b$ in proving the lower bound (1.1) (up to changing the values of c and d). Therefore we may restrict to rationals β such that $f_1, \ldots, f_\mu$, and $\exp(\beta z)$ are linearly independent over $\mathbb{Q}(z)$.

4.2 Application of the effective linear independence measure

As explained in the previous section, we are trying to bound

$$\left|\ln(f_1(1)) - \frac{a}{b}\right|$$

from below, with $a \in \mathbb{Z}$ and $b \in \mathbb{N}^*$. We may assume that a and b are coprime. By the mean value theorem (see Eq. (4.2)), it is essentially equivalent to bounding below

$$|f_1(1) - \exp(a/b)|,$$

which is a $\mathbb{Z}$-linear combination of the values at 1 of the E-functions $f_1(z)$ and $\exp(\beta z)$, with $\beta := a/b$. We point out that the coefficients of this linear combination are only 0, 1 and -1 whereas in general, we are always interested in linear combinations with large coefficients.

As explained at the end of Sect. 4.1, we may assume that $f_1, \ldots, f_\mu$ and $f_{\mu+1}(z) = \exp(\beta z)$ are strict E-functions linearly independent over $\mathbb{Q}(z)$, solution of a first-order differential system without non-zero singularity. Moreover $f_1, \ldots, f_\mu$ are solution of a first-order differential system without non-zero singularity, independent from β: we are in the setting of (3.4).

Proposition 3 yields a constant c, which depends on $f_1, \ldots, f_\mu$ and polynomially on $H(\beta)$, such that with $H := \max(3, n_0^{n_0})$ we have

$$|f_1(1) - \exp(a/b)| > \frac{1}{H^c}.$$

An important feature of this proposition is that this value of H depends only on $f_1, \ldots, f_\mu$, not on β.

If $|f_1(1) - \exp(a/b)| \geq f_1(1)/2$, the lower bound of Theorem 2 holds trivially. Therefore we may assume that

$$\left| f_1(1) - \exp(a/b) \right| < f_1(1)/2. \tag{4.1}$$

Accordingly $|\beta|$ is bounded in terms of $f_1(1)$ only. Since $H(\beta) \leq b \cdot \max(1, |\beta|)$, there exists a polynomial $Q \in \mathbb{R}[X]$ with positive coefficients such that $c \leq Q(b)$; this polynomial Q depends only on $f_1, \dots, f_\mu$. Therefore

$$|f_1(1) - \exp(a/b)| > \frac{1}{H^{Q(b)}}$$

and this concludes the proof of Theorem 2. Indeed by the mean value theorem, for all $(a, b) \in \mathbb{Z}^* \times \mathbb{N}^*$, a, b coprime, there exists $\omega > 0$ in the interval with endpoints $\exp(a/b)$ and $f_1(1)$ such that

$$\left| \ln(f_1(1)) - \frac{a}{b} \right| = \frac{1}{\omega} \left| f_1(1) - \exp(a/b) \right|, \tag{4.2}$$

and finally in this equality, the coefficient ω can be bounded above in terms of $f_1(1)$ only due to Eq. (4.1).

Remark 2 Let us explain why this method is not sufficient to prove that $\ln(f(\xi))$ is non-Liouville, in the context of Theorem 2. Indeed, to conclude using Eq. (3.10) we need to have $\max_{1 \leq j \leq m-1} |L_j| < 1$. In the upper bound in Lemma 4, the factor $C_{11}^n n^{-n(m-1-3\varepsilon_1)}$ has to tend to 0, so n has to be larger than a power of C_{11}. Since $C_{11} \geq 1$ depends polynomially on $H(\xi)$, B, C and D, it follows that n has to depend polynomially on these parameters. Then Eq. (3.11) shows that we cannot obtain anything better than the conclusion of Theorem 2.

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