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Quantum Variance and Ergodicity for the Baker's Map

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Abstract: We prove an Egorov theorem, or quantum-classical correspondence, for the quantised baker's map, valid up to the Ehrenfest time. This yields a logarithmic upper bound for the decay of the quantum variance, and, as a corollary, a quantum ergodic theorem for this map.

1. Introduction

The correspondence principle of quantum mechanics suggests that in the classical limit the behaviour of quantum systems reproduces that of the system's classical dynamics. It is becoming clear that to understand this process fully represents a challenge not only to methods of semiclassical analysis, but also the modern theory of dynamical systems.

For a broad class of smooth Hamiltonian systems it has been proved that if the system is ergodic, then, in the classical limit, almost all eigenfunctions of the corresponding quantum mechanical Hamiltonian operator become equidistributed with respect to the *natural* measure (Liouville) over the energy shell. This is the content of the so-called *quantum ergodicity theorem* [Šni, Zel1, CdV, HMR].

This mathematical result, even if it can be considered quite mild from the physical point of view, still constitutes one of the few rigorous results concerning the properties of quantum eigenfunctions in the classical limit, and it still leaves open the possible existence of *exceptional* subsequences of eigenstates which might converge to other invariant measures. In the last few years a certain number of works have explored this mathematically and physically interesting issue. While exceptional subsequences can be present for some hyperbolic systems with extremely high quantum degeneracies [FDBN], it is believed that they do not exist for a *typical* chaotic system (by chaotic, we generally mean that the system is ergodic and mixing). The uniqueness of the classical limit for the

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quantum diagonal matrix elements is called *quantum unique ergodicity* (QUE) [RudSar, Sar1]. There have been interesting recent results in this direction for Hecke eigenstates of the Laplacian on compact arithmetic surfaces [Lin], using methods which combine rigidity properties of semi-classical measures with purely dynamical systems theory.

The model studied in the present paper is not a Hamiltonian flow, but rather a discrete-time symplectic map on the 2-dimensional torus phase space. In the case of quantised hyperbolic automorphisms of the 2-torus ("quantum cat maps"), QUE has been proven along a subsequence of Planck's constants [DEGI, KR2], and for a certain class of eigenstates (also called "Hecke" eigenstates) [KR1] without restricting Planck's constant. QUE has also been proved in the case of some uniquely ergodic maps [MR, Ros]. Quantum (possibly non-unique) ergodicity has been shown for some ergodic maps which are smooth by parts, with discontinuities on a set of zero Lebesgue measure [DBDE, MO'K, DE$^+$]. Discontinuities generally produce diffraction effects at the quantum level, which need to be taken care of (this problem also appears in the case of Euclidean billiards with non-smooth boundaries [GL, ZZ]). Most proofs of quantum ergodicity consist of showing that the quantum variance defined below (Eq. (1.1)) vanishes in the classical limit.

To state our results we now turn to the specific dynamics considered in the present article. We take as classical dynamical system the baker's map on $\mathbb{T}^2$, the 2-dimensional torus [AA]. For any even positive integer $N \in 2\mathbb{N}$ (N is the inverse of Planck's constant h), this map can be quantised into a unitary operator (propagator) $\hat{B}_N$ acting on an N-dimensional Hilbert space. The *quantum variance* measures the average equidistribution of the eigenfunctions $\{\varphi_{N,j}\}_{j=0}^{N-1}$ of $\hat{B}_N$:

$$S_2(a, N) := \frac{1}{N} \sum_{j=0}^{N-1} \left| \langle \varphi_{N,j}, \mathrm{Op}_N^{\mathrm{W}}(a)\varphi_{N,j} \rangle - \int_{\mathbb{T}^2} a(q,p)\, \mathrm{d}q\mathrm{d}p \right|^2. \tag{1.1}$$

Here a is some smooth function (observable) on $\mathbb{T}^2$ and $\mathrm{Op}_N^{\mathrm{W}}(\cdot)$ is the Weyl quantisation mapping a classical observable to a corresponding quantum operator. The quantised baker's map (or some variant of it) is a well-studied example in the physics literature on quantum chaology [BV, Sa, SaVo, O'CTH, Lak, Kap, ALP$\dot{Z}$], which motivated our desire to provide rigorous proofs for both the quantum-classical correspondence and quantum ergodicity.

In this paper we prove a logarithmic upper bound on the decay of the quantum variance (see Theorem 1 below), which implies quantum ergodicity as a byproduct (Corollary 2). A similar upper bound was first obtained by Zelditch [Zel2] in the case of the geodesic flow of a compact negatively curved Riemannian manifold, and was generalized by Robert [Rob] to more general ergodic Hamiltonian systems. Both are using some control on the rate of classical ergodicity (Zelditch also proved similar upper bounds for higher moments of the matrix elements). The main semiclassical ingredient needed for all proofs of quantum ergodicity is some control on the correspondence between quantum and classical evolutions of observables, namely some *Egorov estimate*. As for billiard flows [Fa], such a correspondence can only hold for observables supported away from the set of discontinuities. We establish this correspondence for the quantum baker's map in Sect. 5.2, generalizing previous results [DBDE] for a subclass of observables (an Egorov theorem was already proven in [RubSal] for a different quantisation of the baker's map). Some related results can be found in [BGP, BR] for the case of smooth Hamiltonian systems. To obtain this Egorov estimate, we study the propagation of coherent states (Gaussian wavepackets): they provide a convenient way to "avoid" the set of discontinuities. The correspondence will hold up to times of the order of the *Ehrenfest time*

$$T_{\mathrm{E}}(N) := \frac{\log N}{\log 2} \tag{1.2}$$

(here log 2 is the positive Lyapunov exponent of the classical baker's map).

Equipped with this estimate, one could apply the general results of [MO'K] to prove that the quantum variance semiclassically vanishes. We prefer to generalise the method of [Schu2] (applied to smooth maps or flows) to our discontinuous baker's map. This method, inspired by some earlier heuristic calculations [FP, Wil, EFK$^+$], yields a logarithmic upper bound for the variance. It relies on the decay of classical correlations (mixing property), which is related, yet not equivalent, with the control on the rate of ergodicity used in [Zel2, Rob].

Our main result is the following theorem.

Theorem 1. *For any observable $a \in C^\infty(\mathbb{T}^2)$, there is a constant $C(a)$ depending only on a, such that the quantum variance over the eigenstates of $\hat{B}_N$ satisfies:*

$$\forall N \in 2\mathbb{N}, \qquad S_2(a, N) \leq \frac{C(a)}{\log N}.$$

We believe that this method can be extended to any piecewise linear map satisfying a fast mixing. We also can speculate that the method would work for non-linear piecewise-smooth maps, although in that case the propagation of coherent states should be analysed in more detail (see Remark 2).

The upper bound in Theorem 1 seems far from being sharp. The heuristic calculations in [FP, Wil, EFK$^+$] suggest that the quantum variance decays like $V(a)\, N^{-1}$, where the prefactor $V(a)$ is the *classical variance* of the observable a, appearing in the central limit theorem. This has been conjectured to be the true decay rate for a "generic" Anosov system. The decay of quantum variance has been studied numerically in [EFK$^+$] for the baker's map and [BSS] for Euclidean billiards; in both cases, the results seem to be compatible with a decay $\asymp N^{-1}$; however, a discrepancy of around 10% was noted between the observed and conjectured prefactors. This was attributed to the low values of N (or energy in the case of billiards) considered. A more recent numerical study of a chaotic billiard, at higher energies, still reveals some (smaller) deviations from the conjectured law [Bar], leaving open the possibility of a decay $\asymp N^{-\gamma}$ with $\gamma \neq 1$.

A decay of the form $\tilde{V}(a)\, N^{-1}$ (with an explicit factor $\tilde{V}(a)$) could be rigorously proven for two particular Anosov systems, using their rich arithmetic structure [KR1, LS, RuSo]. In both cases, the prefactor $\tilde{V}(a)$ generally differs from the classical variance $V(a)$, which is attributed to the arithmetic properties of the systems, which potentially makes them "non-generic". Algebraic decays have also been proven for some uniquely ergodic (non-hyperbolic) maps [MR, Ros], by pushing the Egorov property to times of order $\mathcal{O}(N)$.

The rigorous investigation of the quantum variance thus remains an important open problem in quantum chaology [Sar2].

Quantum ergodicity follows from Theorem 1 as a corollary:

Corollary 2. *For each $N \in 2\mathbb{N}$ there exists a subset $J_N \subset \{1, \ldots, N\}$, with $\frac{\# J_N}{N} \xrightarrow{N\to\infty} 1$, such that for any $a \in C^\infty(\mathbb{T}^2)$ and any sequence $(j_N \in J_N)_{N\in 2\mathbb{N}}$,*

$$\lim_{N\to\infty} \langle \varphi_{N,j_N}, \mathrm{Op}_N^{\mathrm{W}}(a)\varphi_{N,j_N}\rangle = \int_{\mathbb{T}^2} a(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{1.3}$$

This generalises a result of [DBDE] to any observable $a \in C^\infty(\mathbb{T}^2)$ (previously only observables of the form $a = a(q)$ could be handled). The restriction to a subset J_N is the "almost all" clarification in quantum ergodicity.

The paper is organised as follows. In Sect. 2 we briefly describe the classical baker's map on $\mathbb{T}^2$. In Sect. 3, we recall how this map can be quantised [BV] into an $N \times N$ unitary matrix. We then describe the action of the quantised baker map on coherent states (Proposition 5). This is the first step towards the Egorov estimates proven in Sect. 5 (Theorems 12 and 13, which shows the correspondence up to the Ehrenfest time). The first part of that section (Subsect. 5.1) compares the Weyl and anti-Wick quantisations, for observables which become more singular when N grows. This technical step is necessary to obtain Egorov estimates for times $\asymp \log N$. In the final section, we implement the method of [Schu2] to the quantum baker's map, using our Egorov estimates up to logarithmic times, and prove Theorem 1.

2. The Classical Baker's Map

The baker's map[1] is the prototype model for discontinuous hyperbolic systems, and it has been extensively studied in the literature. Standard results may be found in [AA], while the exponential mixing property was analyzed by [Has], and also derives from the results of [Ch]. Here, for the sake of fixing notations, we restrict ourself to recalling the very basic definitions and properties, referring the reader to the above references for more details concerning the ergodic properties of the map.

We identify the torus $\mathbb{T}^2$ with the square $[0, 1) \times [0, 1)$. The first (horizontal) coordinate q represents the "position", while the second (vertical) represents the "momentum". In our notations, $\boldsymbol{x} = (q, p)$ will always represent a phase space point, either on $\mathbb{R}^2$ or on its quotient $\mathbb{T}^2$.

The baker's map is defined as the following piecewise linear bijective transformation on $\mathbb{T}^2$:

$$B(q, p) = (q', p') = \begin{cases} (2q, p/2), & \text{if } q \in [0, 1/2), \\ (2q - 1, (p + 1)/2), & \text{if } q \in [1/2, 1). \end{cases} \tag{2.1}$$

The transformation is discontinuous on the following subset of $\mathbb{T}^2$:

$$\mathcal{S}_1 := \{p = 0\} \cup \{q = 0\} \cup \{q = 1/2\}, \tag{2.2}$$

and smooth everywhere else. If we consider iterates of the map, the discontinuity set becomes larger: for any $n \in \mathbb{N}$, the map B^n is piecewise linear, and discontinuous on the set

$$\mathcal{S}_n := \{p = 0\} \cup \bigcup_{j=0}^{2^n - 1} \left\{ q = \frac{j}{2^n} \right\},$$

while its inverse B^{-n} is discontinuous on the set $\mathcal{S}_{-n}$ obtained from $\mathcal{S}_n$ by exchanging the q and p coordinates. Clearly, the discontinuity set becomes dense in $\mathbb{T}^2$ as $|n| \to \infty$. The map is area preserving and uniformly hyperbolic outside the discontinuity set, with constant Lyapunov exponents $\pm \log 2$ and positive topological entropy (see below). The stable (resp. unstable) manifold is made of vertical (resp. horizontal) segments.

A nice feature of this map lies in a simple symbolic coding for its orbits. Each real number $q \in [0, 1)$ can be associated with a binary expansion

[1] The name refers to the cutting and stretching mechanism in the dynamics of the map which is reminiscent of the procedure for making bread. Hence we write the word "baker" with a lower case "b".

$$q = \cdot\,\epsilon_0\epsilon_1\epsilon_2\ldots \qquad (\epsilon_i \in \{0,1\}).$$

This representation is one-to-one if we forbid expansions of the form $\cdot\epsilon_0\epsilon_1\ldots 111\ldots$. Using the same representation for the p-coordinate:

$$p = \cdot\,\epsilon_{-1}\epsilon_{-2}\ldots,$$

a point $\boldsymbol{x} = (q,p) \in \mathbb{T}^2$ can be represented by the doubly-infinite sequence

$$\boldsymbol{x} = \ldots\epsilon_{-2}\epsilon_{-1}\cdot\epsilon_0\epsilon_1\ldots.$$

Then, one can easily check that the baker's map acts on this representation as a symbolic shift:

$$B(\ldots\epsilon_{-2}\epsilon_{-1}\cdot\epsilon_0\epsilon_1\ldots) = \ldots\epsilon_{-2}\epsilon_{-1}\epsilon_0\cdot\epsilon_1\ldots. \tag{2.3}$$

From this symbolic representation, one gets the Kolmogorov-Sinai entropy of the map with respect to the Lebesgue measure, $h_{\mathrm{KS}} = \log 2$, as well as exponential mixing properties [Ch, Has]: there exists $\Gamma > 0$ and $C > 0$ such that, for any smooth observables $a,\ b$ on $\mathbb{T}^2$, the correlation function

$$\mathcal{K}_{ab}(n) := \int_{\mathbb{T}^2} a(\boldsymbol{x})\, b(B^{-n}\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_{\mathbb{T}^2} a(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\int_{\mathbb{T}^2} b(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \tag{2.4}$$

is bounded as

$$|\mathcal{K}_{ab}(n)| \le C\,\|a\|_{C^1}\,\|b\|_{C^1}\,\mathrm{e}^{-\Gamma|n|}. \tag{2.5}$$

According to [Has], one can take for Γ any number smaller than $\log 2$.

3. Quantised Baker's Map

The quantisation of the 2-torus phase space is now well-known and we refer the reader to [DEG], here describing only the important facts. The quantisation of an area-preserving map on the torus is less straightforward, and in general it contains some arbitrariness. The quantisation of linear symplectomorphisms of the 2-torus (or "generalised Arnold cat maps") was first considered in [HB], and the case of nonlinear perturbations of cat maps was treated in [BdM+] (quantum ergodicity was proven for these maps in [BDB]). The scheme we present below, specific for the baker's map, was introduced in [BV].

We start by defining the quantum Hilbert space associated to the torus phase space. For any $\hbar \in (0,1]$, we consider the quantum translations (elements of the Heisenberg group) $\hat{T}_{\boldsymbol{v}} = \mathrm{e}^{i(v_2\hat{q}-v_1\hat{p})/\hbar}$, $\boldsymbol{v} \in \mathbb{R}^2$, acting on $L^2(\mathbb{R})$ and by extension on $\mathcal{S}'(\mathbb{R})$. We then define the space of distributions

$$\mathcal{H}_\hbar = \{\psi \in \mathcal{S}'(\mathbb{R}),\ \hat{T}_{(1,0)}\psi = \hat{T}_{(0,1)}\psi = \psi\}.$$

These are distributions $\psi(q)$ which are $\mathbb{Z}$-periodic, and such that their $\hbar$-Fourier transform

$$(\hat{F}_\hbar\psi)(p) := \int_{-\infty}^{\infty} \psi(q)\,\mathrm{e}^{-iqp/\hbar}\,\frac{\mathrm{d}q}{\sqrt{2\pi\hbar}} \tag{3.1}$$

is also $\mathbb{Z}$-periodic.

One easily shows that this space is nontrivial iff $(2\pi\hbar)^{-1} = N \in \mathbb{N}$, which we will always assume from now on. This space can be obtained as the image of $L^2(\mathbb{R})$ through the "projector"

$$\hat{P}_{\mathbb{T}^2} = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} (-1)^{Nm_1m_2}\, \hat{T}_{\boldsymbol{m}} = \Big(\sum_{m_2\in\mathbb{Z}} \hat{T}_{0,m_2}\Big)\Big(\sum_{m_1\in\mathbb{Z}} \hat{T}_{m_1,0}\Big). \tag{3.2}$$

$\mathcal{H}_\hbar = \mathcal{H}_N$ then forms an N-dimensional vector space of distributions, admitting a "position representation"

$$\psi(q) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \sum_{\nu\in\mathbb{Z}} \psi_j\, \delta\Big(q - \frac{j}{N} + \nu\Big) =: \sum_{j=0}^{N-1} \psi_j\, \mathbf{q}_j(q), \tag{3.3}$$

where each coefficient $\psi_j \in \mathbb{C}$. Here we have denoted by $\{\mathbf{q}_j\}_{j=0}^{N-1}$ the canonical ("position") basis for $\mathcal{H}_N$.

This space can be naturally equipped with the Hermitian inner product:

$$\langle \mathbf{q}_j, \mathbf{q}_k\rangle = \delta_{jk} \Longrightarrow \langle\psi,\omega\rangle := \sum_{j=0}^{N-1} \overline{\psi_j}\, \omega_j. \tag{3.4}$$

Since $\mathcal{H}_N$ is the image of $\mathcal{S}(\mathbb{R})$ through the "projector" (3.2), any state $\psi \in \mathcal{H}_N$ can be constructed by projecting some Schwartz function $\Psi(q)$. The decomposition on the RHS of (3.2) suggests that we may first periodicise in the q-direction, obtaining a periodic function $\Psi_{\mathcal{C}}(q)$; such a wavefunction describes a state living in the *cylinder* phase space $\mathcal{C} = \mathbb{T}\times\mathbb{R}$. The torus state $\psi(q)$ is finally obtained by periodicising $\Psi_{\mathcal{C}}$ in the Fourier variable; equivalently, the N components of ψ in the basis $\{\mathbf{q}_j\}$ are obtained by sampling this function at the points $q_j = \frac{j}{N}$:

$$\psi_j = \frac{1}{\sqrt{N}}\, \Psi_{\mathcal{C}}\Big(\frac{j}{N}\Big), \qquad 0 \le j < N. \tag{3.5}$$

The $\hbar$-Fourier transform $\hat{F}_\hbar$ (seen as a linear operator on $\mathcal{S}'(\mathbb{R})$) leaves the space $\mathcal{H}_N$ invariant. On the basis $\{\mathbf{q}_j\}$, it acts as an $N\times N$ unitary matrix $\hat{F}_N$ called the "discrete Fourier transform":

$$(\hat{F}_N)_{kj} = \frac{1}{\sqrt{N}}\, \mathrm{e}^{-2i\pi kj/N}, \quad k, j = 0,\dots,N-1. \tag{3.6}$$

$\hat{F}_\hbar$ quantises the rotation by $-\pi/2$ around the origin, $F(q_0,p_0) = (p_0,-q_0)$. As a result, $\hat{F}_N$ maps the "position basis" $\{\mathbf{q}_j\}$ onto the "momentum basis" $\{\mathbf{p}_j\}$:

$$\mathbf{p}_j = \sum_{k=0}^{N-1} (\hat{F}_N^{-1})_{kj}\, \mathbf{q}_k.$$

The quantised baker's map $\hat{B}_N$ was introduced by Balazs and Voros [BV]. They require N to be an *even* integer, and prescribe the following matrix in the basis $\{\mathbf{q}_j\}$:

$$\hat{B}_N := (\hat{F}_N)^{-1}\hat{B}_{N,\mathrm{mix}}, \qquad \text{with} \qquad \hat{B}_{N,\mathrm{mix}} := \begin{pmatrix} \hat{F}_{N/2} & 0 \\ 0 & \hat{F}_{N/2} \end{pmatrix}. \tag{3.7}$$

This definition was slightly modified by Saraceno [Sa], in order to restore the parity symmetry of the classical map. Although we will concentrate on the map (3.7), all our results also apply to this modified setting.

3.1. Notations. Since we will be dealing with quantities depending on Planck's constant N (plus possibly other parameters), all asymptotic notations will refer to the classical limit $N \to \infty$.

The notations $A = \mathcal{O}(B)$ and $A \ll B$ both mean that there exists a constant c such that for any $N \geq 1$, $|A(N)| \leq c|B(N)|$. Writing $A = \mathcal{O}_r(B)$ and $A \ll_r B$ means that the constant c depends on the parameter r. Similarly $A = \mathrm{o}(B)$ and $A << B$ both mean that $\lim_{N\to\infty} \frac{A(N)}{B(N)} = 0$. By $A \asymp B$ we mean that $A \ll B$ and $B \ll A$ simultaneously. We indicate by $A \sim B$ the more precise asymptotics $\lim_{N\to\infty} \frac{A(N)}{B(N)} = 1$.

We use the convention for number sets that $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Also $\mathbb{R}_+ := [0, \infty)$, as usual.

We will use various norms. We denote by $\|\cdot\|_{\mathcal{H}_N}$ the norm on $\mathcal{H}_N$ defined as $\|\psi\|^2_{\mathcal{H}_N} = \langle\psi, \psi\rangle$. Unless stated otherwise, $\|\cdot\|$ will refer to the norm on bounded operators on $\mathcal{H}_N$, also denoted by $\|\cdot\|_{\mathcal{B}(\mathcal{H}_N)}$. The Hilbert-Schmidt scalar product of two operators A, B on $\mathcal{H}_N$ will be denoted by

$$\Big\langle A, B\Big\rangle := \frac{1}{N}\,\mathrm{Tr}(A^\dagger B). \tag{3.8}$$

Other norms describe classical observables (smooth functions f on $\mathbb{T}^2$). The sup-norm will be denoted by $\|f\|_{C^0}$, and for any $j > 0$, the C^j-norm is defined as

$$\|f\|_{C^j} := \sum_{0\leq|\boldsymbol{\gamma}|\leq j} \|\partial^{\boldsymbol{\gamma}} f\|_{C^0}.$$

Here $\boldsymbol{\gamma} = (\gamma_1, \gamma_2) \in \mathbb{N}_0^2$ denotes the multiindex of differentiation: $\partial^{\boldsymbol{\gamma}} = \partial_q^{\gamma_1}\,\partial_p^{\gamma_2}$, and $|\boldsymbol{\gamma}| := \gamma_1 + \gamma_2$.

Because we want to consider large time evolution, namely times $n \asymp \log N$, we need to consider (smooth) functions which depend on Planck's constant $1/N$. Indeed, starting from a given smooth function a, its evolution $a \circ B^{-n}$ fluctuates more and more strongly along the vertical direction, while it is smoother and smoother along the horizontal one as $n \to \infty$ (assuming a is supported away from the discontinuity set $\mathcal{S}_n$). For this reason, we introduce the following spaces of functions [DS, Chap. 7]:

Definition 1. *For any $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}_+^2$, we call $S_{\boldsymbol{\alpha}}(\mathbb{T}^2)$ the space of N-dependent smooth functions $f = f(\cdot, N)$ such that, for any multiindex $\boldsymbol{\gamma} \in \mathbb{N}_0^2$, the quantity*

$$C_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(f) := \sup_{N\in\mathbb{N}} \frac{\|\partial^{\boldsymbol{\gamma}} f(\cdot, N)\|_{C^0}}{N^{\boldsymbol{\alpha}\cdot\boldsymbol{\gamma}}}$$

is finite (here $\boldsymbol{\alpha} \cdot \boldsymbol{\gamma} = \alpha_1\gamma_1 + \alpha_2\gamma_2$). The seminorms $C_{\boldsymbol{\alpha},\boldsymbol{\gamma}}$ ($\boldsymbol{\gamma} \in \mathbb{N}_0^2$) endow $S_{\boldsymbol{\alpha}}(\mathbb{T}^2)$ with the structure of a Fréchet space.

4. Coherent States on $\mathbb{T}^2$

Our proof of the quantum-classical correspondence will use coherent states on $\mathbb{T}^2$. Below we define them, and collect some useful properties. More comprehensive details and proofs may be found in [Fo, Per, LV, BDB, BonDB].

We define a (plane) coherent state at the origin with squeezing $\sigma > 0$ through its wavefunction $\Psi_{0,\sigma} \in \mathcal{S}(\mathbb{R})$ (we will always omit the indication of $\hbar$-dependence):

$$\Psi_{0,\sigma}(q) := \left(\frac{\sigma}{\pi\hbar}\right)^{1/4} e^{-\frac{\sigma q^2}{2\hbar}}. \tag{4.1}$$

The (plane) coherent state at the point $\boldsymbol{x} = (q_0, p_0) \in \mathbb{R}^2$ is obtained by applying a quantum translation $\hat{T}_{\boldsymbol{x}}$ to the state above, which yields:

$$\begin{aligned}\Psi_{\boldsymbol{x},\sigma}(q) &:= \left(\frac{\sigma}{\pi\hbar}\right)^{1/4} e^{-i\frac{p_0 q_0}{2\hbar}}\, e^{i\frac{p_0 q}{\hbar}}\, e^{\frac{-\sigma(q-q_0)^2}{2\hbar}}\\ &= (2N\sigma)^{1/4} e^{-\pi i N q_0 p_0 + 2\pi i N p_0 q - \sigma N\pi(q-q_0)^2}.\end{aligned}$$

(In the second line, we took $\hbar = (2\pi N)^{-1}$, as is required if we want to project on the torus). From here we obtain a coherent state on the cylinder by periodicising along the q-axis:

$$\Psi_{\boldsymbol{x},\sigma,\mathcal{C}}(q) := \sum_{\nu\in\mathbb{Z}} \Psi_{\boldsymbol{x},\sigma}(q+\nu). \tag{4.2}$$

Finally, the coherent state on the torus is obtained by further periodicising in the Fourier variable, or equivalently by sampling this cylinder wavefunction: its coefficients in the canonical basis read

$$\left(\psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}\right)_j = \frac{1}{\sqrt{N}}\Psi_{\boldsymbol{x},\sigma,\mathcal{C}}(j/N), \quad j = 0,\dots,N-1. \tag{4.3}$$

One can check that $\psi_{\boldsymbol{x}+\boldsymbol{m},\sigma,\mathbb{T}^2} \propto \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}$ for any $\boldsymbol{m} \in \mathbb{Z}^2$: up to a phase, the state $\psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}$ depends on the projection on $\mathbb{T}^2$ of the point $\boldsymbol{x}$.

In the classical limit, it will often be useful to approximate a torus (or cylinder) coherent state by the corresponding planar one:

Lemma 3. *Let $q_0 \in (\delta, 1-\delta)$ for some $0 < \delta < 1/2$. Then in the classical limit:*

$$\forall q \in [0,1), \qquad \Psi_{\boldsymbol{x},\sigma,\mathcal{C}}(q) = \Psi_{\boldsymbol{x},\sigma}(q) + \mathcal{O}\big((\sigma N)^{1/4} e^{-\pi N\sigma\delta^2}\big). \tag{4.4}$$

The error estimate is uniform for $\sigma N \geq 1$.

Proof. Extracting the $\nu = 0$ term in (4.2), one gets

$$\forall q \in [0,1), \qquad \Psi_{\boldsymbol{x},\sigma,\mathcal{C}}(q) = \Psi_{\boldsymbol{x},\sigma}(q) + \mathcal{O}\Big((\sigma N)^{1/4}\, e^{-\pi\sigma N \min\{|q-q_0+\nu|^2 : \nu\neq 0\}}\Big).$$

Now, if $q_0 \in (\delta, 1-\delta)$, one has $|q - q_0| \leq 1-\delta$, so that

$$\forall \nu \neq 0, \qquad |q - q_0 - \nu| \geq |\nu| - |q-q_0| \geq 1 - |q-q_0| \geq \delta. \quad \square$$

The next lemma describes how a torus coherent state transforms under the application of the discrete Fourier transform.

Lemma 4. *For any* $\boldsymbol{x} = (q_0, p_0) \in \mathbb{R}^2$*, let* $F\boldsymbol{x} := (p_0, -q_0)$ *denote its rotation by* $-\pi/2$ *around the origin. Then*

$$\forall N \geq 1,\ \forall\sigma > 0, \qquad \hat{F}_N\, \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2} = \psi_{F\boldsymbol{x},1/\sigma,\mathbb{T}^2}. \tag{4.5}$$

Proof. The plane coherent states, which are Gaussian wavefunctions, are obviously covariant through the Fourier transform $\hat{F}_\hbar$: a straightforward computation shows that

$$\forall \boldsymbol{x} \in \mathbb{R}^2, \quad \hat{F}_\hbar\, \psi_{\boldsymbol{x},\sigma} = \psi_{F\boldsymbol{x},1/\sigma}.$$

When $(2\pi\hbar) = N^{-1}$, we apply the projector (3.2) to both sides of this inequality, and remember that $\hat{F}_\hbar$ acts on $\mathcal{H}_N$ as the matrix $\hat{F}_N$: this means $\hat{P}_{\mathbb{T}^2}\, \hat{F}_\hbar = \hat{F}_N\, \hat{P}_{\mathbb{T}^2}$, so the above covariance is carried over to the torus coherent states. □

4.1. Action of $\hat{B}_N$ *on coherent states.* We assume N to be an even integer, and apply the matrix $\hat{B}_N$ to the coherent state $\psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}$, seen as an N-component vector in the basis $\{\mathbf{q}_j\}$. We get nice results if the point $\boldsymbol{x} = (q_0, p_0)$ is "far enough" from the singularity set $\mathcal{S}_1$ (in this case $B\boldsymbol{x}$ is well-defined). More precisely, we define the following subsets of $\mathbb{T}^2$:

Definition 2. *For any* $0 < \delta < 1/4$ *and* $0 < \gamma < 1/2$*, let*

$$\mathcal{D}_{1,\delta,\gamma} := \Big\{(q,p) \in \mathbb{T}^2,\ q \in (\delta, 1/2-\delta) \cup (1/2+\delta, 1-\delta),\ p \in (\gamma, 1-\gamma)\Big\}. \tag{4.6}$$

The evolution of coherent states will be simple for states localised in this set.

Proposition 5. *For some parameters* $\delta,\ \gamma$ *(which may depend on* N*), we consider points* $\boldsymbol{x} = (q_0, p_0) \in \mathbb{T}^2$ *in the set* $\mathcal{D}_{1,\delta,\gamma}$*. We associate to these points the phase*

$$\Theta(\boldsymbol{x}) = \begin{cases} 0, & \text{if } q_0 \in (\delta, 1/2-\delta), \\ q_0 + \dfrac{p_0+1}{2}, & \text{if } q_0 \in (1/2+\delta, 1-\delta). \end{cases} \tag{4.7}$$

We assume that the squeezing σ *may also depend on* N*, remaining in the interval* $\sigma \in [1/N, N]$*. From* $\delta,\ \gamma,\ \sigma$ *we form the parameter*

$$\theta = \theta(\delta,\gamma,\sigma) := \min(\sigma\delta^2, \gamma^2/\sigma). \tag{4.8}$$

Then, in the semiclassical limit, the coherent state $\psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}$ *evolves almost covariantly through the quantum baker's map:*

$$\|\hat{B}_N\, \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2} - \mathrm{e}^{i\pi\Theta(\boldsymbol{x})}\, \psi_{B\boldsymbol{x},\sigma/4,\mathbb{T}^2}\|_{\mathcal{H}_N} = \mathcal{O}(N^{3/4}\sigma^{1/4}\,\mathrm{e}^{-\pi N\theta}). \tag{4.9}$$

The implied constant is uniform with respect to $\delta,\ \gamma,\ \sigma \in [1/N, N]$*, and the point* $\boldsymbol{x} \in \mathcal{D}_{1,\delta,\gamma}$*.*

We notice that the exponential in the above remainder will be small only if $\theta >> 1/N$, which requires both $\sigma >> 1/N$ and $\sigma << N$. In further applications we will always consider squeezings satisfying these properties in the limit $N \to \infty$.

Remark 1. If we extend to the full plane each of the maps given by the two lines of Eq. (2.1), we get two linear symplectic transformation S_0, S_1, which can be quantised on $L^2(\mathbb{R})$ by the metaplectic transformations

$$\hat{S}_{0,\hbar} = \hat{D}_2, \qquad \hat{S}_{1,\hbar} = \hat{T}_{(-1,0)} \circ \hat{D}_2 \circ \hat{T}_{(0,1)}$$

(here $[\hat{D}_2\, \psi](q) = 2^{-1/2}\, \psi(q/2)$ is the unitary dilation by a factor 2). Such metaplectic transformations act covariantly on plane coherent states:

$$\forall \sigma > 0, \quad \forall \boldsymbol{x} = (q_0, p_0) \in \mathbb{R}^2, \qquad \begin{cases} \hat{S}_{0,\hbar}\, \Psi_{\boldsymbol{x},\sigma} = \Psi_{S_0\boldsymbol{x},\sigma/4}, \\ \hat{S}_{1,\hbar}\, \Psi_{\boldsymbol{x},\sigma} = \mathrm{e}^{\frac{i}{2\hbar}(q_0 + \frac{p_0+1}{2})}\, \Psi_{S_1\boldsymbol{x},\sigma/4}. \end{cases}$$

The approximate covariance stated in Proposition 5 is therefore a microlocal version of this exact global covariance.

Remark 2. The fact that the error is exponentially small is due to the piecewise-linear character of the map B. Indeed, for a nonlinear area-preserving map M on $\mathbb{T}^2$, coherent states are also transformed covariantly through $\hat{M}_N$, but the error term is in general of order $\mathcal{O}(N\, \Delta\boldsymbol{x}^3)$, where $\Delta\boldsymbol{x}$ is the "maximal width" of the coherent state (here $\Delta\boldsymbol{x} = \max(\sigma, \sigma^{-1})\, N^{-1/2}$) [Schu1]. Moreover, in general the squeezing σ takes values in the complex half-plane $\{\mathrm{Re}\,(\sigma) > 0\}$: the reason why we can here restrict ourselves to the positive real line is due to the orientation of the baker's dynamics.

Proof of Proposition 5. Since we already know that $\hat{F}_N$ acts covariantly on coherent states, we only need to analyse the action of $\hat{B}_{N,\mathrm{mix}}$ (Eq. (3.7)).

We first consider a coherent state in the "left" strip $(\delta, 1/2-\delta) \times (\gamma, 1-\gamma)$ of $\mathcal{D}_{1,\delta,\gamma}$. In this case, the "relevant" coefficients of $\hat{B}_{N,\mathrm{mix}}\, \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}$ are in the interval $0 \le m < \frac{N}{2}$:

$$\left(\hat{B}_{N,\mathrm{mix}}\, \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}\right)_m = \frac{1}{\sqrt{N}} \sum_{j=0}^{N/2-1} (\hat{F}_{N/2})_{mj}\; \Psi_{\boldsymbol{x},\sigma,\mathcal{C}}\left(\frac{j}{N}\right). \tag{4.10}$$

From the formula (3.6), we have for all $0 \le j, m < N/2$:

$$(\hat{F}_{N/2})_{mj} = \sqrt{2}\, (\hat{F}_N)_{2m\, j}.$$

Since $q_0 \in (\delta, 1/2 - \delta)$, for any $N/2 \le j$ one has $j/N - q_0 \ge \delta$; using Lemma 3, we obtain

$$\forall j \in \{N/2, \ldots, N-1\}, \qquad \Psi_{\boldsymbol{x},\sigma,\mathcal{C}}\left(\frac{j}{N}\right) = \mathcal{O}\big((\sigma N)^{1/4}\, \mathrm{e}^{-\pi N \sigma \delta^2}\big). \tag{4.11}$$

We can therefore extend the range of summation in (4.10) to $j \in \{0, \ldots, N-1\}$, incurring only an exponentially small error:

$$\begin{aligned} \left(\hat{B}_{N,\mathrm{mix}}\, \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}\right)_m &= \sqrt{2} \sum_{j=0}^{N-1} (\hat{F}_N)_{2m\, j} \left(\psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}\right)_j + \mathcal{O}((\sigma N)^{1/4} \mathrm{e}^{-\pi N \sigma \delta^2}) \\ &= \sqrt{2} \left(\psi_{F\boldsymbol{x},1/\sigma,\mathbb{T}^2}\right)_{2m} + \mathcal{O}((\sigma N)^{1/4} \mathrm{e}^{-\pi N \sigma \delta^2}). \end{aligned} \tag{4.12}$$

In the last step, we have used the covariance property of Lemma 4.

Since $p_0 \in (\gamma, 1-\gamma)$ and $N/\sigma \geq 1$, it follows from Lemma 3 and simple manipulations of plane coherent states that for all $q \in [0, 1/2)$,

$$\begin{aligned}\sqrt{2}\,\Psi_{F\boldsymbol{x},1/\sigma,\mathcal{C}}(2q) &= \sqrt{2}\,\Psi_{F\boldsymbol{x},1/\sigma}(2q) + \mathcal{O}\big((N/\sigma)^{1/4}\mathrm{e}^{-\pi N\gamma^2/\sigma}\big)\\ &= \Psi_{(p_0/2,-2q_0),4/\sigma}(q) + \mathcal{O}\big((N/\sigma)^{1/4}\mathrm{e}^{-\pi N\gamma^2/\sigma}\big)\\ &= \Psi_{(p_0/2,-2q_0),4/\sigma,\mathcal{C}}(q) + \mathcal{O}\big((N/\sigma)^{1/4}\,\mathrm{e}^{-\pi N\gamma^2/\sigma}\big).\end{aligned}$$

The identity $(p_0/2, -2q_0) = FB\boldsymbol{x}$ (valid for $\boldsymbol{x}$ in the left strip) inserted in (4.12) yields for all $m \in \{0, \dots, N/2-1\}$,

$$\Big(\hat{B}_{N,\mathrm{mix}}\;\psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}\Big)_m = \big(\psi_{FB\boldsymbol{x},4/\sigma,\mathbb{T}^2}\big)_m + \mathcal{O}((\sigma N)^{1/4}\mathrm{e}^{-\pi N\theta}) \tag{4.13}$$

(θ is defined in (4.8), and we used the assumption $\sigma N > 1$ to simplify the remainder).

The remaining coefficients $N/2 \leq m < N$ are bounded using (4.11):

$$\Big(\hat{B}_{N,\mathrm{mix}}\;\psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}\Big)_m = \frac{1}{\sqrt{N}}\sum_{j=N/2}^{N-1}(\hat{F}_{N/2})_{m\,j}\;\Psi_{\boldsymbol{x},\sigma,\mathcal{C}}\Big(\frac{j}{N}\Big) = \mathcal{O}((\sigma N)^{1/4}\mathrm{e}^{-\pi\sigma N\delta^2}). \tag{4.14}$$

On the other hand, Lemma 3 shows that the coefficients $\big(\psi_{FB\boldsymbol{x},4/\sigma,\mathbb{T}^2}\big)_m$ for $N/2 \leq m < N$ are bounded from above by the same RHS. Hence, Eq. (4.13) holds for all $m = 0, \dots, N-1$. A norm estimate is obtained by multiplying this component-wise estimate by a factor $\sqrt{N}$.

We now apply the inverse Fourier transform and Lemma 4, to obtain the part of the theorem dealing with coherent states in the left strip of $D_{1,\delta,\gamma}$.

A similar computation treats the case of coherent states in the right strip of $D_{1,\delta,\gamma}$. The large components of $\psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}$ are in the interval $j \geq N/2$, so the second block of $\hat{B}_{N,\mathrm{mix}}$ is relevant. The analogue to (4.13) reads, for $m \in \{N/2, \dots, N-1\}$:

$$\Big(\hat{B}_{N,\mathrm{mix}}\;\psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}\Big)_m = \sqrt{\frac{2}{N}}\;\Psi_{F\boldsymbol{x},1/\sigma}\Big(\frac{2m}{N}-1\Big) + \mathcal{O}((\sigma N)^{1/4}\mathrm{e}^{-\pi N\theta}). \tag{4.15}$$

Proceeding as before, we identify for all $q \in [1/2, 1)$,

$$\begin{aligned}\sqrt{2}\,\Psi_{F\boldsymbol{x},1/\sigma}(2q-1) &= \mathrm{e}^{\pi i N(q_0+\frac{p_0+1}{2})}\;\Psi_{((p_0+1)/2,-(2q_0-1)),4/\sigma}(q)\\ &= \mathrm{e}^{\pi i N(q_0+\frac{p_0+1}{2})}\;\Psi_{FB\boldsymbol{x},4/\sigma,\mathcal{C}}(q) + \mathcal{O}((N/\sigma)^{1/4}\,\mathrm{e}^{-\pi N\gamma^2/\sigma}).\end{aligned} \tag{4.16}$$

Applying the inverse Fourier transform we obtain the second part of the theorem. □

5. Egorov Property

Our objective in this section is to control the evolution of quantum observables through $\hat{B}_N$, in terms of the corresponding classical evolution. Namely, we want to prove an Egorov theorem of the type

$$\|\hat{B}_N^n \operatorname{Op}_N(a)\, \hat{B}_N^{-n} - \operatorname{Op}_N(a \circ B^{-n})\| \xrightarrow{N\to\infty} 0. \tag{5.1}$$

Here $\operatorname{Op}_N(a)$ is some quantisation of an observable $a \in C^\infty(\mathbb{T}^2)$. As explained in the introduction, to avoid the diffraction problems due to the discontinuities of B, we will require the function a to be supported away from the set $\mathcal{S}_n$ of discontinuities of B^n. Otherwise, $a \circ B^{-n}$ may be discontinuous, and already its quantisation poses some problems.

An Egorov theorem has been proven in [RubSal] for a different quantisation of the baker's map, also using coherent states. In [DBDE, Cor. 17] an Egorov theorem was obtained for $\hat{B}_N$, but valid only for observables of the form $a(q)$ (or $a(p)$, depending on the direction of time) and restricting the observables to a "good" subspace of $\mathcal{H}_N$ of dimension $N - o(N)$.

Since we control the evolution of coherent states through $\hat{B}_N$ (Proposition 5), it is natural to use a quantisation defined in terms of coherent states, namely the *anti-Wick quantisation* [Per] (see Definition 4 below). However, because the quasi-covariance (4.9) connects a squeezing σ to a squeezing $\sigma/4$, it will be necessary to relate the corresponding quantisations $\operatorname{Op}_N^{\mathrm{AW},\sigma}$ and $\operatorname{Op}_N^{\mathrm{AW},\sigma/4}$ to one another. This will be done in the next subsection, by using the *Weyl quantisation* as a reference.

Besides, we want to control the correspondence (5.1) uniformly with respect to the time n. We already noticed that for $n >> 1$, an observable a supported away from $\mathcal{S}_n$ needs to fluctuate quite strongly along the q-direction, while its dependence in the p variable may remain mild. Likewise, $a \circ B^{-n}$, supported away from $\mathcal{S}_{-n}$, will strongly fluctuate along the p-direction.

All results in this section will be stated for two classes of observables:

- general functions $f \in C^\infty(\mathbb{T}^2)$, without any indication on how f depends on N. This yields a Egorov theorem valid for time $|n| \leq (\frac{1}{6} - \epsilon)\, T_{\mathrm{E}}$ (with $\epsilon > 0$ fixed), which will suffice to prove Theorem 1 ($T_{\mathrm{E}} = T_{\mathrm{E}}(N)$ is the Ehrenfest time (1.2)).
- functions $f \in S_{\boldsymbol{\alpha}}(\mathbb{T}^2)$ for some $\boldsymbol{\alpha} \in \mathbb{R}_+^2$ with $|\boldsymbol{\alpha}| < 1$ (see Definition 1). Here we use more sophisticated methods in order to push the Egorov theorem up to the times $|n| \leq (1-\epsilon) T_{\mathrm{E}}$.

5.1. Weyl vs. anti-Wick quantizations on $\mathbb{T}^2$. In this subsection, we define and compare the Weyl and anti-Wick quantisations on the torus. The main result is Proposition 8, which precisely estimates the discrepancies between these quantisations, in the classical limit. We start by recalling the definition of the Weyl quantisation on the torus [BDB, DEG].

Definition 3. *Any function $f \in C^\infty(\mathbb{T}^2)$ can be Fourier expanded as follows:*

$$f = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \tilde{f}(\boldsymbol{k})\, e_{\boldsymbol{k}}, \quad \textit{where} \quad e_{\boldsymbol{k}}(\boldsymbol{x}) := \mathrm{e}^{2\pi i \boldsymbol{x}\wedge\boldsymbol{k}} = \mathrm{e}^{2\pi i (q k_2 - p k_1)}.$$

The Weyl quantisation of this function is the following operator:

$$\mathrm{Op}_N^{\mathrm{W}}(f) := \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \tilde{f}(\boldsymbol{k})\, T(\boldsymbol{k}), \quad \textit{where} \quad T(\boldsymbol{k}) := \hat{T}_{h\boldsymbol{k}}. \tag{5.2}$$

We use the same notations for translation operators $T(\boldsymbol{k})$ acting on either $\mathcal{H}_N$ or $L^2(\mathbb{R})$; in the latter case, the Weyl-quantised operator will be denoted by $\mathrm{Op}_N^{\mathrm{W},\mathbb{R}^2}(f)$.

The operators $\{T(\boldsymbol{k})\,;\,\boldsymbol{k}\in\mathbb{Z}^2\}$ acting on $L^2(\mathbb{R})$ form an independent set of unitary operators. On the other hand, on $\mathcal{H}_N$ these operators satisfy $T(\boldsymbol{k}+N\boldsymbol{m}) = (-1)^{\boldsymbol{k}\wedge\boldsymbol{m}}\, T(\boldsymbol{k})$. Hence, defining $\mathbb{Z}_N := \{-N/2,\dots,N/2-1\}$, the set $\{T(\boldsymbol{k}),\ \boldsymbol{k}\in\mathbb{Z}_N^2\}$ forms a basis of the space of operators on $\mathcal{H}_N$. This basis is orthonormal with respect to the Hilbert-Schmidt scalar product (3.8).

The Weyl quantisations on $L^2(\mathbb{R})$ and $\mathcal{H}_N$ satisfy the following inequality [BDB, Lemma 3.9]:

$$\forall f\in C^\infty(\mathbb{T}^2),\quad \forall N\in\mathbb{N},\qquad \|\mathrm{Op}_N^{\mathrm{W}}(f)\|_{\mathcal{B}(\mathcal{H}_N)} \le \|\mathrm{Op}_N^{\mathrm{W},\mathbb{R}^2}(f)\|_{\mathcal{B}(L^2(\mathbb{R}))}. \tag{5.3}$$

This will allow us to use results pertaining to the Weyl quantisation of bounded functions on the plane (see the proof of Lemma 9).

We now define a family of anti-Wick quantisations.

Definition 4. *For any squeezing $\sigma > 0$, the anti-Wick quantisation of a function $f\in L^1(\mathbb{T}^2)$ is the operator $\mathrm{Op}_N^{\mathrm{AW},\sigma}(f)$ on $\mathcal{H}_N$ defined as:*

$$\forall\phi,\ \phi'\in\mathcal{H}_N,\qquad \langle\phi, \mathrm{Op}_N^{\mathrm{AW},\sigma}(f)\,\phi'\rangle := N\int_{\mathbb{T}^2} f(\boldsymbol{x})\,\langle\phi,\psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}\rangle\,\langle\psi_{\boldsymbol{x},\sigma,\mathbb{T}^2},\phi'\rangle\,\mathrm{d}\boldsymbol{x}. \tag{5.4}$$

Both Weyl and anti-Wick quantisations map a real observable onto a Hermitian operator. As opposed to the Weyl quantisation, the anti-Wick quantisation enjoys the important property of *positivity*. Namely, if the function a is nonnegative, then for any N, σ, the operator $\mathrm{Op}_N^{\mathrm{AW},\sigma}(a)$ is positive.

These quantisations will be easy to compare once we have expressed the anti-Wick quantisation in terms of the Weyl one [BonDB].

Lemma 6. *Using the quadratic form $Q_\sigma(\boldsymbol{k}) := \sigma\, k_1^2 + \sigma^{-1}\, k_2^2$, one has the following expression for the anti-Wick quantisation:*

$$\forall f\in L^1(\mathbb{T}^2),\qquad \mathrm{Op}_N^{\mathrm{AW},\sigma}(f) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \tilde{f}(\boldsymbol{k})\,\mathrm{e}^{-\frac{\pi}{2N}Q_\sigma(\boldsymbol{k})}\, T(\boldsymbol{k}). \tag{5.5}$$

Equivalently, $\mathrm{Op}_N^{\mathrm{AW},\sigma}(f) = \mathrm{Op}_N^{\mathrm{W}}(f^\sharp)$, where the function $f^\sharp$ is obtained by convolution of f (on $\mathbb{R}^2$) with the Gaussian kernel

$$K_{N,\sigma}(\boldsymbol{x}) := 2N\,\mathrm{e}^{-2\pi N Q_\sigma(\boldsymbol{x})}. \tag{5.6}$$

Proof. To prove this lemma, it is sufficient to show that for any $\boldsymbol{k}_0 \in \mathbb{Z}^2$, the anti-Wick quantisation on $\mathcal{H}_N$ of the Fourier mode $e_{\boldsymbol{k}_0}(\boldsymbol{x})$ reads:

$$\mathrm{Op}_N^{\mathrm{AW},\sigma}(e_{\boldsymbol{k}_0}) = \mathrm{e}^{-\frac{\pi}{2N} Q_\sigma(\boldsymbol{k}_0)}\; T(\boldsymbol{k}_0). \tag{5.7}$$

This formula has been proven in [BonDB, Lemma 2.3 (*ii*)], yet we give here its proof for completeness. The idea is to decompose $\mathrm{Op}_N^{\mathrm{AW},\sigma}(e_{\boldsymbol{k}_0})$ in the basis $\{T(\boldsymbol{k}),\ \boldsymbol{k} \in \mathbb{Z}_N^2\}$, using the Hilbert-Schmidt scalar product (3.8). That is, we need to compute

$$\Big\langle T(\boldsymbol{k}), \mathrm{Op}_N^{\mathrm{AW},\sigma}(e_{\boldsymbol{k}_0})\Big\rangle = \int_{\mathbb{T}^2} e_{\boldsymbol{k}_0}(\boldsymbol{x})\, \langle \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}, T(\boldsymbol{k})^\dagger\, \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}\rangle\, \mathrm{d}\boldsymbol{x}. \tag{5.8}$$

The overlaps between torus coherent states derive from the overlaps between plane coherent states, which are simple Gaussian integrals:

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2, \quad \langle \Psi_{\boldsymbol{y},\sigma}, \Psi_{\boldsymbol{x},\sigma}\rangle_{\mathbb{R}^2} = \mathrm{e}^{i\frac{\boldsymbol{y}\wedge\boldsymbol{x}}{2\hbar}}\, \langle \Psi_{0,\sigma}, \hat{T}_{\boldsymbol{x}-\boldsymbol{y}}\, \Psi_{0,\sigma}\rangle_{\mathbb{R}^2} = \mathrm{e}^{i\frac{\boldsymbol{y}\wedge\boldsymbol{x}}{2\hbar}}\, \mathrm{e}^{-\frac{Q_\sigma(\boldsymbol{x}-\boldsymbol{y})}{4\hbar}}.$$

Using the projector (3.2), we get

$$\begin{aligned}\langle \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}, \hat{T}_{\boldsymbol{k}/N}\, \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}\rangle &= \sum_{\boldsymbol{m}\in\mathbb{Z}^2} (-1)^{Nm_1m_2}\, \langle \Psi_{\boldsymbol{x},\sigma}, \hat{T}_{\boldsymbol{k}/N}\, \hat{T}_{\boldsymbol{m}}\, \Psi_{\boldsymbol{x},\sigma}\rangle_{\mathbb{R}^2}\\ &= \sum_{\boldsymbol{m}\in\mathbb{Z}^2} (-1)^{Nm_1m_2+\boldsymbol{m}\wedge\boldsymbol{k}}\, \mathrm{e}^{2i\pi(\boldsymbol{x}\wedge(\boldsymbol{k}+N\boldsymbol{m}))}\, \mathrm{e}^{-\frac{\pi N}{2} Q_\sigma(\boldsymbol{m}+\boldsymbol{k}/N)}.\end{aligned}$$

We insert this expression in the RHS of (5.8) (and remember that N is even):

$$\Big\langle T(\boldsymbol{k}), \mathrm{Op}_N^{\mathrm{AW},\sigma}(e_{\boldsymbol{k}_0})\Big\rangle = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \delta_{\boldsymbol{k}_0,\boldsymbol{k}+N\boldsymbol{m}}\, (-1)^{\boldsymbol{m}\wedge\boldsymbol{k}}\, \mathrm{e}^{-\frac{\pi N}{2} Q_\sigma(\boldsymbol{m}+\boldsymbol{k}/N)}.$$

This expression vanishes unless $\boldsymbol{k} = \boldsymbol{k}_1$, the unique element of $\mathbb{Z}_N^2$ such that $\boldsymbol{k}_1 = \boldsymbol{k}_0 + N\boldsymbol{m}_1$ for some $\boldsymbol{m}_1 \in \mathbb{Z}^2$. The orthonormality of the basis $\{T(\boldsymbol{k}) : \boldsymbol{k} \in \mathbb{Z}_N^2\}$ gives that $\mathrm{Op}_N^{\mathrm{AW},\sigma}(e_{\boldsymbol{k}_0}) = (-1)^{\boldsymbol{m}_1\wedge\boldsymbol{k}_1}\, \mathrm{e}^{-\frac{\pi N}{2} Q_\sigma(\boldsymbol{k}_0)}\, T(\boldsymbol{k}_1) = \mathrm{e}^{-\frac{\pi N}{2} Q_\sigma(\boldsymbol{k}_0)}\, T(\boldsymbol{k}_0)$. □

A simple property of these quantisations is the semi-classical behaviour of the *traces* of quantized observables:

Lemma 7. *For any integer* $M \geq 3$*,*

$$\forall f \in C^\infty(\mathbb{T}^2), \qquad \frac{1}{N}\,\mathrm{Tr}(\mathrm{Op}_N^{\mathrm{W}}(f)) = \int_{\mathbb{T}^2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}_M\Big(\frac{\|f\|_{C^M}}{N^M}\Big). \tag{5.9}$$

For the anti-Wick quantisation, we have:

$$\forall f \in L^1(\mathbb{T}^2), \qquad \frac{1}{N}\,\mathrm{Tr}(\mathrm{Op}_N^{\mathrm{AW},\sigma}(f)) = \int_{\mathbb{T}^2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}(\|f\|_{L^1}\, \mathrm{e}^{-\frac{\pi N}{2}\min(\sigma,1/\sigma)}). \tag{5.10}$$

Proof. The first identity uses the fact that on the space $\mathcal{H}_N$,

$$\frac{1}{N}\operatorname{Tr} T(\boldsymbol{k}) = \begin{cases} 1 & \text{if } \boldsymbol{k} = N\boldsymbol{m} \text{ for some } \boldsymbol{m} \in \mathbb{Z}^2, \\ 0 & \text{otherwise.} \end{cases}$$

The error term in (5.9) is bounded above by $\sum_{\boldsymbol{m}\in\mathbb{Z}^2\setminus\{0\}} |\tilde{f}(N\boldsymbol{m})|$. Now, the Fourier coefficients of a smooth function satisfy

$$\forall M \geq 1, \quad \forall \boldsymbol{k} \in \mathbb{Z}^2, \qquad |\tilde{f}(\boldsymbol{k})| \ll_M \frac{\|f\|_{C^M}}{(1+|\boldsymbol{k}|)^M}. \tag{5.11}$$

Using this upper bound (with $M \geq 3$) in the above sum yields (5.9).

In the anti-Wick case, each term $|\tilde{f}(N\boldsymbol{m})| \leq \|f\|_{L^1}$ of the sum is multiplied by $\mathrm{e}^{-\frac{\pi N}{2} Q_\sigma(\boldsymbol{m})} \leq \mathrm{e}^{-\frac{\pi N}{2}\min(\sigma,1/\sigma)|\boldsymbol{m}|^2}$, which yields (5.10). □

We will now compare the Weyl and anti-Wick quantisations in the operator norm. We give two estimates, corresponding to the two classes of functions described in the introduction of this section.

Proposition 8. *I) For any $f \in C^\infty(\mathbb{T}^2)$ and $\sigma > 0$,*

$$\|\mathrm{Op}_N^{\mathrm{W}}(f) - \mathrm{Op}_N^{\mathrm{AW},\sigma}(f)\| \ll \|f\|_{C^5} \frac{\max\{\sigma, \sigma^{-1}\}}{N}. \tag{5.12}$$

Here σ may depend arbitrarily on N.

II) Let $\boldsymbol{\alpha} \in \mathbb{R}^2_+$, $|\boldsymbol{\alpha}| < 1$ and assume that $\sigma > 0$ may depend on N such that the quantity

$$\hbar_{\boldsymbol{\alpha}}(N,\sigma) := \max\left(\frac{N^{2\alpha_1-1}}{\sigma}, N^{\alpha_1+\alpha_2-1}, \sigma\, N^{2\alpha_2-1}\right) \tag{5.13}$$

goes to zero as $N \to \infty$. Then there exists a seminorm $\mathcal{N}_{\boldsymbol{\alpha}}$ on the space $S_{\boldsymbol{\alpha}}(\mathbb{T}^2)$ such that, for any $f = f(\cdot, N) \in S_{\boldsymbol{\alpha}}(\mathbb{T}^2)$, one has:

$$\forall N \geq 1, \qquad \|\mathrm{Op}_N^{\mathrm{AW},\sigma}(f(\cdot,N)) - \mathrm{Op}_N^{\mathrm{W}}(f(\cdot,N))\| \ll \mathcal{N}_\alpha(f)\, \hbar_{\boldsymbol{\alpha}}(N,\sigma). \tag{5.14}$$

Remark 3. The effective "small parameter" $\hbar_{\boldsymbol{\alpha}}(N,\sigma)$ will be small as $N \to \infty$ only if three conditions are simultaneously satisfied:

– $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 < 1$,
– $N^{2\alpha_1} << N\sigma$,
– $N^{2\alpha_2} << N/\sigma$.

These conditions mean that the horizontal and vertical widths of the kernel (5.6) must be small compared to the typical scale of fluctuations of f in the respective directions. The conditions $N\sigma \geq 1$, $N/\sigma \geq 1$ assumed in Sect. 4 are therefore automatically satisfied.

Proof of Proposition 8. We start with the first (simple) part. Our main ingredient is Lemma 6. By Taylor's theorem,

$$\forall \boldsymbol{k} \in \mathbb{Z}^2, \qquad \mathrm{e}^{-\frac{\pi}{2N} Q_\sigma(\boldsymbol{k})} = 1 + \mathcal{O}\left(\frac{Q_\sigma(\boldsymbol{k})}{N}\right) = 1 + \mathcal{O}\left(\frac{\max\{\sigma,\sigma^{-1}\}}{N}|\boldsymbol{k}|^2\right), \tag{5.15}$$

where the implied constant is independent of $\boldsymbol{k}$. Substituting (5.15) into (5.5), the first term gives the Weyl quantisation of f. Using the bounds (5.11) with $M = 5$, we obtain the first part of the proposition:

$$\begin{aligned}\|\mathrm{Op}_N^{\mathrm{AW},\sigma}(f) - \mathrm{Op}_N^{\mathrm{W}}(f)\| &\ll \frac{\max\{\sigma, \sigma^{-1}\}}{N} \sum_{\boldsymbol{k}\in\mathbb{Z}^2} |\tilde{f}(\boldsymbol{k})|\, |\boldsymbol{k}|^2 \\ &\ll \frac{\max\{\sigma, \sigma^{-1}\}}{N} \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \frac{\|f\|_{C^5}}{(1+|\boldsymbol{k}|)^5} |\boldsymbol{k}|^2 \\ &\ll \frac{\max\{\sigma, \sigma^{-1}\}}{N} \|f\|_{C^5}.\end{aligned}$$

The second part of the proposition requires more care. We first need to control the norm of the Weyl operator.

Lemma 9. *Take any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}_+^2$ such that $|\boldsymbol{\beta}| = 1$ and $\boldsymbol{\beta} \geq \boldsymbol{\alpha}$ (i.e. $\beta_i \geq \alpha_i$, $i = 1, 2$). Then, for any function $f = f(\cdot, N) \in S_{\boldsymbol{\alpha}}(\mathbb{T}^2)$, we have*

$$\|\mathrm{Op}_N^{\mathrm{W}}(f(\cdot, N))\| \ll \sum_{\gamma_1,\gamma_2=0}^{1} C_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(f)\, N^{-\boldsymbol{\gamma}\cdot(\boldsymbol{\beta}-\boldsymbol{\alpha})}, \tag{5.16}$$

and the implied constant is independent of $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

Proof. This lemma is a simple consequence of the Calderón-Vaillancourt theorem, a sharp form of which was obtained in [Boul]. Assume f is a smooth function on $\mathbb{R}^2$ such that $\partial^{\boldsymbol{\gamma}} f$ is uniformly bounded for all $\boldsymbol{\gamma}$ with $\gamma_1, \gamma_2 \in \{0, 1\}$. Then, its Weyl quantisation on $L^2(\mathbb{R})$ for $\hbar = 1$ is a bounded operator, and

$$\|\mathrm{Op}_{\hbar=1}^{\mathrm{W},\mathbb{R}^2}(f)\| \leq C \sum_{\gamma_1,\gamma_2=0}^{1} \|\partial^{\boldsymbol{\gamma}} f\|_{C^0(\mathbb{R}^2)}. \tag{5.17}$$

Here $\|\cdot\|$ is the norm of bounded operators on $L^2(\mathbb{R})$, and C is independent of f.

Now, we use the scaling properties of the Weyl quantisation[2]. For any $\beta \in [0, 1]$ and $\hbar > 0$ we define

$$f_{\hbar,\beta}(q, p) := f(\hbar^{\beta} q, \hbar^{1-\beta} p).$$

Then, if $U_{\hbar,\beta}$ is the dilation operator $U_{\hbar,\beta}\psi(q) = \hbar^{\beta/2}\psi(\hbar^{\beta} q)$, we have [Ma, p. 60]

$$U_{\hbar,\beta}\, \mathrm{Op}_{\hbar}^{\mathrm{W},\mathbb{R}^2}(f)\, U_{\hbar,\beta}^{-1} = \mathrm{Op}_{\hbar=1}^{\mathrm{W},\mathbb{R}^2}(f_{\hbar,\beta}). \tag{5.18}$$

Applying (5.17) to $f_{\hbar,\beta}$, we obtain

$$\forall \hbar > 0, \qquad \|\mathrm{Op}_{\hbar}^{\mathrm{W},\mathbb{R}^2}(f)\| \leq C \sum_{\gamma_1,\gamma_2=0}^{1} \|\partial^{\boldsymbol{\gamma}} f\|_{C^0(\mathbb{R}^2)}\, \hbar^{\beta\gamma_1 + (1-\beta)\gamma_2}.$$

In the case $\hbar = (2\pi N)^{-1}$ we apply this bound to a function $f \in S_{\boldsymbol{\alpha}}(\mathbb{T}^2)$, selecting $\boldsymbol{\beta} = (\beta, 1-\beta)$ such that $\boldsymbol{\beta} \geq \boldsymbol{\alpha}$: we then obtain the upper bound of (5.16) for $\mathrm{Op}_N^{\mathrm{W},\mathbb{R}^2}(f)$. The inequality (5.3) shows that this bound applies as well to the Weyl operator on $\mathcal{H}_N$. □

[2] We thank N. Anantharaman for pointing out to us this scaling argument.

Equipped with this lemma, we can now prove the second part of Proposition 8. From the Taylor expansion

$$|f(\boldsymbol{x}+\boldsymbol{y})-f(\boldsymbol{x})-(\boldsymbol{y}\cdot\nabla)f(\boldsymbol{x})| \leq \frac{1}{2}\max_{0\leq t\leq 1}\left\{\left|(\boldsymbol{y}\cdot\nabla)^2 f(\boldsymbol{z})\right|,\ \boldsymbol{z}=\boldsymbol{x}+t\boldsymbol{y}\right\}$$

and Lemma 6, one easily checks that for any $f\in C^\infty(\mathbb{T}^2)$,

$$\|f^\sharp - f\|_{C^0} \leq \frac{1}{8\pi N}\Big(\frac{1}{\sigma}\,\|\partial_q^2 f\|_{C^0} + 2\|\partial_q\partial_p f\|_{C^0} + \sigma\,\|\partial_p^2 f\|_{C^0}\Big).$$

Since differentiation commutes with convolution, one controls all derivatives: for all $\boldsymbol{\gamma}\in\mathbb{N}_0^2$,

$$\|\partial^{\boldsymbol{\gamma}}(f^\sharp - f)\|_{C^0} \leq \frac{1}{8\pi N}\Big(\frac{1}{\sigma}\,\|\partial^{\boldsymbol{\gamma}+(2,0)} f\|_{C^0} + 2\|\partial^{\boldsymbol{\gamma}+(1,1)} f\|_{C^0} + \sigma\,\|\partial^{\boldsymbol{\gamma}+(0,2)} f\|_{C^0}\Big). \tag{5.19}$$

For $f=f(\cdot,N)\in S_{\boldsymbol{\alpha}}(\mathbb{T}^2)$, this estimate implies:

$$\begin{aligned}\|\partial^{\boldsymbol{\gamma}}(f^\sharp - f)\|_{C^0} &\leq N^{\boldsymbol{\alpha}\cdot\boldsymbol{\gamma}}\Big(\frac{N^{2\alpha_1-1}}{\sigma}\,C_{\boldsymbol{\alpha},\boldsymbol{\gamma}+(2,0)}(f)\\ &\qquad + N^{\alpha_1+\alpha_2-1}\,C_{\boldsymbol{\alpha},\boldsymbol{\gamma}+(1,1)}(f) + \sigma\,N^{2\alpha_2-1}\,C_{\boldsymbol{\alpha},\boldsymbol{\gamma}+(0,2)}(f)\Big)\\ &\leq N^{\boldsymbol{\alpha}\cdot\boldsymbol{\gamma}}\,\hbar_{\boldsymbol{\alpha}}(N,\sigma)\Big(C_{\boldsymbol{\alpha},\boldsymbol{\gamma}+(2,0)}(f)\\ &\qquad + C_{\boldsymbol{\alpha},\boldsymbol{\gamma}+(1,1)}(f) + C_{\boldsymbol{\alpha},\boldsymbol{\gamma}+(0,2)}(f)\Big).\end{aligned} \tag{5.20}$$

Here we used the parameter $\hbar_{\boldsymbol{\alpha}}(N,\sigma)$ defined in (5.13). This shows that the function $f^{\sharp,\mathrm{rem}}(\cdot,N) := \frac{1}{\hbar_{\boldsymbol{\alpha}}(N,\sigma)}\big(f^\sharp(\cdot,N)-f(\cdot,N)\big)$ is also an element of $S_{\boldsymbol{\alpha}}(\mathbb{T}^2)$, with seminorms dominated by seminorms of f. Applying Lemma 9 to that function and taking any $\boldsymbol{\beta}\geq\boldsymbol{\alpha}$, $|\boldsymbol{\beta}|=1$, we get

$$\|\mathrm{Op}_N^{\mathrm{AW},\sigma}(f(\cdot,N)) - \mathrm{Op}_N^{\mathrm{W}}(f(\cdot,N))\| \ll \hbar_{\boldsymbol{\alpha}}(N,\sigma)\sum_{|\boldsymbol{\gamma}'|\leq 2}\sum_{\gamma_1,\gamma_2=0}^{1} C_{\boldsymbol{\alpha},\boldsymbol{\gamma}+\boldsymbol{\gamma}'}(f).$$

The seminorm stated in the theorem can therefore be defined as

$$\mathcal{N}_{\boldsymbol{\alpha}}(f) := \sum_{|\boldsymbol{\gamma}'|\leq 2}\sum_{\gamma_1,\gamma_2=0}^{1} C_{\boldsymbol{\alpha},\boldsymbol{\gamma}+\boldsymbol{\gamma}'}(f). \tag{5.21}$$

□

5.2. Egorov estimates for the baker's map. We now turn to the proof of the Egorov property (5.1). Let us start with the case $n = 1$. We assume that a is supported in the set $\mathcal{D}_{1,\delta,\gamma}$ defined in Eq. (4.6), away from the discontinuity set $\mathcal{S}_1$ of B.

Proposition 10. *Let $0 < \delta < 1/4$ and $0 < \gamma < 1/2$. Assume that the support of $a \in C^\infty(\mathbb{T}^2)$ is contained in $\mathcal{D}_{1,\delta,\gamma}$. Then, in the classical limit,*

$$\| \hat{B}_N \operatorname{Op}_N^{\mathrm{AW},\sigma}(a)\, \hat{B}_N^{-1} - \operatorname{Op}_N^{\mathrm{AW},\sigma/4}(a \circ B^{-1})\| \ll \|a\|_{C^0} N^{5/4} \sigma^{1/4} \mathrm{e}^{-\pi N\theta},$$

uniformly with respect to $\delta, \gamma, \sigma \in [1/N, N]$. Here we took as before $\theta = \min(\sigma\delta^2, \gamma^2/\sigma)$.

Proof. For any normalised state $\phi \in \mathcal{H}_N$, we consider the matrix element

$$\langle \phi, \hat{B}_N \operatorname{Op}_N^{\mathrm{AW},\sigma}(a) \hat{B}_N^{-1}\phi\rangle = N \int_{\mathbb{T}^2} a(\boldsymbol{x})\, \langle \phi, \hat{B}_N \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}\rangle\, \langle \hat{B}_N \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}, \phi\rangle\, \mathrm{d}\boldsymbol{x}. \tag{5.22}$$

Using the quasi-covariance of coherent states localised in $\mathcal{D}_{1,\delta,\gamma}$ (Proposition 5) and applying the Cauchy-Schwarz inequality, the RHS reads

$$N \int_{\mathbb{T}^2} a(\boldsymbol{x})\, \langle \phi, \psi_{B\boldsymbol{x},\sigma/4,\mathbb{T}^2}\rangle\, \langle \psi_{B\boldsymbol{x},\sigma/4,\mathbb{T}^2}, \phi\rangle\, \mathrm{d}\boldsymbol{x} + \mathcal{O}(\|a\|_{C^0} N^{5/4}\sigma^{1/4}\mathrm{e}^{-\pi N\theta}). \tag{5.23}$$

The remainder is uniform with respect to the state ϕ. Through the variable substitution $\boldsymbol{x} = B^{-1}(\boldsymbol{y})$, this gives

$$\langle \phi, \hat{B}_N \operatorname{Op}_N^{\mathrm{AW},\sigma}(a) \hat{B}_N^{-1}\phi\rangle = \langle \phi, \operatorname{Op}_N^{\mathrm{AW},\sigma/4}(a \circ B^{-1})\phi\rangle + \mathcal{O}(\|a\|_{C^0} N^{5/4}\sigma^{1/4}\mathrm{e}^{-\pi N\theta}). \tag{5.24}$$

Since the operators on both sides are self-adjoint, this identity implies the norm estimate of the proposition. □

Remark 4. Here we used the property that the linear local dynamics is the same at each point $\boldsymbol{x} \in \mathbb{T}^2 \setminus \mathcal{S}_1$ (expansion by a factor 2 along the horizontal, contraction by 1/2 along the vertical). Were this not the case, the state $\hat{B}_N \psi_{\boldsymbol{x},\sigma,\mathbb{T}^2}$ would be close to a coherent state at the point $B\boldsymbol{x}$, but with a squeezing depending on the point $\boldsymbol{x}$. Integrating over $\boldsymbol{x}$, we would get an anti-Wick quantisation of $a \circ B^{-1}$ with $\boldsymbol{x}$-dependent squeezing, the analysis of which would be more complicated (see [Schu1, Chap. 4] for a discussion of such quantisations).

We now generalise to $n > 1$. We assume that a is supported away from the set $\mathcal{S}_n$ of discontinuities of B^n. More precisely, for some $\delta \in (0, 2^{-n-1})$ and $\gamma \in (0, 1/2)$, we define the following open set, generalizing (4.6):

$$\mathcal{D}_{n,\delta,\gamma} := \left\{ (q,p) \in \mathbb{T}^2,\ \forall k \in \mathbb{Z},\ \left| q - \frac{k}{2^n} \right| > \delta,\ p \in (\gamma, 1-\gamma) \right\}.$$

The evolution of the sets $\mathcal{D}_{n,\delta,\gamma}$ through B satisfies:

$$\forall j \in \{0, \dots, n-1\}, \qquad B^j \mathcal{D}_{n,\delta,\gamma} \subset \mathcal{D}_{n-j,2^j\delta,\gamma/2^j}. \tag{5.25}$$

This is illustrated for $n = 2,\ j = 1$ in Fig. 5.1. If a is supported in $\mathcal{D}_{n,\delta,\gamma}$, then the support of $a \circ B^{-j}$ is contained in $\mathcal{D}_{n-j,2^j\delta,\gamma/2^j} \subset \mathcal{D}_{1,2^j\delta,\gamma/2^j}$. So for each $0 \le j < n$, we can apply Proposition 10 to the observable $a \circ B^{-j}$, replacing the parameters δ, γ, σ by their corresponding values at time j; we find that the parameter θ is independent of j. The triangle inequality then yields:

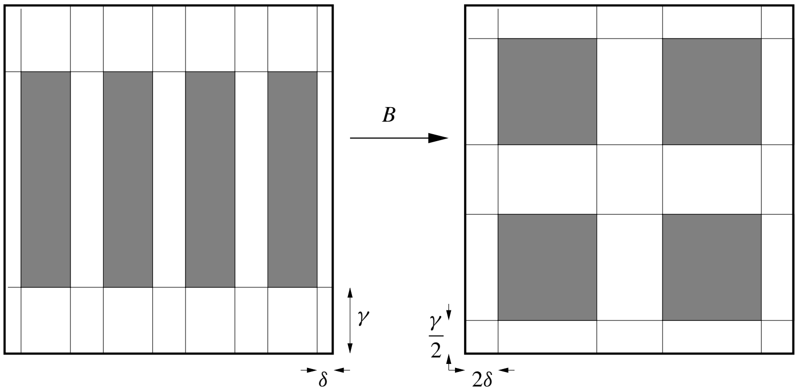

Fig. 5.1. The action of the map B. On the left we show the set $\mathcal{D}_{2,\delta,\gamma}$ (shaded) and on the right is its image under the action of B

Corollary 11. *Let $n > 0$ and for some $\delta \in (0, 2^{-n-1})$, $\gamma \in (0, 1/2)$, let $a \in C^\infty(\mathbb{T}^2)$ have support in $\mathcal{D}_{n,\delta,\gamma}$. Then, as $N \to \infty$,*

$$\| \hat{B}_N^n \, \mathrm{Op}_N^{\mathrm{AW},\sigma}(a) \, \hat{B}_N^{-n} - \mathrm{Op}_N^{\mathrm{AW},\sigma/4^n}(a \circ B^{-n}) \| \ll \|a\|_{C^0} \, N^{5/4} \, \sigma^{1/4} \mathrm{e}^{-\pi N \theta}. \tag{5.26}$$

This estimate is uniform with respect to n, the parameters δ, γ in the above ranges and the squeezing $\sigma \in [\frac{4^n}{N}, N]$.

Remark 5. The requirement $N\theta >> 1$, together with the allowed ranges for δ, γ, impose the restriction $\frac{4^n}{N} << \sigma << N$, $n \le T_{\mathrm{E}}$. This is possible only if $T_{\mathrm{E}} - n >> 1$, where T_{E} is the Ehrenfest time (1.2).

We can reach times $n \sim T_{\mathrm{E}}(1 - \epsilon)$ (with $\epsilon > 0$ fixed) by taking the parameters $\delta = 2^{-n-2} \asymp N^{-1+\epsilon}$, $\gamma \asymp 1$, $\sigma \asymp N^{1-\epsilon}$: in that case, the argument of the exponential in the RHS of Eq. (5.26) satisfies $\pi N \theta \asymp N^\epsilon$, so that the RHS decays in the classical limit.

We wish to obtain Egorov theorems where both terms correspond to a quantisation with the same parameter σ, or the Weyl quantisation. To do so, we will use Proposition 8 to replace the anti-Wick quantisations by the Weyl quantisation. Using the first statement of that proposition, we easily obtain the following Egorov theorem:

Theorem 12. *Let $n > 0$ and for some $\delta \in (0, 2^{-n-1})$, $\gamma \in (0, 1/2)$, let $a \in C^\infty(\mathbb{T}^2)$ have support in $\mathcal{D}_{n,\delta,\gamma}$. Then, in the limit $N \to \infty$, and for any squeezing parameter $\sigma \in [\frac{4^n}{N}, N]$, we have*

$$\begin{aligned} \| \hat{B}_N^n \, \mathrm{Op}_N^{\mathrm{W}}(a) \, \hat{B}_N^{-n} - \mathrm{Op}_N^{\mathrm{W}}(a \circ B^{-n}) \| &\ll \|a\|_{C^0} \, N^{5/4} \, \sigma^{1/4} \, \mathrm{e}^{-\pi N \theta} \\ &+ \frac{1}{N} \Big(\max(\sigma, \sigma^{-1}) \, \|a\|_{C^5} + \max\Big(\frac{\sigma}{4^n}, \frac{4^n}{\sigma} \Big) \, \|a \circ B^{-n}\|_{C^5} \Big). \end{aligned} \tag{5.27}$$

The implied constants are uniform in n, σ, δ, γ.

If $n,\ \delta,\ \gamma$ and the observable a supported on $\mathcal{D}_{n,\delta,\gamma}$ are independent of N, the RHS semi-classically converges to zero if we simply take $\sigma = 1$. This is the "finite-time" Egorov theorem.

On the other hand, if we let n grow with N, the function a needs to change with N as well (at least because its support needs to change). In the next subsection we construct a specific family of functions $\{a_n\}_{n\geq 1}$, each one supported away from $\mathcal{S}_n$, and compute the estimate (5.27) for this family.

Remark 6. The same estimate holds if we replace n by $-n$ on the LHS of (5.27), and replace σ by σ^{-1} on the RHS, including the definition of θ. Now, the function a must be supported in the set $\mathcal{D}_{-n,\delta,\gamma}$ obtained from $\mathcal{D}_{n,\delta,\gamma}$ by exchanging the roles of q and p.

Indeed, using the unitarity of $\hat{B}_N$, we may interpret the estimate (4.9) as the quasi-covariant evolution of the coherent state $\psi_{\boldsymbol{y},\sigma',\mathbb{T}^2}$ (where $\boldsymbol{y} = B\boldsymbol{x}$, $\sigma' = \sigma/4$) into the state $\psi_{B^{-1}\boldsymbol{y},4\sigma',\mathbb{T}^2}$, and the rest of the proof identically follows.

5.3. Egorov estimates for truncated observables.

5.3.1. A family of admissible functions. For future purposes (see the proof of Theorem 1 in the next section), and in order to understand better the bound (5.27), we explicitly construct a sequence of functions $\{a_n\}_{n\geq 0}$, each function being supported away from $\mathcal{S}_n$. This sequence is simply obtained by taking the products of a fixed observable $a \in C^\infty(\mathbb{T}^2)$ with cutoff functions $\chi_{\delta,n}$, which we now describe.

Definition 5. *For some* $0 < \delta < 1/4$*, we consider a* $\mathbb{Z}$*-periodic function* $\tilde{\chi}_\delta \in C^\infty(\mathbb{R})$ *which vanishes for* $x \in [-\delta, \delta] \bmod \mathbb{Z}$ *and takes value* 1 *for* $x \in [2\delta, 1-2\delta] \bmod \mathbb{Z}$*.*

For any $n \geq 0$*, we then define the following cutoff functions on* $\mathbb{T}^2$*:*

$$\chi_{\delta,n}(\boldsymbol{x}) := \tilde{\chi}_\delta(2^n\, q)\, \tilde{\chi}_\delta(p),$$
$$\chi_{\delta,-n}(\boldsymbol{x}) := \tilde{\chi}_\delta(2^n\, p)\, \tilde{\chi}_\delta(q).$$

For any $n \in \mathbb{Z}$*, we split the observable* $a \in C^\infty(\mathbb{T}^2)$ *into its "good part"* $a_n(\boldsymbol{x}) := a(\boldsymbol{x})\, \chi_{\delta,n}(\boldsymbol{x})$ *and its "bad part"* $a_n^{\mathrm{bad}}(\boldsymbol{x}) = a(\boldsymbol{x})\,(1 - \chi_{\delta,n}(\boldsymbol{x}))$*.*

One easily checks that a_n is supported on $\mathcal{D}_{n,\delta/2^n,\delta}$, while a_n^{bad} is supported on a neighbourhood of $\mathcal{S}_n$ of area $\mathcal{O}(\delta)$.

In light of Remark 6 we can, without loss of generality, consider only times $n > 0$. For any multiindex $\boldsymbol{\gamma} \in \mathbb{N}_0^2$, we have

$$\|\partial^{\boldsymbol{\gamma}} a_n\|_{C^0} \ll_{\boldsymbol{\gamma}} \|a\|_{C^{|\boldsymbol{\gamma}|}}\, 2^{n\gamma_1}\, \delta^{-|\boldsymbol{\gamma}|}. \tag{5.28}$$

When evolving a_n through the map B, the derivatives grow along p and decrease along q; after n iterations, $a_n \circ B^{-n}$ is still smooth, and

$$\|\partial^{\boldsymbol{\gamma}}(a_n \circ B^{-n})\|_{C^0} \ll_{\boldsymbol{\gamma}} \|a\|_{C^{|\boldsymbol{\gamma}|}}\, 2^{n\gamma_2}\, \delta^{-|\boldsymbol{\gamma}|}. \tag{5.29}$$

These estimates show that the C^5-norms of a_n and $a_n \circ B^{-n}$ (appearing on the RHS of Eq. (5.27)) are both of order $2^{5n}/\delta^5$. With our conventions, the parameter θ appearing in the RHS of (5.26) reads $\theta = \frac{\delta^2}{\max(\sigma, 4^n/\sigma)}$. We maximise it by selecting $\sigma = 2^n$. With this choice, the upper bound (5.27) reads

$$\|\hat{B}_N^n \operatorname{Op}_N^{\mathrm{W}}(a_n)\, \hat{B}_N^{-n} - \operatorname{Op}_N^{\mathrm{W}}(a_n \circ B^{-n})\| \ll \|a\|_{C^0}\, N^{5/4}\, 2^{n/4}\, \mathrm{e}^{-\pi N \delta^2/2^n} + \frac{2^{6n}\, \|a\|_{C^5}}{N\, \delta^5}. \tag{5.30}$$

Using Remark 6, the same estimate holds if we replace n by $-n$ on the LHS.

The last term of the RHS in (5.30) can semiclassically vanish only if $|n| < \frac{T_{\mathrm{E}}}{6}$. This time window, although not optimal (see the following subsection), will be sufficient to prove Theorem 1 in Sect. 6.

Before that, in the last part of this section we will sharpen this estimate by using the second part of Proposition 8: this will allow us to prove a Egorov property up to times $|n| \leq (1-\epsilon)T_{\mathrm{E}}$, for any $\epsilon > 0$.

5.3.2. Optimised Egorov estimates. In this subsection we prove the following "optimal" Egorov theorem.

Theorem 13. *Choose $\epsilon > 0$ arbitrarily small, and consider any observable $a \in C^\infty(\mathbb{T}^2)$. For any $N \geq 1$ and $n \in \mathbb{Z}$, construct the "good part" a_n of that observable using Definition 5 with a width $\delta(N) \geq \min(N^{-\epsilon/4}, 1/10)$.*

Then, the following Egorov estimate holds: there exists $C > 0$ (independent of a, ϵ) and $N(\epsilon) > 0$ such that for any $N \geq N(\epsilon)$ and any time $|n| \leq (1-\epsilon)T_{\mathrm{E}}$,

$$\|\hat{B}_N^n \operatorname{Op}_N^{\mathrm{W}}(a_n)\, \hat{B}_N^{-n} - \operatorname{Op}_N^{\mathrm{W}}(a_n \circ B^{-n})\| \leq C\Big(\|a\|_{C^0}\, N^{3/2}\, \mathrm{e}^{-\pi N^{\epsilon/2}} + \frac{\|a\|_{C^4}}{N^{\epsilon/2}}\Big). \tag{5.31}$$

Proof. We only treat the case $n \geq 0$, finally invoking the time-reversal symmetry as in Remark 6.

We consider $\epsilon > 0$ fixed, and define $N(\epsilon)$ through the equation $N(\epsilon)^{-\epsilon/4} = 1/10$. We then take $N \geq N(\epsilon)$ and consider any positive time $n \leq (1-\epsilon)T_{\mathrm{E}}$.

The improvement over Theorem 12 will be a sharper bound for the norms $\|\operatorname{Op}_N^{\mathrm{AW},\sigma}(a_n) - \operatorname{Op}_N^{\mathrm{W}}(a_n)\|$ and $\|\operatorname{Op}_N^{\mathrm{AW},\sigma/4^n}(a_n \circ B^{-n}) - \operatorname{Op}_N^{\mathrm{W}}(a_n \circ B^{-n})\|$. Using the rescaled time $t = \frac{n}{T_{\mathrm{E}}}$ and the property $\delta(N) \geq N^{-\epsilon/4}$, the bound (5.28) on derivatives of a_n reads:

$$\|\partial^{\boldsymbol{\gamma}} a_n\|_{C^0} \ll_{\boldsymbol{\gamma}} \|a\|_{C^{|\gamma|}}\, 2^{n\gamma_1}\, N^{\frac{\epsilon}{4}|\boldsymbol{\gamma}|} = \|a\|_{C^{|\gamma|}}\, N^{t\gamma_1}\, N^{\frac{\epsilon}{4}|\boldsymbol{\gamma}|}.$$

Thus, the derivatives of a_n scale as those of an N-dependent function in the space $S_{\boldsymbol{\alpha}_t}(\mathbb{T}^2)$, where $\boldsymbol{\alpha}_t := (t + \epsilon/4, \epsilon/4)$. As in the former subsection, we must take $\sigma = 2^n = N^t$ to minimise the remainder. The second part of Proposition 8 applied to a function in $S_{\boldsymbol{\alpha}_t}(\mathbb{T}^2)$ yields a "small parameter" $\hbar_{\boldsymbol{\alpha}_t}(N, 2^n) = N^{t+\epsilon/2-1}$, so that the difference between the two quantisations of a_n is bounded as

$$\|\operatorname{Op}_N^{\mathrm{W}}(a_n) - \operatorname{Op}_N^{\mathrm{AW},2^n}(a_n)\| \ll \|a\|_{C^4}\, N^{t+\epsilon/2-1}.$$

Similar considerations using (5.29) show that $\|\operatorname{Op}_N^{\mathrm{W}}(a \circ B^{-n}) - \operatorname{Op}_N^{\mathrm{AW},2^{-n}}(a \circ B^{-n})\|$ is bounded by the same quantity. The argument of the exponential in Eq. (5.26) takes the value $N\theta = N\delta^2/2^n \geq N^{1-t-\epsilon/2}$, so that the full estimate reads:

$$\|\hat{B}_N^n \operatorname{Op}_N^{\mathrm{W}}(a_n)\, \hat{B}_N^{-n} - \operatorname{Op}_N^{\mathrm{W}}(a_n \circ B^{-n})\| \ll \|a\|_{C^0}\, N^{3/2}\, \mathrm{e}^{-\pi N^{1-t-\epsilon/2}} + \frac{\|a\|_{C^4}}{N^{1-t-\epsilon/2}}.$$

We obtain the bound (5.31) uniform in n by noticing that for the time window we consider, $N^{1-t-\epsilon/2} \geq N^{\epsilon/2}$. □

Our reason for believing that this estimate is "optimal" lies in Remark 5: we evolve states which stay away from the discontinuity set $\mathcal{S}_1$ along their evolution. Since any state satisfies $\Delta q \Delta p \gtrsim \frac{1}{2}\hbar$ due to Heisenberg's uncertainty principle, and Δq doubles at each time step, it is impossible for such a state to remain away from $\mathcal{S}_1$ during a time window larger than T_{E}.

Besides, at the time T_{E} the "good part" a_n oscillates on a scale $\approx \hbar$ in the q direction, so it behaves more like a Fourier integral operator than an observable (pseudo-differential operator).

6. Quantum Ergodicity

For any even N, we denote by $\{\varphi_{N,j}\}$ the eigenvectors of $\hat{B}_N$ (if some eigenvalues happen to be degenerate, which seems to be ruled out by numerical simulations, take an arbitrary orthonormal eigenbasis). Let us consider a fixed real-valued observable $a \in C^\infty(\mathbb{T}^2)$ satisfying $\int_{\mathbb{T}^2} a(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$. Quantum ergodicity follows if we prove that the quantum variance

$$S_2(a, N) = \frac{1}{N}\sum_{j=1}^{N} |\langle \varphi_{N,j}, \mathrm{Op}_N^{\mathrm{W}}(a)\, \varphi_{N,j}\rangle|^2 \xrightarrow{N\to\infty} 0. \tag{6.1}$$

One method to prove this limit for our quantised baker's map would be to apply the methods of [MO'K]: one only needs the Egorov property (Theorem 12) for finite times n, and the classical *ergodicity* of B. However, this method seems unable to give information about the rate of decay of the variance.

In order to prove the upper bound stated in Theorem 1, we will rather adapt the method used in [Zel2, Schu2] to our discontinuous map. This method requires the *correlation functions* of the classical map to decay sufficiently fast, which is the case here (Eq. 2.5).

Proof of Theorem 1. To begin with, we consider the function

$$g(x) := 2\left(\frac{1-\cos x}{x^2}\right) \tag{6.2}$$

and its Fourier transform

$$\hat{g}(k) = \int_{-\infty}^{\infty} g(x)\,\mathrm{e}^{-2\pi i k x}\,\mathrm{d}x = \begin{cases} 2\pi(1-|k|), & \text{for } -1 \leq k \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

For any $T \geq 1$, we use it to construct the following periodic function:

$$f_T(\theta) := \sum_{m\in\mathbb{Z}} g(T(\theta + m)).$$

f_T admits the Fourier decomposition $f_T(\theta) = \sum_{k\in\mathbb{Z}} \hat{f}_T(k)\,\mathrm{e}^{2\pi i k\theta}$, where

$$\hat{f}_T(k) = \begin{cases} \frac{2\pi}{T}\left(1 - \frac{|k|}{T}\right) & \text{for } -T \leq k \leq T, \\ 0 & \text{for } |k| > T. \end{cases}$$

Using this function, one may easily prove the following lemma [Schu2].

Lemma 14. *With notations described above, for any even* $N \geq 2$ *and* $T \geq 1$ *one has*

$$S_2(a, N) \leq \sum_{n \in \mathbb{Z}} \hat{f}_T(n)\, \frac{1}{N} \operatorname{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\, \hat{B}_N^n\, \mathrm{Op}_N^{\mathrm{W}}(a)\, \hat{B}_N^{-n}\Big).$$

Notice that the terms in the sum on the RHS vanish for $|n| > T$.

Proof. Let $\{\varphi_j\}$ be the eigenbasis of $\hat{B}_N$, with $\hat{B}_N \varphi_j = \mathrm{e}^{2\pi i \theta_j}\, \varphi_j$. Then one has

$$\operatorname{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\, \hat{B}_N^n\, \mathrm{Op}_N^{\mathrm{W}}(a)\, \hat{B}_N^{-n}\Big) = \sum_{j,k=0}^{N-1} \mathrm{e}^{2\pi i n(\theta_k - \theta_j)}\, |\langle \mathrm{Op}_N^{\mathrm{W}}(a)\varphi_j, \varphi_k\rangle|^2.$$

Multiplying by $\hat{f}_T(n)$ and summing over n, we get,

$$\begin{aligned}
&\sum_{n \in \mathbb{Z}} \hat{f}_T(n)\, \operatorname{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\, \hat{B}_N^n\, \mathrm{Op}_N^{\mathrm{W}}(a)\, \hat{B}_N^{-n}\Big)\\
&\quad = \sum_{j,k=0}^{N-1} f_T(\theta_k - \theta_j)\, |\langle \mathrm{Op}_N^{\mathrm{W}}(a)\varphi_j, \varphi_k\rangle|^2\\
&\quad = \sum_{j=0}^{N-1} f_T(0)\, |\langle \mathrm{Op}_N^{\mathrm{W}}(a)\varphi_j, \varphi_j\rangle|^2 + \sum_{j \neq k} f_T(\theta_k - \theta_j)\, |\langle \mathrm{Op}_N^{\mathrm{W}}(a)\varphi_j, \varphi_k\rangle|^2\\
&\quad \geq N\, S_2(a, N).
\end{aligned}$$

The final inequality follows from the positivity of f_T and the property $f_T(0) \geq 1$. $\square$

To prove the theorem we will estimate the traces appearing in Lemma 14. Due to the support properties of $\hat{f}_T$, only the terms with $n \in [-T, T]$ will be needed. We take the time T depending on N, precisely as

$$T = T(N) := \frac{T_{\mathrm{E}}}{11},$$

where T_{E} is the Ehrenfest time (1.2). For each $n \in \mathbb{Z} \cap [-T, T]$, we will apply the Egorov Theorem 12. We first decompose a into a "good" part a_n and "bad" part a_n^{bad}, as described in Definition 5:

$$a = a_n + a_n^{\mathrm{bad}}, \quad a_n := a.\chi_{\delta,n}. \tag{6.3}$$

We let the width $\delta > 0$ depend on N as $\delta \asymp (\log N)^{-1}$. Therefore, for any $n \in [-T, T]$ we will have $\frac{2^{|n|}}{\delta} \ll N^{1/10}$. As a result, the bounds (5.28) for the derivatives of a_n read:

$$\forall n \in \mathbb{Z} \cap [-T, T], \qquad \|\partial^\gamma a_n\|_{C^0} \ll_\gamma \|a\|_{C^{|\gamma|}}\, N^{\frac{|\gamma|}{10}}. \tag{6.4}$$

Furthermore, the same bounds are satisfied by the derivatives of a_n^{bad} and $a_n \circ B^{-n}$.

We decompose the traces of Lemma 14 according to the splitting (6.3):

$$\begin{aligned}
\operatorname{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\, \hat{B}_N^n\, \mathrm{Op}_N^{\mathrm{W}}(a)\, \hat{B}_N^{-n}\Big) &= \operatorname{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\, \hat{B}_N^n\, \mathrm{Op}_N^{\mathrm{W}}(a_n)\, \hat{B}_N^{-n}\Big)\\
&\quad + \operatorname{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\, \hat{B}_N^n\, \mathrm{Op}_N^{\mathrm{W}}(a_n^{\mathrm{bad}})\, \hat{B}_N^{-n}\Big).
\end{aligned} \tag{6.5}$$

The second term in the RHS will be controlled by replacing $\mathrm{Op}_N^{\mathrm{W}}(a_n^{\mathrm{bad}})$ by its anti-Wick quantisation:

$$\mathrm{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\,\hat{B}_N^n\,\mathrm{Op}_N^{\mathrm{W}}(a_n^{\mathrm{bad}})\,\hat{B}_N^{-n}\Big) = \mathrm{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\,\hat{B}_N^n\,\mathrm{Op}_N^{\mathrm{AW},1}(a_n^{\mathrm{bad}})\,\hat{B}_N^{-n} + \mathcal{R}_N(n)\Big). \tag{6.6}$$

The remainder $\mathcal{R}_N(n)$ is dealt with using part I of Proposition 8, together with the bounds (6.4) applied to a_n^{bad}:

$$\begin{aligned}\|\mathcal{R}_N(n)\| &\le \|\mathrm{Op}_N^{\mathrm{W}}(a)\|\,\|\mathrm{Op}_N^{\mathrm{W}}(a_n^{\mathrm{bad}}) - \mathrm{Op}_N^{\mathrm{AW},1}(a_n^{\mathrm{bad}})\| \\ &\ll \|\mathrm{Op}_N^{\mathrm{W}}(a)\|\,\frac{\|a_n^{\mathrm{bad}}\|_{C^5}}{N} \ll \|\mathrm{Op}_N^{\mathrm{W}}(a)\|\,\frac{\|a\|_{C^5}}{N^{1/2}}.\end{aligned} \tag{6.7}$$

In order to compute $\mathrm{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\,\hat{B}_N^n\,\mathrm{Op}_N^{\mathrm{AW},1}(a_n^{\mathrm{bad}})\,\hat{B}_N^{-n}\Big)$, we split the function a_n^{bad} into its positive and negative parts, $a_n^{\mathrm{bad}} = a_{n,+}^{\mathrm{bad}} - a_{n,-}^{\mathrm{bad}}$, where $a_{n,\pm}^{\mathrm{bad}} \ge 0$. We then use the following (standard) linear algebra lemma to estimate the trace:

Lemma 15. *Let A, B be self-adjoint operators on $\mathcal{H}_N$, and assume B is positive. Then*

$$|\mathrm{Tr}(AB)| \le \|A\|\,\mathrm{Tr}(B). \tag{6.8}$$

Since the anti-Wick operator $\mathrm{Op}_N^{\mathrm{AW},1}(a_{n,+}^{\mathrm{bad}})$ is positive, this lemma yields:

$$\Big|\mathrm{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\,\hat{B}_N^n\,\mathrm{Op}_N^{\mathrm{AW},1}(a_{n,+}^{\mathrm{bad}})\,\hat{B}_N^{-n}\Big)\Big| \le \|\mathrm{Op}_N^{\mathrm{W}}(a)\|\,\mathrm{Tr}\big(\mathrm{Op}_N^{\mathrm{AW},1}(a_{n,+}^{\mathrm{bad}})\big),$$

and similarly by replacing $a_{n,+}^{\mathrm{bad}}$ by $a_{n,-}^{\mathrm{bad}}$. By linearity and $a_{n,+}^{\mathrm{bad}} + a_{n,-}^{\mathrm{bad}} = |a_n^{\mathrm{bad}}|$, we get

$$\Big|\mathrm{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\,\hat{B}_N^n\,\mathrm{Op}_N^{\mathrm{AW},1}(a_n^{\mathrm{bad}})\,\hat{B}_N^{-n}\Big)\Big| \le \|\mathrm{Op}_N^{\mathrm{W}}(a)\|\,\mathrm{Tr}\big(\mathrm{Op}_N^{\mathrm{AW},1}(|a_n^{\mathrm{bad}}|)\big).$$

From Eq. (5.10), the trace on the RHS is equal to $N \cdot \|a_n^{\mathrm{bad}}\|_{L_1(\mathbb{T}^2)}\big(1 + \mathcal{O}(\mathrm{e}^{-\pi N/2})\big)$. Since a_n^{bad} is supported on a neighbourhood of $\mathcal{S}_n$ of area $\mathcal{O}(\delta)$, its L^1 norm is of order $\mathcal{O}(\delta\,\|a\|_{C^0})$. Using the Calderón-Vaillancourt estimate $\|\mathrm{Op}_N^{\mathrm{W}}(a)\| \le C\,\|a\|_{C^2}$, we have thus proven the following bound for the second term in (6.5):

$$\frac{1}{N}\mathrm{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\,\hat{B}_N^n\,\mathrm{Op}_N^{\mathrm{W}}(a_n^{\mathrm{bad}})\,\hat{B}_N^{-n}\Big) \ll \|a\|_{C^2}\left(\delta\,\|a\|_{C^0} + \frac{\|a\|_{C^5}}{N^{1/2}}\right). \tag{6.9}$$

We now estimate the first term in (6.5). We write

$$\mathrm{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\,\hat{B}_N^n\mathrm{Op}_N^{\mathrm{W}}(a_n)\hat{B}_N^{-n}\Big) = \mathrm{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\mathrm{Op}_N^{\mathrm{W}}(a_n \circ B^{-n}) + \mathcal{R}'_N(n)\Big), \tag{6.10}$$

and control the remainder $\mathcal{R}'_N(n)$ with the Egorov estimate (5.30), remembering that $n \le T_{\mathrm{E}}/11$:

$$\begin{aligned}\|\mathcal{R}'_N(n)\| &\ll \|\mathrm{Op}_N^{\mathrm{W}}(a)\|\left(\|a\|_{C^0}\,N^{5/4}\,2^{n/4}\,\mathrm{e}^{-\pi N\delta^2/2^n} + \frac{2^{6n}\,\|a\|_{C^5}}{N\,\delta^5}\right) \\ &\ll \frac{\|a\|_{C^5}^2}{N^{2/5}}.\end{aligned} \tag{6.11}$$

The following lemma (proved in [MO'K, Lemma 3.1]) will allow us to replace the quantum product by a classical one.

Lemma 16. *There exists $C > 0$ such that, for any pair $a, b \in C^\infty(\mathbb{T}^2)$,*

$$\forall N \geq 1, \qquad \|\mathrm{Op}_N^{\mathrm{W}}(a)\,\mathrm{Op}_N^{\mathrm{W}}(b) - \mathrm{Op}_N^{\mathrm{W}}(ab)\| \leq C\,\frac{\|a\|_{C^4}\,\|b\|_{C^4}}{N}. \tag{6.12}$$

Using this lemma and the bounds (6.4), we get

$$\mathrm{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}(a)\,\mathrm{Op}_N^{\mathrm{W}}(a_n \circ B^{-n})\Big) = \mathrm{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}\big(a(a_n \circ B^{-n})\big) + \mathcal{R}''_N(n)\Big),$$
$$\text{with} \qquad \|\mathcal{R}''_N(n)\| \ll \frac{\|a_n\|_{C^4}\,\|a_n \circ B^{-n}\|_{C^4}}{N} \ll \frac{\|a\|_{C^4}^2}{N^{1/5}}. \tag{6.13}$$

To finally estimate the trace of $\mathrm{Op}_N^{\mathrm{W}}\big(a(a_n \circ B^{-n})\big)$, we use Eq. (5.9) together with the estimates (6.4):

$$\frac{1}{N}\mathrm{Tr}\Big(\mathrm{Op}_N^{\mathrm{W}}\big(a(a_n \circ B^{-n})\big)\Big) = \int_{\mathbb{T}^2} a(a_n \circ B^{-n})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}\Big(\frac{\|a\|_{C^3}^2}{N^2}\Big).$$

It remains to compute the integral on the RHS. We split it in two integrals, according to $a_n = a - a_n^{\mathrm{bad}}$. The second integral can be bounded by

$$\left|\int_{\mathbb{T}^2} a(\boldsymbol{x})\,a_n^{\mathrm{bad}}(B^{-n}\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| \leq \|a\|_{C^0}\,\|a_n^{\mathrm{bad}}\|_{L^1} \ll \|a\|_{C^0}^2\,\delta, \tag{6.14}$$

while the first one reads

$$\int_{\mathbb{T}^2} a(\boldsymbol{x})\,a(B^{-n}\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \mathcal{K}_{a\,a}(n). \tag{6.15}$$

This integral is the classical autocorrelation function for the observable $a(\boldsymbol{x})$, a purely classical quantity. At this point we must use the dynamical properties of the classical baker's map B, namely its fast mixing properties (see the end of Sect. 2): for some $\Gamma > 0$, the autocorrelation decays (when $n \to \infty$) as

$$\mathcal{K}_{a\,a}(n) \ll \|a\|_{C^1}^2\,\mathrm{e}^{-\Gamma|n|}.$$

Collecting all terms and using the properties of the function $\hat{f}_T$, Lemma 14 finally yields the following upper bound:

$$\begin{aligned} S_2(a, N) &\ll \|a\|_{C^5}^2 \sum_{n\in[-T,T]} |\hat{f}_T(n)|\,\Big(\mathrm{e}^{-\Gamma|n|} + \delta + \frac{1}{N^{1/5}}\Big) \\ &\ll \|a\|_{C^5}^2\,\Big(\frac{1}{T} + \delta\Big). \end{aligned}$$

Since we took $T \asymp \log N$ and $\delta \asymp (\log N)^{-1}$, this concludes the proof of Theorem 1. □

Proof of Corollary 2. We start by picking an observable $a \in C^\infty(\mathbb{T}^2)$, assuming $\int a(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$. For any decreasing sequence $\alpha(N) \xrightarrow{N\to\infty} 0$, Chebychev's inequality yields an upper bound on the number of eigenvectors of $\hat{B}_N$ for which $|\langle\varphi_{N,j}, \mathrm{Op}_N^{\mathrm{W}}(a)\varphi_{N,j}\rangle| > \alpha(N)$:

$$\frac{\#\big\{j \in \{1,\ldots,N\}\,:\,|\langle\varphi_{N,j}, \mathrm{Op}_N^{\mathrm{W}}(a)\,\varphi_{N,j}\rangle| > \alpha(N)\big\}}{N} \leq \frac{S_2(a, N)}{\alpha(N)^2}. \tag{6.16}$$

From Theorem 1, if we take $\alpha(N) >> (\log N)^{-1/2}$, the above fraction converges to zero. Defining $J_N(a)$ as the complement of the set in the above numerator, we obtain a sequence of subsets $J_N(a) \subset \{1, \dots, N\}$ satisfying $\frac{\#J_N(a)}{N} \to 1$, such that the eigenstates φ_{N,j_N} with $j_N \in J_N(a)$ satisfy (1.3).

Using a standard diagonal argument [CdV, HMR, Zel1], one can then extract subsets $J_N \subset \{1, \dots, N\}$ independent of the observable $a \in C^\infty(\mathbb{T}^2)$, with $\frac{\#J_N}{N} \to 1$, such that (1.3) is satisfied for any $a \in C^\infty(\mathbb{T}^2)$ if one takes $j_N \in J_N$. □

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