Introduction to the spectral theory

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Notation

We list some notations used throughout the text.

The symbol $\mathbb{N}$ denotes the set of the natural numbers starting from 0.

If $(M, \mathcal{T}, \mu)$ is a measure space and $f : M \to \mathbb{C}$ is a measurable function, then we denote the *essential range* and the *essential supremum* of f w.r.t. the measure μ:

$$\begin{aligned} \operatorname{ess}_\mu \operatorname{ran} f &:= \Big\{ z \in \mathbb{C} : \, \mu\big\{ m \in M : \big|z - f(m)\big| < \varepsilon \big\} > 0 \text{ for all } \varepsilon > 0 \Big\}, \\ \operatorname{ess}_\mu \sup |f| &:= \inf \Big\{ a \in \mathbb{R} : \, \mu\big\{ m \in M : \big|f(m)\big| > a \big\} = 0 \Big\}. \end{aligned}$$

If the measure μ is obvious in the context, we will omit to indicate it in the notations.

In what follows the term *Hilbert space* will mean a *separable complex Hilbert space*. The symbol $\mathcal{H}$ will implicitly denote such a Hilbert space. For two elements $x, y \in \mathcal{H}$, then $\langle x, y \rangle$ will denote the sesquilinear scalar product of x and y. If several Hilbet spaces are considered in the problem, we will specify the scalar product with the notation $\langle x, y \rangle_{\mathcal{H}}$. To respect the convention in quantum mechanics, our scalar products will always be linear with respect to the *second* argument, and as antilinear with respect to the first one:

$$\forall \alpha \in \mathbb{C} \quad \langle x, \alpha y \rangle = \langle \overline{\alpha} x, y \rangle = \alpha \langle x, y \rangle.$$

For example, that the scalar product in the Lebesgue space $L^2(\mathbb{R})$ is defined by

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) \, dx.$$

If A is a finite or countable set, $\ell^2(A)$ denotes the vector space of square-summable functions $x : A \to \mathbb{C}$:

$$\sum_{a \in A} \big|\xi(a)\big|^2 < \infty.$$

Such functions x are sometimes written using subscripts: $x(a) = x_a$, in particular when $A = \mathbb{N}$ or $A = \mathbb{Z}$.

This is a Hilbert space, equipped with the scalar product

$$\langle x, y \rangle = \sum_{a \in A} \overline{x(a)} y(a).$$

If $\mathcal{H}$ and $\mathcal{G}$ are two Hilbert spaces, then by $\mathcal{L}(\mathcal{H}, \mathcal{G})$ and $\mathcal{K}(\mathcal{H}, \mathcal{G})$ denote the spaces of bounded linear operators, respectively of compact operators from $\mathcal{H}$ and $\mathcal{G}$. Furtheremore, $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$ and $\mathcal{K}(\mathcal{H}) := \mathcal{K}(\mathcal{H}, \mathcal{H})$.

If $\Omega \subset \mathbb{R}^d$ is an open set and $k \in \mathbb{N}$, then $H^k(\Omega)$ denotes the kth Sobolev space, i.e. the space of L^2 functions whose partial derivatives up to order k are also in $L^2(\Omega)$. By $H^k_0(\Omega)$ we denote the completion in $H^k(\Omega)$ of the subspace $C^\infty_c(\Omega)$ (with respect to the norm of $H^k(\Omega)$). The symbol $C^k(\Omega)$ denotes the space of functions on Ω whose partial derivatives up to order k are continuous; in particular, the set of the continuous functions is denoted as $C^0(\Omega)$. This should not be confused with the notation $C_0(\mathbb{R}^d)$ for the space of continuous functions $f : \mathbb{R}^d \to \mathbb{C}$ vanishing at infinity: $\lim_{|x|\to\infty} f(x) = 0$. The subscript $_{\text{comp}}$ on a functional space indicates that its elements have compact supports: for instance $H^1_{\text{comp}}(\mathbb{R}^d)$ is the space of functions in $H^1(\mathbb{R}^d)$ having compact supports.

Recommended books

During the preparation of the notes I used a part of the text by Bernard Helffer which is available online [9]; an extended version was recently published as a book [10]. Other recommended books are the one E. B. Davies [5] and the book of G. Teschl [17] (available online).

Additional references for particular topics will be given throughout the text.

1 Unbounded operators

1.1 Closed operators

A *linear operator* T in $\mathcal{H}$ is a linear map from a subspace (the domain of T) $D(T) \subset \mathcal{H}$ to $\mathcal{H}$. The range of T is the set $\operatorname{ran} T := \{Tx : x \in D(T)\}$. We say that a linear operator T is *bounded* if the quantity

$$\mu(T) := \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

is finite. In what follows, the word combination "an unbounded operator" should be understood as "an operator which is not assumed to be bounded". If $D(T) = \mathcal{H}$ and T is bounded, we arrive at the notion of a continuous linear operator in $\mathcal{H}$; the space of such operators is denoted by $\mathcal{L}(\mathcal{H})$. This is a Banach space equipped with the norm $\|T\| := \mu(T)$.

During the whole course, by introducing a linear operator we always assume that its domain is dense, if the contrary is not stated explicitly.

If T is a bounded operator in $\mathcal{H}$, it can be uniquely extended to a continuous linear operator. Let us discuss a similar idea for unbounded operators.

The *graph* of a linear operator T in $\mathcal{H}$ is the set

$$\operatorname{gr} T := \big\{(x, Tx) : x \in D(T)\big\} \subset \mathcal{H} \times \mathcal{H}.$$

For two linear operators T_1 and T_2 in $\mathcal{H}$ we write $T_1 \subset T_2$ if $\operatorname{gr} T_1 \subset \operatorname{gr} T_2$. I.e. $T_1 \subset T_2$ means that $D(T_1) \subset D(T_2)$ and that $T_2 x = T_1 x$ for all $x \in D(T_1)$; the operator T_2 is then called an extension of T_1 and T_1 is called a restriction of T_2.

Definition 1.1 (Closed operator, closable operator).

- A linear operator T in $\mathcal{H}$ is called *closed* if its graph is a closed subspace in $\mathcal{H} \times \mathcal{H}$.
- A linear operator T in $\mathcal{H}$ is called *closable*, if the closure $\overline{\operatorname{gr} T}$ of the graph of T in $\mathcal{H} \times \mathcal{H}$ is still the graph of a certain operator $\overline{T}$. This operator $\overline{T}$ with $\operatorname{gr} \overline{T} = \overline{\operatorname{gr} T}$ is called the *closure* of T. □

The following proposition is obvious:

Proposition 1.2. *A linear operator T in $\mathcal{H}$ is closed if and only if the three conditions*

- $x_n \in D(T)$,
- x_n *converge to* x *in* $\mathcal{H}$,
- Tx_n *converge to* y *in* $\mathcal{H}$

imply the inclusion $x \in D(T)$ and the equality $y = Tx$.

Definition 1.3 (Graph norm). Let T be a linear operator in $\mathcal{H}$. Define on $D(T)$ a new scalar product by $\langle x, y\rangle_T = \langle x, y\rangle + \langle Tx, Ty\rangle$. The associated norm $\|x\|_T := \sqrt{\langle x, x, \rangle_T} = \sqrt{\|x\|^2 + \|Tx\|^2}$ is called the *graph norm* for T. $\square$

The following assertion is also evident.

Proposition 1.4. *Let T be a linear operator in $\mathcal{H}$.*

- *T is closed iff $D(T)$ is complete in the graph norm.*
- *If T is closable, then $D(\overline{T})$ is exactly the completion of $D(T)$ with respect to the graph norm.*

Informally, one could say that $D(\overline{T})$ consists of those x for which there is a unique candidate for $\overline{T}x$ if one tries to extend T by density. I.e., a vector $x \in \mathcal{H}$ belongs to $D(\overline{T})$ iff:

- there exists a sequence $(x_n) \subset D(T)$ converging to x,
- their exists the limit of Tx_n,
- this limit is the same for any sequence x_n satisfying the previous two properties.

Let us consider some examples.

Example 1.5 (Bounded linear operators are closed). By the closed graph theorem, a linear operator T in $\mathcal{H}$ with $D(T) = \mathcal{H}$ is closed if and only if it is bounded. In this course we consider mostly unbounded closed operators.

Example 1.6 (Multiplication operator). Take again $\mathcal{H} = L^2(\mathbb{R}^d)$ and pick $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. Introduce a linear operator M_f in $\mathcal{H}$ as follows:

$$D(M_f) = \{u \in L^2(\mathbb{R}^d) : fu \in L^2(\mathbb{R}^d)\} \text{ and } \quad M_f u = fu \text{ for } u \in D(M_f).$$

It can be easily seen that $D(M_f)$ equipped with the graph norm coincides with the weighted space $L^2\big(\mathbb{R}^d, (1 + |f|^2)dx\big)$, which is complete. This shows that M_f is closed.

An interested reader may generalize this example by considering multiplications operators in measure spaces. $\square$

Example 1.7 (Laplacians in $\mathbb{R}^d$). Take $\mathcal{H} = L^2(\mathbb{R}^d)$ and consider two operators in $\mathcal{H}$:

$$\begin{aligned} T_0 u &= -\Delta u, & D(T_0) &= C_c^\infty(\mathbb{R}^d),\\ T_1 u &= -\Delta u, & D(T_1) &= H^2(\mathbb{R}^d). \end{aligned}$$

We are going to show that $\overline{T}_0 = T_1$ (it follows that T_1 is closed and T_0 is not closed). We prove first the following equality:

$$H^2(\mathbb{R}^d) = \big\{f \in L^2(\mathbb{R}^d) : \, -\Delta f \in L^2(\mathbb{R}^d)\big\} \tag{1.1}$$

Clearly, $H^2(\mathbb{R}^d)$ is included into the set on the right-hand side. Now, let f belong to the set on the right-hand side, we have that $\widehat{f} \in L^2(\mathbb{R}^d)$ and $p^2\widehat{f} \in L^2(\mathbb{R}^d)$. From

$$\Big|p_j p_k \widehat{f}\Big| \le \frac{p_j^2 + p_k^2}{2}|\widehat{f}| \le p^2|\widehat{f}|, \quad \Big|p_j \widehat{f}\Big| \le \frac{1 + p_j^2}{2}|\widehat{f}| \le (1+p^2)|\widehat{f}|$$

it follows that $p_j\widehat{f} \in L^2$, $p_j p_k \widehat{f} \in L^2$. In summary, $\partial^\alpha f \in L^2(\mathbb{R}^d)$ for $|\alpha| \le 2$, which proves the inclusion $f \in H^2(\mathbb{R}^d)$ and the equality (1.1)

Denote by $F : \mathcal{H} \to \mathcal{H}$ the Fourier transform in $L^2(\mathbb{R}^d)$ and consider the following operator $\widehat{T}$ in $\mathcal{H}$:

$$D(\widehat{T}) = \{f \in L^2 : \, p \mapsto p^2 f(p) \in L^2\}, \quad \widehat{T}f(p) = p^2 f(p).$$

Indeed, $\widehat{T}$ is closed operator, as this is just a multiplication operator, see Example 1.6. On the other hand, for $f \in \mathcal{H}$ one has the following equivalence: $f \in D(T_1)$ iff $Ff \in D(\widehat{T})$, and in that case $FT_1 f(p) = \widehat{T}Ff(p)$. In other words, one can represent

$$\operatorname{gr} T_1 = \{(F^{-1}u, F^{-1}\widehat{T}u) : \, u \in D(\widehat{T})\} = K(\operatorname{gr}\widehat{T}),$$

where K is the linear operator in $\mathcal{H} \times \mathcal{H}$ defined by $K(x,y) = (F^{-1}x, F^{-1}y)$. As F is a unitary operator, so is K, which means, in particular, that K maps closed sets to closed sets. As $\operatorname{gr}\widehat{T}$ is closed, the graph $\operatorname{gr} T_1$ is also closed, and T_1 is a closed operator.

As we have the inclusion $T_0 \subset T_1$ and T_1 is closed, it follows that T_0 is at least closable, and the domain $D(\overline{T}_0)$ is the completion $D(T_0)$ in the graph norm of T_0 (Proposition 1.4) or, equivalently, $D(T_0)$ is the closure of $D(\overline{T}_0)$ in the Hilbert space $\big(D(T_1), \langle\cdot,\cdot\rangle_{T_1}\big)$. Actually we have shown above that the graph norm of T_1 is equivalent to the norm of $H^2(\mathbb{R}^d)$, and hence $D(T_0) = C_c^\infty(\mathbb{R}^d)$ is dense in $\big(D(T_1), \langle\cdot,\cdot\rangle_{T_1}\big)$. This shows that $D(\overline{T}_0) = H^2(\mathbb{R}^d)$.

Furthermore, $\overline{T}_0 \subset T_1$, which means that for any $u \in D(\overline{T}_0)$ we have $\overline{T}_0 u = T_1 u = -\Delta u$. □

Example 1.8 (Non-closable operator). Take $\mathcal{H} = L^2(\mathbb{R})$ and pick a $g \in \mathcal{H}$ with $g \neq 0$. Consider the operator L defined on $D(L) = C^0(\mathbb{R}) \cap L^2(\mathbb{R})$ by $Lf = f(0)g$. Assume that there exists the closure $\overline{L}$ and let $f \in D(\overline{L})$. One can find two sequences $(f_n), (g_n)$ in $D(L)$ such that both converge in the L^2 norm to f but such that $f_n(0) = 0$ and $g_n(0) = 1$ for all n. Then $Lf_n = 0$, $Lg_n = g$ for all n, and both sequences Lf_n and Lg_n converge, but to different limits. This contradicts to the closedness of $\overline{L}$. Hence L is not closed. □

Example 1.9 (Partial differential operators). Let Ω be an open subset of $\mathbb{R}^d$ and $P(x, D_x)$ be a partial differential expression with C^∞ coefficients. Introduce in $\mathcal{H} := L^2(\Omega)$ a linear operator P by: $D(P) = C_c^\infty(\Omega)$, $Pu(x) = P(x, D_x)u(x)$. Like in the previous example one shows the inclusion

$$\overline{\operatorname{gr} P} \subset \big\{(u, f) \in \mathcal{H} \times \mathcal{H} : \, P(x, D_x)u = f \text{ in } \mathcal{D}'(\Omega)\big\},$$

hence $\overline{\operatorname{gr} P}$ is still a graph, and P is closable.

So we see that we naturally associate with the differential expression $P(x, D_x)$ several linear operators (besides P), in particular, the *minimal operator* $P_{\min} := \overline{P}$, which is always closed, and the *maximal operator* $P_{\max}$ defined by $P_{\max}u = P(x, D_x)u$ on the domain

$$D(P_{\max}) := \big\{u \in \mathcal{H} : \, P(x, D_x)u \in \mathcal{H}\big\},$$

where $P(x, D_x)u$ is understood in the sense of distributions. Clearly, one always has the inclusion $P_{\min} \subset P_{\max}$, and we saw in example 1.7 that one can have $P_{\min} = P_{\max}$. But one can easily find examples where this equality does not hold. For example, for $P(x, D_x) = d/dx$ and $\Omega = (0, +\infty)$ we have $D(P_{\min}) = H_0^1(0, \infty)$ and $D(P_{\max}) = H^1(0, \infty)$. In general, one may expect that $P_{\min} \neq P_{\max}$ if Ω has a boundary.

Such questions become more involved if one studies the partial differential operators with more singular coefficients (e.g. with coefficients which are not smooth but just belong to some L^p). During the course we will see how to deal with some special classes of such operators. □

1.2 Adjoint operators

Recall that for $T \in \mathcal{L}(\mathcal{H})$ its adjoint T^* is defined by the relation

$$\langle x, Ty \rangle = \langle T^*x, y \rangle \text{ for all } x, y \in \mathcal{H}.$$

The proof of the existence comes from the Riesz representation theorem: for each $x \in \mathcal{H}$ the map $\mathcal{H} \ni y \mapsto \langle x, Ty \rangle \in \mathbb{C}$ is a continuous linear functional, which means that there exists a unique vector, denoted by T^*x with $\langle x, Ty \rangle = \langle T^*x, y \rangle$ for all $y \in \mathcal{H}$. One can then show that the map $x \mapsto T^*x$ is linear, and by estimating the scalar product one shows that T^* is also continuous. Let us use the same idea for unbounded operators.

Definition 1.10 (Adjoint operator). If T be a linear operator in $\mathcal{H}$, then its *adjoint* T^* is defined as follows. The domain $D(T^*)$ consists of the vectors $u \in \mathcal{H}$ for which the map $D(T) \ni v \mapsto \langle u, Tv \rangle \in \mathbb{C}$ is bounded with respect to the $\mathcal{H}$-norm. For such u there exists, by the Riesz theorem, a unique vector denoted by T^*u such that $\langle u, Tv \rangle = \langle T^*u, v \rangle$ for all $v \in D(T)$. □

We note that the implicit assumption $\overline{D(T)} = \mathcal{H}$ is important here: if it is not satisfied, then the value T^*u is not uniquely determined, one can add to T^*u an arbitrary vector from $D(T)^\perp$.

Let us give a geometric interpretation of the adjoint operator. Consider a unitary linear operator

$$J : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}, \quad J(x,y) = (y, -x)$$

and note that J commutes with the operator of the orthogonal complement in $\mathcal{H}\times\mathcal{H}$, i.e. $J(V)^\perp = J(V^\perp)$ for any $V \subset \mathcal{H} \times \mathcal{H}$. This will be used several times during the course.

Proposition 1.11 (Geometric interpretation of the adjoint). *Let T be a linear operator in $\mathcal{H}$. The following two assertions are equivalent:*

- *$u \in D(T^*)$ and $f = T^* u$,*
- *$\big\langle (u, T^*u), J(v, Tv)\big\rangle_{\mathcal{H}\times\mathcal{H}} = 0$ for all $v \in D(T)$.*

In other words,

$$\operatorname{gr} T^* = J(\operatorname{gr} T)^\perp. \tag{1.2}$$

As a simple application we obtain

Proposition 1.12. *One has $(\,\overline{T}\,)^* = T^*$, and T^* is a closed operator.*

Proof. Follows from (1.2): the orthogonal complement is always closed, and $J(\operatorname{gr} T)^\perp = J(\overline{\operatorname{gr} T})^\perp$. $\square$

Up to now we do not know if the domain of the adjoint contains non-zero vectors. This is discussed in the following proposition.

Proposition 1.13 (Domain of the adjoint). *Let T be a closable operator $\mathcal{H}$, then:*

(i) $D(T^*)$ is a dense subspace of $\mathcal{H}$,

(ii) $T^{**} := (T^*)^* = \overline{T}$.

Proof. The item (ii) easily follows from (i) and (1.2): one applies the same operations again and remark that $J^2 = -1$ and that taking twice the orthogonal complement results in taking the closure.

Now let us prove the item (i). Let a vector $w \in \mathcal{H}$ be orthogonal to $D(T^*)$: $\langle u, w\rangle = 0$ for all $u \in D(T^*)$. Then one has $\langle J(u, T^*u), (0, w)\rangle_{\mathcal{H}\times\mathcal{H}} \equiv \langle u, w\rangle + \langle T^*u, 0\rangle = 0$ for all $u \in D(T^*)$, which means that $(0, w) \in J(\operatorname{gr} T^*)^\perp = \overline{\operatorname{gr} T}$. As the operator T is closable, the set $\overline{\operatorname{gr} T}$ must be a graph, which means that $w = 0$. $\square$

Let us look at some examples.

Example 1.14 (Adjoint for bounded operators). The general definition of the adjoint operator is compatible with the one for continuous linear operators.

Example 1.15 (Laplacians in $\mathbb{R}^d$). Let us look again at the operators T_0 and T_1 from example 1.7. We claim that $T_0^* = T_1$. To see this, let us describe the adjoint T_0^* using the definition. The domain $D(T_0^*)$ consists of the functions $u \in L^2(\mathbb{R}^d)$ for which there exists a vector $f \in L^2(\mathbb{R}^d)$ such that the equality

$$\int_{\mathbb{R}^d} \overline{u(x)}(-\Delta v)(x)dx = \int_{\mathbb{R}^d} \overline{f(x)}v(x)dx$$

holds for all $v \in D(T_0) \equiv C_c^\infty(\mathbb{R}^d)$. This means that one should have $f = -\Delta u$ in the sense of distributions. Therefore, $D(T_0^*)$ consists of the functions $u \in L^2$ such that $-\Delta u \in L^2$. By (1.1) we have $u \in H^2(\mathbb{R}^d) = D(T_1)$. □

Definition 1.16 (Free Laplacian in $\mathbb{R}^d$). The operator T in $L^2(\mathbb{R}^d)$ defined by

$$D(T) = H^2(\mathbb{R}^d), \quad Tu = -\Delta u,$$

is called the *free Laplacian in* $\mathbb{R}^d$. By Example 1.15, is is a self-adjoint operator.

Example 1.17. As an exercise, one can show that for the multiplication operator M_f from example 1.6 one has $(M_f)^* = M_{\overline{f}}$. □

The following definition introduces two classes of linear operator that will be studied throughout the course.

Definition 1.18 (Symmetric, self-adjoint, essentially self-adjoint operators). We say that a linear operator T in $\mathcal{H}$ is *symmetric* (or *Hermitian*) if

$$\langle u, Tv \rangle = \langle Tu, v \rangle \quad \text{for all } u, v \in D(T),$$

or, equivalently, if $T \subset T^*$. Furthermore:

- T is called *self-adjoint* if $T = T^*$,
- T is called *essentially self-adjoint* if $\overline{T}$ is self-adjoint. □

An important feature of symmetric operators is their closability:

Proposition 1.19. *Symmetric operators are closable.*

Proof. Indeed for a symmetric operator T we have $\operatorname{gr} T \subset \operatorname{gr} T^*$ and, due to the closedness of T^*, $\overline{\operatorname{gr} T} \subset \operatorname{gr} T^*$. □

Example 1.20 (Self-adjoint Laplacian in $\mathbb{R}^d$). For the laplacian T_1 from example 1.7 one has $T_1 = T_1^*$. Indeed, $T_1 = T_0^*$, then $T_1^* = T_0^{**} = \overline{T_0} = T_1$. □

Example 1.21 (Bounded symmetric operators are self-adjoint). Note that for $T \in \mathcal{L}(\mathcal{H})$ the fact to be symmetric is equivalent to the fact to be self-adjoint, but it is not the case for unbounded operators! □

Example 1.22 (Self-adjoint multiplication operators). As follows from example 1.17, the multiplication operator M_f from example 1.6 is self-adjoint iff $f(x) \in \mathbb{R}$ for a.e. $x \in \mathbb{R}^d$. □

A large class of self-adjoint operators comes from the following proposition.

Proposition 1.23. *Let T be an* injective *self-adjoint operator, then its inverse is also self-adjoint.*

Proof. We show first that $D(T^{-1}) := \operatorname{ran} T$ is dense in $\mathcal{H}$. Let $u \perp \operatorname{ran} T$, then $\langle u, Tv\rangle = 0$ for all $v \in D(T)$. This can be rewritten as $\langle u, Tv\rangle = \langle 0, v\rangle$ for all $v \in D(T)$, which shows that $u \in D(T^*)$ and $T^*u = 0$. As $T^* = T$, we have $u \in D(T)$ and $Tu = 0$. As T in injective, one has $u = 0$
Now consider the operator $S : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ given by $S(x, y) = (y, x)$. One has then $\operatorname{gr} T^{-1} = S(\operatorname{gr} T)$. We conclude the proof by noting that S commutes with J and with the operation of the orthogonal complement in $\mathcal{H} \times \mathcal{H}$. □

1.3 Exercises

Exercise 1.24. (a) In the Hilbert space $H = L^2(0, 1)$ consider the operator T acting on the domain $D(T) = H(0, 1)$ by $Tf = if'$.
Is T closed? symmetric? self-adjoint? semibounded from below?
(b) The same questions for the operator T_1 given by the same expression but on the domain $D(T_1) = H^1_0(0, 1)$.

Exercise 1.25. (a) Let H_1 and H_2 be Hilbert spaces. Let A be a linear operator in H_1, B be a linear operator in H_2. Assume that there exists a unitary operator $U : H_2 \to H_1$ such that $D(A) = UD(B)$ and that $U^*AUf = Bf$ for all $f \in D(B)$; such A and B are called *unitary equivalent*.
Let two operators A and B be unitarily equivalent. Show that A is closed/symmetric/self-adjoint iff B has the same property.
(b) Let (λ_n) be an arbitrary sequence of complex numbers, $n \in \mathbb{N}$. In the Hilbert space $\ell^2(\mathbb{N})$ consider the operator S:

$$D(S) = \big\{(x_n) : \text{there exists } N \text{ such that } x_n = 0 \text{ for } n > N\big\}, \quad S(x_n) = (\lambda_n x_n).$$

Describe the closure of S.
(c) Now let H be a separable Hilbert space and T be a linear operator in H with the following property there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H with $e_n \in D(T)$ and $Te_n = \lambda_n e_n$ for all $n \in \mathbb{N}$, where λ_n are some complex numbers.

1. Describe the closure $\overline{T}$ of T. Hint: one may use (a) and (b).
2. Describe the adjoint T^* of T.
3. Let all λ_n be real. Show that the operator $\overline{T}$ is self-adjoint.

Exercise 1.26. Let A and B be self-adjoint operators in a Hilbert space H such that $D(A) \subset D(B)$ and $Au = Bu$ for all $u \in D(A)$. Show that $D(A) = D(B)$. (This property is called the *maximality* of self-adjoint operators.)

Exercise 1.27. In *this exercise*, by the sum $A + B$ of a linear operator B with a *continuous* operator B; both acting in a Hilbert space H, we mean the operator S defined by $D(S) = D(A)$, $Su := Au + Bu$. 5We note that defining the sum of two operators becomes a non-trivial task if unbounded operators are involved.)

(a) Let A be a closed and B be continuous. Show that $A + B$ is closed.

(b) Assume, in addition, that A is densely defined. Show that $(A + B)^* = A^* + B^*$.

Exercise 1.28. Let $H := L^2(0,1)$. For $\alpha \in \mathbb{C}$ consider the operator T_α acting as $T_\alpha f = if$ on the domain

$$D(T_\alpha) = \Big\{ f \in C^\infty\big([0,1]\big) : f(1) = \alpha f(0) \Big\}.$$

(a) Find the adjoint of T_α.

(b) Find the closure $S_\alpha := \overline{T_\alpha}$.

(c) Find all α for which S_α is self-adjoint.

2 Operators and quadratic forms

2.1 Operators defined by quadratic forms

A *sesquilinear form* t in a Hilbert space $\mathcal{H}$ with the domain $D(t) \subset \mathcal{H}$ is a map $t : \mathcal{H} \times \mathcal{H} \supset D(t) \times D(t) \to \mathbb{C}$ which is linear with respect to the second argument and is antilinear with respect to the first one. By default we assume that $D(t)$ is a dense subset of $\mathcal{H}$. (In the literature, one uses also the terms *bilinear form* and *quadratic form.*) We will consider the following classes of sesquilinear forms: a form t is called

- *bounded*, if $D(t) = \mathcal{H}$ and there exists $M > 0$ such that $\big|t(u,v)\big| \le M\|u\| \cdot \|v\|$ for all $u, v \in \mathcal{H}$,
- *elliptic*, if it is bounded and there exists $\alpha > 0$ such that $\big|t(u,u)\big| \ge \alpha\|u\|^2$ for all $u \in \mathcal{H}$,
- *symmetric* if $t(u,v) = \overline{t(u,v)}$ for all $u, v \in D(t)$,
- *semibounded from below* if t is symmetric and for some $c \in \mathbb{R}$ one has $t(u,u) \ge c\|u\|^2$ for all $u \in D(t)$; in this case we write $t \ge c$;
- *positive* or *non-negative*, if one can take $c = 0$ in the previous item,
- *positively definite* ot *strictly positive*, if one can take $c > 0$.

Now let $\mathcal{V}$ be a Hilbert space and let t be a bounded sesquilinear form in $\mathcal{V}$. It is known that there is a uniquely determined operator $A \in \mathcal{L}(V)$ such that $t(u, v) = \langle u, Av\rangle$ for all $u, v \in \mathcal{V}$. Let us recall the following classical result:

Theorem 2.1 (Lax-Milgram theorem). *If t is elliptic, then the associated operator A is an isomorphism of $\mathcal{V}$, that is, A is invertible and $A^{-1} \in \mathcal{L}(\mathcal{V})$.*

Proof. By assumption, one can find two constants $\alpha, C > 0$ such that

$$\alpha\|v\|^2 \leq \big|\, t(v, v)\big| \leq C\|v\|^2 \text{ for all } v \in \mathcal{V}.$$

This implies $\alpha\|v\|^2 \leq \big|a(v, v)\big| = \big|\langle v, Av\rangle\big| \leq \|v\| \cdot \|Av\|$. Hence,

$$\|Av\| \geq \alpha\|v\| \text{ for all } v \in \mathcal{V}. \tag{2.1}$$

Step 1. We can see that A is injective, because $Av = 0$ implies $v = 0$ by (2.1).

Step 2. Let us show that $\operatorname{ran} A$ is closed. Assume that $f_n \in \operatorname{ran} A$ and that f_n converge to f in the norm of $\mathcal{V}$. By the result of step 1 there are uniquely determined vectors $v_n \in \mathcal{V}$ with $f_n = Av_n$. The sequence $(f_n) = (Av_n)$ is convergent and is then a Cauchy one. By (2.1), the sequence (v_n) is also a Cauchy one and, due to the completeness of $\mathcal{V}$, converges to some $v \in \mathcal{V}$. As A is continuous, Av_n converges to Av. Hence, $f = Av$, which shows that $f \in \operatorname{ran} A$

Step 3. Let us show finally that $\operatorname{ran} A = \mathcal{V}$. As we showed already that $\operatorname{ran} A$ is closed, it is sufficient to show that $(\operatorname{ran} A)^\perp = \{0\}$. Let $u \perp \operatorname{ran} A$, then $t(u, v) = \langle u, Av\rangle = 0$ for all $v \in V$. Taking $v = u$ we obtain $a(u, u) = 0$, and $u = 0$ by the ellipticity. □

Let us extend the above construction to unbounded operators and forms.

Definition 2.2 (Operator defined by a form).
Let $\mathcal{V}$ and t be as in Theorem 2.1. Moreover, assume that $\mathcal{V}$ is a dense subset of another Hilbert space $\mathcal{H}$ and that there exists a constant $c > 0$ such that $\|u\|_{\mathcal{H}} \leq c\|u\|_{\mathcal{V}}$ for all $u \in \mathcal{V}$. Introduce an operator T defined by t as follows. The domain $D(T)$ consists of the vectors $v \in \mathcal{V}$ for which the map $\mathcal{V} \ni u \mapsto t(u, v)$ can be extended to a continuous antilinear map from $\mathcal{H}$ to $\mathbb{C}$. By the Riesz theorem, for such a v there exists a uniquely defined $f_v \in \mathcal{H}$ such that $t(u, v) = \langle u, f_v\rangle_{\mathcal{H}}$ for all $u \in \mathcal{V}$, and we set $Tv := f_v$. □

Note that one can associate an operator T to any sesquilinear form t but the properties of this operator are quite unpredictable if one does not assume any additional properties of the form t.

Theorem 2.3. *In the situation of definition 2.2 one has*

- *the domain of T is dense in $\mathcal{H}$,*
- *$T : D(T) \to \mathcal{H}$ is bijective,*

- $T^{-1} \in \mathcal{L}(\mathcal{H})$.

Proof. Let $v \in D(T)$. Using the $\mathcal{V}$-ellipticity we have the following inequalities:

$$\alpha\|v\|_{\mathcal{H}}^2 \leq \alpha c^2\|v\|_{\mathcal{V}}^2 \leq c^2\big|t(v,v)\big| \leq c^2\big|\langle v, Tv\rangle_{\mathcal{H}}\big| \leq c^2\|v\|_{\mathcal{H}} \cdot \|Tv\|_{\mathcal{H}},$$

showing that

$$\|Tv\|_{\mathcal{H}} \geq \frac{\alpha}{c^2}\|v\|_{\mathcal{H}}. \tag{2.2}$$

We see immediately that T in injective.

Let us show that T is surjective. Let $h \in \mathcal{H}$ and let $A \in \mathcal{L}(\mathcal{V})$ be the operator associated with t. The map $\mathcal{V} \ni u \mapsto \langle u, h\rangle_{\mathcal{H}}$ is a continuous antilinear map from $\mathcal{V}$ to $\mathbb{C}$, so one can find $w \in \mathcal{V}$ such that $\langle u, h\rangle_{\mathcal{H}} = \langle u, w\rangle_{\mathcal{V}}$ for all $u \in \mathcal{V}$. Denote $v := A^{-1}w$, then $\langle u, h\rangle_{\mathcal{H}} = \langle u, Av\rangle_{\mathcal{V}} = a(u,v)$. By definition this means that $v \in D(T)$ and $h = Tv$.

Hence, T is surjective and injective, and the inverse is bounded by (2.2). It remains to show that the domain of T is dense in $\mathcal{H}$. Let $h \in \mathcal{H}$ with $\langle u, h\rangle_{\mathcal{H}} = 0$ for all $u \in D(T)$. As T is surjective, there exists $v \in D(T)$ with $h = Tv$. Taking now $u = v$ we obtain $\langle v, Tv\rangle_{\mathcal{H}} = 0$, and the $\mathcal{V}$-ellipticity gives $v = 0$ and $h = 0$. □

If the form t has some additional properties, then the associated operators T also enjoys some additional properties.

Theorem 2.4 (Self-adjoint operators defined by forms). *Let T be the operator associated with a* symmetric *sesqulinear form t in the sense of definition 2.2, then*

1. *T is a self-adjoint operator in $\mathcal{H}$,*
2. *$D(T)$ is a dense subspace of the Hilbert space $\mathcal{V}$.*

Proof. For any $u, v \in D(T)$ we have:

$$\langle u, Tv\rangle_{\mathcal{H}} = t(u,v) = \overline{t(v,u)} = \overline{\langle v, Tu\rangle_{\mathcal{H}}} = \langle Tu, v\rangle_{\mathcal{H}}.$$

Therefore, T is at least symmetric and $T \subset T^*$. Let $v \in D(T^*)$. We know from the previous theorem that T is surjective. This means that we can find $v_0 \in D(T)$ such that $Tv_0 = T^*v$. Then for all $u \in D(T)$ we have:

$$\langle Tu, v\rangle_{\mathcal{H}} = \langle u, T^*v\rangle_{\mathcal{H}} = \langle u, Tv_0\rangle_{\mathcal{H}} = \langle Tu, v_0\rangle_{\mathcal{H}}.$$

As T is surjective, this imples $v = v_0$ and then $T = T^*$.

Let us show the density of $D(T)$ in $\mathcal{V}$. Let $h \in \mathcal{V}$ such that $\langle v, h\rangle_{\mathcal{V}} = 0$ for all $v \in D(T)$. There exists $f \in \mathcal{V}$ such that $h = Af$, where $A \in \mathcal{L}(\mathcal{V})$ is the operator associated with t. We have then $0 = \langle v, h\rangle_{\mathcal{V}} = \langle v, Af\rangle_{\mathcal{V}} = t(v,f) = \overline{t(f,v)} = \overline{\langle f, Tv\rangle_{\mathcal{H}}} = \langle Tv, f\rangle_{\mathcal{H}}$. As the vectors Tv cover the whole of $\mathcal{H}$ as v runs through $D(T)$, this imples $f = 0$ and $h = Af = 0$, and we see that the orthogonal complement of $D(T)$ in $\mathcal{V}$ is $\{0\}$. □

An important point in the above consideration is the presence of a certain auxiliary Hilbert space $\mathcal{V}$. As a set, $\mathcal{V}$ coincides with the domain of the form t. This motivates the following definition:

Definition 2.5 (Closed forms). A sesquilinear form t in a Hilbert space H with a dense domain $D(t)$ is called *closed* if the following properties are satisfied:

- t is symmetric,
- t is semibounded from below: $t(u,u) \geq -C\|u\|^2_{\mathcal{H}}$ for all $u \in D(t)$ for some $C \in \mathbb{R}$,
- The domain $D(t)$ equipped with the scalar product

$$\langle u, v\rangle_t := t(u,v) + (C+1)\langle u, v\rangle$$

  is a Hilbert space.

The previous considerations imply the following result:

Proposition 2.6 (Operators defined by closed forms). *Let t be a closed sesquilinear form in $\mathcal{H}$, then the associated linear operator T in $\mathcal{H}$ is self-adjoint.*

Proof. One simply takes $\big(D(t), \langle\cdot,\cdot\rangle_t\big)$ as $\mathcal{V}$, then $t : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is bounded and elliptic. □

Definition 2.7 (Closable form). Let us introduce another important notion. We say that a symmetric sesqulinear form t is *closable*, if there exists a closed form in $\mathcal{H}$ which extends t. The closed sesqulinear form with the above property and with the minimal domain is called the *closure* of t and is denoted $\overline{t}$.

The following proposition is rather obvious.

Proposition 2.8 (Domain of the closure of a form). *If $t \geq -c$, $c \in \mathbb{R}$, and t is a closable form, then $D(\overline{t})$ is exactly the completion of $D(t)$ with respect to the scalar product $\langle u, v\rangle_t := t(u,v) + (c+1)\langle u, v\rangle$.*

It is time to look at examples!

Example 2.9 (Non-closable form). Take $\mathcal{H} = L^2(\mathbb{R})$ and consider the form $a(u,v) = \overline{u(0)}v(0)$ defined on $D(a) = L^2(\mathbb{R}) \cap C^0(\mathbb{R})$. This form is densely defined, symmetric and positive. Let us show that it is not closable. By contradiction, assume that $\overline{a}$ is the closure of a. One should then have the following property: if (u_n) is a sequence of vectors from $D(a)$ which is Cauchy with respect to $\langle\cdot,\cdot\rangle_a$, then it converges to some $u \in D(\overline{a}) \subset \mathcal{H}$ and $a(u_n, u_n)$ converges to $\overline{a}(u,u)$. But one can construct two sequences (u_n) and (v_n) in $D(a)$ such that

- both converge to u in the L^2-norm,

- $u_n(0) = 1$ and $v_n(0) = 0$ for all n.

Then both sequences are a-Cauchy, but the limits of $a(u_n, u_n)$ and $a(v_n, v_n)$ are different. This shows that $\overline{a}$ cannot exist. $\square$

Let us give some "canonical" examples of operators defined by forms.

Example 2.10 (Laplacian). Consider the Hilbert space $H = L^2(\mathbb{R}^d)$ and the form

$$t(u, v) = \int_{\mathbb{R}^d} \overline{\nabla u} \nabla v \, dx, \quad D(t) = H^1(\mathbb{R}^d),$$

which is clearly closed. Let us find the associated operator T. We know from the very beginning that T is self-adjoint.
Let $f \in D(T)$ and $g := Tf$, then for any $u \in H^1(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \overline{\nabla u} \nabla f \, dx = \int_{\mathbb{R}^d} \overline{u} g \, dx.$$

In particular, this equality holds for $u \in C_c^\infty(\mathbb{R}^d)$, which gives

$$\int_{\mathbb{R}^d} \overline{u} g \, dx = \int_{\mathbb{R}^d} \overline{\nabla u} \nabla f \, dx = \int_{\mathbb{R}^d} \overline{(-\Delta u)} f \, dx.$$

It follows that $g = -\Delta f$ in $\mathcal{D}'(\mathbb{R}^d)$. Therefore, for each $f \in D(T)$ we have $\Delta f \in L^2(\mathbb{R}^d)$, which by (1.1) means that $D(T) \subset H^2(\mathbb{R}^d)$. In other words, we have $T \subset T_1$, where T_1 is the free Laplacian in $\mathbb{R}^d$ (see Definition 1.16). As both T and T_1 are self-adjoint, we have $T = T_1$ (Exercise 1.26).

Example 2.11 (Neumann boundary condition on the halfline). Take $\mathcal{H} = L^2(0, \infty)$. Consider the form

$$t(u, v) = \int_0^\infty \overline{u'(x)} v'(x) dx, \quad D(t) = H^1(0, \infty).$$

The form is semibounded below and closed (which is in fact just equivalent to the completeness of H^1 in the respective Sobolev norm). Let us describe the associated operator T.
Let $v \in D(T)$, then there exists $f_v \in \mathcal{H}$ such that

$$\int_0^\infty \overline{u'(x)} v'(x) dx = \int_0^\infty \overline{u(x)} f_v(x) dx$$

for all $u \in H^1$. Taking here $u \in C_c^\infty$ we obtain just the definition of the derivative in the sense of distributions: $f_v := -(v')' = -v''$. As $f_v \in L^2$, the function v must be in $H^2(0, \infty)$ and $Tv = -v''$.
Now note that for $v \in H^2(0, \infty)$ and $u \in H^1(0, \infty)$ there holds

$$\int_0^\infty \overline{u'(x)} v'(x) dx = \overline{u(x)} v'(x) \Big|_{x=0}^{x=\infty} - \int_0^\infty \overline{u(x)} v''(x) dx.$$

Hence, in order to obtain the requested inequality $t(u,v) = \langle u, Tv\rangle_{\mathcal{H}}$ the boundary term must vanish, which gives the additional condition $v'(0) = 0$.

Therefore, the associated operator is $T_N := T$ acts as $T_N v = -v''$ on the domain $D(T_N) = \big\{v \in H^2(0,\infty) : v'(0) = 0\big\}$. It will be referred as the (positive) Laplacian with the Neumann boundary condition or simply the Neumann Laplacian. □

Example 2.12 (Dirichlet boundary condition on the halfline). Take $\mathcal{H} = L^2(0,\infty)$. Consider the form which is a restriction of the previous one,

$$t_0(u,v) = \int_0^\infty \overline{u'(x)} v'(x) dx, \quad D(t_0) = H_0^1(0,\infty).$$

The form is still semibounded below and closed (as H_0^1 is still complete with respect to the H^1-norm), and the boundary term does not appear when integrating by parts, which means that the associated operator $T_D = T$ acts as $T_D v = -v''$ on the domain $D(T_D) = H^2(0,\infty) \cap H_0^1(0,\infty) = \{v \in H^2(0,\infty) : v(0) = 0\}$. It will be referred to as the (positive) Laplacian with the Dirichlet boundary condition or the Dirichlet Laplacian. □

Remark 2.13. In the two previous examples we see an important feature: the fact that one closed form extends another closed form does *not* imply the same relation for the associated operators. □

Example 2.14 (Neumann/Dirichlet Laplacians: general case). The two previous examples can be generalized to the multidimensional case. Let Ω be an open subset of $\mathbb{R}^d$ with a sufficiently regular boundary $\partial\Omega$ (for example, a compact Lipschitz one). In $\mathcal{H} = L^2(\Omega)$ consider two sesqulinear forms:

$$\begin{aligned} t_0(u,v) &= \int_\Omega \overline{\nabla u}\, \nabla v dx, & D(t_0) &= H_0^1(\Omega), \\ t(u,v) &= \int_\Omega \overline{\nabla u}\, \nabla v dx, & D(t) &= H^1(\Omega). \end{aligned}$$

Both these forms are closed and semibounded from below, and one can easily show that the respective operators A and A_0 act both as $u \mapsto -\Delta u$. By a more careful analysis and, for example, for a smooth $\partial\Omega$, one can show that

$$\begin{aligned} D(A_0) &= H^2(\Omega) \cap H_0^1(\Omega) = \{u \in H^2(\Omega) : u|_{\partial\Omega} = 0\}, \\ D(A) &= \{u \in H^2(\Omega) : \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}, \end{aligned}$$

where n denotes the outward pointing unit normal vector on $\partial\Omega$, and the restrictions to the boundary should be understood as the respective traces. If the boundary is not regular, the domains become more complicated, in particular, the domains of A and A_0 are not included in $H^2(\Omega)$, see e.g. the book [7]. Nevertheless, the operator A_0 is called the *Dirichlet Laplacian* in Ω and A is called the *Neumann Laplacian*. Indeed, the whole construction makes sense if the boundary of Ω is non-empty. For example, $A = A_0$ if $\Omega = \mathbb{R}^d$, as $H^1(\mathbb{R}^d) = H_0^1(\mathbb{R}^d)$.

2.2 Semibounded operators and Friedrichs extensions

Definition 2.15 (Semibounded operator). We say that a symmetric operator T in $\mathcal{H}$ is *semibounded from below* if there exists a constant $C \in \mathbb{R}$ such that

$$\langle u, Tu \rangle \geq -C\langle u, u \rangle \text{ for all } u \in D(T),$$

and in that case we write $T \geq -C$. $\square$

Now assume that an operator T is semibounded from below and consider the induced sesqulinear form t in $\mathcal{H}$,

$$t(u, v) = \langle u, Tv \rangle, \quad D(t) = D(T).$$

Proposition 2.16. *The sesqulinear form t is semibounded from below and closable.*

Proof. The semiboundedness of t from below follows directly from the analogous property for T. To show the closability we remark that without loss of generality one can assume $T \geq 1$. By proposition 2.8, the domain $\mathcal{V}$ of the closure of t must be the completion of $D(T)$ with respect to the norm $p(x) = \sqrt{t(x, x)}$. More concretely, a vector $u \in \mathcal{H}$ belongs to $\mathcal{V}$ iff there exists a sequence $u_n \in D(T)$ which is p-Cauchy such that u_n converges to u in $\mathcal{H}$. A natural candidate for the norm of u is $p(u) = \lim p(u_n)$. Actually we just need to show that this limit is independent of the choice of the sequence u_n. Using the standard arguments we are reduced to prove the following:

Assertion. If $(u_n) \subset D(t)$ is a p-Cauchy sequence converging to zero in $\mathcal{H}$, then $\lim p(u_n) = 0$.

To prove this assertion we observe first that $p(x_n)$ is a non-negative Cauchy sequence, and is convergent to some limit $\alpha \geq 0$. Suppose by contradiction that $\alpha > 0$. Now let us remark that $t(u_n, u_m) = t(u_n, u_n) + t(u_n, u_m - u_n)$. Moreover, by the Cauchy-Schwartz inequality for p we have $\left|t(u_n, u_m - u_n)\right| \leq p(u_n)p(u_m - u_n)$. Combining the two preceding expressions with the fact that u_n is p-Cauchy, we see that for any $\varepsilon > 0$ there exists $N > 0$ such that $\left|t(u_n, u_m) - \alpha^2\right| \leq \varepsilon$ for all $n, m > N$. Take $\varepsilon = \alpha^2/2$ and take the associated N, then for $n, m > N$ we have

$$\left|\langle u_n, Tu_m \rangle\right| \equiv \left|t(u_n, u_m)\right| \geq \frac{\alpha^2}{2}.$$

On the other hand, the term on the left-hand side goes to 0 as $n \to \infty$. So we obtain a contradiction, and the assertion is proved. $\square$

Definition 2.17 (Friedrichs extensions). Let T be a linear operator in $\mathcal{H}$ which is semibounded from below. Consider the above sesqulinear form t and its closure $\bar{t}$. The self-adjoint operator T_F associated with the form $\bar{t}$ is called the *Friedrichs extension* of T. $\square$

Corollary 2.18. *A semibounded operator always has a self-adjoint extension.*

Remark 2.19 (Form domain). If T is a self-adjoint operator and is semibounded from below, then it is the Friedrichs extension of itself. The domain of the associated form $\bar{t}$ is usually called the *form domain* of T and is denoted $Q(T)$. The form domain plays an important role in the analysis of self-adjoint operators, see e.g. Section 8. For $f, g \in Q(T)$ one uses sometimes a slightly abusive writing $\langle f, Tg\rangle$ instead of $\bar{t}(f, g)$. $\square$

Example 2.20 (Schrödinger operators). A basic example for the Friedrichs extension is delivered by Schrödinger operators with semibounded potentials. Let $W \in L^2_{\text{loc}}(\mathbb{R}^d)$ and $W \geq -C$, $C \in \mathbb{R}$ (i.e. W is semibounded from below). In $\mathcal{H} = L^2(\mathbb{R}^d)$ consider the operator T acting as $Tu(x) = -\Delta u(x) + W(x)u(x)$ on the domain $D(T) = C_c^\infty(\mathbb{R}^d)$. One has clearly $T \geq -C$. The Friedrichs extension T_F of T will be called the *Schrödinger operator* with the *potential* W. Note that the sesqulinear form t associated with T is given by

$$t(u, v) = \int_{\mathbb{R}^d} \overline{\nabla u} \nabla v dx + \int_{\mathbb{R}^d} W \bar{u} v dx.$$

Denote by $\bar{t}$ the closure of the form t. One can easily show the inclusion

$$D(\bar{t}) \subset H^1_W(\mathbb{R}^d) := \big\{u \in H^1(\mathbb{R}^d) : \sqrt{|W|}u \in L^2(\mathbb{R}^d)\big\}.$$

Note that actually we have the equality $D(\widetilde{t}) = H^1_W(\mathbb{R}^d)$, see Theorem 8.2.1 in the book [5] for a rather technical proof, but the inclusion will be sufficient for our purposes.

Let us extend the above example by including a class of potentials which are not semibounded from below.

Proposition 2.21 (Hardy inequality). *Let $d \geq 3$ and $u \in C_c^\infty(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \big|\nabla u(x)\big|^2 dx \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{\big|u(x)\big|^2}{|x|^2} dx.$$

Proof. For any $\gamma \in \mathbb{R}$ one has

$$\int_{\mathbb{R}^d} \Big|\nabla u(x) + \gamma \frac{xu(x)}{|x|^2} dx\Big|^2 dx \geq 0,$$

which may be rewritten in the form

$$\begin{aligned}\int_{\mathbb{R}^d} \Big|\nabla u(x)\Big|^2 dx + \gamma^2 \int_{\mathbb{R}^d} \frac{\big|u(x)\big|^2}{|x|^2} dx&\\ \geq -\gamma \int_{\mathbb{R}^d} \Big(x \cdot \overline{\nabla u(x)}\, \frac{u(x)}{|x|^2} + x \cdot \nabla u(x)\, \frac{\overline{u(x)}}{|x|^2}\Big) dx.&\end{aligned} \tag{2.3}$$

Using the identities

$$\nabla |u|^2 = \bar{u}\, \nabla u + u \overline{\nabla u}, \quad \operatorname{div} \frac{x}{|x|^2} = \frac{d-2}{|x|^2}$$

and the integration by parts we obtain

$$\int_{\mathbb{R}^d}\Big(x\cdot\overline{\nabla u(x)}\,\frac{u(x)}{|x|^2}+x\cdot\nabla u(x)\,\frac{\overline{u(x)}}{|x|^2}\Big)\,dx=\int_{\mathbb{R}^d}\nabla\big|u(x)\big|^2\cdot\frac{x}{|x|^2}\,dx$$
$$=-\int_{\mathbb{R}^d}\big|u(x)\big|^2\operatorname{div}\frac{x}{|x|^2}\,dx=-(d-2)\int_{\mathbb{R}^d}\frac{\big|u(x)\big|^2}{|x|^2}\,dx.$$

Inserting this equality into (2.3) gives

$$\int_{\mathbb{R}^d}\big|\nabla u(x)\big|^2dx\geq\gamma\big((d-2)-\gamma\big)\int_{\mathbb{R}^d}\frac{\big|u(x)\big|^2}{|x|^2}\,dx,$$

and in order to optimize the coefficient before the integral on the right-hand side we take $\gamma=(d-2)/2$. $\square$

Note that the integral in the right-hand side of the Hardy inequality is not defined for $d\leq 2$, because the function $x\mapsto|x|^{-1}$ does not belong to L^2_{loc} anymore.

Corollary 2.22. *Let $d\geq 3$ and $W\in L^2_{\text{loc}}(\mathbb{R}^d)$ be real-valued with $W(x)\geq\frac{(d-2)^2}{4|x|^2}$, then the operator $T=-\Delta+W$ defined on $C_c^\infty(\mathbb{R}^d)$ is semibounded from below and, hence, has a self-adjoint extension.*

Example 2.23 (Coulomb potential). We would like to show that the operator $T=-\Delta+q/|x|$ in $L^2(\mathbb{R}^3)$ is semibounded from below for any real q. For $q\geq 0$ we are in the situation of Example 2.20. For $q<0$ we are going to use the Hardy inequality. For any $u\in C_c^\infty(\mathbb{R}^3)$ and any $p\in\mathbb{R}\setminus\{0\}$ we have:

$$\int_{\mathbb{R}^3}\frac{|u|^2}{|x|}dx=\int_{\mathbb{R}^3}p|u|\frac{|u|}{p|x|}dx$$
$$\leq\frac{p^2}{2}\int_{\mathbb{R}^3}|u|^2dx+\frac{1}{2p^2}\int_{\mathbb{R}^3}\frac{|u|^2}{|x|^2}dx$$
$$\leq\frac{p^2}{2}\int_{\mathbb{R}^3}|u|^2dx+\frac{1}{8p^2}\int_{\mathbb{R}^3}|\nabla u|^2dx,$$

and

$$\langle u,Tu\rangle=\int_{\mathbb{R}^3}|\nabla u|^2dx+q\int_{\mathbb{R}^3}\frac{|u|^2}{|x|}dx\geq\int_{\mathbb{R}^3}|\nabla u|^2dx-|q|\int_{\mathbb{R}^3}\frac{|u|^2}{|x|}dx$$
$$\geq\Big(1-\frac{|q|}{8p^2}\Big)\int_{\mathbb{R}^3}|\nabla u|^2dx-\frac{|q|p^2}{2}\int_{\mathbb{R}^3}|u|^2dx,$$

and is sufficient to pick any p with $8p^2\geq|q|$.

Therefore, for any $q\in\mathbb{R}$ the above operator T has a self-adjoint extension (Friedrichs extension). Actually we will see below that this self-adjoint extension is unique. $\square$

2.3 Exercises

Exercise 2.24. Show that the following sesquilinear forms t are closed and semibounded from below and describe the associated self-adjoint operators in H ($\alpha \in \mathbb{R}$ is a fixed parameter):

(a) $H = L^2(0,\infty)$, $D(t) = H^1(0,\infty)$, $t(u,v) = \displaystyle\int_0^\infty \overline{u'(s)}v'(s)\,ds + \alpha\overline{u(0)}v(0)$.

(b) $H = L^2(\mathbb{R})$, $D(t) = H^1(\mathbb{R})$, $t(u,v) = \displaystyle\int_{\mathbb{R}} \overline{u'(s)}v'(s)\,ds + \alpha\overline{u(0)}\,v(0)$.

(c) $H = L^2(0,1)$, $D(t) = \big\{u \in H^1(0,1) : u(0) = u(1)\big\}$, $t(u,v) = \displaystyle\int_0^1 \overline{u'(s)}v'(s)\,ds$.

Exercise 2.25. This exercise shows a possible way of constructing the sum of two unbounded operators under the assumption that one of them is "smaller" that the other one. In a sense, we are going to extend the construction of Exercise 1.4.

Let H be a Hilbert space, t be a symmetric sesquilinear form in H which is densely defined, closed and semibounded below. Let T be the self-adjoint operator in H associated with t. Let B be a symmetric linear operator in H such that $D(t) \subset D(B)$ and such that there exist $\alpha, \beta > 0$ with $\|Bu\|^2 \le \alpha t(u,u) + \beta\|u\|^2$ for all $u \in D(t)$. Consider the operator S on $D(S) = D(T)$ defined by $Su = Tu + Bu$. We are going to show that S is self-adjoint.

(a) Consider the sesquilinear form $s(u,v) = t(u,v) + \langle u, Bv\rangle$, $D(s) = D(t)$. Show that s is symmetric, closed and semibounded from below.

(b) Let $\widetilde{S}$ be the operator associated with s. Show that $D(\widetilde{S}) = D(T)$ and that $\widetilde{S}u = Tu + Bu$ for all $u \in D(T)$.

(c) Show that S is self-adjoint.

Exercise 2.26. In the examples below the Sobolev embedding theorem and the previous exercise can be of use.

(a) Let $v \in L^2(\mathbb{R})$ be real-valued. Show that the operator A having as domain $D(A) = H^2(\mathbb{R})$ and acting by $Af(x) = -f''(x) + v(x)f(x)$ is a self-adjoint operator in $L^2(\mathbb{R})$.

(b) Let $v \in L^2_{\text{loc}}(\mathbb{R})$ be real-valued and 1-periodic, i.e. $v(x+1) = v(x)$ for all $x \in \mathbb{R}$. Show that the operator A with the domain $D(A) = H^2(\mathbb{R})$ acting by $Af(x) = -f''(x) + v(x)f(x)$ is self-adjoint.

(c) Let $H = L^2(\mathbb{R}^3)$. Suggest a class of unbounded potentials $v : \mathbb{R}^3 \to \mathbb{R}$ such that the operator A, $Af(x) = -\Delta f(x) + v(x)f(x)$, with the domain $D(A) = H^2(\mathbb{R}^3)$ is self-adjoint in H.

Exercise 2.27. (a) Let H be a Hilbert space and A be a closed densely defined operator in H (not necessarily symmetric). Consider the operator L given by

$$Lu = A^*Au, \quad u \in D(L) = \big\{u \in D(A) :\ Au \in D(A^*)\big\}.$$

We will write simply $L = A^*A$ having in mind the above precise definition. While the above is a natural definition of the product of two operators, it is not clear if the domain $D(L)$ is sufficiently large. We are going to study this question.

1. Consider the sesquilinear form $b(u, v) = \langle Au, Av \rangle + \langle u, v \rangle$ in H defined on $D(b) = D(A)$. Show that this form is closed, densely defined and semibounded from below.

2. Let B be the self-adjoint operator associated with the form b. Find a relation between L and B and show that L is densely defined, self-adjoint and positive.

3. Let A_0 denote the restriction of A to $D(L)$. Show that $\overline{A_0} = A$.

(b) A linear operator A acting in a Hilbert space H is called *normal* if $D(A) = D(A^*)$ and $\|Ax\| = \|A^*x\|$ for all $x \in D(A)$.

1. Show that any normal operator is closed.

2. Let A be a closed operator. Show: A is normal iff A^* is normal.

3. Let A be a normal operator. Show: $\langle Ax, Ay \rangle = \langle A^*x, A^*y \rangle$ for all $x, y \in D(A) \equiv D(A^*)$.

4. Let A be a closed operator. Show: A is normal iff $AA^* = A^*A$. Here the both operators are defined as in (a), the operator AA^* being understood as $(A^*)^*A^*$.

3 Spectrum and resolvent

3.1 Definitions

Actually most definitions of this chapter can be introduced in the Banach spaces, but we prefer to concentrate on the Hilbertian case.

Definition 3.1 (Resolvent set, spectrum, point spectrum). Let T be a linear operator in a Hilbert space $\mathcal{H}$. The *resolvent set* $\operatorname{res} T$ consists of the complex z for which the operator $T - z : D(T) \to \mathcal{H}$ is bijective and the inverse $(T - z)^{-1}$ is bounded. The *spectrum* $\operatorname{spec} T$ of T is defined by $\operatorname{spec} T := \mathbb{C} \setminus \operatorname{res} T$. The *point spectrum* $\operatorname{spec}_p T$ is the set of the eigenvalues of T. □

Note that very often the resolvent set and the spectrum of T are often denoted by $\rho(T)$ and $\sigma(T)$, respectively.

Proposition 3.2. *If* $\operatorname{res} T \neq \emptyset$, *then* T *is a closed operator.*

Proof. Let $z \in \operatorname{res} T$, then $\operatorname{gr}(T - z)^{-1}$ is closed by the closed graph theorem, but then the graph of $T - z$ is also closed, as $\operatorname{gr}(T - z)$ and $\operatorname{gr}(T - z)^{-1}$ are isometric in $\mathcal{H} \times \mathcal{H}$. □

Proposition 3.3. *For a* closed *operator* T *one has the following equivalence:*

$$z \in \operatorname{res} T \quad \textit{iff} \quad \begin{cases} \ker(T - z) = \{0\}, \\ \operatorname{ran}(T - z) = \mathcal{H}. \end{cases}$$

Proof. The $\Rightarrow$ direction follows from the definition.

Now let T be closed and $z \in \mathbb{C}$ with $\ker(T - z) = \{0\}$ and $\operatorname{ran}(T - z) = \mathcal{H}$. The inverse $(T - z)^{-1}$ is then defined everywhere and has a closed graph (as the graph of $T - z$ is closed), and is then bounded by the closed graph theorem. □

Proposition 3.4 (Properties of the resolvent). *The set* $\operatorname{res} T$ *is open and the set* $\operatorname{spec} T$ *is closed. The operator function*

$$\operatorname{res} T \ni z \mapsto R_T(z) := (T - z)^{-1} \in \mathcal{L}(\mathcal{H})$$

called the resolvent *of* T *is holomorphic and satisfies the identities*

$$R_T(z_1) - R_T(z_2) = (z_1 - z_2) R_T(z_1) R_T(z_2), \tag{3.1}$$

$$R_T(z_1) R_T(z_2) = R_T(z_2) R_T(z_1), \tag{3.2}$$

$$\frac{d}{dz} R_T(z) = R_T(z)^2 \tag{3.3}$$

for all $z, z_1, z_2 \in \operatorname{res} T$.

Proof. Let $z_0 \in \operatorname{res} H$. We have the equality

$$T - z = (T - z_0)\big(1 - (z - z_0) R_T(z_0)\big).$$

If $|z - z_0| < 1/\|R_T(z_0)\|$, then the operator on the right had sinde has a bounded inverse, which means that $z \in \operatorname{res} T$. Moreover, one has the series representation

$$R_T(z) = \big(1 - (z - z_0) R_T(z_0)\big)^{-1} R_T(z_0) = \sum_{j=0} (z - z_0)^j R_T(z_0)^{j+1}, \tag{3.4}$$

which shows that R_T is holomorphic. The remaining properties can be proved in a similar way. □

3.2 Examples

Let us consider a series of examples showing several situations where an explicit calculation of the spectrum is possible. We emphasize that the point spectrum is not the same as the spectrum!

Example 3.5. Consider the multiplication operator M_f from Example 1.6. Recall that the *essential range* of a function f is defined by

$$\operatorname{ess\,ran} f = \Big\{\lambda : \mu\big\{x : |f(x) - \lambda| < \epsilon\big\} > 0 \text{ for all } \epsilon > 0\Big\}.$$

Clearly, this notion makes sense in any measure space. For a continuous function f and the Lebesgue measure μ, the essential range coincides with the closure of the usual range.

Proposition 3.6 (Spectrum of the multiplication operator). *There holds*

$$\operatorname{spec} M_f = \operatorname{ess\,ran} f,$$
$$\operatorname{spec}_p M_f = \big\{\lambda : \mu\{x : f(x) = \lambda\} > 0\big\}.$$

Proof. Let $\lambda \notin \operatorname{ess\,ran} f$, then the operator $M_{1/(f-\lambda)}$ is bounded, and one easily checks that this is the inverse for $M_f - \lambda$. On the other hand, let $\lambda \in \operatorname{ess\,ran} f$. For any $m \in \mathbb{N}$ denote

$$\widetilde{S}_m := \big\{x : |f(x) - \lambda| < 2^{-m}\big\}$$

and choose a subset $S_m \subset \widetilde{S}_m$ of strictly positive but finite measure. If ϕ_m is the characteristic function of S_m, one has

$$\big\|(M_f - \lambda)\phi_m\big\|^2 = \int_{S_m} \big|f(x) - \lambda\big|^2 \big|\phi_m(x)\big|^2 dx \le 2^{-2m} \big\|\phi_m\big\|^2,$$

and the operator $(M_f - \lambda)^{-1}$ cannot be bounded.

To prove the second assertion we remark that the condition $\lambda \in \operatorname{spec}_p M_f$ is equivalent to the existence of $\phi \in L^2(\mathbb{R}^d)$ such that $\big(f(x) - \lambda\big)\phi(x) = 0$ for a.e. x. This means that $\phi(x) = 0$ for a.e. x with $f(x) \neq \lambda$, and $\operatorname{spec}_p M_f = \emptyset$ if $\mu\{x : f(x) = \lambda\} = 0$. On the other hand, if $\mu\{x : f(x) = \lambda\} > 0$, one can choose a subset $\Sigma \subset \{x : f(x) = \lambda\}$ of a strictly positive but finite measure, then the characteristic function ϕ of Σ is an eigenfunction of M_f corresponding to the eigenvalue λ. □

Example 3.7. It can be shown that the spectrum is invariant under unitary transformations (see Exercise 3.21):

Proposition 3.8 (Spectrum and unitary equivalence). *Let two operators A an B be unitarily equivalent, then* $\operatorname{spec} A = \operatorname{spec} B$ *and* $\operatorname{spec}_{\mathrm{p}} A = \operatorname{spec}_{\mathrm{p}} B$.

Example 3.9. Let T be the free Laplacian in $\mathbb{R}^d$ (see Definition 1.16). As seen above, T is unitarily equivalent to the multiplication operator $f(p) \mapsto p^2 f(p)$ in $L^2(\mathbb{R}^d)$. By Propositions 3.6 and 3.8 there holds $\operatorname{spec} T = [0, +\infty)$ and $\operatorname{spec}_{\mathrm{p}} T = \emptyset$.

Example 3.10 (Discrete multiplication operator). Take $\mathcal{H} = \ell^2(\mathbb{Z})$. Consider an aribtrary function $a : \mathbb{Z} \to \mathbb{C}$, $n \mapsto a_n$, and the associated operator T:

$$D(T) = \big\{(\xi_n) \in \ell^2(\mathbb{Z}) :\, (a_n \xi_n) \in \ell^2(\mathbb{Z})\big\}, \quad (T\xi)_n = a_n \xi_n.$$

Similarly to examples 1.6 and 3.6 one can show that T is a closed operator and that

$$\operatorname{spec} T := \overline{\{a_n :\, n \in \mathbb{Z}\}}, \quad \operatorname{spec}_{\mathrm{p}} T := \{a_n :\, n \in \mathbb{Z}\}.$$

Example 3.11 (Harmonic oscillator). Let $\mathcal{H} = L^2(\mathbb{R})$. Consider the operator $T_0 = -d^2/dx^2 + x^2$ defined on $\mathcal{S}(\mathbb{R})$. We see that this operator is semibounded from below and denote by T its Friedrichs extension. The operator T is called the *harmonic oscillator*; it is one of the basic models appearing in quantum mechanics.

One can easily see that the numbers $\lambda_n = 2n-1$ are eigenvalues of T_0, $n \in \mathbb{N}$, and the associated eigenfunctions ϕ_n are given by $\phi_n(x) = c_n(-d/dx + x)^{n-1}\phi_1(x)$, $\phi_1(x) = c_1 \exp(-x^2/2)$, where c_n are normalizing constants. It is known that the functions (ϕ_n) (called *Hermite functions*) form an orthonormal basis in $L^2(\mathbb{R})$. We remark that $\phi_n \in D(T_0)$ for all n, hence, T_0 is essentially self-adjoint (see Exercise 1.25c). This means, in particular, that $T = \overline{T_0}$.

Furthermore, using the unitary map $U : L^2(\mathbb{R}) \to \ell^2(\mathbb{N})$, $Uf(n) = \langle \phi_n, f\rangle$, one easily checks that the operator T is unitarily equivalent to the operator of multiplication by (λ_n) in $\ell^2(\mathbb{N})$, cf. Example 3.10, which gives

$$\operatorname{spec} T = \operatorname{spec}_{\mathrm{p}} T = \{2n-1 : \quad n \in \mathbb{N}\}.$$

Example 3.12 (A finite-difference operator). Consider again the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$ and the operator T in $\mathcal{H}$ acting as $(Tu)(n) = u(n-1)+u(n+1)$. Clearly, $T \in \mathcal{L}(\mathcal{H})$. To find its spectrum consider the map

$$\Phi : \ell^2(\mathbb{Z}) \to L^2(0,1), \quad (\Phi u)(x) = \sum_{n\in\mathbb{Z}} u(n) e^{2\pi i n x},$$

where the sum on the right hand side should be understood as a series in L^2. It is known that Φ is a unitary map. On the other hand, for any $u \in \ell^2(\mathbb{Z})$ supported at a finite number of points we have

$$\begin{aligned}
\Phi(Tu)(x) &= \sum_n (Tu)(n) e^{2\pi i n x}\\
&= \sum_n u(n-1) e^{2\pi i n x} + \sum_n u(n+1) e^{2\pi i n x}\\
&= \sum_n u(n) e^{2\pi i (n+1) x} + \sum_n u(n) e^{2\pi i (n-1) x}\\
&= e^{2\pi i x}\sum_n u(n) e^{2\pi i n x} + e^{-2\pi i x}\sum_n u(n) e^{2\pi i n x}\\
&= 2\cos(2\pi x)(\Phi u)(x).
\end{aligned}$$

Using the density argument we show that the operator $\Phi T \Phi^*$ is exactly the multilication by $f(x) = 2\cos(2\pi x)$ in the space $L^2(0,1)$, and its spectrum coincides with the segment $[-2,2]$, i.e. with the essential range of f. So we have $\operatorname{spec} T = [-2,2]$ and $\operatorname{spec}_{\mathrm{p}} T = \emptyset$.

Example 3.13 (Empty spectrum). Take $\mathcal{H} = L^2(0,1)$ and consider the operator T acting as $Tf = f'$ on the domain $D(T) = \big\{f \in H^1(0,1) : f(0) = 0\big\}$. One can easily see that for any $g \in L^2(0,1)$ and any $z \in \mathbb{C}$ the equation $(T-z)f = g$ has the unique solution explicitly given by

$$f(x) = \int_0^x e^{z(x-t)} g(t)\, dt,$$

and the map $g \mapsto f$ is bounded in the norm of $\mathcal{H}$. So we obtained $\operatorname{res} T = \mathbb{C}$ and $\operatorname{spec} T = \emptyset$.

Example 3.14 (Empty resolvent set). Let us modify the previous example. Take $\mathcal{H} = L^2(0,1)$ and consider the operator T acting as $Tf = f'$ on the domain $D(T) = H^1(0,1)$. Now for any $z \in \mathbb{C}$ we see that the function $\phi_z(x) = e^{zx}$ belongs to $D(T)$ and satisfies $(T - z)\phi_z = 0$. Therefore, $\operatorname{spec}_p T = \operatorname{spec} T = \mathbb{C}$.

As we can see in the two last examples, for general operators one cannot say much on the location of the spectrum. In what follows we will study mostly self-adjoint operators, whose spectral theory is now understood much better than for the non-self-adjoint case.

3.3 Basic facts on the spectra of self-adjoint operators

The following two propositions will be of importance during the whole course.

Proposition 3.15. *Let T be a closable operator in a Hilbert space $\mathcal{H}$ and $z \in \mathbb{C}$, then*

$$\ker(T^* - \overline{z}) = \operatorname{ran}(T - z)^\perp, \tag{3.5}$$

$$\overline{\operatorname{ran}(T - z)} = \ker(T^* - \overline{z})^\perp. \tag{3.6}$$

Proof. Note that the second equality can be obtained from the first one by taking the orthogonal complement in the both parts. Let us prove the first equality. As $D(T)$ is dense, the condition $f \in \ker(T^* - \overline{z})$ is equivalent to $\langle (T^* - \overline{z})f, g\rangle = 0$ for all $g \in D(T)$, which can be also rewritten as

$$\langle T^* f, g\rangle = z\langle f, g\rangle \text{ for all } g \in D(T).$$

By the definition of T^*, one has $\langle T^* f, g\rangle = \langle f, Tg\rangle$ and

$$\langle f, Tg\rangle - z\langle f, g\rangle \equiv \langle f, (T - z)g\rangle = 0 \text{ for all } g \in D(T),$$

i.e. $f \perp \operatorname{ran}(T - z)$. □

Proposition 3.16 (Spectrum of a self-adjoint operator is real). *Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$, then* $\operatorname{spec} T \subset \mathbb{R}$, *and for any $z \in \mathbb{C} \setminus \mathbb{R}$ there holds*

$$\big\|(T - z)^{-1}\big\| \le \frac{1}{|\Im z|}. \tag{3.7}$$

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$ and $u \in D(T)$. We have

$$\langle u, (T - z)u\rangle = \langle u, Tu\rangle - \Re z\langle u, u\rangle - i\Im z\langle u, u\rangle.$$

As T is self-adjoint, the number $\langle u, Tu\rangle$ is real. Therefore,

$$|\Im z|\|u\|^2 \le \big|\langle u, (T - z)u\rangle\big| \le \big\|(T - z)u\big\| \cdot \|u\|,$$

which shows that

$$\big\|(T - z)u\big\| \ge |\Im z| \cdot \|u\|. \tag{3.8}$$

It follows from here that $\operatorname{ran}(T - z)$ is closed, that $\ker(T - z) = \{0\}$ and, by proposition 3.15, than $\operatorname{ran}(T - z) = \mathcal{H}$. Therefore, $(T - z)^{-1} \in \mathcal{L}(\mathcal{H})$, and the estimate (3.7) follows from (3.8). □

The following proposition is of importance when studying bounded operators.

Proposition 3.17 (Spectrum of a continuous operator). *Let $T \in \mathcal{L}(\mathcal{H})$, then* $\operatorname{spec} T$ *is a non-empty subset of* $\big\{z \in \mathbb{C} : |z| \le \|T\|\big\}$.

Proof. Let $z \in \mathbb{C}$ with $|z| > \|T\|$. Represent $T - z = -z(1 - T/z)$. As $\|T/z\| < 1$, the inverse to $T - z$ is defined by the series,

$$(T - z)^{-1} = -\sum_{n=0}^{\infty} T^n z^{-n-1}.$$

and $z \in \operatorname{res} T$. This implies the sought inclusion.

Let us show that the spectrum is non-empty. Assume that it is not the case. Then for any $f, g \in \mathcal{H}$ the function $\mathbb{C} \ni z \mapsto F(z) := \langle f, R_T(z) g\rangle \in \mathbb{C}$ is holomorphic in $\mathbb{C}$ by proposition 3.4. On the other hand, it follows from the above series representation for the resolvent that for large z the norm of $R_T(z)$ tends to zero. It follows that $F(z) \to 0$ as $|z| \to \infty$ and that F is bounded. By Liouville's theorem, F is constant, and, moreover, $F(z) = \lim_{|z|\to+\infty} F(z) = 0$. Therefore, $\langle f, R_T(z)g\rangle = 0$ for all $z \in \mathbb{C}$ and $f, g \in \mathcal{H}$, which means that $R_T(z) = 0$. This contradicts the definition of the resolvent and shows that the spectrum of T must be non-empty. □

Proposition 3.18 (Location of spectrum of self-adjoint operators). *Let $T = T^* \in \mathcal{L}(\mathcal{H})$. Denote*

$$m = m(T) = \inf_{u \neq 0} \frac{\langle u, Tu\rangle}{\langle u, u\rangle}, \quad M = M(T) = \sup_{u \neq 0} \frac{\langle u, Tu\rangle}{\langle u, u\rangle},$$

then $\operatorname{spec} T \subset [m, M]$ *and* $\{m, M\} \subset \operatorname{spec} T$.

Proof. We proved already that $\operatorname{spec} T \subset \mathbb{R}$. For $\lambda \in (M, +\infty)$ we have

$$\big|\langle u, (\lambda - T)u\rangle\big| \ge (\lambda - M)\|u\|^2,$$

and $(T - \lambda)^{-1} \in \mathcal{L}(\mathcal{H})$ by the Lax-Milgram theorem. In the same way one shows that $\operatorname{spec} T \cap (-\infty, m) = \emptyset$.

Let us show that $M \in \operatorname{spec} T$ (for m the proof is similar). Using the Cauchy-Schwarz inequality for the semi-scalar product $(u, v) \mapsto \langle u, (M - T)v\rangle$ we obtain

$$\big|\langle u, (M - T)v\rangle\big|^2 \le \langle u, (M - T)u\rangle \cdot \langle v, (M - T)v\rangle.$$

Taking the supremum over all $u \in \mathcal{H}$ with $\|u\| \le 1$ we arrive at

$$\big\|(M - T)v\big\| \le \|M - T\| \cdot \big\langle v, (M - T)v\big\rangle.$$

By assumption, one can construct a sequence (u_n) with $\|u_n\| = 1$ such that $\langle u_n, Tu_n\rangle \to M$ as $n \to \infty$. By the above inequality we have then $(M - T)u_n \to 0$, and the operator $M - T$ cannot have bounded inverse. Thus $M \in \operatorname{spec} T$. □

Corollary 3.19. *If $T = T^* \in \mathcal{L}(\mathcal{H})$ and $\operatorname{spec} T = \{0\}$, then $T = 0$.*

Proof. By proposition 3.18 we have $m(T) = M(T) = 0$. This means that $\langle x, Tx\rangle = 0$ for all $x \in \mathcal{H}$, and the polar identity shows that $\langle x, Ty\rangle = 0$ for all $x, y \in \mathcal{H}$. $\square$

Let us combine all of the above to show the following fundamental fact.

Theorem 3.20 (Non-emptiness of spectrum). *The spectrum of a self-adjoint operator in a Hilbert space is a non-empty closed subset of the real line.*

Proof. In view of the preceding discussion, it remains to show the non-emptyness of the spectrum. Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. By contradiction, assume that $\operatorname{spec} T = \emptyset$. Then, first of all, $T^{-1} \in \mathcal{L}(\mathcal{H})$. Let $\lambda \in \mathbb{C} \setminus \{0\}$. One can easily show that the operator

$$L_\lambda := -\frac{T}{\lambda}\Big(T - \frac{1}{\lambda}\Big)^{-1} \equiv -\frac{1}{\lambda} - \frac{1}{\lambda^2}\Big(T - \frac{1}{\lambda}\Big)^{-1}$$

belongs to $\mathcal{L}(\mathcal{H})$ and that $(T^{-1} - \lambda)L_\lambda = \mathrm{Id}_\mathcal{H}$ and $L_\lambda(T^{-1} - \lambda) = \mathrm{Id}_\mathcal{H}$. Therefore, $\lambda \in \operatorname{res}(T^{-1})$. As λ was an arbitrary non-zero complex number, we have $\operatorname{spec}(T^{-1}) \subset \{0\}$. As T^{-1} is bounded, its spectrum is non-empty, hence, $\operatorname{spec} T^{-1} = \{0\}$. On the other hand, T^{-1} is self-adjoint by Proposition 1.23, and $T^{-1} = 0$ by Corollary 3.19, which contradicts the definition of the inverse operator. $\square$

3.4 Exercises

Exercise 3.21. 1. Let two operators A and B be unitarily equivalent (see Exercise 1.25). Show that the $\operatorname{spec} A = \operatorname{spec} B$ and $\operatorname{spec}_\mathrm{p} A = \operatorname{spec}_\mathrm{p} B$.

2. Let $\mu \in \operatorname{res} A \cap \operatorname{res} B$. Show that A and B are unitarily equivalent iff their resolvents $R_A(\mu)$ and $R_B(\mu)$ are unitarily equivalent.

3. Let A be a closed operator. Show that $\operatorname{spec} A^* = \{\overline{z} : z \in \operatorname{spec} A\}$ and that the resolvent identity $R_A(z)^* = R_{A^*}(\overline{z})$ holds for any $z \in \operatorname{res} A$.

4. Let $k \in L^1(\mathbb{R})$. Consider in $L^2(\mathbb{R})$ the operator A, $Af(x) = \int_\mathbb{R} k(x-y)f(y)\,dy$. Show: (i) the operator A is well-defined and bounded, (ii) the spectrum of A is a connected set.

Exercise 3.22. 1. Let $\Omega \subset \mathbb{R}^n$ be a non-empty open set and let $L : \Omega \to M_2(\mathbb{C})$ be a continuous matrix function such that $L(x)^* = L(x)$ for all $x \in \Omega$. Define an operator A in $H = L^2(\Omega, \mathbb{C}^2)$ by

$$Af(x) = L(x)f(x), \quad D(A) = \Big\{f \in H : \int_\Omega \|L(x)f(x)\|^2_{\mathbb{C}^2} dx < +\infty\Big\}.$$

Show that A is self-adjoint and explain how to calculate its spectrum using the eigenvalues of $L(x)$.

Hint: For each $x \in \Omega$, let $\xi_1(x)$ and $\xi_2(x)$ be suitably chosen eigenvectors of $L(x)$ forming an orthonormal basis of $\mathbb{C}^2$. Consider the map

$$U : H \to H, \quad Uf(x) = \begin{pmatrix} \big\langle \xi_1(x), f(x) \big\rangle_{\mathbb{C}^2} \\ \big\langle \xi_2(x), f(x) \big\rangle_{\mathbb{C}^2} \end{pmatrix}$$

and the operator $M = UAU^*$.

2. In $H = l^2(\mathbb{Z})$ consider the operator T given by

$$Tf(n) = f(n-1) + f(n+1) + V(n)f(n), \quad V(n) = \begin{cases} 4 & \text{if } n \text{ is even}, \\ -2 & \text{if } n \text{ is odd}. \end{cases}$$

Calculate its spectrum.

Hint: Consider the operators

$$U : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}, \mathbb{C}^2), \quad Uf(n) := \begin{pmatrix} f(2n) \\ f(2n+1) \end{pmatrix}, \quad n \in \mathbb{Z},$$
$$F : \ell^2(\mathbb{Z}, \mathbb{C}^2) \to L^2\big((0,1), \mathbb{C}^2\big), \quad (Ff)(\theta) = \sum_{n \in \mathbb{Z}} f(n) e^{2\pi i n \theta}.$$

Write explicit expressions for the operators $S := UTU^*$ and $\widehat{S} := FSF^*$ and use the item (1).

Exercise 3.23. Let A be a semibounded from below self-adjoint operator. Show:

1. $\inf \operatorname{spec} A = \inf\limits_{\substack{x \in D(A) \\ x \neq 0}} \dfrac{\langle x, Ax \rangle}{\langle x, x \rangle}$.

2. $\inf \operatorname{spec} A = \inf\limits_{\substack{x \in Q(A) \\ x \neq 0}} \dfrac{\langle x, Ax \rangle}{\langle x, x \rangle}$, where $Q(A)$ is the form domain of A.

4 Spectral theory of compact operators

4.1 Fredholm's alternative and spectra of compact operators

It is assumed that the the reader already has some knowledge of compact operators. We recall briefly the key points. Recall first that any Hilbert space is locally compact in the weak topology, which means that any bounded sequence contains a weakly convergent subsequence.

A linear operator T acting from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$ is called compact, if the image of the unit ball in $\mathcal{H}_1$ is relatively compact in $\mathcal{H}_2$. We denote by $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all such operators. The definition can also be reformulated as follows: an operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is compact iff any bounded sequence $(x_n) \subset \mathcal{H}_1$ has a subsequence (x_{n_k}) such that Tx_{n_k} converges in $\mathcal{H}_2$.

Recall also that any compact operator is continuous. If A is a continuous operator and B is a compact one, then the products AB and BA are compact. It is also known the norm limit of a sequence compact operators is compact, and that any finite-dimensional operator (i.e. an operator having a finite-dimensional range) is compact. It is also known that the adjoint of a compact operator is compact (Schauder's theorem).

Proposition 4.1. *Let $T \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$, then:*

(a) If $x_n, x \in \mathcal{H}_1$ and x_n converge weakly to x, then Tx_n converges to Tx in the norm of $\mathcal{H}_2$,

(b) $\operatorname{ran} T$ and $(\ker T)^\perp$ are separable.

Proof. (a) Clearly, it is sufficient to consider the case $x = 0$.

Let us prove first the following assertion: (A) If (u_n) is a sequence which converges weakly to 0, then it contains a subsequence (u_{n_k}) converging to 0 in the norm.

As (u_n) is bounded (being weakly convergent) and T is compact, one can extract a subsequence (u_{n_k}) such that Tu_{n_k} converges to some $v \in \mathcal{H}_2$. For any $g \in \mathcal{H}_2$ we have

$$\langle v, g\rangle = \lim\langle Tu_{n_k}, g\rangle = \langle u_{n_k}, T^*g\rangle = 0,$$

i.e. $v = 0$, and the assertion (A) is proved.

Now assume by contradiction that Tx_n does not converge to 0, then there exists a subsequence (x_{n_k}) and $\epsilon > 0$ with $\|Tx_{n_k}\| \geq \epsilon$, which contradicts the assertion (A).

(b) Let $(e_\alpha)_{\alpha\in A}$ be a total orthonormal family in $(\ker T)^\perp$. Take any injection $\mathbb{N} \ni n \mapsto \alpha_n \in A$, then the sequence (e_{α_n}) converges to 0 weakly, and Te_{α_n} converges to 0. It follows that for any $\epsilon > 0$ the set

$$A_\epsilon := \big\{\alpha \in A : \|Te_\alpha\| > \varepsilon\big\}.$$

is finite, and the representation

$$A = \bigcup_{n\in\mathbb{N}} A_{1/n}$$

shows that A is a countable and, finally, that $(\ker T)^\perp$ is separable. On the other hand, the linear hull of the vectors Te_α, $\alpha \in A$, is dense in $\operatorname{ran} T$, which means that $\operatorname{ran} T$ is also separable. $\square$

The previous proposition shows that it is sufficient to consider compact operators in separable Hilbert spaces, which is assumed from now on.

Below we will consider some examples, but we prefer to discuss first some basic questions of the spectral theory.

Theorem 4.2 (Fredholm's alternative). *Let T be a compact operator in a Hilbert space $\mathcal{H}$, then*

(a) $\ker(1-T)$ *has a finite dimension,*

(b) $\operatorname{ran}(1-T)$ *is closed and of finite codimension,*

(c) $\operatorname{ran}(1-T) = \mathcal{H}$ *if and only if* $\ker(1-T) = \{0\}$.

Proof. We give the proof for the case $T = T^*$ only. An interested reader may refer to Section VI.5 in [12] for the proof of the general case.

To show (a) let us recall the Riesz theorem: if E is a normed vector space such that the unit ball is relatively compact, then E is finite-dimensional. Let us apply this to $E = \ker(1-T)$ with the same norm as in $\mathcal{H}$. For every $u \in E$ we have $Tu = u$. As the unit ball B in E is bounded, it is weakly compact. As T is compact, $B = T(B)$ is a relatively compact set, which means that E is finite-dimensional.

Let us prove (b). Show first that $\operatorname{ran}(1-T)$ is closed. Let $(y_n) \subset \operatorname{ran}(1-T)$ such that y_n converges to y in the norm of $\mathcal{H}$. To show that $y \in \operatorname{ran}(1-T)$ we choose $x_n \in \ker(1-T)^\perp$ with $y_n = (1-T)x_n$.

We show first that the sequence (x_n) is bounded. Assume by contradiction that it is not the case, then one can choose a subsequence with norms growing to $+\infty$. To keep simple notation we denote the subsequence again by x_n and denote $u_n := x_n/\|x_n\|$, then

$$u_n - Tu_n = \frac{(1-T)x_n}{\|x_n\|} = \frac{y_n}{\|x_n\|}.$$

As the norms of y_n are bounded, the vectors $u_n - Tu_n$ converge to 0. As the sequence u_n is bounded, one can choose a subsequence u_{n_j} which is weakly convergent, then the sequence Tu_{n_j} is convergent with respect to the norm to some $u \in \mathcal{H}$ due to the compactness of T. On the other hand, as shown above, $u_{n_j} - Tu_{n_j}$ converge to 0, which means that $u - Tu = 0$ and $u \in \ker(1-T)$. On the other hand, we have $u_n \in \ker(1-T)^\perp$, which means that $u \in \ker(1-T)^\perp$ too. This shows that $u = 0$, but this contradicts to $\|u_n\| = 1$. This contradiction shows that (x_n) is a bounded sequence.

As (x_n) is bounded, one can find a subsequence x_{n_j} which converges weakly to some $x_\infty \in \mathcal{H}$, and then Tx_{n_j} converge in the norm to Tx_∞. Now we have $x_{n_j} = y_{n_j} + Tx_{n_j}$, both sequences on the right-hand side are convergent with respect to the norm, so x_{n_j} is also convergent to x_∞ in the norm. Finally we obtain $x_\infty = y + Tx_\infty$, or $y = (1-T)x_\infty$, which means that $y \in \operatorname{ran}(1-T)$. So we proved that $\operatorname{ran}(1-T)$ is closed.

For our particular case $T = T^*$ we have $\operatorname{ran}(1-T) = \ker(1-T)^\perp$ by Proposition 3.15. Combining this with (a) we complete the proof of (b), and the item (c) is proved too. □

In a sense, the Fredholm alternative show that the operators $1 - T$ with compact T behave like operators in finite dimensional spaces. We know that a linear operator in a finite-dimensional space is injective if and only if it is surjective, and we see a similar feature in the case under consideration. We remark that the Fredholm alternative also holds for compact operators in Banach spaces, but we are not discussing this direction.

Theorem 4.3 (Spectrum of compact operator). *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and $T \in \mathcal{K}(\mathcal{H})$, then*

(a) $0 \in \operatorname{spec} T$,

(b) $\operatorname{spec} T \setminus \{0\} = \operatorname{spec}_p T \setminus \{0\}$,

(c) we are in one and only one of the following situations:

- $\operatorname{spec} T \setminus \{0\} = \emptyset$,
- $\operatorname{spec} T \setminus \{0\}$ *is a finite set,*
- $\operatorname{spec} T \setminus \{0\}$ *is a sequence convergent to* 0.

(d) Each $\lambda \in \operatorname{spec} T \setminus \{0\}$ is isolated (i.e. has a neighbodhood containing no other values of the spectrum), and $\dim\ker(T-\lambda) < \infty$.

Proof. (a) Assume that $0 \notin \operatorname{spec} T$, then $T^{-1} \in \mathcal{L}(\mathcal{H})$, and the operator $\mathrm{Id} = T^{-1}T$ is compact. This is possible only if $\mathcal{H}$ is finite-dimensional.

(b) If $\lambda \neq 0$ we have $T-\lambda = -\lambda(1-T/\lambda)$, and by Fredholm alternative the condition $\lambda \in \operatorname{spec} T$ is equivalent to $\ker(1-T/\lambda) \equiv \ker(T-\lambda) \neq \{0\}$.

(c) Here we actually need to prove the following assertion: if (λ_j) is a sequence distinct non-zero eigenvalues of T converging to some $\lambda \in \mathbb{C}$, then $\lambda = 0$. For the proof, assume by contradiction that $\lambda \neq 0$. Let (e_j) be the normalized eigenvectors associated with the eigenvalues λ_j, $Te_j = \lambda_j e_j$. One checks that the vectors e_n are linearly indepedent. Denote by E_n the linear subspace spanned by $e_1, \dots, e_n$, then $E_n \subset E_{n+1}$ and $E_n \neq E_{n+1}$. For any n we choose $u_n \in E_n \cap E_{n-1}^{\perp}$ with $\|u_n\| = 1$. As T is compact and (u_n) is bounded, one can extract a subsequence u_{n_k} such that the sequence (Tu_{n_k}) converges, and then the sequence Tu_{n_k}/λ_{n_k} is also convergent. Let $j > k \geq 2$. We can write

$$\left\| \frac{Tu_{n_j}}{\lambda_{n_j}} - \frac{Tu_{n_k}}{\lambda_{n_k}} \right\|^2 = \left\| \frac{(T-\lambda_{n_j})u_{n_j}}{\lambda_{n_j}} - \frac{(T-\lambda_{n_k})u_{n_k}}{\lambda_{n_k}} + u_{n_j} - u_{n_k} \right\|^2 \tag{4.1}$$

Note that for any k we have $(T-\lambda_{n_k})E_{n_k} \subset E_{n_k-1}$. On the right-hand side of (4.1) one has $u_{n_j} \in E_{n_j}$ and all the other vectors are in the strictly smaller subspace E_{n_j-1}. Therefore, u_{n_j} is orthogonal to the other three vectors, which gives the estimate

$$\left\| \frac{Tu_{n_j}}{\lambda_{n_j}} - \frac{Tu_{n_k}}{\lambda_{n_k}} \right\|^2 \geq \|u_{n_j}\|^2 = 1.$$

Therefore, (Tu_{n_j}/λ_{n_j}) cannot be a Cauchy sequence, which shows that $\lambda = 0$.

The item (d) easily follows from (c) and from the part (a) of the Fredholm alternative. □

Finally let us apply all of the above to show the main result on the spectra of of compact self-adjoint operators.

Theorem 4.4 (Spectrum of compact self-adjoint operator). *Let $T = T^* \in \mathcal{K}(\mathcal{H})$, then can construct an orthonormal basis consisting of eigenvectors of T, and the respective eigenvalues form a real sequence convergent to 0.*

Proof. Let $(\lambda_n)_{n\geq 1}$ be the distinct non-zero eigenvalues of T. As T is self-adjoint, these eigenvalues are real. Set $\lambda_0 = 0$, and for $n \geq 0$ denote $E_n := \ker(T - \lambda_n)$. One can easily see that $E_n \perp E_m$ for $n \neq m$. Denote by F the linear hull of $\cup_{n\geq 0} E_n$. We are going to show that F is dense in $\mathcal{H}$.

Clearly, we have $T(F) \subset F$. Due to the self-adjointness of T we also have $T(F^\perp) \subset F^\perp$. Denote by $\widetilde{T}$ the restriction of T to $F^\perp$, then $\widetilde{T}$ is compact, self-adjoint, and its spectrum equals $\{0\}$, so $\widetilde{T} = 0$. But this means that $F^\perp \subset \ker T = E_0 \subset F$ and shows that $F^\perp = \{0\}$. Therefore F is dense in $\mathcal{H}$.

Now taking an orthonormal basis in each subspace E_n we obtain an orthonormal basis in the whole space $\mathcal{H}$. □

4.2 Integral and Hilbert-Schmidt operators

An important class of compact operators is delivered by integral operators. For simplicity we restrict our attention to the case $\mathcal{H} = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is an open set. An interested reader may generalize all the considerations to more general measure spaces.

Let $K \in L^1_{\text{loc}}(\Omega \times \Omega)$. Consider the operator T_K defined by

$$T_K f(x) = \int_\Omega K(x, y) f(y)\, dy \tag{4.2}$$

on bounded functions with compact supports. We would like to understand first under which conditions the expression (4.2) defines a bounded operator in $\mathcal{H}$. A standard result in this direction is delivered by the following theorem.

Theorem 4.5 (Schur's test). *Assume that*

$$M_1 = \sup_{x\in\Omega} \int_\Omega \big|K(x,y)\big| dy < \infty, \quad M_2 = \sup_{y\in\Omega} \int_\Omega \big|K(x,y)\big| dx < \infty,$$

then the above expression (4.2) *defines a continuous linear operator T_K with the norm $\|T_K\| \leq \sqrt{M_1 M_2}$.*

Proof. We have, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
|T_K u(x)|^2 &\le \Big(\int_\Omega \sqrt{|K(x,y)|}\sqrt{|K(x,y)|}\cdot|u(y)|dy\Big)^2\\
&\le \int_\Omega |K(x,y)|dy \int_\Omega |K(x,y)|\cdot|u(y)|^2dy\\
&\le M_1\int_\Omega |K(x,y)|\cdot|u(y)|^2dy,\\
\text{and } \|T_K u\|^2 &\le M_1 \int_\Omega\int_\Omega \big|K(x,y)\big||u(y)|^2dydx \le M_1M_2\|u\|^2. \qquad \square
\end{aligned}$$

To obtain a class of compact integral operators we introduce the following notion. We say that an operator $T\in\mathcal{L}(\mathcal{H})$ is *Hilbert-Schmidt* if for some orthonormal basis (e_n) of $\mathcal{H}$ the sum

$$\|T\|_2^2 = \sum_n \|Te_n\|^2 \tag{4.3}$$

is finite.

Proposition 4.6 (Hilbert-Schmidt norm). *Let T be a Hilbert-Schmidt operator, then the quantity $\|T\|_2$ (called the Hilbert-Schmidt norm of T) does not depend on the choice of the basis, and $\|T\|\le\|T\|_2$. Moreover, the adjoint operator T^* is also Hilbert-Schmidt with $\|T^*\|_2=\|T\|_2$.*

Proof. Let (e_n) and (f_n) be two orthonormal bases. Using the Parseval identity we have

$$\sum_n \|Te_n\|^2 = \sum_n\sum_m \big|\langle f_m, Te_n\rangle\big|^2 = \sum_m\sum_m \big|\langle T^*f_m, e_n\rangle\big|^2 = \sum_m \|T^*f_m\|^2.$$

This shows that the expression (4.3) is independent of the choice of (e_n) and that $\|T^*\|_2=\|T\|_2$. To show $\|T\|\le\|T\|_2$, let $x\in\mathcal{H}$ with $x_n:=\langle e_n,x\rangle$, then

$$\|Tx\|^2 = \Big\|\sum_n x_nTe_n\Big\|^2 \le \Big(\sum_n |x_n|\|Te_n\|\Big)^2 \le \sum_n |x_n|^2\sum_n\|Te_n\|^2 = \|T\|_2^2\|x\|^2.$$

$\square$

Proposition 4.7. *Any Hilbert-Schmidt operator is compact.*

Proof. For any $x\in\mathcal{H}$ we have $Tx=\sum_{n=1}^\infty\langle e_n,x\rangle Te_n$. For $N\ge1$ introduce the operators T_N by $T_Nx=\sum_{n=1}^N\langle e_n,x\rangle Te_n$. One has

$$\|T-T_N\|^2\le\|T-T_N\|_2^2 = \sum_{n\ge N+1}\|Te_n\|^2 \xrightarrow{N\to\infty} 0,$$

and T is compact being the norm-limit of the finite-dimensional operators T_N. $\square$

The following proposition describes the class of integral operators which are Hilbert-Schmidt and makes a link with integral operators.

Proposition 4.8 (Integral Hilbert-Schmidt operators). *Let $\mathcal{H} = L^2(\Omega)$. An operator T in $\mathcal{H}$ is Hilbert-Schmidt iff there exists an integral kernel $K \in L^2(\Omega \times \Omega)$ such that $T = T_K$, cf. Eq. (4.2), and in that case $\|T_K\|_2 = \|K\|_{L^2(\Omega\times\Omega)}$.*

Proof. Let first $K \in L^2(\Omega \times \Omega)$. Let us show that the associated operator T_K is Hilbert-Schmidt. Let (e_n) be an orthonormal basis in $\mathcal{H}$, then the functions $e_{m,n}(x, y) = e_m(x)\overline{e_n(y)}$ form an orthonormal basis in $\mathcal{H} \otimes \mathcal{H} \simeq L^2(\Omega \times \Omega)$. There holds

$$\sum_n \|T_K e_n\|^2 = \sum_{m,n} \big|\langle e_m, T_K e_n\rangle\big|^2 = \sum_{m,n} \Big| \int_\Omega \overline{e_m(x)} \Big(\int_\Omega K(x, y)e_n(y)dy\Big)\, dx\Big|^2$$
$$= \sum_{m,n} \Big| \int_\Omega \int_\Omega \overline{e_m(x)} e_n(y)K(x, y)dx\, dy\Big|^2 = \sum_{m,n} \big|\langle e_{m,n}, K\rangle\big|^2 = \|K\|^2_{L^2(\Omega\times\Omega)}.$$

Now let T be an arbitrary Hilbert-Schmidt operator in $\mathcal{H}$. We have, for any $u \in \mathcal{H}$ and with $u_n := \langle e_n, u\rangle$,

$$Tu = \sum_n \langle e_n, u\rangle Te_n = \sum_{m,n} \langle e_n, u\rangle\langle e_m, Te_n\rangle e_m.$$

Take

$$K(x, y) = \sum_{m,n} \overline{e_n(y)}\langle e_m, Te_n\rangle e_m(x) = \sum_{m,n} \langle e_m, Te_n\rangle e_{m,n}(x, y).$$

One easily checks that $K \in L^2(\Omega \times \Omega)$, that $T = T_K$, and that the remaining properties hold as well. $\square$

One can easily see that the operator T_K is self-adjoint iff $K(x, y) = \overline{K(y, x)}$ for a.e. $(x, y) \in \Omega \times \Omega$. Together the Hilbert-Schmidt condition (Proposition 4.8) this gives an important class of self-adjoint compact operators to which the previous considerations can be applied. Taking an orthonormal basis consisting of eigenfunctions we see that a compact self-adjoint operator T is a Hilbert-Schmidt one iff the series

$$\sum_n \lambda_n^2 = \|T\|_2^2$$

is convergent, where λ_n denote the non-zero eigenvalues of T taking according to their multiplicities. Moreover, by Proposition 4.8, for $T = T_K$ one has the exact equality (trace formula)

$$\sum_n \lambda_n^2 = \|K\|^2_{L^2(\Omega\times\Omega)},$$

which may be used to estimate the eigenvalues using the integral kernel.

4.3 Operators with compact resolvent

Let us continue the discussion of operators defined by forms, see Section 2.

Proposition 4.9. *In the situation of Theorem 2.3 one has $T^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{V})$.*

Proof. For any $u \in D(T)$ we have:

$$\|u\|_{\mathcal{H}} \|Tu\|_{\mathcal{H}} \geq \big|\langle u, Tu\rangle_{\mathcal{H}}\big| = |a(u,u)| \geq \alpha \|u\|_V^2 \geq C\alpha \|u\|_V \|u\|_{\mathcal{H}},$$

i.e. $\|Tu\|_{\mathcal{H}} \geq C\alpha\|u\|_V$ and $\|T^{-1}u\|_{\mathcal{V}} \leq (C\alpha)^{-1}\|u\|_{\mathcal{H}}$. □

This gives an important consequence:

Corollary 4.10. *In the situation of Theorem 2.3, assume that the embedding $j : \mathcal{V} \to \mathcal{H}$ is compact, then T^{-1} is a compact operator.*

Proof. Indeed we have $T^{-1} = jL$, where L is the operator T^{-1} viewed as an operator from $\mathcal{H}$ to $\mathcal{V}$. Hence T^{-1} is compact as a composition of a bounded operator and a compact one. □

The above can be applied to a variety of cases. For example, take the Dirichlet Laplacian A_0 defined in example 2.14. If Ω is relatively compact, then the embedding of $\mathcal{V} = H_0^1(\Omega)$ to $\mathcal{H} = L^2(\Omega)$ is compact. Therefore, the operator $L = (A_0 + 1)^{-1}$ is compact. Moreover, it is self-adjoint due to the previous considerations. By Theorem 4.4 there exists an orthonormal basis (e_n) of $L^2(\Omega)$ such that $Le_n = \lambda_n e_n$, where λ_n is a real-valued sequenece converging to 0. By elementary operations, $e_n \in D(A_0)$ and $A_0 e_n = \mu_n e_n$ with

$$\mu_n = \frac{1}{\lambda_n} - 1.$$

It is an easy exercise to show that the spectrum of A_0 is exactly the union of all the μ_n and that $\mu_n \to +\infty$ as $n \to +\infty$.

The values μ_n are called the Dirichlet eigenvalues of the domain Ω. It is an important domain of the modern analysis to study the relations between the geometric and topological properties of Ω and its Dirichlet eigenvalues.

The preceding example can be easily generalized. More precisely, we say that an operator A with $\operatorname{res} A \neq \emptyset$ has a compact resolvent if $R_\lambda(A)$ is a compact operator for all $\lambda \in \operatorname{res} A$. One can easily check that it is sufficient to check this property at a single value λ.

Similar to the preceding constructions one can show:

Proposition 4.11 (Spectra of operators with compact resolvents). *Let T be a self-adjoint operator with a compact resolvent in an infinite-dimensional Hilbert space, then:*

- $\operatorname{spec} T = \operatorname{spec}_p T$,
- *the eigenvalues of T form a sequence converging to ∞.*

The proof is completely the same as for the Dirichlet Laplacian if we manage to show that $\operatorname{spec} T \neq \mathbb{R}$. In principle this can be done in a direct way, but we prefer to show it later using the spectral theorem, see Example 5.23 below.

4.4 Schrödinger operators with growing potentials

Let us discuss a particular class of operators with compact resolvents.

Recall the following classical criterion of compactness in $L^2(\mathbb{R}^d)$ (sometimes referred to as the Riesz-Kolmogorov-Tamarkin criterion):

Proposition 4.12. *A subset $A \subset L^2(\mathbb{R}^d)$ is relatively compact in $L^2(\mathbb{R}^d)$ if and only if the following three conditions are satisfied:*

(a) A is bounded,

(b) there holds

$$\int_{|x|\geq R} |u(x)|^2 dx \to 0 \text{ as } R \to \infty$$

uniformly for $u \in A$,

(c) $\|u_h - u\| \to 0$ as $h \to 0$ uniformly for $u \in A$. Here, for $h \in \mathbb{R}^d$ and $v \in L^2(\mathbb{R}^d)$, the symbol v_h denote the function defined by $v_h(x) = v(x+h)$.

An interested reader may refer to [8] for the proof and various generalizations.

Now let $W \in L^2_{\text{loc}}(\mathbb{R}^d)$ and $W \geq 0$. Consider the operator $T = -\Delta + W$ defined as the Friedrichs extension starting from $C_c^\infty(\mathbb{R}^d)$and discussed in Example 2.20. We know already that T is a self-adjoint and semibounded from below operator in $\mathcal{H} = L^2(\mathbb{R}^d)$. We would like to identify a reasonable large class of potentials W for which T has a compact resolvent.

Theorem 4.13. *For $r \geq 0$ denote*

$$w(r) := \inf_{|x|\geq r} W(x).$$

If $\lim_{r\to+\infty} w(r) = +\infty$, then the associated Schrödinger operator $T = -\Delta + W$ has a compact resolvent.

Proof. As follows from Example 2.20, it is sufficient to show that the embedding of $\mathcal{V} = H^1_W(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ is a compact operator, where $\mathcal{V}$ is equipped with the norm $\|u\|_W^2 = \|u\|_{H^1}^2 + \|\sqrt{W}u\|_{L^2}$. Let B be the unit ball in $\mathcal{V}$. We will show that B is relatively compact in $L^2(\mathbb{R}^d)$ using Proposition 4.12.

The condition (a) holds due to the inequality $\|u\|_{L_2} \leq \|u\|_W$. The condition (b) follows from

$$\int_{|x|\geq R} |u(x)|^2 dx \leq \frac{1}{w(R)} \int_{|x|\geq R} W(x)|u(x)|^2 \leq \frac{\|\sqrt{W}u\|_{L^2}^2}{w(R)} \leq \frac{\|u\|_W^2}{w(R)}.$$

For the condition (c) we have:

$$\int_{\mathbb{R}^d} \big|u(x+h)-u(x)\big|^2 dx = \int_{\mathbb{R}^d} \Big| \int_0^1 \frac{d}{dt} u(x+th)dt\Big|^2 dx$$
$$= \int_{\mathbb{R}^d} \Big| \int_0^1 h\cdot \nabla u(x+th)dt\Big|^2 dx \le h^2 \int_{\mathbb{R}^d} \int_0^1 \big|\nabla u(x+th)\big|^2 dt\, dx$$
$$\le h^2 \int_0^1 \int_{\mathbb{R}^d} \big|\nabla u(x+th)\big|^2 dx\, dt = h^2 \|\nabla u\|^2_{L^2} \le h^2 \|u\|^2_W. \quad \square$$

The assumption of Theorem 4.13 is rather easy to check, but this condition in not optimal one. For example, it is known that the operator $-\Delta + W$ with $W(x_1,x_2) = x_1^2 x_2^2$ has a compact resolvent, while the condition cleraly fails.

A rather simple necessary and sufficient condition is known in the one-dimensional case:

Proposition 4.14 (Molchanov criterium). *The operator $T = -d^2/dx^2 + W$ has a compact resolvent iff*

$$\lim_{x\to\infty} \int_x^{x+\delta} W(s)ds = +\infty$$

for any $\delta > 0$.

Necessary and sufficient conditions are also available for the multi-dimensional case, but their form is much more complicated. An advanced reader may refer to the paper [14] for the discussion of such questions.

5 Spectral theorem

Some points in this section are just sketched to avoid technicalities. A more detailed presentation can be found in [5, Chapter 2] or in [16, Section 12.7].

The aim of the present section is to define, for a given self-adjoint operator T, the operators $f(T)$, where f are sufficiently general functions.

To be provided with a certain motivation, let T be either a compact self-adjoint operator or a self-adjoint operator with a compact resolvent in a Hilbert space $\mathcal{H}$. As shown in the previous section, there exists an orthonormal basis (e_n) in $\mathcal{H}$ and real numbers λ_n such that, with

$$Tx = \sum_n \lambda_n \langle e_n, x\rangle e_n \quad \text{for all } x \in D(T),$$

and the domain $D(T)$ is characterized by

$$D(T) = \Big\{ x \in \mathcal{H} : \sum_n \lambda_n^2 \big|\langle e_n, x\rangle\big|^2 < \infty \Big\}.$$

For $f \in C_0(\mathbb{R})$ one can define an operator $f(T) \in \mathcal{L}(\mathcal{H})$ by

$$f(T)x = \sum_n f(\lambda_n)\langle e_n, x\rangle e_n.$$

This map $f \mapsto f(T)$ enjoys a number of properties. For example, $(fg)(T) = f(T)g(T)$, $\overline{f}(T) = f(T)^*$, $\operatorname{spec} f(T) = \overline{f(\operatorname{spec} T)}$ etc. The existence of such a construction allows one to write rather explicit expressions for solutions of some equations. For example, one can easily show that the initial value problem

$$-ix'(t) = Tx(t),\, x(0) = y \in D(T), \quad x : \mathbb{R} \to D(T),$$

has a solution that can be written as $x(t) = f_t(T)y$ with $f_t(x) = e^{itx}$. Informally speaking, for a large class of equations involving the operator T one may first assume that T is a real constant and obtain a formula for the solution, and then one can give this formula an operator-valued meaning using the above map $f \mapsto f(T)$.

Moreover, if we introduce the map $U : \mathcal{H} \to \ell^2(\mathbb{N})$ defined by $Ux =: (x_n)$, $x_n = \langle e_n, x\rangle$, then the operator UTU^* becomes a multiplication operator $(x_n) \mapsto (\lambda_n x_n)$, cf. Example 3.10.

At this point, all the preceding facts are proved for compact self-adjoint operators and self-adjoint operator with a compact resolvent only. The aim of the present section is to develop a similar theory for general self-adjoint operators.

To avoid potential misunderstanding let us recall that $C_0(\mathbb{R})$ denotes the class of the continuous functions $f : \mathbb{R} \to \mathbb{C}$ with $\lim_{|x|\to+\infty} f(x) = 0$ equipped with the sup-norm. This should not be confused with the set $C^0(\mathbb{R})$ of the continuous functions on $\mathbb{R}$.

5.1 Continuous functional calculus

We say that a function $f : \mathbb{C} \to \mathbb{C}$ belongs to $C^\infty(\mathbb{C})$ if the function of two real variables $\mathbb{R}^2 \ni (x, y) \mapsto f(x + iy) \in \mathbb{C}$ belongs to $C^\infty(\mathbb{R}^2)$. In the similar way one defines the classes $C_c^\infty(\mathbb{C})$, $C^k(\mathbb{C})$ etc. In what follows we always use the notation $\Re z =: x$, $\Im z =: y$ for $z \in \mathbb{C}$. Using $x = \dfrac{z + \overline{z}}{2}$ and $y = \dfrac{z - \overline{z}}{2i}$, for $f \in C^1(\mathbb{C})$ one defines the derivative

$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

Clearly, $\partial g/\partial\overline{z} = 0$ if g is a holomorphic function. Recall the *Stokes formula* written in this notation: if $f \in C^\infty(\mathbb{C})$ and $\Omega \subset \mathbb{C}$ is a domain with a sufficiently regular boundary, then

$$\iint_\Omega \frac{\partial f}{\partial \overline{z}}\, dx\, dy = \frac{1}{2i}\oint_{\partial\Omega} f\, dz.$$

The following fact is actually known, but is presented in a slightly unusual form.

Lemma 5.1 (Cauchy integral formula). *Let $f \in C_c^\infty(\mathbb{C})$, then for any $w \in \mathbb{C}$ we have*

$$\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}} \frac{1}{w - z}\, dx\, dy = f(w).$$

Proof. We note first that the singularity $1/z$ is integrable in two dimensions, and the integral is well-defined. Let Ω be a large ball containing the support of f and the point w. For small $\varepsilon > 0$ denote $B_\varepsilon := \{z \in \mathbb{C} : |z - w| \le \varepsilon\}$, and set $\Omega_\varepsilon := \Omega \setminus B_\varepsilon$. Using the Stokes formula we have:

$$\begin{aligned}
\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}} \frac{1}{w - z}\, dx\, dy &= \frac{1}{\pi}\iint_{\Omega} \frac{\partial f}{\partial \overline{z}} \frac{1}{w - z}\, dx\, dy \\
&= \lim_{\varepsilon\to 0}\frac{1}{\pi}\iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{z}} \frac{1}{w - z}\, dx\, dy = \lim_{\varepsilon\to 0}\frac{1}{\pi}\iint_{\Omega_\varepsilon} \frac{\partial}{\partial \overline{z}}\Big(f(z)\, \frac{1}{w - z}\Big)\, dx\, dy \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\oint_{\partial\Omega_\varepsilon} f(z)\frac{1}{w-z}\, dz \\
&= \frac{1}{2\pi i}\oint_{\partial\Omega} f(z)\frac{1}{w-z}\, dz - \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\oint_{|z-w|=\varepsilon} f(z)\frac{1}{w-z}\, dz.
\end{aligned}$$

The first term on the right-hand side is zero, because f vanishes at the boundary of Ω. The second term can be calculated explicitly:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\oint_{|z-w|=\varepsilon} f(z)\frac{1}{w-z}\, dz &= \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_0^{2\pi} f(w + \varepsilon e^{it})\frac{i\varepsilon e^{it} dt}{w - (w + \varepsilon e^{it})} \\
&= -\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi} f(w + \varepsilon e^{it}) dt = -f(w),
\end{aligned}$$

which gives the result. □

The main idea of the subsequent presentation is to define the operators $f(T)$, for a self-adjoint operator T, using an operator-valued generalization of the Cauchy integral formula.

Introduce first some notation. For $z \in \mathbb{C}$ we write

$$\langle z \rangle := \sqrt{1+|z|^2}.$$

For $\beta < 0$ denote by $\mathcal{S}_\beta$ the set of the smooth functions $f : \mathbb{R} \to \mathbb{C}$ satisfying the estimates

$$\left|f^{(n)}(x)\right| \le c_n \langle x \rangle^{\beta-n}$$

for any $n \ge 0$ and $x \in \mathbb{R}$, where the positive constant c_n may depend on f. Set $\mathcal{A} := \bigcup_{\beta<0} \mathcal{S}_\beta$; one can show that $\mathcal{A}$ is an alebra. Moreover, if $f = P/Q$, where P and Q are polynomials with $\deg P < \deg Q$ and $Q(x) \neq 0$ for $x \in \mathbb{R}$, then $f \in \mathcal{A}$. For any $n \ge 1$ one can introduce the norms on $\mathcal{A}$:

$$\|f\|_n := \sum_{r=0}^{n} \int_{\mathbb{R}} \left|f^{(r)}(x)\right| \langle x \rangle^{r-1} dx.$$

One can easily see that the above norms on $\mathcal{A}$ induce continuous embeddings $\mathcal{A} \to C_0(\mathbb{R})$. Moreover, one can prove that $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{A}$ with respect to any norm $\|\cdot\|_n$.

Now let $f \in C^\infty(\mathbb{R})$. Pick $n \in \mathbb{N}$ and a smooth function $\tau : \mathbb{R} \to \mathbb{R}$ such that $\tau(s) = 1$ for $|s| < 1$ and $\tau(s) = 0$ for $|s| > 2$. For $x, y \in \mathbb{R}$ set $\sigma(x,y) := \tau(y/\langle x \rangle)$. Define $\widetilde{f} \in C^\infty(\mathbb{C})$ by

$$\widetilde{f}(z) = \left[\sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r!}\right] \sigma(x,y).$$

Clearly, for $x \in \mathbb{R}$ we have $\widetilde{f}(x) = f(x)$, so $\widetilde{f}$ is an extension of f. One can check the following identity:

$$\frac{\partial \widetilde{f}}{\partial \overline{z}} = \frac{1}{2}\left[\sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r!}\right] \Big(\sigma_x + i\sigma_y\Big) + \frac{1}{2} f^{(n+1)}(x) \frac{(iy)^n}{n!} \sigma. \tag{5.1}$$

Now let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. For $f \in \mathcal{A}$ define an operator $f(T)$ in $\mathcal{H}$ by

$$f(T) := \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial \widetilde{f}}{\partial \overline{z}} \, (T-z)^{-1} \, dx \, dy. \tag{5.2}$$

This integral expression is called the *Helffer-Sjöstrand formula.* We need to show several points: that the integral is well-defined, that it does not depend in the choice of σ and n etc. This will be done is a series of lemmas.

Note first that, as shown in Proposition 3.16, we have the norm estimate $\|(T-z)^{-1}\| \le 1/|\Im z|$, and one can see from (5.1) that $\widetilde{\partial f}/\partial \overline{z}(x+iy) = O(y^n)$ for any fixed x, so the subintegral function in (5.2) is locally bounded. By additional technical efforts one can show that the integral is convergent and defines an continuous operator with $\|f(T)\| \le c\|f\|_{n+1}$ for some $c > 0$. Using this observation and the density of $C_c^\infty(\mathbb{R})$ in $\mathcal{A}$ the most proofs will be provided for $f \in C_c^\infty$ and extended to $\mathcal{A}$ and larger spaces using the standard density arguments.

Lemma 5.2. *If $F \in C_c^\infty(\mathbb{C})$ and $F(z) = O(y^2)$ as $y \to 0$, then*

$$A := \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial F}{\partial \overline{z}} (T - z)^{-1}\, dx\, dy = 0.$$

Proof. By choosing a sufficiently large $N > 0$ one may assyme that the support of F is contained in $\Omega := \{z \in \mathbb{C} : |x| < N, |y| < N\}$. For small $\varepsilon > 0$ define $\Omega_\varepsilon := \{z \in \mathbb{C} : |x| < N, \varepsilon < |y| < N\}$. Using the Stokes formula we have

$$A = \lim_{\varepsilon \to 0} \frac{1}{\pi} \iint_{\Omega_\varepsilon} \frac{\partial F}{\partial \overline{z}} (T - z)^{-1}\, dx\, dy = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \oint_{\partial \Omega_\varepsilon} F(z)\, (T - z)^{-1}\, dz.$$

The boundary $\partial\Omega_\varepsilon$ consists of eight segments. The integral over the vertical segments and over the horizontal segments with $y = \pm N$ are equal to 0 because the function F vanishes on these segments. It remains to estimate the integrals over the segments with $y = \pm\varepsilon$. Here we have $\|(T - z)^{-1}\| \le \varepsilon^{-1}$ and

$$\|A\| \le \lim_{\varepsilon \to 0} \frac{1}{2\pi} \oint_{\partial \Omega_\varepsilon} \Big(|F(x + i\varepsilon)| + |F(x - i\varepsilon)| \Big) \varepsilon^{-1} dx = 0. \qquad \square$$

Corollary 5.3. *For $f \in \mathcal{A}$ the integral in* (5.2) *is independent of the choice of $n \ge 1$ and σ.*

Proof. For $f \in C_c^\infty(\mathbb{C})$ the assertion follows from the definition of $\widetilde{f}$ and Lemma 5.2. This is extended to $\mathcal{A}$ using the density arguments. $\square$

Lemma 5.4. *Let $f \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \cap \operatorname{spec} T = \emptyset$, then $f(T) = 0$.*

Proof. If $f \in C_c^\infty(\mathbb{R})$, then obviously $\widetilde{f} \in C_c^\infty(\mathbb{C})$. One can find a finite family of closed curves γ_r which do not meet the spectrum of T and enclose a domain U containing $\operatorname{supp} \widetilde{f}$. Using the Stokes formula we have

$$f(T) = \frac{1}{\pi} \iint_U \frac{\partial \widetilde{f}}{\partial \overline{z}} (T - z)^{-1}\, dx\, dy = \sum_r \frac{1}{2\pi i} \oint_{\gamma_r} \widetilde{f}(z)\, (T - z)^{-1} dz.$$

All the terms in the sum are zero, because $\widetilde{f}$ vanishes on γ_r. $\square$

Lemma 5.5. *For $f, g \in \mathcal{A}$ one has $(fg)(T) = f(T)g(T)$.*

Proof. By the density arguments is it sufficient to consider the case $f, g \in C_c^\infty(\mathbb{R})$. Let K and L be large balls containing the supports of $\widetilde{f}$ and $\widetilde{g}$ respectively. Using the notation $w = u + iv$, $u, v \in \mathbb{R}$, one can write:

$$f(T)g(T) = \frac{1}{\pi^2} \iiiint\limits_{K \times L} \frac{\partial \widetilde{f}}{\partial \overline{z}} \frac{\partial \widetilde{g}}{\partial \overline{w}} (T - z)^{-1} (T - w)^{-1}\, dx\, dy\, du\, dv.$$

Using the resolvent identity

$$(T-z)^{-1}(T-w)^{-1} = \frac{1}{w-z}(T-w)^{-1} - \frac{1}{w-z}(T-z)^{-1}$$

we rewrite the preceding integral in the form

$$\begin{aligned} f(T)g(T) = \frac{1}{\pi^2}\iint_L \frac{\partial \widetilde{g}}{\partial \overline{w}}(T-w)^{-1}\Big(\iint_K \frac{\partial \widetilde{f}}{\partial \overline{z}}\frac{1}{w-z}dx\,dy\Big)du\,dv \\ - \frac{1}{\pi^2}\iint_K \frac{\partial \widetilde{f}}{\partial \overline{z}}(T-z)^{-1}\Big(\iint_L \frac{\partial \widetilde{g}}{\partial \overline{w}}\frac{1}{w-z}du\,dv\Big)dx\,dy. \end{aligned}$$

By Lemma 5.1 we have

$$\iint_K \frac{\partial \widetilde{f}}{\partial \overline{z}}\frac{1}{w-z}dx\,dy = \pi f(w), \qquad \iint_L \frac{\partial \widetilde{g}}{\partial \overline{w}}\frac{1}{w-z}du\,dv = -\pi g(z),$$

and we arrive at

$$\begin{aligned} f(T)g(T) &= \frac{1}{\pi}\iint_L \widetilde{f}(w)\frac{\partial \widetilde{g}}{\partial \overline{w}}(T-w)^{-1}du\,dv + \frac{1}{\pi}\iint_K \widetilde{g}(z)\frac{\partial \widetilde{f}}{\partial \overline{z}}(T-z)^{-1}dx\,dy \\ &= \frac{1}{\pi}\iint_{K\cup L} \frac{\partial(\widetilde{f}\widetilde{g})}{\partial \overline{z}}(T-z)^{-1}dx\,dy \\ &= \frac{1}{\pi}\iint_{\mathbb{C}} \frac{\partial \widetilde{fg}}{\partial \overline{z}}(T-z)^{-1}dx\,dy + \frac{1}{\pi}\iint_{\mathbb{C}} \frac{\partial(\widetilde{f}\widetilde{g}-\widetilde{fg})}{\partial \overline{z}}(T-z)^{-1}dx\,dy \\ &= (fg)(T) + \frac{1}{\pi}\iint_{\mathbb{C}} \frac{\partial(\widetilde{f}\widetilde{g}-\widetilde{fg})}{\partial \overline{z}}(T-z)^{-1}dx\,dy. \end{aligned}$$

By direct calculation one can see that $(\widetilde{fg}-\widetilde{f}\widetilde{g})(z) = O(y^2)$ for small y, and Lemma 5.2 shows that the second integral is zero. $\square$

Lemma 5.6. *Let $w \in \mathbb{C}\setminus\mathbb{R}$. Define a function r_w by $r_w(z) = (z-w)^{-1}$. Then $r_w(T) = (T-w)^{-1}$.*

Proof. We provide just the main line of the proof without technical details (they can be easily recovered). Use first the independence of n and σ. We take $n=1$ and put $\sigma(z) = \tau(\lambda y/\langle x\rangle)$ where $\lambda > 0$ is sufficiently large, to have $w \notin \operatorname{supp}\sigma$. Without loss of generality we assume $\Im w > 0$. For large $m > 0$ consider the region

$$\Omega_m := \{z \in \mathbb{C}: \quad |x| < m, \quad \frac{\langle x\rangle}{m} < y < 2m\}.$$

Using the definition and the Stokes formula we have

$$r_w(T) = \lim_{m\to\infty}\frac{1}{\pi}\iint_{\Omega_m}\frac{\partial \widetilde{r}_w}{\partial \overline{z}}\,(T-z)^{-1}\,dx\,dy = \lim_{m\to\infty}\frac{1}{2\pi i}\oint_{\partial\Omega_m}\widetilde{r}_w(z)\,(T-z)^{-1}\,dz.$$

By rather technical explicit estimates (which are omitted here) one can show that

$$\lim_{m\to\infty} \oint_{\partial\Omega_m} \Big(\widetilde{r}_w(z) - r_w(z)\Big)\,(T-z)^{-1}\,dz = 0.$$

and we arrive at

$$r_w(T) = \frac{1}{2\pi i} \lim_{m\to\infty} \oint_{\partial\Omega_m} \frac{1}{z-w}\,(T-z)^{-1}\,dz.$$

For sufficiently large m one has the inclusion $w \in \Omega_m$. For any $f, g \in \mathcal{H}$ the function $\mathbb{C} \ni z \mapsto \langle f, (T-z)^{-1} g\rangle \in \mathbb{C}$ is holomorphic in Ω_m, so applying the Cauchy formula, for large m we have

$$\frac{1}{2\pi i} \oint_{\partial\Omega_m} \frac{1}{z-w}\,\big\langle f, (T-z)^{-1} g\big\rangle\,dz = \big\langle f, (T-w)^{-1} g\big\rangle,$$

which shows that $r_w(T) = (T-w)^{-1}$. □

Lemma 5.7. *For any $f \in \mathcal{A}$ we have:*

(a) $\overline{f}(T) = f(T)^*$,

(b) $\big\|f(T)\big\| \le \|f\|_\infty$.

Proof. The item (a) follows directly from the equalities

$$\big((T-z)^{-1}\big)^* = (T-\overline{z})^{-1}, \quad \overline{\widetilde{f}(z)} = \widetilde{\overline{f}}(\overline{z}).$$

To show (b), take an arbitrary $c > \|f\|_\infty$ and define $g(s) := c - \sqrt{c^2 - |f(s)|^2}$. One can show that $g \in \mathcal{A}$. There holds $\overline{f}f - 2cg + g^2 = 0$, and using the preceding lemmas we obtain $f(T)^* f(T) - cg(T) - cg(T)^* + g(T)^* g(T) = 0$, and

$$f(T)^* f(T) + \big(c - g(T)\big)^* \big(c - g(T)\big) = c^2.$$

Let $\psi \in \mathcal{H}$. Using the preceding equality we have:

$$\begin{aligned}
\big\|f(T)\psi\big\|^2 &\le \big\|f(T)\psi\big\|^2 + \Big\|\big(c - g(T)\big)\psi\Big\|^2 \\
&= \Big\langle \psi, f(T)^* f(T)\psi \Big\rangle + \Big\langle \psi, \big(c - g(T)\big)^* \big(c - g(T)\big)\psi \Big\rangle \\
&= c^2 \|\psi\|^2.
\end{aligned}$$

As $c > \|f\|_\infty$ was arbitrary, this concludes the proof. □

All the preceding lemmas put together lead us to the following fundamental result.

Theorem 5.8 (Spectral theorem, continuous functional calculus). *Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. There exists a unique linear map*

$$C_0(\mathbb{R}) \ni f \mapsto f(T) \in \mathcal{L}(\mathcal{H})$$

with the following properties:

- $f \mapsto f(T)$ *is an algebra homomorphism,*
- $\overline{f}(T) = f(T)^*$,
- $\|f(T)\| \le \|f\|_\infty$,
- *if* $w \notin \mathbb{R}$ *and* $r_w(s) = (s-w)^{-1}$, *then* $r_w(T) = (T-w)^{-1}$,
- *if* $\operatorname{supp} f$ *does not meet* $\operatorname{spec} T$, *then* $f(T) = 0$.

Proof. Existence. If one replaces C_0 by $\mathcal{A}$, everything is already proved. But $\mathcal{A}$ is dense in $C_0(\mathbb{R})$ in the sup-norm, because $C_c^\infty(\mathbb{R}) \subset \mathcal{A}$, so we can use the density argument.

Uniqueness. If we have two such maps, they coincide on the functions f which are linear combinations of r_w, $w \in \mathbb{C} \setminus \mathbb{R}$. But such functions are dense in C_0 by the Stone-Weierstrass theorem, so by the density argument both maps coincide on C_0. □

Remark 5.9.
- One may wonder why to introduce the class of functions $\mathcal{A}$: one could just start by C_c^∞ which is also dense in C_0. The reason in that we have no intuition on how the operator $f(T)$ should look like if $f \in C_c^\infty$. On the other hand, it is naturally expected that for $r_w(s) = (s-w)^{-1}$ we should have $r_w(T) = (T-w)^{-1}$, otherwise there are no reasons why we use the notation $r_w(T)$. So it is important to have an explicit formula for a sufficiently large class of functions containing all such r_w.
- The approach based on the Helffer-Sjöstrand formula, which is presented here, is relatively new, and it allows one to consider bounded and unbounded self-adjoint operators simultaneously. The same results can be obtained by other methods, starting e.g. with polynomials instead of the resolvents, which is a more traditional approach, see, for example, Sections VII.1 and VIII.3 in the book [12]. □

5.2 Borelian functional calculus and L^2 representation

Now we would like to extend the functional calculus to more general functions, not necessarily continuous and not necessarily vanishing at infinity.

Definition 5.10 (Invariant and cyclic subspaces). Let $\mathcal{H}$ be a Hilbert space, L be a closed linear subspace of $\mathcal{H}$, and T be a self-adjoint linear operator in $\mathcal{H}$.

Let T be bounded. We say that L is an *invariant subspace* of T (or just T-invariant) if $T(L) \subset L$. We say that L is a *cyclic subspace* of T with cyclic vector v if L coincides with the closed linear hull of all vectors $p(T)v$, where p are polynomials.

Let T be general. We say that L is an *invariant subspace* of T (or just T-invariant) if $(T-z)^{-1}(L) \subset L$ for all $z \notin \mathbb{R}$. We say that L is a *cyclic subspace* of T with cyclic vector v if L coincides with the closed linear space of all vectors $(T-z)^{-1}v$ with $z \notin \mathbb{R}$. □

Clearly, if L is T-invariant, then $L^\perp$ is also T-invariant.

Proposition 5.11. *Both definitions of an invariant/cyclic subspace are equivalent for bounded self-adjoint operators.*

Proof. Let $T = T^* \in \mathcal{L}(\mathcal{H})$. We note first that $\operatorname{res} T$ is a connected set.

Let a closed subspace L be T-invariant in the sense of the definition for bounded operators. If $z \in \mathbb{C}$ and $|z| > \|T\|$, then $z \notin \operatorname{spec} T$ and

$$(T - z)^{-1} = -z\Big(1 - \frac{T}{z}\Big)^{-1} = \sum_{n=0}^{\infty} z^{-n-1} T^n.$$

If $x \in L$, then $T^n x \in L$ for any n. As the series on the right hand side converges in the operator norm sense and as L is closed, $(T-z)^{-1}x$ belongs to L.

Let us denote $W = \big\{z \in \operatorname{res} T : (T-z)^{-1}(L) \subset L\big\}$. As just shown, W is non-empty. On the other hand, W is closed in $\operatorname{res} T$ in the relative topology: if $x \in L$, $z_n \in W$ and z_n converge to $z \in W$, then $(T - z_n)^{-1}x \in L$ and $(T - z_n)^{-1}x$ converge to $(T - z)^{-1}x$. On the other hand, W is open: if $z_0 \in W$ and $|z - z_0|$ is sufficiently small, then

$$(T - z)^{-1} = \sum_{n=0} (z - z_0)^n (T - z_0)^{-n-1},$$

see (3.4), and $(T - z)^{-1}L \subset L$. Therefore, $W = \operatorname{res} T$, which shows that L is T-invariant in the sense of the definition for general operators.

Now let $T = T^* \in \mathcal{L}(\mathcal{H})$, and assume that L is T-invariant in the sense of the definition for general operators, i.e. $(T - z)^{-1}(L) \subset L$ for any $z \notin \mathbb{R}$. Pick any $z \notin \mathbb{R}$ and any $f \in L$. We can represent $Tf = g + h$, where $g \in L$ and $h \in L^\perp$ are uniquely defined vectors. As $L^\perp$ is T-invariant, $(T - z)^{-1}h \subset L^\perp$. On the other hand

$$\begin{aligned}(T - z)^{-1}h &= (T - z)^{-1}(Tf - g)\\ &= (T - z)^{-1}\big((T - z)f + zf - g\big)\\ &= f + (T - z)^{-1}(zf - g).\end{aligned}$$

As $zf - g \in L$, both vectors on the right-hand side are in L. Therefore, $(T-z)^{-1}h \in L$, and finally $(T - z)^{-1}h = 0$ and $h = 0$, which shows that $Tf = g \in L$. The equivalence of the two definitions of an invariant subspace is proved.

On the other hand, for both definitions, L is T-cyclic with cyclic vector v iff L is the smallest T-invariant subspace containing v. Therefore, both definitions of a cyclic subspace also coincide for bounded self-adjoint operators. $\square$

Theorem 5.12 (L^2 representation, cyclic case). *Let T be a self-adjoint linear operator in $\mathcal{H}$ and let $S := \operatorname{spec} T$. Assume that $\mathcal{H}$ is a cyclic subspace for T with a cyclic vector v, then there exists a bounded measure μ on S with $\mu(S) \le \|v\|^2$ and a unitary map $U : \mathcal{H} \to L^2(S, d\mu)$ with the following properties:*

- *a vector $x \in \mathcal{H}$ is in $D(T)$ iff $hUx \in L^2(S, d\mu)$, where h is the function on S given by $h(s) = s$,*
- *for any $\psi \in U\big(D(T)\big)$ there holds $UTU^{-1}\psi = h\psi$.*

In other words, T is unitarily equivalent to the operator M_h of the multiplciation by h in $L^2(S, d\mu)$.

Proof. Step 1. Consider the map $\phi : C_0(\mathbb{R}) \to \mathbb{C}$ defined by $\phi(f) = \big\langle v, f(T)v\big\rangle$. Let us list the properties of this map:

- ϕ is linear,
- $\phi(\overline{f}) = \overline{\phi(f)}$,
- if $f \geq 0$, then $\phi(f) \geq 0$. This follows from

$$\phi(f) = \big\langle v, f(T)v\big\rangle = \big\langle v, \sqrt{f}(T)\sqrt{f}(T)v\big\rangle = \big\|\sqrt{f}(T)v\big\|^2.$$

- $\big|\phi(f)\big| \leq \|f\|_\infty \|v\|^2$.

By the Riesz representation theorem there exists a uniquely defined regular Borel measure μ such that

$$\phi(f) = \int_{\mathbb{R}} f d\mu \text{ for all } f \in C_0(\mathbb{R}).$$

Moreover, for $\operatorname{supp} f \cap S = \emptyset$ we have $f(T) = 0$ and $\phi(f) = 0$, which means that $\operatorname{supp} \mu \subset S$, and we can write the above as

$$\big\langle v, f(T)v\big\rangle = \int_S f d\mu \text{ for all } f \in C_0(\mathbb{R}). \tag{5.3}$$

Step 2. Consider the map $\Theta : C_0(\mathbb{R}) \to L^2(S, d\mu)$ defined by $\Theta f = f$. We have

$$\begin{aligned}\langle \Theta f, \Theta g\rangle &= \int_S \overline{f} g \, d\mu = \phi(\overline{f} g) \\ &= \big\langle v, f(T)^* g(T)v\big\rangle = \big\langle f(T)v, g(T)v\big\rangle.\end{aligned}$$

Denote $\mathcal{M} := \big\{f(T)v : f \in C_0(\mathbb{R})\big\} \subset \mathcal{H}$, then the preceding equality means that the map

$$U : \mathcal{H} \supset \mathcal{M} \to C_0(\mathbb{R}) \subset L^2(S, d\mu), \quad U\big(f(T)v\big) = f,$$

is one-to-one and isometric. Moreover, $\mathcal{M}$ is dense in $\mathcal{H}$, because v is a cyclic vector. Furthermore, $C_0(\mathbb{R})$ is a dense subspace of $L^2(S, d\mu)$, as μ is regular. Therefore, U is uniquely extended to a unitary map from $\mathcal{H}$ to $L^2(S, d\mu)$, and we denote this extension by the same symbol.

Step 3. Let $f, f_j \in C_0(\mathbb{R})$ and $\psi_j := f_j(T)v$, $j = 1, 2$. There holds

$$\begin{aligned}
\big\langle \psi_1, f(T)\psi_2 \big\rangle &= \big\langle f_1(T)v, f(T)f_2(T)v \big\rangle \\
&= \big\langle v, (\overline{f}_1 f f_2)(T)v \big\rangle \\
&= \int_S f\overline{f}_1 f_2 \, d\mu \\
&= \langle U\psi_1, M_f U\psi_2 \rangle,
\end{aligned}$$

where M_f is the operator of the multiplication by f in $L^2(S, d\mu)$. In particular, for any $w \notin \mathbb{R}$ and $r_w(s) = (s - w)^{-1}$ we obtain $Ur_w(T)U^*\xi = r_w\xi$ for all $\xi \in L^2(S, d\mu)$. The operator U maps the set $\operatorname{ran} r_w(T) \equiv D(T)$ to the range of M_{r_w}. In other words, U is a bijection from $D(T)$ to

$$\operatorname{ran} M_{r_w} = \big\{ \phi \in L^2(S, d\mu) : x \mapsto x\phi(x) \in L^2(S, d\mu) \big\} = D(M_h).$$

Therefore, if $\xi \in L^2(S, d\mu)$, then $\psi := r_w\xi \in D(M_h)$,

$$Tr_w(T)U^*\xi = (T - w)r_w(T)U^*\xi + wr_w(T)U^*\xi = U^*\xi + wr_w(T)U^*\xi$$

and, finally,

$$\begin{aligned}
UTU^*\psi = UTU^*r_w\xi = UTr_w(T)U^*\xi = U\big(U^*\xi + wr_w(T)U^*\xi\big) \\
= \xi + wr_w\xi = h\psi. \quad \square
\end{aligned}$$

Theorem 5.13 (L^2 representation). *Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$ with $\operatorname{spec} T =: S$. Then there exists $N \subset \mathbb{N}$, a finite measure μ on $S \times N$ and a unitary operator $U : \mathcal{H} \to L^2(S \times N, d\mu)$ with the following properties.*

- *Let $h : S \times N \to \mathbb{R}$ be given by $h(s, n) = s$. A vector $x \in \mathcal{H}$ belongs to $D(T)$ iff $hUx \in L^2(S \times N, d\mu)$,*
- *for any $\psi \in U\big(D(T)\big)$ there holds $UTU^{-1}\psi = h\psi$.*

Proof. Using the induction one can find $N \subset \mathbb{N}$ and non-empty closed mutually orthogonal subspaces $\mathcal{H}_n \subset \mathcal{H}$ with the following properties:

- $\mathcal{H} = \bigoplus_{n\in N} \mathcal{H}_n$,
- each $\mathcal{H}_n$ is a cyclic subspace of T with cyclic vector v_n satisfying $\|v_n\| \le 2^{-n}$.

The restriction T_n of T to $\mathcal{H}_n$ is a self-adjoint operator in $\mathcal{H}_n$, and one can apply to all these operators Theorem 5.12, which gives associated measures μ_n with $\mu(S) \le 4^{-n}$, and unitary maps $U_n : \mathcal{H}_n \to L^2(S, d\mu_n)$. Now one can define a measure μ on $S \times N$ by $\mu\big(\Omega \times \{n\}\big) = \mu_n(\Omega)$, and a unitary map

$$U : \mathcal{H} \equiv \bigoplus_{n\in N} \mathcal{H}_n \to L^2(S \times N, d\mu) \equiv \bigoplus_{n\in N} L^2(S, d\mu_n)$$

by $U(\psi_n) = (U_n\psi_n)$, and one can easily check that all the properties are verified. $\square$

Remark 5.14. • The previous theorem shows that any self-adjoint operator is unitarily equivalent to a multiplication operator in some L^2 space, and this multiplication operator is sometimes called a *spectral representation* of T. Clearly, such a representation is not unique, for example, the decomposition of the Hilbert space in cyclic subspaces is not unique.

- The cardinality of the set N is not invariant. The minimal cardinality among all possible N is called the *spectral multiplicity* of T, and it generalizes the notion of the multiplicity for eigenvalues. Calculating the spectral multiplicity is a non-trivial problem. □

Theorem 5.13 can be used to improve the result of Theorem 5.8. In the rest of the section we use the function h and the measure μ from Theorem 5.13 without further specifications.

Introduce the set $\mathcal{B}_\infty$ consisting of the bounded Borel functions $f : \mathbb{R} \to \mathbb{C}$. In what follows, we say that $f_n \in \mathcal{B}_\infty$ converges to $f \in \mathcal{B}_\infty$ and write $f_n \xrightarrow{\mathcal{B}_\infty} f$ if the following two conditions hold:

- there exists $c > 0$ such that $\|f_n\|_\infty \le c$,
- $f_n(x) \to f(x)$ for all x.

Definition 5.15 (Strong convergence). Wa say that a sequence $A_n \in \mathcal{L}(\mathcal{H})$ converges strongly to $A \in \mathcal{L}(\mathcal{H})$ and write $A = \mathrm{s} - \lim A_n$ if $Ax = \lim A_n x$ for any $x \in \mathcal{H}$. □

Theorem 5.16 (Borel functional calculus). *(a) Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. There exists a map $\mathcal{B}_\infty \ni f \mapsto f(T) \in \mathcal{L}(\mathcal{H})$ extending the map from Theorem 5.8 and satisfying the same properties except that one can improve the estimate $\|f(T)\| \le \|f\|_\infty$ by $\|f(T)\| \le \|f\|_{\infty,T}$.*

(b) This extension is unique if we assume that the condition $f_n \xrightarrow{\mathcal{B}_\infty} f$ implies $f(T) = \mathrm{s} - \lim f_n(T)$.

Proof. Consider the map U from Theorem 5.8. Then it is sufficient to define $f(T) := U^* M_{f\circ h} U$, then one routinely check that all the properties hold, and (a) is proved.

To prove (b) we remark first that the map just defined satisfies the requested condition: If $x \in L^2(S, d\mu)$ and $f_n \xrightarrow{\mathcal{B}_\infty} f$, then $f_n(h)x$ converges to $f(h)x$ in the norm of $L^2(S \times N, d\mu)$ by the dominated convergence. But this means exactly that $f(T) = \mathrm{s} - \lim f_n(T)$.

On the other hand, $C_0(\mathbb{R})$ is obviously dense in $\mathcal{B}_\infty$ with respect to the $\mathcal{B}_\infty$ convergence, which proves the uniqueness of the extension. □

We have a series of important corollaries, whose proof is an elementary modification of the constructions given for the multiplication operator in Example 3.6.

Corollary 5.17. • $\operatorname{spec} T = \operatorname{ess}_\mu \operatorname{ran} h$,

- *for any* $f \in \mathcal{B}_\infty$ *one has* $\operatorname{spec} f(T) = \operatorname{ess}_\mu \operatorname{ran} f \circ h$,
- *in particular,* $\|f(T)\| = \operatorname{ess}_\mu \sup |f \circ h|$.

Example 5.18. One can also define the operators $\varphi(T)$ with unbounded functions φ by $\varphi(T) = U^* M_{\varphi \circ h} U$. These operators are in general unbounded, but they are self-adjoint for real-valued φ; this follows from the self-adjointness of the multiplication operators $M_{\varphi \circ h}$.

Example 5.19. The usual Fourier transform is a classical example of a spectral representation. For example, Take $\mathcal{H} = L^2(\mathbb{R})$ and $T = -id/dx$ with the natural domain $D(T) = H^1(\mathbb{R})$. If $\mathcal{F}$ is the Fourier transform, then $\mathcal{F}T\mathcal{F}$ is exactly the operator of multiplication $x \mapsto xf(x)$, and $\operatorname{spec} T = \mathbb{R}$.

In particular, for bounded Borel functions $f : \mathbb{R} \to \mathbb{C}$ one can define the operators $f(T)$ by $f(T)h = \mathcal{F}^* M_f \mathcal{F}$, where M_f is the operator of multiplication by f, i.e. in general one obtains a pseudodifferential operator.

Let us look at some particular examples. Consider the shift operator A in $\mathcal{H}$ which is defined by $Af(x) = f(x+1)$. It is a bounded operator, and for any $u \in \mathcal{S}(\mathbb{R})$ we have $\mathcal{F}A\mathcal{F}^* u(p) = e^{ip} u(p)$. This means that $A = e^{iT}$, and this gives the relation $\operatorname{spec} A = \{z : |z| = 1\}$. On may also look at the operator B defined by

$$Bf(x) = \int_{x-1}^{x+1} f(t)dt.$$

Using the Fourier transform one can show that $B = \varphi(T)$, where $\varphi(x) = 2 \sin x / x$ with $\operatorname{spec} B = \overline{\varphi(\mathbb{R})}$.

Example 5.20. For practical computations one does not need to have the canonical representation from Theorem 5.13 to construct the Borel functional calculus. It is sufficient to represent $T = U^* M_f U$, where $U : \mathcal{H} \to L^2(X, d\mu)$ and M_f is the multiplcation operator by some function f. Then for any Borel function φ one can put $\varphi(T) = U^* M_{\varphi \circ f} U$.

For example, for the free Laplacian T in $\mathcal{H} = L^2(\mathbb{R}^d)$ the above is realized with $X = \mathbb{R}^d$ and U being the Fourier transform, and with $f(p) = p^2$. This means that the operators $\varphi(T)$ act by

$$\varphi(T)f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi(p^2) \widehat{f}(p) e^{ipx}\, dx.$$

For example,

$$\sqrt{-\Delta + 1} f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \sqrt{1 + p^2} \widehat{f}(p) e^{ipx}\, dx$$

and one can show that $D(\sqrt{-\Delta + 1}) = H^1(\mathbb{R}^d)$.

Example 5.21. Another classical example is provided by the Fourier series. Take $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ and let a function $t : \mathbb{Z}^d \to \mathbb{R}$ satisfy $t(-m) = \overline{t(m)}$ and $\big|t(m)\big| \le c_1 e^{-c_2|m|}$ with some $c_1, c_2 > 0$. Define T by

$$Tu(m) = \sum_{n\in\mathbb{Z}^d} t(m-n)u(n).$$

One can easily see that T is bounded. If one introduces the unitary map $\Phi : \mathcal{H} \to L^2(\mathbb{T}^d)$, $\mathbb{T} := \mathbb{R}/\mathbb{Z}$,

$$\Phi u(x) = \sum_{m\in\mathbb{Z}^d} e^{2\pi i m x} u(m), \quad mx := m_1x_1 + \cdots + m_dx_d,$$

then $T = \Phi^* M_\tau \Phi$ with

$$\tau(x) = \sum_{m\in\mathbb{Z}^d} t(m)e^{2\pi i m x}.$$

□

Example 5.22. A less obvious example is given by the Neumann Laplacian T_N on the half-line defined in Example 2.11.

Let T be the free Laplacian in $L^2(\mathbb{R})$. Denote by $\mathcal{G} := L^2_p(\mathbb{R})$ the subspace of $L^2(\mathbb{R})$ consisting of the even functions. Clearly, $\mathcal{G}$ is an invariant subspace for T (the second derivate of an even function is also an even function), and the restriction of T to $\mathcal{G}$ is a self-adjoint operator; denote this restriction by A. Moreover, $\mathcal{G}$ is an invariant subspace of the Fourier transform $\mathcal{F}$ (the Fourier image of an even function is also an even function). Introduce now the a map $\Phi : L^2(\mathbb{R}_+) \to \mathcal{G}$ by $\Phi f(x) = 2^{-1/2} f\big(|x|\big)$. One checks easily that Φ is unitary and that $D(A) = \Phi\big(D(T_N)\big)$.

So we have $T_N = \Phi^* A \Phi$ and $A = \mathcal{F}^* \widetilde{M}_h \mathcal{F}$, where $\widetilde{M}_h$ is the multiplication by the function $h(p) = p^2$ in $\mathcal{G}$. Finally, $\widetilde{M}_h = \Phi M_h \Phi^*$, where M_h is the multiplication by h in $L^2(\mathbb{R}_+)$.

At the end of the day we have $T_N = U^* M_h U$ with $U = \Phi^* \mathcal{F} \Phi$, and U is unitary being a composition of three unitary operators. By direct calculation, for $f \in L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ one has

$$Uf(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(px) f(x)\, dx.$$

This transform U is sometimes called the cos-Fourier transform. Roughly speaking, U is just the Fourier transform restricted to the even functions together with some identifications.

An interested reader may adapt the preceding constructions to the Dirichlet Laplacian T_D on the half-line, see Example 2.12. □

Example 5.23 (Operators with compact resolvents). Let us fill the gap which was left open in Subsection 4.3. Namely let us show that if a self-adjoint T has a compact resolvent, then $\operatorname{spec} T \neq \mathbb{R}$.

Assume that $\operatorname{spec} T = \mathbb{R}$ and consider the function g given by $g(x) = (x - i)^{-1}$. Then $g(T) = (T - i)^{-1}$ is a compact operator, and its spectrum has at most one accumulation point. On the other hand, using Corollary 5.17 and the continuity of g one has the equality $\operatorname{spec} g(T) = \overline{g(\operatorname{spec} T)} = \overline{g(\mathbb{R})}$, and this set has no isolated points.

6 Some applications of spectral theorem

In this chapter we discuss some direct applications of the spectral theorem to the estimates of the spectra of self-adjoint operators. We still use without special notification the measure μ and the function h from Theorem 5.13.

Theorem 6.1 (Distance to spectrum). *Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$, and $0 \neq x \in D(T)$, then for any $\lambda \in \mathbb{C}$ one has the estimate*

$$\operatorname{dist}(\lambda, \operatorname{spec} T) \leq \frac{\big\|(T-\lambda)x\big\|}{\|x\|}.$$

Proof. If $\lambda \in \operatorname{spec} T$, then the left-hand side is zero, and the inequality is valid. Assume now that $\lambda \notin \operatorname{spec} T$. By Corollary 5.17, one has, with $\rho(x) = (x-\lambda)^{-1}$,

$$\|(T-\lambda)^{-1}\| = \operatorname{ess}_\mu \sup |\rho \circ h| = \frac{1}{\operatorname{dist}(\lambda, \operatorname{spec} T)},$$

which gives

$$\|x\| = \|(T-\lambda)^{-1}(T-\lambda)x\| \leq \frac{1}{\operatorname{dist}(\lambda, \operatorname{spec} T)} \|(T-\lambda)x\|. \qquad \square$$

Remark 6.2. The previous theorem is one of the basic tools for the constructing approximations of the spectrum of the self-adjoint operators. It is important to understand that the resolvent estimate obtained in Theorem 6.1 uses in an essential way the self-adjointness of the operator T. For non-self-adjoint operators the estimate fails even in the finite-dimensional case. For example, take $\mathcal{H} = \mathbb{C}^2$ and

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then $\operatorname{spec} T = \{0\}$, and for $z \neq 0$ we have

$$(T-z)^{-1} = -\frac{1}{z^2}\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}.$$

For the vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ one has $\langle e_1, (T-z)^{-1}e_2\rangle = -z^{-2}$, which shows that the norm of the resolvent near $z = 0$ is of order z^{-2}. In the infinite dimensional-case one can construct examples with $\|(T-z)^{-1}\| \sim \operatorname{dist}(z, \operatorname{spec} T)^{-n}$ for any power n. $\square$

6.1 Spectral projections

Definition 6.3 (Spectral projection). Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and $\Omega \subset \mathbb{R}$ be a Borel subset. The *spectral projection* of T on Ω is the operator $E_T(\Omega) := 1_\Omega(T)$, where 1_Ω is the characteristic function of Ω.

This exchange between the index and the argument is due to the fact that the spectral projections are usually considered as functions of subsets Ω (with a fixed operator T). $\square$

The following proposition summarizes the most important properties of the spectral projections.

Proposition 6.4. *For any self-adjoint operator T acting a in a Hilbert space there holds:*

1. *for any Borel subset $\Omega \subset \mathbb{R}$ the associated spectral projection $E_T(\Omega)$ is an orthogonal projection commuting with T. In particular, $E_T(\Omega)D(T) \subset D(T)$.*
2. *$E_T\big((a,b)\big) = 0$ if and only if $\operatorname{spec} T \cap (a,b) = \emptyset$.*
3. *for any $\lambda \in \mathbb{R}$ there holds $\operatorname{ran} E_T(\{\lambda\}) = \ker(T-\lambda)$, and $f \in \ker(T-\lambda)$ iff $f = E_T(\{\lambda\})f$.*
4. *$\operatorname{spec} T = \{\lambda \in \mathbb{R} : E_T\big((\lambda-\varepsilon, \lambda+\varepsilon)\big) \neq 0$ for all $\varepsilon > 0\}$.*

Proof. To prove (1) we remark that $1_\Omega^2 = 1_\Omega$ and $1_\Omega = \overline{1_\Omega}$, which gives $E_T(\Omega)E_T(\Omega) = E_T(\Omega)$ and $E_T(\Omega) = E_T(\Omega)^*$ and shows that $E_T(\Omega)$ is an orthogonal projection. To prove the commuting with T we restrict ourselves by considering T realized as a multiplication operator from Theorem 5.8. Let $x \in D(T)$, then $hx \in L^2(S, \times N, \mu)$ and, subsequently, $h \cdot 1_\Omega \circ h \cdot x \in L^2$, which means that $1_\Omega x \in D(T)$. The commuting follows now from $h \cdot 1_\Omega \circ h \cdot x = 1_\Omega \circ h \cdot h \cdot x$.
To prove (2) we note that the condition $E_T\big((a,b)\big) = 0$ is, by definition, equivalent to $1_{(a,b)} \circ h = 0$ μ-e.a., which in turn means that $(a,b) \cap \operatorname{ess}_\mu \operatorname{ran} h = \emptyset$, and it remains to recall that $\operatorname{ess}_\mu \operatorname{ran} h = \operatorname{spec} T$, see Corollary 5.17.
The items (3) and (4) are left as elementary exercises. $\square$

As an important corollary of the assertion (4) one has the following description of the spectra of self-adjoint operators, whose proof is another simple exercise.

Corollary 6.5. *Let T be self-adjoint, then $\lambda \in \operatorname{spec} T$ if and only if there exists a sequence $x_n \in D(T)$ with $\|x_n\| \geq 1$ such that $(T-\lambda)x_n$ converge to 0.*

One can see from Proposition 6.4 that the spectral projections contains a lot of useful information about the spectrum. Therefore, it is a good idea to understand how to calculate them at least for simple sets Ω.

Proposition 6.6 (Spectral projection to a point). *For any $\lambda \in \mathbb{R}$ there holds*

$$E_T(\{\lambda\}) = -i \operatorname{s-}\lim_{\varepsilon \to 0+} \varepsilon (T - \lambda - i\varepsilon)^{-1}.$$

Proof. For $\varepsilon > 0$ consider the function

$$f_\varepsilon(x) := -\frac{i\varepsilon}{x - \lambda - i\varepsilon}.$$

One has the following properties:

- $|f_\varepsilon| \leq 1$,

- $f_\varepsilon(\lambda) = 1$,
- if $x \neq \lambda$, then $f_\varepsilon(x)$ tends to 0 as ε tends to 0.

This means that $f_\varepsilon \xrightarrow{\mathcal{B}_\infty} 1_{\{\lambda\}}$. By Theorem 5.16, $E_T(\{\lambda\}) = \mathrm{s}-\lim_{\varepsilon\to 0+} f_\varepsilon(T)$, and it remains to note that $f_\varepsilon(T) = (T-\lambda-i\varepsilon)^{-1}$ by Theorem 5.8. □

Proposition 6.7 (Stone formula). *For $a < b$ one has:*

$$\frac{1}{2}\Big(E_T\big((a,b)\big) + E_T\big([a,b]\big)\Big) = \frac{1}{\pi}\,\mathrm{s}-\lim_{\varepsilon\to 0+} \int_a^b \Im R(\lambda + i\varepsilon)\,d\lambda.$$

Proof. For $\varepsilon > 0$ consider the function

$$f_\varepsilon(x) = \frac{1}{\pi}\int_a^b \Im \frac{1}{x-\lambda-i\varepsilon}\,d\lambda.$$

By direct computation we have

$$f_\varepsilon(x) = \frac{1}{\pi}\int_a^b \frac{\varepsilon}{(\lambda-x)^2+\varepsilon^2}\,d\lambda = \frac{1}{\pi}\bigg(\arctan\frac{b-x}{\varepsilon} - \arctan\frac{a-x}{\varepsilon}\bigg).$$

Therefore, $|f_\varepsilon| \le 1$, and

$$\lim_{\varepsilon\to 0+} f_\varepsilon(x) = \begin{cases} 0, & x \notin [a,b],\\ 1, & x\in(a,b),\\ \dfrac{1}{2}, & x\in\{a,b\},\end{cases} = \frac{1}{2}\Big(1_{(a,b)}(x) + 1_{[a,b]}(x)\Big),$$

and the rest follows as in the previous proposition. □

Finally the following formula can be useful for the computation of spectral projections on isolated components of the spectrum.

Proposition 6.8 (Spectral projection on isolated part of spectrum). *Let $\Gamma\subset\mathbb{C}$ be a smooth closed curve oriented in the anti-clockwise sense which does not meet* $\operatorname{spec} T$*, and let Ω be the intersection of the interior of* Γ *with* $\mathbb{R}$*, then*

$$E_T(\Omega) = \frac{1}{2\pi i}\oint_\Gamma (T-z)^{-1}dz.$$

Proof. If x is an intersection point of Γ with $\mathbb{R}$, then, by assumption $x \notin \operatorname{ess}_\mu \operatorname{ran} h$. On the other hand, for $x \in \mathbb{R}\setminus\Gamma$ there holds, using the Cauchy formula,

$$\frac{1}{2\pi i}\oint_\Gamma (x-z)^{-1}dz = \begin{cases} 1, & x \text{ is inside } \Gamma,\\ 0, & x \text{ is outside } \Gamma.\end{cases}$$

Therefore, μ-a.e. one has

$$\frac{1}{2\pi i}\oint_\Gamma (h-z)^{-1}dz = 1_\Omega\circ h,$$

and one can replace h by T using Theorem 5.16. □

As a final remark we mention that the map $\Omega \mapsto E_T(\Omega)$ can be viewed an operator-valued measure, and one can integrate reasonable scalar function (bounded Borel ones or even unbounded) with respect to this measure using e.g. the Lebesgue integral sums. Then one obtains the integral representations,

$$T = \int_{\mathbb{R}} \lambda dE_T(\lambda), \quad f(T) = \int_{\mathbb{R}} f(\lambda) dE_T(\lambda),$$

and the associated integral sums can be viewed as certain approximations of the respective operators.

6.2 Generalized eigenfunctions

Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Let $\mathcal{H}_+$ be another Hilbert space which is continuously and densely embedded in $\mathcal{H}$, i.e. that there exists a continuous linear operator $j : \mathcal{H}_+ \to \mathcal{H}$ whose range is dense. Denote by $\mathcal{H}_-$ the space of continuous linear functionals on $\mathcal{H}_+$. The action of $h_- \in \mathcal{H}_+$ on $h_+ \in \mathcal{H}_+$ will be denoted by $\langle h_-, h_+ \rangle$. The space $\mathcal{H}_-$ has the natural vector structure. There is a natural linear map $j^* : \mathcal{H} \to \mathcal{H}_-$ which assign to each $h \in \mathcal{H}$ the functional $j^* h$ given by $\langle j^* h, h_+ \rangle = \langle h, jh_+ \rangle$. If one introduces the norm in $\mathcal{H}_-$ by

$$\|h_-\|_- = \sup_{\|h_+\|=1} \big| \langle h_-, h_+ \rangle \big|,$$

then $j^* \in \mathcal{L}(\mathcal{H}, \mathcal{H}_-)$ and, moreover, $\ker j^* = \{0\}$, i.e. j^* is an embedding. The triple $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is usually referred to as a *Gelfand triple* or *rigging of $\mathcal{H}$*.

Definition 6.9 (Generalized eigenfunction). We say that a vector $\psi \in \mathcal{H}_-$ is a *generalized eigenfunction* of T with the generalized eigenvalue $\lambda \in \mathbb{R}$ if $\big\langle \psi, (T - \lambda)\varphi \big\rangle = 0$ for all $\varphi \in D(T) \cap \mathcal{H}_+$ with $T\varphi \in \mathcal{H}_+$. □

One has the following fundamental result.

Theorem 6.10 (Existence of expansion in generalized eigenfunctions). *One can find a rigging such that:*

- *the set $\mathcal{D} := \{\varphi \in D(T) \cap \mathcal{H}_+ : T\varphi \in \mathcal{H}_+\}$ is a core of T, e.g. $\overline{T|_{\mathcal{D}}} = T$,*
- *there exists a measure space (M, μ) and a map $\Phi : M \to \mathcal{H}_-$ with the following properties:*
  - *the map $\mathcal{H}_+ \ni h \mapsto \widehat{h} \in L^2(M, d\mu)$ defined by $\widehat{h}(m) := \langle \Phi(m), h \rangle$ extends to a unitary operator from $\mathcal{H}$ to $L^2(M, d\mu)$,*
  - *there exists a measurable function $a : M \to \mathbb{R}$ such that $\Phi(m)$ is a generalized eigenfunction of T with the generalized eigenvalue $a(m)$ for μ-a.e. $m \in M$.*

We are not giving any proof here, an interested reader may refer to a good concise discussion in Supplement S1.2 of the book [2] or to the detailed study in the book [1].

Example 6.11 (Generalized eigenfunctions of Laplacian). Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and $T = -\Delta$. One can take $\mathcal{H}_+ = H^2(\mathbb{R}^n)$, then $\mathcal{H}_- = H^{-2}(\mathbb{R}^n)$. One can easily show that for any $p \in \mathbb{R}^3$ the function ψ, $\psi(x) = e^{ipx}$ is a generalized eigenfunction of T with the generalized eigenvalue p^2. The associated map $h \mapsto \widehat{h}$ is the usual Fourier transform, and $(M, d\mu)$ is just $\mathbb{R}^n$ with the Lebesgue measure.

Theorem 6.10 just gives a special form of a unitary transform U from Theorem 5.13. Informally speaking, the theorem says that calculating the spectrum is in a sense equivalent to solving the eigenvalue problem $T\psi = \lambda\psi$ but in a certain larger space $\mathcal{H}_-$. On the main difficulties in applying such an approach is that in concrete examples the spaces $\mathcal{H}_\pm$ are described in a rather implicit way, and it difficult to decide if a given vector/distribution belongs to this space or not. Some particular cases are indeed well-studied. For example, one has the following nice description of the spectrum for Schrödinger operators, which we state without proof (see e.g. [4], Chapter 2):

Theorem 6.12 (Shnol theorem). *Let $\mathcal{H} = L^2(\mathbb{R}^n)$, $V \in L^2_{loc}$, $V \geq 0$, $T = -\Delta + V$ (we take the operator defined by the Fridrichs extension). Denote by Σ the set of the real numbers λ for which there exists a non-zero solution u to the differential equation $(-\Delta + V)u = \lambda u$ with the subexponential growth, i.e. such that for any $a > 0$ there exists $C > 0$ such that $\big|u(x)\big| \leq Ce^{a|x|}$ for all $x \in \mathbb{R}^n$. Then the spectrum of T coincides with the closure of Σ.*

Note that there are various versions of the above result for differential operators on manifolds and other related spaces, then the subexponentional growth condition should be replaced by a suitable relation comparing the growth of generalized eigenfunctions with the growth of the volume of balls at infinity.

Example 6.13. One can look again at the operator $T = -d^2/dx^2$ in $\mathcal{H} = L^2(\mathbb{R})$. for any $\lambda \in \mathbb{R}$ the equation $-u'' = \lambda u$ has two linearly independent solutions. For $\lambda < 0$ all non-zero solutions are exponentially growing for $x \to +\infty$ or for $x \to -\infty$, and such values λ do not belong to the spectrum. For $\lambda = 0$ one has either a constant or a linear function, and for $\lambda > 0$ the both solutions are bounded, which gives again $\operatorname{spec} T = [0, +\infty)$.

6.3 Tensor products

A more detailed discussion of tensor products can be found e.g. in [12, Sections II.4 and VIII.10] or in [17, Sections 1.4 and 4.5].

Let A_j be self-adjoint operators in Hilbert spaces $\mathcal{H}_j$, $j = 1, \dots, n$. With any monomial $\lambda_1^{m_1} \cdot \ldots \lambda_n^{m_n}$, $m_j \in \mathbb{N}$, one can associate the operator $A_1^{m_1} \otimes \cdot A_n^{m_n}$ in $\mathcal{H} := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ defined by

$$(A_1^{m_1} \otimes \cdots \otimes A_n^{m_n})(\psi_1 \otimes \cdots \otimes \psi_n) = A_1^{m_1}\psi_1 \otimes \cdots \otimes A_n^{m_n}\psi_n, \quad \psi_j \in D(A_j^{m_j}),$$

and then extended by linearity; here the zero powers A_j^0 equal the identity operators in the respective spaces.

Remark 6.14. For an operator A in a Hilbert space $\mathcal{H}$ the domain $D(A^n)$ is usually defined in a recursive way:

$$D(A^0) = \mathcal{H} \text{ and } D(A^n) = \big\{x \in D(A) :\ Ax \in D(A^{n-1})\big\} \text{ for } n \in \mathbb{N}.$$

As an exercise one can show that for a self-adjoint A one has $D(A^n) = \operatorname{ran} R_A(z)^n$ with any $z \in \operatorname{res} A$ and that $D(A^n)$ is dense in $\mathcal{H}$ for any n. □

Using the above construction one can associate with any real-valued polynomial P of $\lambda_1, \dots, \lambda_n$ of degree N a linear operator $P(A_1, \dots, A_n)$ in $\mathcal{H}$ defined on the set $\mathcal{H}$ consisting of the linear combinations of the vectors of the form $\psi_1 \otimes \dots \otimes \psi_n$ with $\psi_j \in D(A_j^N)$.

Theorem 6.15 (Spectrum of tensor product). *Denote by B the closure of the above operator $P(A_1, \dots, A_n)$, then B is self-adjoint, and*

$$\operatorname{spec} B = \overline{\big\{P(\lambda_1, \dots, \lambda_n) :\ \lambda_j \in \operatorname{spec} A_j\big\}}.$$

Sketch of the proof. The complete proof involves a number of technicalities, see e.g. Section III.10 in [12], but the main idea is rather simple. By the spectral theorem, it is sufficient to consider the case when A_j is the multiplication by a certain function f_j in $\mathcal{H}_j := L^2(M_j, d\mu_j)$. Then

$$\mathcal{H} = L^2(M, d\mu), \quad M = M_1 \times \dots \times M_n, \quad \mu = \mu_1 \otimes \dots \otimes \mu_n,$$

and $P(A_1, \dots, A_n)$ acts in $\mathcal{H}$ as the multiplication by p, $p(x_1, \dots, x_n) = P\big(f_1(x_1), \dots, f_n(x_n)\big)$, and its domain includes at least all the linear combinations of the functions $\psi_1 \otimes \dots \otimes \psi_n$ where ψ_j are L^2 with compact supports. It is a routine to show that the closure of this operator is just the usual multiplication operator by p, which gives the sought relation. □

Example 6.16 (Laplacian in rectangle). A typical example of the above construction is given by the Laplacians in rectangles. Namely, let $a, b > 0$ and $\Omega = (0, a) \times (0, b) \subset \mathbb{R}^2$, $\mathcal{H} = L^2(\Omega)$, and T be the Dirichlet Laplacian in Ω. One can show that T can be obtained using the above procedure using the representation

$$T = L_a \otimes 1 + 1 \otimes L_b,$$

where by L_a we denote the Dirichlet Laplacian in $\mathcal{H}_a := L^2(0, a)$, i.e.

$$L_a f = -f'', \quad D(L_a) = H^2(0, a) \cap H_0^1(0, a).$$

It is known (from the exercises) that the spectrum of L_a consists of the simple eigenvalues $(\pi n/a)^2$, $n \in \mathbb{N}$, with the eigenfunctions $x \mapsto \sin(\pi n x/a)$, and this means that the spectrum of T consists of the eigenvalues

$$\lambda_{m,n}(a, b) = \Big(\frac{\pi m}{a}\Big)^2 + \Big(\frac{\pi n}{b}\Big)^2, \quad m, n \in \mathbb{N},$$

and the associated eigenfunctions are the products of the respective eigenfunctions for L_a and L_b. The multiplicity of each eigenvalue λ is exactly the number of pairs $(m,n) \in \mathbb{N}^2$ for which $\lambda = \lambda_{m,n}$.

Note that the closure of the set $\{\lambda_{m,n}\}$ can be omitted as this is a discrete set.

The same constructions hold for the Neumann Laplacians, one obtains the same formula for the eigenvalues but now with $m,n \in \mathbb{N} \cup \{0\}$. $\square$

7 Perturbations

7.1 Kato-Rellich theorem

We have seen since the beginning of the course that one needs to pay a great attention to the domains when dealing with unbounded operators. The aim of the present subsection is to describe some classes of operators in which such problems can be avoided.

Definition 7.1 (Essentially self-adjoint operator). We say that a linear operator T is essentially self-adjoint on a subspace $\mathcal{D} \subset D(T)$ if the closure of the restriction of T to $\mathcal{D}$ is a self-adjoint operator. If the closure of T is self-adjoint, then we simply say that T is essentially self-adjoint. □

Proposition 7.2. *An essentially self-adjoint operator has a unique self-adjoint extension.*

Proof. Let T be an essentially self-adjoint operator, and let S be a self-adjoint extension of T. As S is closed, the inclusion $T \subset S$ implies $\overline{T} \subset S$. On the other hand, $S = S^* \subset (\overline{T})^* = \overline{T}$ (as $\overline{T}$ is self-adjoint). This shows that $S = \overline{T}$. □

Theorem 7.3 (Self-adjointness criterion). *Let T be a closed symmetric operator in a Hilbert space $\mathcal{H}$, then the following three assertions are equivalent:*

1. *T is self-adjoint,*
2. $\ker(T^* + i) = \ker(T^* - i) = \{0\}$,
3. $\operatorname{ran}(T + i) = \operatorname{ran}(T - i) = \mathcal{H}$.

Proof. The implication $1 \Rightarrow 2$ is obvious: a self-adjoint operator cannot have non-real eigenvalues.

To show the implication $2 \Rightarrow 3$ recall first that $\ker(T^* \pm i) = \operatorname{ran}(T \mp i)^\perp$. Therefore, it is sufficient to show that the subspaces $\operatorname{ran}(T \pm i)$ are closed. For any $f \in D(T)$ we have:

$$\big\|(T \pm i)f\big\|^2 = \big\langle (T \pm i)f, (T \pm i)f \big\rangle = \langle Tf, Tf\rangle + \langle f, f\rangle \pm i\big(\langle Tf, f\rangle - \langle f, Tf\rangle\big)$$
$$= \|Tf\|^2 + \|f\|^2.$$

Let $f_n \in \operatorname{ran}(T \pm i)$ such that f_n converge to some $f \in \mathcal{H}$. Find $\varphi_n \in D(T)$ with $f_n = (T \pm i)\varphi_n$, then due to the preceding equality (φ_n) and $(T\varphi_n)$ are Cauchy sequences. As T is closed, φ_n converge to some $\varphi \in D(T)$ and $T\varphi_n$ converge to $T\varphi$, and then $f_n = (T \pm i)\varphi_n$ converge to $(T \pm i)\varphi = f$ and $f \in \operatorname{ran}(T \pm i)$.

It remains to the prove the implication $3 \Rightarrow 1$. Let $\varphi \in D(T^*)$. Due to the surjectivity of $T - i$ one can find $\psi \in D(T)$ with $(T - i)\psi = (T^* - i)\varphi$. As $T \subset T^*$, we have $(T^* - i)(\psi - \varphi) = 0$. On the other hand, due to $\operatorname{ran}(T + i) = \mathcal{H}$ we have $\ker(T^* - i) = 0$, which means that $\varphi = \psi \in D(T)$. □

Note that during the proof we obtained the following simple fact:

Proposition 7.4. *Let T be a symmetric operator, then* $\overline{\operatorname{ran}(T \pm i)} = \operatorname{ran}(\overline{T} \pm i)$.

This leads as to the following assertion:

Corollary 7.5 (Essential self-adjointness criterion). *Let T be a symmetric operator in a Hilbert space $\mathcal{H}$, then the following three assertions are equivalent:*

1. *T is essentially self-adjoint,*
2. $\ker(T^* + i) = \ker(T^* - i) = \{0\}$,
3. $\operatorname{ran}(T + i)$ *and* $\operatorname{ran}(T - i)$ *are dense in* $\mathcal{H}$.

Remark 7.6. The above theorem can be modified in several ways. For example, it still holds if one replaces $T \pm i$ by $T \pm i\lambda$ with any $\lambda \in \mathbb{R} \setminus \{0\}$. For semibounded operators we have an alternative version:

Theorem 7.7 (Self-adjointness criterion for semibounded operators). *Let T be a closed symmetric operator in a Hilbert space $\mathcal{H}$ and $T \geq 0$ and let $a > 0$, then the following three assertions are equivalent.*

1. *T is self-adjoint,*
2. $\ker(T^* + a) = \{0\}$,
3. $\operatorname{ran}(T + a) = \mathcal{H}$.

This is left as an exercise. The analogues of Proposition 7.4 and Corollary 7.5 hold as well. □

Now we would like to apply the above assertions to the study of some perturbations of self-adjoint operators.

Definition 7.8 (Relative boundedness). Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and B be a linear operator with $D(A) \subset D(B)$. Assume that there exist $a, b > 0$ such that

$$\|Bf\| \leq a\|Af\| + b\|f\| \quad \text{for all } f \in D(A),$$

then B is called *relatively bounded* with respect to A or, for short, A-*bounded*. The infimum of all possible values a is called the *relative bound* of B with respect to A. If the relative bound is equal to 0, then B is called *infinitesimally small* with respect to A. □

Theorem 7.9 (Kato-Rellich). *Let A be a self-adjoint operator in $\mathcal{H}$ and let B be a symmetric operator in $\mathcal{H}$ which is A-bounded with a relative bound < 1, then the operator $A + B$ with the domain $D(A + B) = D(A)$ is self-adjoint. Moreover, if A is essentially self-adjoint on some $\mathcal{D} \subset D(A)$, then $A + B$ is essentially self-adjoint on $\mathcal{D}$ too.*

Proof. By assumption, one can find $a \in (0,1)$ and $b > 0$ such that

$$\|Bu\| \le a\|Au\| + b\|u\|, \quad \text{for all} \quad u \in D(A). \tag{7.1}$$

Step 1. As seen many times, for any $\lambda > 0$ one has

$$\big\|(A+B\pm i\lambda)u\big\|^2 = \big\|(A+B)u\big\|^2 + \lambda^2\|u\|^2.$$

Therefore, for all $u \in D(A)$ one can estimate

$$\begin{aligned} 2\big\|(A+B\pm i\lambda)u\big\| \ge \big\|(A+B)u\big\| + \lambda\|u\| &\ge \|Au\| - \|Bu\| + \lambda\|u\| \\ &= (1-a)\|Au\| + (\lambda - b)\|u\|. \end{aligned} \tag{7.2}$$

Let us pick some $\lambda > b$.

Step 2. Let us show that $A+B$ with the domain equal to $D(A)$ is a closed operator. Let $(u_n) \subset D(A)$ and $f_n := (A+B)u_n$ such that both u_n and f_n converge in $\mathcal{H}$. By (7.2), Au_n is a Cauchy sequence. As A is closed, u_n converge to som $u \in D(A)$ and Au_n converge to Au. By (7.1), Bu_n is a Cauchy sequence and is hence convergent to some $v \in \mathcal{H}$. Let us show that Bu_n converge exactly to u_n (actually this would follow from the closedness of B, but we did not assume that B is closed or closable!). Take any $h \in D(A)$, then $\langle v, h\rangle = \lim\langle Bu_n, h\rangle = \lim\langle u_n, Bh\rangle = \langle u, Bh\rangle = \langle Bu, h\rangle$. So finally $(A+B)u_n$ converge to $(A+B)u$. This shows that $A+B$ is closed.

Step 3. Let us show that the operators $A+B\pm i\lambda : D(A) \to \mathcal{H}$ are bijective at least for large λ. As previously, we have $\|(A\pm i\lambda)u\|^2 = \|Au\|^2 + \lambda^2\|u\|^2$. Then

$$\|Bu\| \le a\|Au\| + b\|u\| \le a\big\|(A\pm i\lambda)u\big\| + \frac{b}{|\lambda|}\big\|(A\pm i\lambda)u\big\| = \big(a + \frac{b}{|\lambda|}\big)\big\|(A\pm i\lambda)u\big\|.$$

As $a \in (0,1)$, we can choose λ sufficiently large to have $a + b/|\lambda| < 1$. This means that for such λ we have $\big\|B(A\pm i\lambda)^{-1}\big\| < 1$. Now we can represent

$$A + B \pm i\lambda = \Big(1 + B(A\pm i\lambda)^{-1}\Big)(A\pm i\lambda).$$

As A is self-adjoint, the operators $A\pm i\lambda : D(A) \to \mathcal{H}$ are bijections, and $1 + B(A\pm i\lambda)^{-1}$ is a bijection from $\mathcal{H}$ to itself. Therefore, $A+B\pm i\lambda$ are bijective, in particular, $\operatorname{ran}(A+B\pm i\lambda) = \mathcal{H}$. By Theorem 7.3 and Remark 7.6, $A+B$ is self-adjoint.

The part concerning the essential self-adjointness is reduced to the proof of the relation $\overline{A+B} = \overline{A} + B$, which is an elementary exercise. □

7.2 Essential self-adjointness of Schrödinger operators

The Kato-Rellich theorem is one of the tools used to simplify the consideration of the Schrödinger operators.

Theorem 7.10. *Let $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ be real-valued with $p = 2$ if $d \le 3$ and $p > d/2$ if $d > 3$, then the operator $T = -\Delta + V$ with the domain $D(T =)H^2(\mathbb{R}^d)$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$, and it is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$.*

Proof. We give the proof only for the dimension $d \le 3$. For all $f \in \mathcal{S}(\mathbb{R}^d)$ and $\lambda > 0$ we have the representation

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ipx} \widehat{f}(p)\, dp \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{1}{p^2+\lambda} (p^2+\lambda)\widehat{f}(p) dp \\ &\le \frac{1}{(2\pi)^{d/2}} \Big\| \frac{1}{p^2+\lambda} \Big\| \cdot \Big\| (p^2+\lambda)\widehat{f}(p) \Big\| \\ &\le \frac{1}{(2\pi)^{d/2}} \Big\| \frac{1}{p^2+\lambda} \Big\| \cdot \Big(\|p^2 \widehat{f}(p)\| + \lambda \|\widehat{f}\| \Big) = a_\lambda \|\Delta f\| + b_\lambda \|f\| \end{aligned}$$

with

$$a_\lambda = \frac{1}{(2\pi)^{d/2}} \Big\| \frac{1}{p^2+\lambda} \Big\|, \quad b_\lambda = \frac{\lambda}{(2\pi)^{d/2}} \Big\| \frac{1}{p^2+\lambda} \Big\|.$$

By density, for all $f \in H^2(\mathbb{R}^d)$ and all $\lambda > 0$ we have

$$\|f\|_\infty \le a_\lambda \|\Delta f\| + b_\lambda \|f\|.$$

By assumption we can represent $V = V_1 + V_2$ with $V_1 \in L^2(\mathbb{R}^d)$ and $V_2 \in L^\infty(\mathbb{R}^d)$. Using the preceding estimate we arrive at

$$\|Vf\| \le \|V_1 f\| + \|V_2 f\| \le \|V_1\|_2 \|f\|_\infty + \|V_2\| \|f\| \le \widetilde{a}_\lambda \|\Delta f\| + \widetilde{b}_\lambda \|f\|, \quad f \in H^2(\mathbb{R}^d),$$

with $\widetilde{a}_\lambda = \|V_1\|_2 a_\lambda$ and $\widetilde{b}_\lambda = \|V_1\|_2 b_\lambda + \|V_2\|_\infty$. As a_λ can be made arbitrary small by a suitable choice of λ, we see that the multiplication operator V is infinitesimally small with respect to the free Laplacian, and the result follows from the Kato-Rellich theorem.

The above proof does not work for $d \ge 3$ as the function $p \mapsto (p^2+\lambda)^{-1}$ does not belong to $L^2(\mathbb{R}^d)$ anymore. The respective parts of argument should be replaced by suitable Sobolev embedding theorems. □

Example 7.11 (Coulomb potential). Consider the three-dimensional case and the potential $V(x) = \alpha/|x|$, $\alpha \in \mathbb{R}$. For any bounded open set Ω containing the origin, one has $1_\Omega V \in L^2(\mathbb{R}^3)$ and $(1 - 1_\Omega)V \in L^\infty(\mathbb{R}^3)$, and finally $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. This means that the operator $T = -\Delta + \alpha/|x|$ is self-adjoint on the domain $H^2(\mathbb{R}^d)$.

Let us mention some other conditions guaranteeing the essential self-adjointness of the Schrödinger operators for other types of potentials.

Theorem 7.12. *Let $\mathcal{H} = L^2(\mathbb{R}^d)$ and let $V \in C^0(\mathbb{R}^d)$ be real-valued such that for some $c \in \mathbb{R}$ one has the inequality*

$$\langle u, (-\Delta + V)u \rangle \ge c\|u\|^2$$

for all $u \in C_c^\infty(\mathbb{R}^d)$. Then the operator $T = -\Delta + V$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$.

Proof. By adding a constant to the potential V one can assume that $T \geq 1$. In other words, using the integration by parts,

$$\int_{\mathbb{R}^d} \big|\nabla u(x)\big|^2 dx + \int_{\mathbb{R}^d} V(x)\big|u(x)\big|^2 dx \geq \int_{\mathbb{R}^d} \big|u(x)\big|^2 dx \tag{7.3}$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, and this extends by density at least to all $u \in H^1_{\mathrm{comp}}(\mathbb{R}^d)$. By Theorem 7.7 it is sufficient to show that the range of T is dense.

Let $f \in L^2(\mathbb{R}^d)$ such that $\big\langle f, (-\Delta + V)u\big\rangle = 0$ for all $u \in C_c^\infty(\mathbb{R}^d)$. Note that T preserve the real-valuedness, and we can suppose without loss of generality that f is real-valued. We have at least $(-\Delta + V)f = 0$ in the sense of $\mathcal{D}'(\mathbb{R}^d)$, and $\Delta f = Vf$. As V is locally bounded, the function Vf is in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$, and the elliptic regularity gives $f \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$.

Let us pick a real-valued function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi(x) = 1$ for $|x| \leq 1$, that $\varphi(x) = 0$ for $|x| \geq 2$ and that $0 \leq \varphi \leq 1$, and introduce functions φ_n by $\varphi_n(x) = \varphi(x/n)$. For any $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ we have, by a standard computation:

$$\begin{aligned}\int_{\mathbb{R}^d} \nabla(\varphi_n f)\nabla(\varphi_n u) dx &+ \int_{\mathbb{R}^d} V \varphi_n f \varphi_n u dx \\ &= \int_{\mathbb{R}^d} \big|\nabla\varphi_n\big|^2 f u\, dx + \sum_{j=1}^d \int_{\mathbb{R}^d} \Big(f\frac{\partial u}{\partial x_j} - u\frac{\partial f}{\partial x_j}\Big)\varphi_n \frac{\partial \varphi_n}{\partial x_j}\, dx + \langle f, T\varphi_n^2 u\rangle. \end{aligned}\tag{7.4}$$

As $\varphi_n^2 u \in C_c^\infty(\mathbb{R}^d)$, the last term vanishes. Taking now $u = f$ and using (7.3) we arrive at

$$\int_{\mathbb{R}^d} \big|\nabla\varphi_n\big|^2 f^2\, dx \geq \int_{\mathbb{R}^d} \varphi_n^2 f^2 dx \geq \int_\Omega \varphi_n^2 f^2 dx,$$

where Ω is any ball. As n tends to infinity, the left-hand side goes to 0. On the other side, the restriction of $\varphi_n f$ to Ω coincides with f for sufficiently large n, and this means that f vanishes in Ω. As Ω is arbitrary, $f = 0$. □

Another condition, which complements the preceding theorems, is given without proof (as it needs some advanced PDE machinery).

Theorem 7.13. *Let $V \in L_{\mathrm{loc}}(\mathbb{R}^d)$ be non-negative, then the operator $-\Delta + V$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$.*

7.3 Discrete and essential spectra

Up to now we just distinguished between the whole spectrum and the point spectrum, i.e. the set of the eigenvalues. Let us introduce another classification of spectra, which is useful when studying various perturbations.

Definition 7.14 (Discrete spectrum, essential spectrum). Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. We define its *discrete spectrum* $\operatorname{spec}_{\mathrm{disc}} T$ by

$$\operatorname{spec}_{\mathrm{disc}} T := \Big\{\lambda \in \operatorname{spec} T : \exists \varepsilon > 0 \text{ with } \dim \operatorname{ran} E_T\big((\lambda - \varepsilon, \lambda + \varepsilon)\big) < \infty\Big\}.$$

The set $\operatorname{spec}_{\mathrm{ess}} T := \operatorname{spec} T \setminus \operatorname{spec}_{\mathrm{disc}} T$ is called the *essential spectrum* of T. $\square$

The following proposition gives an alternative description of the discrete spectrum.

Proposition 7.15. *A real λ belongs to $\operatorname{spec}_{\mathrm{disc}} T$ iff it is an isolated eigenvalue of T of finite multiplicity.*

Proof. Let $\lambda \in \operatorname{spec}_{\mathrm{disc}} T$, then there exists $\varepsilon_0 > 0$ such that the operators $E_T\big((\lambda-\varepsilon, \lambda+\varepsilon)\big)$ do not depend on ε if $\varepsilon \in (0, \varepsilon_0)$. On the the other hand, this limit operator is non-zero, as $\lambda \in \operatorname{spec} T$. This means $E_T\big(\{\lambda\}\big) = \mathrm{s}-\lim_{\varepsilon\to 0+} E_T\big((\lambda-\varepsilon, \lambda+\varepsilon)\big) \neq 0$, and $\lambda \in \operatorname{spec}_{\mathrm{p}} T$ by Proposition 6.4(3). At the same time, $E_T\big((\lambda-\varepsilon_0, \lambda)\big) = E_T\big((\lambda, \lambda+\varepsilon_0)\big) = 0$, and Proposition 6.4(2) show that λ is an isolated point of the spectrum.

Now let λ be an isolated eigenvalue of finite multiplicity. Then there exists $\varepsilon_0 > 0$ such that $E_T\big((\lambda-\varepsilon_0, \lambda)\big) = E_T\big((\lambda, \lambda+\varepsilon_0)\big) = 0$, and $\dim \operatorname{ran} E_T(\{\lambda\}) = \dim\ker(T-\lambda) < \infty$. Therefore,

$$\begin{aligned}\dim \operatorname{ran} E_T\big((\lambda-\varepsilon_0, \lambda+\varepsilon_0)\big) &= \dim \operatorname{ran} E_T\big((\lambda-\varepsilon_0, \lambda)\big)\\ &\quad + \dim \operatorname{ran} E_T\big((\lambda, \lambda+\varepsilon_0)\big) + \dim \operatorname{ran} E_T(\{\lambda\}) < \infty. \quad \square\end{aligned}$$

Therefore, we arrive at the following direct description of the essential spectrum

Proposition 7.16. *A value $\lambda \in \operatorname{spec} T$ belongs to $\operatorname{spec}_{\mathrm{ess}} T$ iff at least one of the following three conditions holds:*

- $\lambda \notin \operatorname{spec}_{\mathrm{p}} T$,
- *λ is an accumulation point of $\operatorname{spec}_{\mathrm{p}} T$,*
- $\dim\ker(T-\lambda) = \infty$.

Furthermore, the essential spectrum is a closed set.

Proof. The first part just describes the points of the spectrum which are not isolated eigenvalues of finite multiplicity.

For the second part we note that $\operatorname{spec}_{\mathrm{ess}} T$ is obtained from the closed set $\operatorname{spec} T$ by removing some isolated points. As the removing an isolated point does not change the property to be closed, $\operatorname{spec}_{\mathrm{ess}} T$ is also closed. $\square$

Let us list some examples.

Proposition 7.17 (Essential spectrum for compact operators). *Let T be a compact self-adjoint operator in an infinite-dimensional space $\mathcal{H}$, then $\operatorname{spec}_{\mathrm{ess}} T = \{0\}$.*

Proof. By Theorem 4.3, for any $\varepsilon > 0$ the set $\operatorname{spec} T \setminus (-\varepsilon, \varepsilon)$ consists of a finite number of eigenvalues of finite multiplicity, hence we have: $\operatorname{spec}_{\text{ess}} T \setminus (-\varepsilon, \varepsilon) = \emptyset$ and $\dim \operatorname{ran} E_T\big(\mathbb{R} \setminus (-\varepsilon, \varepsilon)\big) < \infty$. On the other hand, $\dim \mathcal{H} = \dim \operatorname{ran} E_T\big(\mathbb{R} \setminus (-\varepsilon, \varepsilon)\big) + \dim \operatorname{ran} E_T\big((-\varepsilon, \varepsilon)\big)$, and $\dim \operatorname{ran} E_T\big((-\varepsilon, \varepsilon)\big)$ must be infinite for any $\varepsilon > 0$, which means that $0 \in \operatorname{spec}_{\text{ess}} T$. □

Proposition 7.18 (Essential spectrum of operators with compact resolvents). *The essential spectrum of a self-adjoint operator is empty if and only if the operator has a compact resolvent.*

Proof is left as an exercise.

Sometimes one uses the following terminology:

Definition 7.19 (Purely discrete spectrum). We say that a self-adjoint operator T has a *purely discrete spectrum* in some interval (a, b) if $\operatorname{spec}_{\text{ess}} T \cap (a, b) = \emptyset$. If one has simply $\operatorname{spec}_{\text{ess}} T = \emptyset$, then we say simply that the spectrum of T is *purely discrete*. As follows from the previous proposition, this exactly means that T has a compact resolvent. □

Example 7.20. As seen several times, the free Laplacian in $L^2(\mathbb{R}^d)$ has the spectrum $[0, +\infty)$. This set has no isolated points, so this operator has no discrete spectrum.

The main difference between the discrete and the essential spectra comes from their behavior with respect to perturbations. This will be discussed in the following sections.

7.4 Weyl criterion and relatively compact perturbations

Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$.

The following proposition is an exercise.

Proposition 7.21. *Let λ be an isolated eigenvalue of T, then there exists $c > 0$ such that $\|(T - \lambda)u\| \geq c\|u\|$ for all $u \perp \ker(T - \lambda)$.*

The following theorem gives a description of the essential spectrum using approximating sequences.

Theorem 7.22 (Weyl criterion). *The condition $\lambda \in \operatorname{spec}_{\text{ess}} T$ is equivalent to the existence of a sequence $(u_n) \subset D(T)$ satisfying the following three properties:*

1. $\|u_n\| \geq 1$,
2. *u_n converge weakly to 0,*
3. *$(T - \lambda)u_n$ converge to 0 in the norm of $\mathcal{H}$.*

Such a sequence will be called a singular Weyl sequence *for λ. Moreover, as will be shown in the proof, one can replace the conditions (1) and (2) just by:*

1'. u_n form an orthonormal sequence.

Proof. Denote by $W(T)$ the set of all real numbers λ for which one can find a singular Weyl sequence.

Show first the inclusion $W(T) \subset \operatorname{spec}_{\text{ess}} T$. Let $\lambda \in W(T)$ and let (u_n) be an associated singular Weyl sequence, then we have at least $\lambda \in \operatorname{spec} T$. Assume by contradiction that $\lambda \in \operatorname{spec}_{\text{disc}} T$ and denote by Π the orthogonal projection to $\ker(T - \lambda)$ in $\mathcal{H}$. As Π is a finite-rank operator, it is compact, and the sequence Πu_n converge to 0. Therefore, the norms of the vectors $w_n := (1 - \Pi)u_n$ satisfy $\|w_n\| \geq 1/2$ for large n. On the other hand, the vectors $(T-\lambda)w_n = (1-\Pi)(T-\lambda)u_n$ converge to 0, which contradicts to Proposition 7.21.

Conversely, if $\lambda \in \operatorname{spec}_{\text{ess}} T$, then $\dim \operatorname{ran} E_T\big((\lambda - \varepsilon, \lambda + \varepsilon)\big) = \infty$ for all $\varepsilon > 0$. In particular, one can find a strictly decreasing to 0 sequence (ε_n) with $E_T(I_n \setminus I_{n+1}) \neq 0$, where $I_n := (\lambda - \varepsilon_n, \lambda + \varepsilon_n)$. Now we can choose u_n with $\|u_n\| = 1$ and $E_T(I_n \setminus I_{n+1})u_n = u_n$. These vectors form an orthonormal sequence and, in particular, converge weakly to 0. On the other hand,

$$\|(T - \lambda)u_n\| = \|(T - \lambda)E_T(I_n \setminus I_{n+1})u_n\| \leq \varepsilon_n \|u_n\| = \varepsilon_n,$$

which shows that the vectors $(T - \lambda)u_n$ converge to 0. Therefore, (u_n) is a singular Weyl sequence, and $\operatorname{spec}_{\text{ess}} T \subset W(T)$. □

The following theorem provides a starting point to the study of perturbations of self-adjoint operators.

Theorem 7.23 (Stability of essential spectrum). *Let A and B be self-adjoint operators such that for some $z \in \operatorname{res} A \cap \operatorname{res} B$ the difference of their resolvents $K(z) := (A - z)^{-1} - (B - z)^{-1}$ is a compact operator, then* $\operatorname{spec}_{\text{ess}} A = \operatorname{spec}_{\text{ess}} B$.

Proof. One can easily see, using the resolvent identities (Proposition 3.4), that $K(z)$ is compact for *all* $z \in \operatorname{res} A \cap \operatorname{res} B$.

Let $\lambda \in \operatorname{spec}_{\text{ess}} A$ and let (u_n) be an associated singular Weyl sequence. Without loss of generality assume that $\|u_n\| = 1$ for all n. We have

$$\lim \Big((A - z)^{-1} - \frac{1}{\lambda - z}\Big)u_n = \lim \frac{1}{z - \lambda}(A - z)^{-1}(A - \lambda)u_n = 0. \tag{7.5}$$

On the other hand, as $K(z)$ is compact, the sequence $K(z)u_n$ converges to 0 with respect to the norm, and

$$\begin{aligned}\lim \frac{1}{z - \lambda}(B - \lambda)(B - z)^{-1}u_n &= \lim \Big((B - z)^{-1} - \frac{1}{\lambda - z}\Big)u_n \\ &= \lim \Big((A - z)^{-1} - \frac{1}{\lambda - z}\Big)u_n - \lim K(z)u_n = 0.\end{aligned}$$

Now denote $v_n := (B-z)^{-1}u_n$. The preceding equality shows that $(B-\lambda)v_n$ converge to 0, and one can easily show that v_n converge weakly to 0. It follows again from

(7.5) and from the compactness of $K(z)$ that $\lim \|v_n\| = |\lambda - z|^{-1}$. Therefore, (v_n) is a singular Weyl sequence for B and λ, and $\lambda \in \operatorname{spec}_{\mathrm{ess}} B$. So we have shown the inclusion $\operatorname{spec}_{\mathrm{ess}} A \subset \operatorname{spec}_{\mathrm{ess}} B$. As the participation of A and B is symmetric, we have also $\operatorname{spec}_{\mathrm{ess}} A \supset \operatorname{spec}_{\mathrm{ess}} B$. □

Let us describe a class of perturbations which can be studied using the preceding theorem.

Definition 7.24 (Relatively compact operators). Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$, and let B a closable linear operator in $\mathcal{H}$ with $D(A) \subset D(B)$. We say that B is *compact with respect to* A (or simply A*-compact*) if $B(A-z)^{-1}$ is compact for at least one $z \in \operatorname{res} A$. (It follows from the resolvent identitites that this holds then for all $z \in \operatorname{res} A$. □

Proposition 7.25. *Let B be A-compact, then B is infinitesimally small with respect to A.*

Proof. We show first that

$$\lim_{\lambda \to +\infty} \big\| B(A - i\lambda)^{-1} \big\| = 0 \tag{7.6}$$

Assume that (7.6) is false. Then one can find a constant $\alpha > 0$, non-zero vectors u_n and a positive sequence λ_n with $\lim \lambda_n = +\infty$ such that $\big\|B(A-i\lambda)^{-1}u_n\big\| > \alpha\|u_n\|$ for all n. Set $v_n := (A-i\lambda)^{-1}u_n$. Using $\|u_n\|^2 = \big\|(A-i\lambda_n)v_n\big\|^2 = \|Av_n\|^2 + \lambda_n^2\|v_n\|^2$ we obtain

$$\|Bv_n\|^2 > \alpha^2\|Av_n\|^2 + \alpha^2\lambda_n^2\|v_n\|^2.$$

Without loss of generality one may assume the normalization $\|Bv_n\| = 1$, then the sequence Av_n is bounded and v_n converge to 0. Let $z \in \operatorname{res} A$, then $(A-z)v_n$ is also bounded, one can extract a weakly convergent subsequence $(A-z)v_{n_k}$. Due to the compactness, the vectors $B(A-z)^{-1} \cdot (A-z)v_{n_k} = Bv_{n_k}$ converge to some $w \in \mathcal{H}$ with $\|w\| = 1$. On the other hand, as shown above, v_{n_k} converge to 0, and the closability of B shows that $w = 0$. This contradiction shows that (7.6) is true.

Now, for any $a > 0$ one can find $\lambda > 0$ such that $\|B(A-i\lambda)^{-1}u\| \le a\|u\|$ for all $u \in \mathcal{H}$. Denoting $v := (A-i\lambda)^{-1}u$ and noting that $(A-i\lambda)^{-1}$ is a bijection between $\mathcal{H}$ and $D(A)$ we see that

$$\|Bv\| \le a\|(A-i\lambda)v\| \le a\|Av\| + a\lambda\|v\|$$

for all $v \in D(A)$. As $a > 0$ is arbitrary, we get the result. □

So a combination of the preceding assertions leads us to the following observation:

Theorem 7.26 (Relatively compact perturbations). *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and let B be symmetric and A-compact, then the operator $A+B$ with $D(A+B) = D(A)$ is self-adjoint, and the essential spectra of A and $A+B$ coincide.*

Proof. The self-adjointness of $A+B$ follows from the Kato-Rellich theorem, and it remains to show that the difference of the resolvents of $A+B$ and A is compact. This follows directly from the obvious identity

$$(A-z)^{-1}-(A+B-z)^{-1}=(A+B-z)^{-1}B(A-z)^{-1},$$

which holds at least for all $z \notin \mathbb{R}$. □

As an exercise one can show the following assertion, which can be useful in some situations.

Proposition 7.27. *Let A be self-adjoint, B be symmetric and A-bounded with a relative bound <1, and C be A-compact, then C is also $(A+B)$-compact.*

7.5 Essential spectra for Schrödinger operators

Definition 7.28 (Kato class potential). We say that a measurable function $V:\mathbb{R}^d\to\mathbb{R}$ belongs to the Kato class if for any $\varepsilon>0$ one can find real-valued $V_\varepsilon\in L^p(\mathbb{R}^d)$ and $V_{\infty,\varepsilon}\in L^\infty(\mathbb{R}^d)$ such that $V_\varepsilon+V_{\infty,\varepsilon}=V$ and $\|V_{\infty,\varepsilon}\|_\infty<\varepsilon$. Here $p=2$ for $d\le 3$ and $p>d/2$ for $d\ge 4$. □

Theorem 7.29. *If V is a Kato class potential in $\mathbb{R}^d$, then V is compact with respect to the free Laplacian $T=-\Delta$ in $L^2(\mathbb{R}^d)$, and the essential spectrum of $-\Delta+V$ is equal to $[0,\infty)$.*

Proof. We give the proof for $d\le 3$ only. Let $\mathcal{F}$ denote the Fourier transform, then for any $f\in L^2(\mathbb{R}^d)$ and $z\in\operatorname{res}T$ we have

$$\big(\mathcal{F}(T-z)^{-1}f\big)(p)=(p^2-z)^{-1}\mathcal{F}f(p).$$

This means that $(T-z)^{-1}f=g_z\star f$, where g_z is the L^2 function with $\mathcal{F}g_z(p)=(p^2-z)^{-1}$, and $\star$ stands for the convolution product. In other words,

$$(T-z)^{-1}f=\int_{\mathbb{R}^d}g_z(x-y)f(y)dy.$$

Let $\varepsilon>0$ and let V_ε and $V_{\infty,\varepsilon}$ be as in Definition 7.28. The operator $V_\varepsilon(T-z)^{-1}$ is an integral one with the integral kernel $K(x,y)=V_\varepsilon(x)g_z(x-y)$, i.e.

$$V_\varepsilon(T-z)^{-1}f(x)=\int_{\mathbb{R}^d}K(x,y)f(y)\,dy.$$

One has

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big|K(x,y)\big|^2dxdy&=\int_{\mathbb{R}^d}\big|V_\varepsilon(x)\big|^2dx\int_{\mathbb{R}^d}\big|g_z(y)\big|^2dy\\&=\|V_\varepsilon\|_2^2\|g_z\|^2<\infty,\end{aligned}$$

which means that $V_\varepsilon(T-z)^{-1}$ is a Hilbert-Schmidt operator and, therefore, is compact, see Subsection 4.2. At the same time we have the estimate

$$\|V_{\infty,\varepsilon}(T-z)^{-1}\| \le \varepsilon\|(T-z)^{-1}\|.$$

Therefore, the operator $V(T-z)^{-1}$ is compact as it can be represented as the norm limit of the compact operators $V_\varepsilon(T-z)^{-1}$ as ε tends to 0. $\square$

Example 7.30 (Coulomb potential). The previous theorem easily applies e.g. to the operators $-\Delta + \alpha/|x|$. It is sufficient to represent

$$\frac{1}{|x|} = \frac{1_R(x)}{|x|} + \frac{1 - 1_R(x)}{|x|},$$

where 1_R is the characteristic function of the ball of radius $R > 0$ and centered at the origin with a sufficiently large R. So the essential spectrum of $-\Delta + \alpha/|x|$ is always the same as for the free Laplacian, i.e. $[0, +\infty)$.

Another typical application of the Weyl criterion can be illustrated as follows.

Theorem 7.31. *Let $V \in L^\infty(\mathbb{R}^d)$ be real-valued. Assume that there exists $\alpha \in \mathbb{R}$ such that the set $\Omega := \{x \in \mathbb{R}^d : V(x) < \alpha\}$ has a finite Lebesgue measure, then $-\Delta + V$ has a purely discrete spectrum in $(-\infty, \alpha)$.*

Proof. Let 1_Ω be the characteristic function of Ω. Denote $U := (V - \alpha)1_\Omega$ and $W := V - U$. Then $U \in L^p(\mathbb{R}^d)$ for any $p \ge 1$ (as U is bounded and supported by a set of finite measure), in particular, U is of Kato class. At the same time, $W \in L^\infty(\mathbb{R}^d)$ and $W \ge \alpha$. By Proposition 7.27, U is $(-\Delta + W)$-compact,

$$\begin{aligned} \operatorname{spec}_{\text{ess}}(-\Delta + V) \cap (-\infty, \alpha) &= \operatorname{spec}_{\text{ess}}(-\Delta + W + U) \cap (-\infty, \alpha) \\ &= \operatorname{spec}_{\text{ess}}(-\Delta + W) \cap (-\infty, \alpha). \end{aligned}$$

On the other hand, $\operatorname{spec}_{\text{ess}}(-\Delta + W) \subset \operatorname{spec}(-\Delta + W) \subset [\alpha, +\infty)$.
Note that we have no trouble with the domains, as all the operators $-\Delta + V$, $(-\Delta + W) + U$ and $-\Delta + W$ are defined on the same domain $H^2(\mathbb{R}^d)$ due to the boundedness of the potentials. $\square$

Remark 7.32. In the physics literature, the situation of Theorem 7.31 is sometimes referred to as a *potential well* below α. The same result holds without assumptions on V outside Ω (i.e. for unbounded potentials), but the proof would then require a slightly different machinery. $\square$

8 Variational principle for eigenvalues

8.1 Max-min and min-max principles

Throughout the subsection we denote by T a self-adjoint operator in an *infinite-dimensional* Hilbert space $\mathcal{H}$, and we assume that T is semibounded from below. If $\operatorname{spec}_{\mathrm{ess}} T = \emptyset$, we denote $\Sigma := +\infty$, otherwise we put $\Sigma := \inf \operatorname{spec}_{\mathrm{ess}} T$.

Theorem 8.1 (Max-min principle). *For $n \in \mathbb{N}$ introduce the following numbers:*

$$\mu_n = \mu_n(T) = \sup_{\psi_1,\dots,\psi_{n-1}\in\mathcal{H}} \inf_{\substack{\varphi\in D(T),\,\varphi\neq 0\\ \varphi\perp\psi_j,\, j=1,\dots,n-1}} \frac{\langle\varphi, T\varphi\rangle}{\langle\varphi,\varphi\rangle},$$

then we are in one and only one of the following situations:

(a) μ_n is the nth eigenvalue of T (when ordering all the eigenvalues in the non-decreasing order according to their multiplicities), and T has a purely discrete spectrum in $(-\infty, \mu_n)$.

(b) $\mu_n = \Sigma$, and $\mu_j = \mu_n$ for all $j \geq n$.

Proof. Step 1. Let us prove first two preliminary assertions:

$$\dim\operatorname{ran} E_T\big((-\infty, a)\big) < n \text{ for } a < \mu_n, \tag{8.1}$$

$$\dim\operatorname{ran} E_T\big((-\infty, a)\big) \geq n \text{ for } a > \mu_n. \tag{8.2}$$

Proof of (8.1). Assume that the assertion is false, then $\dim\operatorname{ran} E_T\big((-\infty,a)\big) \geq n$ for some $a < \mu_n$, and one can find an $n-$dimensional subspace $V \subset \operatorname{ran} E_T\big((-\infty,a)\big)$. As T is semibounded, $V \subset D(T)$. By dimension considerations, for any vectors $\psi_1, \dots, \psi_{n-1}$ there exists a non-zero vector $\varphi \in V$ orthogonal to all ψ_j, $j = 1, \dots, n-1$, and the inclusion $\varphi \in \operatorname{ran} E_T\big((-\infty,a)\big)$ implies $\langle\varphi, T\varphi\rangle \leq a\langle\varphi,\varphi\rangle$. Therefore, for any choice of ψ_j the infimum in the definition of μ_n is not greater than a, which gives the inequality $\mu_n \leq a$, which contradicts the assumption. Eq. (8.1) is proved.

Proof of (8.2). Again, assume by contradiction that the assertion is false, then for some $a > \mu_n$ we have $\dim\operatorname{ran} E_T\big((-\infty,a)\big) \leq n-1$. Let $\psi_1, \dots, \psi_{n-1}$ be some vectors spanning $\operatorname{ran} E_T\big((-\infty,a)\big)$. Due to the equality $E_T\big((-\infty,a)\big) + E_T\big([a,+\infty)\big) = \mathrm{Id}$, for every $\varphi \in D(T)$ with $\varphi \perp \psi_j$, $j = 1, \dots, n-1$, one has $\varphi = E_T\big([a,+\infty)\big)\varphi$ and $\langle\varphi, T\varphi\rangle \geq a\langle\varphi,\varphi\rangle$, which shows that $\mu_n \geq a$. This contradiction proves (8.2).

Step 2. Let us prove that $\mu_n < +\infty$ for any n (note that the equality $\mu_n > -\infty$ follows from the semiboundedness of T). Assume that $\mu_n = +\infty$, then, by (8.1), one has $\dim\operatorname{ran} E_T\big((-\infty,a)\big) < n$ for any $a \in \mathbb{R}$, and $\dim\mathcal{H} \leq n$, which contradicts to the assumption.

Now we have two possibilities: either $\dim\operatorname{ran} E_T\big((-\infty,\mu_n+\varepsilon)\big) = \infty$ for all $\varepsilon > 0$ or $\dim\operatorname{ran} E_T\big((-\infty,\mu_n+\varepsilon)\big) < \infty$ for some $\varepsilon > 0$. Let us consider them separately.

Step 3. Assume that $\dim\operatorname{ran} E_T\big((-\infty,\mu_n+\varepsilon)\big) = \infty$ for all $\varepsilon > 0$. We are going to show that the case (b) of the theorem is realized. Due to (8.1), one has

$\dim \operatorname{ran} E_T\big((\mu_n-\varepsilon,\mu_n+\varepsilon)\big)=\infty$ for all $\varepsilon>0$, and $\mu_n\in\operatorname{spec}_{\mathrm{ess}} T$. On the other hand, again by (8.1), $\operatorname{spec}_{\mathrm{ess}} T\cap(-\infty,\mu_n-\varepsilon)=\emptyset$ for all $\varepsilon>0$, which proves that $\mu_n=\Sigma$. It remains to show that $\mu_{n+1}=\mu_n$. Assume that $\mu_{n+1}>\mu_n$, then, by (8.1), for any $\varepsilon<\mu_{n+1}-\mu_n$ we have $\dim\operatorname{ran} E_T\big((-\infty,\mu_n+\varepsilon)\big)\le n+1$, which contradicts the assumption. Therefore, $\mu_{n+1}=\mu_n$.

Step 4. Assume now that $\dim\operatorname{ran} E_T\big((-\infty,\mu_n+\varepsilon)\big)<\infty$ for some $\varepsilon>0$. It follows directly that the spectrum of T is purely discrete in $(-\infty,\mu_n+\varepsilon)$. Moreover, one can find $\varepsilon_1>0$ such that $E_T\big((-\infty,\mu_n]\big)=E_T\big((-\infty,\mu_n+\varepsilon_1)\big)$. As $\dim\operatorname{ran} E_T\big((-\infty,\mu_n+\varepsilon_1)\big)\ge n$ by (8.1), we have $\dim\operatorname{ran} E_T\big((-\infty,\mu_n]\big)\ge n$, which means that T has at least n eigenvalues $\lambda_1\le\dots\le\lambda_n$ (counting with multiplicities) in $(-\infty,\mu_n]$. If $\lambda_n<\mu_n$, then $\dim\operatorname{ran} E_T\big(-\infty,\lambda_n]\big)\ge n$, which contradicts to (8.1). This proves the equality $\mu_n=\lambda_n$. □

Remark 8.2. Following the convention of Remark 2.19, one may replace the above definition of the numbers μ_n by

$$\mu_n=\sup_{\psi_1,\dots,\psi_{n-1}\in\mathcal{H}}\ \inf_{\substack{\varphi\in Q(T),\,\varphi\neq0\\ \varphi\perp\psi_j,\,j=1,\dots,n-1}}\frac{\langle\varphi,T\psi\rangle}{\langle\varphi,\varphi\rangle}.$$

This follows from the fact that $D(T)$ is dense in $Q(T)$, see Theorem 2.4 and the subsequent discussion.

Another elementary observation is given in the following corollary.

Corollary 8.3. *If there exists $\varphi\in Q(T)$ with $\langle\varphi,T\varphi\rangle<\Sigma\|\varphi\|^2$, then T has at least one eigenvalue in $(-\infty,\Sigma)$.*

Indeed, in this case one has $\mu_1<\Sigma$, which means that μ_1 is an eigenvalue. □

By similar considerations one can obtain another variational formula for the eigenvalues, one may refer to Section 4.5 in [5] for its proof:

Theorem 8.4 (Min-max principle). *All the assertions of Theorem 8.1 hold with*

$$\mu_n:=\inf_{\substack{L\subset D(T)\\ \dim L=n}}\ \sup_{\substack{\varphi\in L\\ \varphi\neq0}}\frac{\langle\varphi,T\varphi\rangle}{\langle\varphi,\varphi\rangle}=\inf_{\substack{L\subset Q(T)\\ \dim L=n}}\ \sup_{\substack{\varphi\in L\\ \varphi\neq0}}\frac{\langle\varphi,T\varphi\rangle}{\langle\varphi,\varphi\rangle}.$$

The max-min and min-max principles are powerful tools for the analysis of the behavior of the eigenvalues with respect to various parameters. As a basic example we mention the following situation, which will be applied later to some specific operators:

Definition 8.5. Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$, both semibouneded from below. We write $A\le B$ if $Q(A)\supset Q(B)$ and $\langle u,Au\rangle\le\langle u,Bu\rangle$ for all $u\in Q(B)$. □

As a direct corollary of the max-min principle we obtain:

Corollary 8.6. *Let A and B be self-adjoint, and $A\le B$. In addition, assume that A and B have compact resolvents. If $\lambda_j(A)$ and $\lambda_j(B)$, $j\in\mathbb{N}$, denote their eigenvalues taken with their multiplicities and enumerated in the non-decreasing order, then $\lambda_j(A)\le\lambda_j(B)$ for all $j\in\mathbb{N}$.*

8.2 Negative eigenvalues of Schrödinger operators

As seen above in Proposition 7.27, if V is a Kato class potential in $\mathbb{R}^d$, then the associated Schrödinger operator $T = -\Delta + V$ acting in $\mathcal{H} = L^2(\mathbb{R}^d)$ has the same essential spectrum as the free Laplacian, i.e. $\operatorname{spec}_{\text{ess}} T = [0, +\infty)$ and $\Sigma = 0$. In the present section we would like to discuss the question on the existence of negative eigenvalues.

We have rather a simple sufficient condition for the one- and two-dimensional cases.

Theorem 8.7. *Let $d \in \{1, 2\}$ and $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be real-valued such that*

$$V_0 := \int_{\mathbb{R}^d} V(x)dx < 0,$$

then the associated Schrödinger operator $T = -\Delta + V$ has at least one negative eigenvalue.

Proof. We assumed the boundedness of the potential just to avoid additional technical issues concerning the domains. It is clear that $V \in L^2(\mathbb{R}^d)$, and $\operatorname{spec}_{\text{ess}} T = [0, +\infty)$ in virtue of Theorem 7.29. By Corollary 8.3 it is now sufficient to show that one can find a non-zero $\varphi \in H^1(\mathbb{R}^d)$ with

$$\tau(\varphi) := \int_{\mathbb{R}^d} \big|\nabla\varphi(x)\big|^2 dx + \int_{\mathbb{R}^d} V(x)\big|\varphi(x)\big|^2 dx < 0.$$

Consider first the case $d = 1$. Take any $\varepsilon > 0$ and consider the function φ_ε given by $\varphi_\varepsilon(x) := e^{-\varepsilon|x|/2}$. Clearly, $\varphi_\varepsilon \in H^1(\mathbb{R})$ for any $\varepsilon > 0$, and the direct computation shows that

$$\int_{\mathbb{R}} \big|\varphi_\varepsilon'(x)\big|^2 dx = \frac{\varepsilon}{2} \quad \text{and} \quad \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^d} V(x)\big|\varphi_\varepsilon(x)\big|^2 dx = V_0 < 0.$$

Therefore, for sufficiently small ε one obtains $\tau(\varphi_\varepsilon) < 0$.

Now let $d = 2$. Take $\varepsilon > 0$ and consider $\varphi_\varepsilon(x)$ defined by $\varphi_\varepsilon(x) = e^{-|x|^\varepsilon/2}$. We have

$$\begin{aligned}
\nabla\varphi_\varepsilon(x) &= -\frac{\varepsilon x|x|^{\varepsilon-2}}{2}\, e^{-|x|^\varepsilon/2},\\
\int_{\mathbb{R}^2} \big|\nabla\varphi_\varepsilon(x)\big|^2 dx = \frac{\varepsilon^2}{4} \int_{\mathbb{R}^2} |x|^{2\varepsilon-2}\, e^{-|x|^\varepsilon}\, dx &= \frac{\pi\varepsilon^2}{2} \int_0^\infty r^{2\varepsilon-1} e^{-r^\varepsilon}\, dr\\
&= \frac{\pi\varepsilon}{2} \int_0^\infty u e^{-u}\, du = \frac{\pi\varepsilon}{2},
\end{aligned}$$

and, as previously,

$$\lim_{\varepsilon \to 0+} \int_{\mathbb{R}^d} V(x)\big|\varphi_\varepsilon(x)\big|^2 dx = V_0 < 0,$$

and for sufficiently small ε we have again $\tau(\varphi_\varepsilon) < 0$. $\square$

We see already in the above proof that finding suitable test functions for proving the existence of eigenvalues may become very tricky and depending on various parameters. One may easily check that the analog of φ_ε for $d=1$ does not work for $d=2$ and vice versa. It is a remarkable fact that the analog of Theorem 8.7 does not hold for the higher dimensions due to the Hardy inequality (Proposition 2.21):

Proposition 8.8. *Let $d\geq 3$ and let $V:\mathbb{R}^d\to\mathbb{R}$ be bounded with a compact support. For $\lambda\in\mathbb{R}$ consider the Schrödinger operators $T_\lambda:=-\Delta+\lambda V$, then there exists $\lambda_0>0$ such that* $\operatorname{spec} T_\lambda=[0,+\infty)$ *for all $\lambda\in(-\lambda_0,+\infty)$.*

Proof. Due to the compactness of $\operatorname{supp} V$ one can find $\lambda_0>0$ in such a way that

$$\lambda_0\big|V(x)\big|\leq\frac{(d-2)^2}{4|x|^2}\text{ for all } x\in\mathbb{R}^d.$$

Using the Hardy inequality, for any $u\in C_c^\infty(\mathbb{R}^d)$ and any $\lambda\in(-\lambda_0,+\infty)$ we have

$$\begin{aligned}\langle u,T_\lambda u\rangle&=\int_{\mathbb{R}^d}\big|\nabla u(x)\big|^2dx+\lambda\int_{\mathbb{R}^d}V(x)\big|u(x)\big|^2dx\\&\geq\int_{\mathbb{R}^d}\big|\nabla u(x)\big|^2dx-\lambda_0\int_{\mathbb{R}^d}\big|V(x)\big|\cdot\big|u(x)\big|^2dx\\&\geq\int_{\mathbb{R}^d}\big|\nabla u(x)\big|^2dx-\frac{(d-2)^2}{4}\int_{\mathbb{R}^d}\frac{\big|u(x)\big|^2}{|x|^2}dx\geq 0.\end{aligned}$$

As T_λ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$, see Theorem 7.10, this inequality extends to all $u\in D(T_\lambda)$, and we obtain $T_\lambda\geq 0$, and this means that $\operatorname{spec} T_\lambda\subset[0,+\infty)$. On the other hand, $\operatorname{spec}_{\text{ess}} T_\lambda=[0,+\infty)$ as λV is of Kato class (see Theorem 7.29). □

9 Laplacian eigenvalues for bounded domains

9.1 Dirichlet and Neumann eigenvalues

In this section we discuss some application of the general spectral theory to the eigenvalues of the Dirichlet and Neumann Laplacians in bounded domains. Let us recall the setting. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with a regular boundary (for example, piecewise smooth and lipschitzian); all the domains appearing in this section will be supposed to have a regular boundary without further specifications. Then the embedding of $H^1(\Omega)$ into $\mathcal{H} := L^2(\Omega)$ is a compact operator. By definition, the Dirichlet Laplacian $T_D = -\Delta_D$ and the Neumann Laplacian $T_N = -\Delta_N$ are the self-adjoint operators in $\mathcal{H}$ associated with the sesqulinear forms t_D and t_N respectively,

$$t_D(u,v) = \int_\Omega \overline{\nabla u(x)} \cdot \nabla v(x)\, dx, \quad D(t_D) = Q(T_D) = H_0^1(\Omega),$$
$$t_N(u,v) = \int_\Omega \overline{\nabla u(x)} \cdot \nabla v(x)\, dx, \quad D(t_N) = Q(T_N) = H^1(\Omega).$$

We know that both T_D and T_N have compact resolvents, and their spectra are purely discrete (see Section 4.3). Denote by $\lambda_j^{D/N} = \lambda_j^{D/N}(\Omega)$, $j \in \mathbb{N}$, the eigenvalues of $T_{D/N}$ repeated according to their multiplicities and enumerated in the non-decreasing order. The eigenvalues are clearly non-negative, and they are usually referred to as the Dirichlet/Neumann eigenvalues of the domain Ω (the presence of the Laplacian is assumed implicitly). Let us discuss some basic properties of these eigenvalues.

Proposition 9.1. *(a) $\lambda_1^N = 0$. If Ω is connected, then $\ker T_N$ is spanned by the constant function $u(x) = 1$.*

(b) $\lambda_1^D > 0$.

Proof. (a) Note that $u = 1$ is clearly an eigenfunction of T_N with the eigenvalue 0. As all the eigenvalues are non-negative, $\lambda_1^N = 0$. Now let $u \in \ker T_N$, then

$$0 = \langle u, T_N u\rangle = t_N(u,u) = \int_\Omega \big|\nabla u(x)\big|^2 dx,$$

which shows that $\nabla u = 0$. Therefore, v is constant on each maximal connected component of Ω.
(b) We have at least $\lambda_1^D \geq 0$. Assume that $\lambda_1^D = 0$ and let v be an associated eigenfunction. We have as above $\nabla v = 0$, so v must be constant on each maximal connected component of Ω. But the restriction of v to the boundary of Ω must vanish, which gives $v = 0$. □

A direct application of Corollary 8.6 based on the comparison $T_N \leq T_D$ gives

Proposition 9.2. *For any $j \in \mathbb{N}$ one has $\lambda_j^N(\Omega) \leq \lambda_j^D(\Omega)$.*

Remark 9.3. Actually a stronger version holds: If the embedding $H^1(\Omega) \to L^2(\Omega)$ is compact, then $\lambda_{j+1}^N(\Omega) \le \lambda_j^D(\Omega)$ for all j. The proof is, of course, less elementary. $\square$

Another important aspect is the dependence of the eigenvalues on the domain.

Proposition 9.4 (Monotonicity with respect to domain, Dirichlet case). *Let $\Omega \subset \widetilde{\Omega}$, then $\lambda_n^D(\widetilde{\Omega}) \le \lambda_n^D(\Omega)$ for all $n \in \mathbb{N}$.*

Proof. We observe first that if $f \in H_0^1(\Omega)$, then its extension $\widetilde{f}$ to $\widetilde{\Omega}$ by zero belongs to $H_0^1(\widetilde{\Omega})$. This allows one to write the following chain of equalities and inequalities:

$$\begin{aligned}
\lambda_n^D(\widetilde{\Omega}) = &\sup_{\psi_1,\dots,\psi_{n-1}\in L^2(\widetilde{\Omega})} \inf_{\substack{f\in H_0^1(\widetilde{\Omega}), f\neq 0\\ \langle f,\psi_j\rangle_{L^2(\widetilde{\Omega})}=0}} \frac{\|\nabla f\|^2_{L^2(\widetilde{\Omega})}}{\|f\|^2_{L^2(\widetilde{\Omega})}}\\
&\le \sup_{\psi_1,\dots,\psi_{n-1}\in L^2(\widetilde{\Omega})} \inf_{\substack{f\in H_0^1(\Omega), f\neq 0\\ \langle \widetilde{f},\psi_j\rangle_{L^2(\widetilde{\Omega})}=0}} \frac{\|\nabla \widetilde{f}\|^2_{L^2(\widetilde{\Omega})}}{\|\widetilde{f}\|^2_{L^2(\widetilde{\Omega})}} = \sup_{\psi_1,\dots,\psi_{n-1}\in L^2(\widetilde{\Omega})} \inf_{\substack{f\in H_0^1(\Omega), f\neq 0\\ \langle f,\psi_j\rangle_{L^2(\Omega)}=0}} \frac{\|\nabla f\|^2_{L^2(\Omega)}}{\|f\|^2_{L^2(\Omega)}}\\
&= \sup_{\psi_1,\dots,\psi_{n-1}\in L^2(\Omega)} \inf_{\substack{f\in H_0^1(\Omega), f\neq 0\\ \langle f,\psi_j\rangle_{L^2(\Omega)}=0}} \frac{\|\nabla f\|^2_{L^2(\Omega)}}{\|f\|^2_{L^2(\Omega)}} = \lambda_n^D(\Omega). \quad \square
\end{aligned}$$

Note that there is no easy generalization of this result to the Neumann case. The reason can be understood at a certain abstract level. As can be seen from the proof, for $\Omega \subset \widetilde{\Omega}$ there exists an obvious embedding $\tau : H_0^1(\Omega) \to H_0^1(\widetilde{\Omega})$ (extension by zero) such that $\|\tau u\| = \|u\|$ for all $u \in H_0^1(\Omega)$. If one replaces the spaces H_0^1 by H^1, then the existence of a bounded embedding and the estimates for its norm in terms of the two domains become non-trivial. We mention at least one important case where a kind of the monotonicity can be proved.

Proposition 9.5 (Neumann eigenvalues of composed domains). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain with a regular boundary, and let Ω_1 and Ω_2 be non-intersecting open subsets of Ω with regular boundaries such that $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$, then $\lambda_n^N(\Omega_1 \cup \Omega_2) \le \lambda_j^N(\Omega)$ for any $j \in \mathbb{N}$.*

Proof. Under the assumptions made, any function $f \in H^1(\Omega)$ belongs to $H^1(\Omega_1 \cup$

Ω_2), while the spaces $L^2(\Omega)$ and $L^2(\Omega_1 \cup \Omega_2)$ coincide, and we have

$$\begin{aligned}
\lambda_n^N(\Omega_1 \cup \Omega_2) &= \sup_{\psi_1,\dots,\psi_{n-1}\in L^2(\Omega_1\cup\Omega_2)} \inf_{\substack{f\in H^1(\Omega_1\cup\Omega_2), f\neq 0\\ \langle f,\psi_j\rangle_{L^2(\Omega_1\cup\Omega_2)}=0}} \frac{\|\nabla f\|^2_{L^2(\Omega_1\cup\Omega_2)}}{\|f\|^2_{L^2(\Omega_1\cup\Omega_2)}}\\
&\le \sup_{\psi_1,\dots,\psi_{n-1}\in L^2(\Omega_1\cup\Omega_2)} \inf_{\substack{f\in H^1(\Omega), f\neq 0\\ \langle f,\psi_j\rangle_{L^2(\Omega_1\cup\Omega_2)}=0}} \frac{\|\nabla f\|^2_{L^2(\Omega_1\cup\Omega_2)}}{\|f\|^2_{L^2(\Omega_1\cup\Omega_2)}}\\
&= \sup_{\psi_1,\dots,\psi_{n-1}\in L^2(\Omega)} \inf_{\substack{f\in H^1(\Omega), f\neq 0\\ \langle f,\psi_j\rangle_{L^2(\Omega)}=0}} \frac{\|\nabla f\|^2_{L^2(\Omega_)}}{\|f\|^2_{L^2(\Omega)}} = \lambda_n^N(\Omega). \quad \square
\end{aligned}$$

Remark 9.6. Under the assumptions of proposition 9.5 for any $n \in \mathbb{N}$ we have $\lambda_n^D(\Omega) \le \lambda_n^D(\Omega_1 \cup \Omega_2)$, which follows from the inclusion $\Omega_1 \cup \Omega_2 \subset \Omega$. Therefore, for any $n \in \mathbb{N}$ one has the chain

$$\lambda^N(\Omega_1 \cup \Omega_2) \le \lambda^N(\Omega) \le \lambda^D(\Omega) \le \lambda_n^D(\Omega_1 \cup \Omega_2),$$

and this is the key argument of the so-called Dirichlet-Neumann bracketing which is used e.g. for estimating the asymptotic behavior of the eigenvalues (see below). $\square$

We complete the discussion by proving the continuity of the Dirichlet eigenvalues with respect to domain.

Proposition 9.7 (Continuity with respect to domain, Dirichlet). *If $\Omega_j \subset \Omega_{j+1}$ for all $j \in \mathbb{N}$, and $\Omega = \bigcup_{j=1}^\infty \Omega_j$, then $\lambda_n^D(\Omega) = \lim_{j\to\infty} \lambda_n^D(\Omega_j)$ for any $n \in \mathbb{N}$.*

Proof. Let us pick $n \in \mathbb{N}$, and let $f_1, \dots f_n$ be the mutually orthogonal normalized eigenfunctions associated with the eigenvalues $\lambda_1^D(\Omega), \dots, \lambda_n^D(\Omega)$. If U denotes the subspace spanned by $f_1, \dots, f_n$, then for any $f \in U$ one has the estimate $\|\nabla f\|^2 \le \lambda_n^D(\Omega)\|f\|^2$.

Now take an arbitrary $\varepsilon > 0$. Using the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$ one can approximate every f_j by $u_j \in C_c^\infty(\Omega)$ in such a way that $u_1, \dots, u_n$ will be linearly independent and that $\|\nabla u\|^2_{L^2(\Omega)} \le \big(\lambda_n^D(\Omega)+\varepsilon\big)\|u\|^2_{L^2(\Omega)}$ for all u from the n-dimensional subspace V spanned by $u_1, \dots, u_n$. Let $K \subset \Omega$ be a compact subset containing the supports of all u_j and, as a consequence, the supports of all functions from V. One can find $M \in \mathbb{N}$ such that $K \subset \Omega_m$ for all $m \ge M$, and then for all $m \ge M$ we have $V \subset H_0^1(\Omega_m)$. Now let $\psi_1, \dots, \psi_{n-1}$ be arbitrary functions from $L^2(\Omega_m)$. As V is n-dimensional, there exists a non-zero $v \in V$ which is orthogonal to all ψ_j. This means that

$$\inf_{\substack{u\in H_0^1(\Omega_m), u\neq 0\\ u\perp\psi_1,\dots\psi_{n-1}}} \frac{\|\nabla u\|^2_{L^2(\Omega_m)}}{\|u\|^2_{L^2(\Omega_m)}} \le \frac{\|\nabla v\|^2_{L^2(\Omega_m)}}{\|v\|^2_{L^2(\Omega_m)}} = \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}} \le \lambda_n^D(\Omega) + \varepsilon.$$

Due to the arbitrariness of ψ_j we have $\lambda_n^D(\Omega_m) \le \lambda_n^D(\Omega) + \varepsilon$ for all $m \ge M$. On the other hand, $\lambda_n^D(\Omega) \le \lambda_n^D(\Omega_m)$ by monotonicity. $\square$

9.2 Weyl asymptotics

In this subsection we will discuss some aspects of the asymptotic behavior of the Laplacian eigenvalues. We introduce the Dirichlet/Neumann counting functions $N_{D/N}(\lambda,\Omega)$ by

$$N_{D/N}(\lambda,\Omega) = \text{the number of } j\in\mathbb{N} \text{ for which } \lambda_j^{D/N}(\Omega)\in(-\infty,\lambda].$$

Clearly, $N_{D/N}(\lambda,\Omega)$ is finite for any λ, and it has a jump at each eigenvalue; the jump is equal to the multiplicity. We emphasize the following obvious properties:

$$N_D(\lambda,\Omega)\le N_N(\lambda,\Omega) \tag{9.1}$$

$$N^{D/N}(\lambda,\Omega_1\cup\Omega_2) = N^{D/N}(\lambda,\Omega_1)+N^{D/N}(\lambda,\Omega_2) \text{ for } \Omega_1\cap\Omega=\emptyset. \tag{9.2}$$

$$N_D(\lambda,\Omega)\le N_D(\widetilde{\Omega}) \text{ for } \Omega\subset\widetilde{\Omega}. \tag{9.3}$$

We are going to discuss the following rather general result on the behavior of the counting functions N_D $\lambda\to+\infty$:

Theorem 9.8 (Weyl asymptotics). *We have*

$$\lim_{\lambda\to+\infty}\frac{N_D(\lambda,\Omega)}{\lambda^{d/2}} = \frac{\omega_d}{(2\pi)^d}\,\mathrm{vol}(\Omega),$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

To keep simple notation we proceed with the proof for the case $d=2$ only. Due to $\omega_2=\pi$ we are reduced to prove

$$\lim_{\lambda\to+\infty}\frac{N_D(\lambda,\Omega)}{\lambda} = \frac{\mathrm{area}(\Omega)}{4\pi}. \tag{9.4}$$

The proof consists of several steps.

Lemma 9.9. *The Weyl asymptotics is valid for rectangles, for both N_N and N_D.*

Proof. Let $\Omega=(0,a)\times(0,b)$, $a,b>0$. As shown in Example 6.16, the Neumann eigenvalues of Ω are the numbers

$$\lambda(m,n) := \Big(\frac{\pi m}{a}\Big)^2+\Big(\frac{\pi n}{b}\Big)^2$$

with $m,n\in\mathbb{N}_0:=\{0\}\cup\mathbb{N}$, and the Dirichlet spectrum consists of the eigenvalues $\lambda(m,n)$ with $m,n\in\mathbb{N}$. Denote

$$D(\lambda) := \Big\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2}+\frac{y^2}{b^2}\le\frac{\lambda}{\pi^2},\ x\ge 0, y\ge 0\Big\},$$

then $N_D(\lambda,\Omega) = \#D(\lambda)\cap\big(\mathbb{N}\times\mathbb{N}\big)$ and $N_N(\lambda,\Omega)=\#D(\lambda)\cap\big(\mathbb{N}_0\times\mathbb{N}_0\big)$, where $\#$ denotes the cardinality.

First, counting the points $(n,0)$ and $(0,n)$ with $n \in \mathbb{N}_0$ inside $D(\lambda)$ we obtain the majoration

$$N_N(\lambda) - N_D(\lambda) \le \big(\frac{a+b}{\pi} + 2\big)\sqrt{\lambda}, \quad \lambda > 0.$$

At the same time, $D(\lambda)$ contains the union of the unit cubes $[m-1, m] \times [n-1, n]$ with $(m,n) \in D(\lambda) \cap \big(\mathbb{N} \times \mathbb{N}\big)$. As there are exactly $N_D(\lambda, \Omega)$ such cubes, we have

$$N_D(\lambda, \Omega) \le \operatorname{area} D(\lambda) = \frac{\lambda ab}{4\pi}.$$

We also observe that $D(\lambda)$ is contained in the union of the unit cubes $[m, m+1] \times [n, n+1]$ with $(m,n) \in D(\lambda) \cap \big(\mathbb{N}_0 \times \mathbb{N}_0\big)$. As the number of such cubes is exactly $N_N(\lambda, \Omega)$, this gives

$$N_N(\lambda, \Omega) \ge \operatorname{area} D(\lambda) = \frac{\lambda ab}{4\pi}.$$

Putting all together we arrive at

$$\frac{\lambda ab}{4\pi} \le N_N(\lambda, \Omega) \le N^D(\lambda, \Omega) + \big(\frac{a+b}{\pi} + 2\big)\sqrt{\lambda} \le \frac{\lambda ab}{4\pi} + \big(\frac{a+b}{\pi} + 2\big)\sqrt{\lambda},$$

and it remains to recall that $\operatorname{area}(\Omega) = ab$. □

Definition 9.10 (Domains composed from rectangles). We say that a domain Ω with a regular boundary is composed from rectangles if there exists a finite family of non-intersecting open rectangles Ω_j, $j = 1, \dots, k$, with $\overline{\Omega} = \overline{\bigcup_{j=1}^k \Omega_j}$. □

Lemma 9.11. *The Weyl asymptotics holds for domains composed from rectangles.*

Proof. Let Ω be a domain composed from rectangles, ant let Ω_j, $j = 1, \dots, k$, be the rectangles as in Definition 9.10. Using Remark 9.6 and the equality (9.2) we obtain the chain

$$\begin{aligned}\frac{N_D(\lambda, \Omega_1) + \dots + N_N(\lambda, \Omega_k)}{\lambda} &= \frac{N_D(\lambda, \Omega_1 \cup \dots \cup \Omega_k)}{\lambda} \le \frac{N_D(\lambda, \Omega)}{\lambda} \\ &\le \frac{N_N(\lambda, \Omega)}{\lambda} \le \frac{N_N(\lambda, \Omega_1 \cup \dots \cup \Omega_k)}{\lambda} = \frac{N_N(\lambda, \Omega_1) + \dots + N_D(\lambda, \Omega_k)}{\lambda},\end{aligned}$$

and the result is obtained by applying Lemma 9.9 to the quotients $N_{D/N}(\lambda, \Omega_j)/\lambda$ and by noting that $\operatorname{area}(\Omega) = \operatorname{area}(\Omega_1) + \dots + \operatorname{area}(\Omega_k)$. □

Proof of Theorem 9.8. Let Ω be a domain with a regular boundary. It is a standard result of the analysis that for any $\varepsilon > 0$ one can find two domains Ω_ε and $\widetilde{\Omega}_\varepsilon$ such that:

- both Ω_ε and $\widetilde{\Omega}_\varepsilon$ are composed from rectangles,
- $\Omega_\varepsilon \subset \Omega \subset \widetilde{\Omega}_\varepsilon$,
- $\operatorname{area}(\widetilde{\Omega}_\varepsilon \setminus \Omega_\varepsilon) < \varepsilon$.

Using (9.1) and the monotonicity of the Dirichlet eigenvalues with respect to domain we have:

$$\frac{N_D(\lambda, \Omega_\varepsilon)}{\lambda} \le \frac{N_D(\lambda, \Omega)}{\lambda} \le \frac{N_D(\lambda, \widetilde{\Omega}_\varepsilon)}{\lambda} \le \frac{N_N(\lambda, \widetilde{\Omega}_\varepsilon)}{\lambda}.$$

By Lemma 9.11, we can find $\lambda_\varepsilon > 0$ such that

$$\frac{\operatorname{area}(\Omega_\varepsilon) - \varepsilon}{4\pi} \le \frac{N_D(\lambda, \Omega)}{\lambda} \le \frac{\operatorname{area}(\widetilde{\Omega}_\varepsilon) + \varepsilon}{4\pi} \quad \text{for } \lambda > \lambda_\varepsilon.$$

At the same time, $\operatorname{area}(\Omega_\varepsilon) \ge \operatorname{area}(\Omega) - \varepsilon$ and $\operatorname{area}(\widetilde{\Omega}_\varepsilon) \le \operatorname{area}(\Omega) + \varepsilon$, so for $\lambda > \lambda_\varepsilon$ we have

$$\frac{\operatorname{area}(\Omega) - 2\varepsilon}{4\pi} \le \frac{N_D(\lambda, \Omega)}{\lambda} \le \frac{\operatorname{area}(\Omega) + 2\varepsilon}{4\pi},$$

which gives the sought result. □

We note that the Weyl asymptotics also holds for the Neumann Laplacian if the domain is sufficiently smooth, which can be proved using suitable extension theorem for Sobolev spaces. The Weyl asymptotics is one of the basic results on the relations between the Dirichlet/Neumann eigenvalues and the geometric properties of the domain. It states, in particular, that the spectrum of the domain contains the information on its dimension and its volume. There are various refinements involving lower order terms with respect to λ, and the respective coefficients contains some information on the topology of the domain, on its boundary etc.

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