Commun. Math. Phys. 245, 201–214 (2004)
Digital Object Identifier (DOI) 10.1007/s00220-003-1019-x

Communications in
Mathematical Physics

On the Maximal Scarring for Quantum Cat Map Eigenstates

Frédéric Faure[1], **Stéphane Nonnenmacher**[2,3]

[1] Laboratoire de Physique et Modélisation des Milieux Condensés (LPM2C), BP 166, 38042 Grenoble Cédex 9, France. E-mail: frederic.faure@ujf-grenoble.fr
[2] Service de Physique Théorique, CEA/DSM/PhT Unité de Recherche Associée au CNRS, CEA/Saclay, 91191 Gif-sur-Yvette Cédex, France. E-mail: nonnen@spht.saclay.cea.fr
[3] Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720-5070, USA

Received: 23 April 2003 / Accepted: 29 July 2003
Published online: 16 January 2004 –

Abstract: We consider the quantized hyperbolic automorphisms on the 2-dimensional torus (or generalized quantum cat maps), and study the localization properties of their eigenstates in phase space, in the semiclassical limit. We prove that if the semiclassical measure corresponding to a sequence of normalized eigenstates has a pure point component (phenomenon of "strong scarring"), then the weight of this component cannot be larger than the weight of the Lebesgue component, and therefore admits the sharp upper bound $1/2$.

1. Introduction

We are interested in the semiclassical properties of quantum maps on the two-dimensional torus $\mathbb{T}^2$, that is the unitary transformations $\hat{M}_h$ which quantize a symplectic map M on $\mathbb{T}^2$ ($h = 2\pi\hbar$ is Planck's constant). More precisely, we focus on maps M having a chaotic dynamics (that is, at least ergodic w.r.t. to the Lebesgue measure), and investigate the phase space properties of the eigenstates of $\hat{M}_h$ in the semiclassical limit $h \to 0$. To any sequence of eigenstates $\{|\psi_h\rangle\}_{h\to 0}$ corresponds a sequence of probability measures on the torus $\{\mu_h\}_{h\to 0}$. In the weak-$*$ topology, the set of Borel probability measures on the torus is compact so the sequence $\{\mu_h\}_{h\to 0}$ admits at least one accumulation point μ; such a μ is called a "semiclassical measure" (also a "quantum limit") of the map M, related with the sequence $\{|\psi_h\rangle\}_{h\to 0}$. From Egorov's theorem, the measure μ is invariant through the classical map M. A natural question is the following:

> «For any M-invariant probability measure ν, does there exist a sequence of eigenstates of the quantized map $\hat{M}_h$ admitting ν as a semiclassical measure ?»

If the answer is negative, then one wants to determine the set of semiclassical measures generated by all possible sequences of eigenstates.

If the map M is ergodic (w.r. to the Lebesgue measure on $\mathbb{T}^2$), it admits a unique absolutely continuous invariant measure, namely the Lebesgue measure itself (which is also the Liouville measure on the symplectic manifold $\mathbb{T}^2$). On the other hand, each periodic orbit of M supports an invariant Dirac (atomic, pure point) measure. As a result, if M is an Anosov map (i.e. uniformly hyperbolic on $\mathbb{T}^2$), the space of invariant pure point measures is infinite-dimensional (its closure yields the full set of invariant probability measures $\mathfrak{M}_M$ [19]).

We now review some results obtained so far on this issue. Schnirelman's theorem provides a partial answer to the above question, in the case of an ergodic map: "almost all sequences" of eigenstates admit for semiclassical measure the Lebesgue measure [18, 5, 21, 3]. This phenomenon is called "quantum ergodicity" in the mathematics literature. Still, this theorem does not exclude "exceptional sequences" of eigenstates converging to a different semiclassical measure.

Extensive numerical studies have shown that many eigenstates of quantum hyperbolic systems show an enhanced concentration on one or several (unstable) periodic orbits [9]. Still, it is commonly believed that this enhancement (called "scarring") is weak enough to allow the concerned eigenstates to converge (in the weak-$*$ sense) to the Liouville measure. Thus, "scarring" must not be mistaken with "strong scarring", that is the existence of a sequence of eigenstates, the semiclassical measure of which contains a pure point component on some periodic orbit.

"Quantum unique ergodicity", that is, the absence of any exceptional sequence of eigenstates, was proven for some families of ergodic linear parabolic maps on $\mathbb{T}^2$ [3, 15], using the fact that for these maps the Lebesgue measure is the unique invariant measure. On the other hand, a special class of ergodic piecewise affine transformations on $\mathbb{T}^2$ have been studied and quantized in [4], for which every classical invariant measure is a semiclassical measure. In both these cases, the maps are only "weakly chaotic", in particular they are not mixing and have no periodic point.

In the continuous-time framework, precise results have been obtained for the eigenstates of the Laplace-Beltrami operator on arithmetic manifolds of constant negative curvature in dimension 2 or 3 [17, 13, 16, 10]; the corresponding classical dynamics, namely the geodesic flows on the manifolds, are known to be of Anosov type. E. Lindenstrauss [12] recently proved quantum unique ergodicity for sequences of joint eigenstates of the Laplacian and the Hecke operators on arithmetic surfaces (all eigenstates of the Laplacian are conjectured to be of this type).

In this paper, we restrict ourselves to a very special family of Anosov maps on the torus, namely the linear hyperbolic automorphisms of the 2-torus, also called generalized Arnold's cat maps. For any automorphism A of the form $A \equiv \mathrm{Id} \bmod 4$ and any value of h, Rudnick and Kurlberg have defined a family of "Hecke operators" commuting with the quantum map $\hat{A}_h$, and proven unique quantum ergodicity for the sequences of joint eigenstates [11]. However, as opposed to the case of arithmetic surfaces, many eigenstates of $\hat{A}_h$ are not Hecke eigenstates, leaving open the possibility of exceptional sequences. In [2], Bonechi and De Bièvre have shown that for any automorphism A, a semiclassical measure of A cannot be completely localized (or completely "scarred"), in that the weight of its pure point component is bounded above by $\left(\sqrt{5}-1\right)/2 \simeq 0.62$ (their proof also applies to ergodic automorphisms on higher-dimensional symplectic tori). In [7], sequences of eigenstates of $\hat{A}_h$ were explicitly constructed, for which the semiclassical measure has a pure point component of weight $1/2$ localized on a finite set of periodic orbits, the remaining part of the measure being Lebesgue. In this paper we improve the results of [2] as follows:

Theorem 1.1. *Let A be a hyperbolic automorphism of $\mathbb{T}^2$, and μ be a normalized semiclassical measure of A. Splitting μ into its pure point, Lebesgue and singular continuous components, $\mu = \mu_{pp} + \mu_{Leb} + \mu_{sc}$, the following inequalities hold between the weights of these components: $\mu_{Leb}(\mathbb{T}^2) \geq \mu_{pp}(\mathbb{T}^2)$, which implies $\mu_{pp}(\mathbb{T}^2) \leq 1/2$.*

The states constructed in [7] saturate this upper bound: they are "maximally scarred".

After recalling the definition of the quantized automorphisms (Sect. 2), we prove in Sect. 3 two "dynamical" propositions (the first one was proven in [2, Sect. 5] in a more general context). They both deal with the evolution through $\hat{A}_h$ up to the "Ehrenfest time" $T \sim |\log h|/\lambda$, of quantum states with prescribed initial localization properties. Using these propositions, we then show in Sect. 4 that for any finite union $\mathcal{S}$ of periodic orbits, any semiclassical measure μ of A satisfies $\mu(\mathcal{S}) \leq \mu_{Leb}(\mathbb{T}^2)$ (Theorem 4.1), from where the above theorem is a straightforward corollary. In final remarks, we draw consequences of the above theorem, concerning the determination of the set $\mathfrak{M}_{A,\mathrm{SC}}$ of semiclassical measures for the automorphism A. We also discuss possible extensions of these results to a broader class of Anosov systems.

Let us mention that our methods are quite similar to the one used in [2]. The reason why we obtain a sharper bound lies in a cautious use of the Cauchy-Schwarz inequality. In the proof of [2, Thm. 1.2], this inequality is directly used to estimate the localization properties of the eigenstate, thereby introducing a loss of information. On the other hand, we apply this inequality in Sect. 3 to obtain sharp estimates on the localization of "partial eigenstates", while Eqs. 4.5–4.7 dealing with the full eigenstate remain equalities.

2. Quantum Hyperbolic Automorphisms on $\mathbb{T}^2$

2.1. Quantum mechanics on $\mathbb{T}^2$. We briefly describe the quantum mechanics on the 2-torus phase space as defined in [8, 6]. The Hilbert space of quantum states corresponding to Planck's constant h will be called $\mathcal{H}_h$: quantum states can be identified with distributions $\psi(q) \in \mathcal{S}'(\mathbb{R})$ such that $\psi(q+1) = \psi(q)$ and $(\mathcal{F}\psi)(k+h^{-1}) = (\mathcal{F}\psi)(k)$, where $\mathcal{F}$ is the Fourier transform. $\mathcal{H}_h$ is a nontrivial vector space iff $h^{-1} \in \mathbb{N}^*$, and then $\mathcal{H}_h \simeq \mathbb{C}^{(h^{-1})}$. In what follows, h will always be taken of that form; the semiclassical limit is therefore defined as the limit $h \to 0$, $h^{-1} \in \mathbb{N}$.

For any classical observable $f \in C^\infty(\mathbb{T}^2)$, we note respectively $\hat{f} = Op_h(f)$ its Weyl quantization and $\hat{f}^{AW} = Op_h^{AW}(f)$ its anti-Wick quantization on $\mathcal{H}_h$ [3]. The anti-Wick quantized operator satisfies the property $\|\hat{f}^{AW}\| \leq \|f\|_\infty = \sup_{\mathbb{T}^2}(|f|)$. To any state $|\psi_h\rangle \in \mathcal{H}_h$ correspond the Wigner and Husimi measures $\tilde{\mu}_h, \mu_h$ defined as [3]

$$\tilde{\mu}_h(f) = \int_{\mathbb{T}^2} dx\, W_{\psi_h}(x)\, f(x) = \langle \psi_h | \hat{f} | \psi_h \rangle, \tag{2.1}$$

$$\mu_h(f) = \int_{\mathbb{T}^2} dx\, H_{\psi_h}(x)\, f(x) = \int_{\mathbb{T}^2} \frac{dx}{h} |\langle \psi_h | x \rangle|^2\, f(x) = \langle \psi_h | \hat{f}^{AW} | \psi_h \rangle. \tag{2.2}$$

The ket $|x\rangle$ denotes the (asymptotically normalized) coherent state in $\mathcal{H}_h$ localized at the point $x = (q, p)$, with widths $\Delta q \sim \Delta p \sim \sqrt{\hbar/2}$. While the Husimi density $H_{\psi_h}(x)$ is a non-negative smooth function on $\mathbb{T}^2$, the "Wigner function" $W_{\psi_h}(x)$ is a finite combination of delta peaks with real (possibly negative) coefficients.

We now consider an infinite sequence of states $\{|\psi_h\rangle \in \mathcal{H}_h\}_{h\to 0}$. By definition, the corresponding sequence of Husimi measures $\{\mu_h\}$ weak-$*$ converges to μ iff for any smooth observable f, one has $\mu_h(f) \xrightarrow{h\to 0} \mu(f)$. The sequence of signed measures $\{\tilde{\mu}_h\}$ then admits the same weak-$*$ limit (μ_h is the convolution of $\tilde{\mu}_h$ by a Gaussian kernel of width $\sim \sqrt{\hbar}$). The limit (Borel) measure μ is then called the **semiclassical measure** of the sequence $\{|\psi_h\rangle\}_{h\to 0}$ (by a slight abuse of language, we also say that the sequence of states $\{|\psi_h\rangle\}_{h\to 0}$ converges to μ).

If the states $|\psi_h\rangle$ are (normalized) eigenstates of a quantized map $\hat{M}_h$, then μ is called a **semiclassical measure for** M. In that case, Egorov's property, that is the semiclassical commutation of time evolution and quantization:

$$\forall t \in \mathbb{Z}, \quad \|\hat{M}_h^{-t}\, Op_h(f)\hat{M}_h^{t} - Op_h(f \circ M^t)\| \xrightarrow{h\to 0} 0, \tag{2.3}$$

implies that μ is an invariant measure for the classical map M.

A sequence of states $\{|\psi_h\rangle\}_{h\to 0}$ is said to be **equidistributed** if it converges semiclassically to (a multiple of) the Lebesgue measure on $\mathbb{T}^2$ (noted dx), i.e. iff for a certain $C > 0$ and for any observable f, $\mu_h(f)$ (equivalently $\tilde{\mu}_h(f)$) converges to $C\int_{\mathbb{T}^2} dx\, f(x)$. On the other hand, a sequence $\{|\psi_h\rangle\}_{h\to 0}$ is called **localized** iff it converges to a pure point measure on $\mathbb{T}^2$, that is if there exists a countable set of points $\{x_i\}$ and weights $\alpha_i > 0$, $\sum_i \alpha_i < \infty$ such that for any observable f, $\mu_h(f) \xrightarrow{h\to 0} \sum_i \alpha_i f(x_i)$.

2.2. Quantum yperbolic automorphisms. An automorphism of $\mathbb{T}^2$, or generalized cat map, is the diffeomorphism on $\mathbb{T}^2$ defined by the action of a matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ on the point $x = \binom{q}{p} \in \mathbb{T}^2$. The map itself will also be denoted by A. It is uniformly hyperbolic on $\mathbb{T}^2$ (therefore of Anosov type) iff $|\mathrm{tr}(A)| > 2$. Depending on the sign of the trace, the eigenvalues of A are of the form $\{\pm e^{\lambda}, \pm e^{-\lambda}\}$, where $\lambda > 0$ is the Lyapounov exponent. The corresponding eigenaxes define the unstable/stable directions at any point $x \in \mathbb{T}^2$, and their projections on the torus make up the unstable and stable manifolds of the origin (which is an obvious fixed point). We will use the property that the slopes of these axes are irrational, so that both manifolds are dense on $\mathbb{T}^2$.

Under the condition $A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ mod 2, the linear automorphism A can be quantized on any $\mathcal{H}_h$, yielding a unitary matrix $\hat{A}_h$ of dimension h^{-1} [8, 6]. For simplicity, we will assume in the following sections that A is of that form. Yet, this restriction can easily be lifted: any matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ can be quantized on $\mathcal{H}_h$ if we restrict h^{-1} to take *even* values, or extend the definition of $\mathcal{H}_h$ to allow nontrivial "Bloch angles" [6, 3]. All our results can be straightforwardly generalized to those cases.

Let us call $T_{\mathbf{v}}$ the classical translation by the vector $\mathbf{v} \in \mathbb{R}^2$. It can be naturally quantized as a unitary matrix $\hat{T}_{\mathbf{v}}$ on $\mathcal{H}_h$ iff $\mathbf{v}$ belongs to the square lattice $(h\mathbb{Z})^2$, that is iff $\mathbf{v} = h\mathbf{k}$ for a certain $\mathbf{k} \in \mathbb{Z}^2$. The operator $\hat{T}_{h\mathbf{k}}$ can also be interpreted as the Weyl quantization $Op_h(F_{\mathbf{k}})$ of the complex-valued observable $F_{\mathbf{k}}(q,p) = \exp\big(-2\pi i(k_1 p - k_2 q)\big)$ on $\mathbb{T}^2$.

For any h, the following intertwining relation holds between the quantum automorphism $\hat{A}_h$ and the quantized translations:

$$\forall \mathbf{k} \in \mathbb{Z}^2, \qquad \hat{A}_h^{-1}\hat{T}_{h\mathbf{k}}\hat{A}_h = \hat{T}_{hA^{-1}\mathbf{k}} \Longleftrightarrow \hat{A}_h^{-1}\, Op_h(F_{\mathbf{k}})\, \hat{A}_h = Op_h(F_{\mathbf{k}} \circ A). \tag{2.4}$$

Comparing with Eq. (2.3), we see that the above identity realizes an *exact* Egorov property: quantization exactly commutes with time evolution, for arbitrary times t and arbitrary h [8]. This exact equality is *characteristic of linear maps*, and will be crucial in the next sections.

3. Localized States Evolve into Equidistributed States

We define the Ehrenfest time or log-time corresponding to the quantum map $\hat{A}_h$ by

$$T = \left[\frac{|\log h|}{\lambda}\right],$$

where [.] denotes the integer part (we will always omit indicating the h-dependence of T).

Consider a set of n points $\mathcal{S} = \{x_i\}_{i=1\to n}$ on $\mathbb{T}^2$, such that for a certain integer $d > 0$, all vectors $(x_i - x_j)$ belong to the lattice $\frac{1}{d}\mathbb{Z}^2$. Assign to each point x_i a weight $\alpha_i > 0$, with $\sum_i \alpha_i = 1$, and define the Dirac probability measure $\delta_{\mathcal{S},\alpha} = \sum_{i=1}^n \alpha_i\, \delta_{x_i}$ on the torus.

Proposition 3.1. *Suppose that a sequence of states $\{|\varphi_h\rangle \in \mathcal{H}_h\}_{h\to 0}$ converges to the measure $\delta_{\mathcal{S},\alpha}$. Then the sequence of states $\{|\varphi'_h\rangle = \hat{A}_h^T|\varphi_h\rangle\}$ is semiclassically equidistributed.*

This property has been proven in a more general setting (higher dimensional automorphisms, time T replaced by a time interval) in [2, Theorem 5.1]. To be self-contained, we give below a "fast" proof of this proposition for our case.

Proof. As a first step, we draw consequences from the assumption $\mu_h \xrightarrow{h\to 0} \delta_{\mathcal{S},\alpha}$, where μ_h is the Husimi measure of $|\varphi_h\rangle$. Denoting by $D(x_i, a)$ the disk of radius $a > 0$ centered at x_i, and by $D(\mathcal{S}, a) \stackrel{\text{def}}{=} \cup_{i=1}^n D(x_i, a)$ the corresponding neighbourhood of $\mathcal{S}$, the weak-$*$ convergence of μ_h implies that $\mu_h(\mathbb{T}^2) \xrightarrow{h\to 0} 1$ and that for any $a > 0$, $\mu_h\big(\mathbb{T}^2 \setminus D(\mathcal{S}, a)\big) \xrightarrow{h\to 0} 0$. From a standard diagonal argument, one can then construct a decreasing sequence $a_h > 0$, $a_h \xrightarrow{h\to 0} 0$ such that

$$|\mu_h\big(\mathbb{T}^2\big) - 1| \le a_h \quad \text{and} \quad \mu_h\big(\mathbb{T}^2 \setminus D(\mathcal{S}, a_h)\big) \le a_h. \tag{3.1}$$

In a second step, we remark that proving the equidistribution of the sequence $\big\{|\varphi'_h\rangle\big\}$ amounts to prove that for any fixed $\mathbf{k} \in \mathbb{Z}^2$,

$$\tilde{\mu}'_h(F_{\mathbf{k}}) = \langle\varphi'_h|\hat{T}_{h\mathbf{k}}|\varphi'_h\rangle \xrightarrow{h\to 0} \int_{\mathbb{T}^2} F_{\mathbf{k}}(x)\, dx = \delta_{\mathbf{k},\mathbf{0}}. \tag{3.2}$$

The case $\mathbf{k} = \mathbf{0}$ is obvious since $\tilde{\mu}'_h(\mathbb{T}^2) = \tilde{\mu}_h(\mathbb{T}^2) = \mu_h(\mathbb{T}^2) \xrightarrow{h\to 0} 1$ by assumption. We now select a fixed wave vector $\mathbf{0} \neq \mathbf{k} \in \mathbb{Z}^2$, and study the above LHS. The intertwining relation (2.4) allows us to rewrite it as

$$\langle\varphi'_h|\hat{T}_{h\mathbf{k}}|\varphi'_h\rangle = \langle\varphi_h|\hat{A}_h^{-T}\hat{T}_{h\mathbf{k}}\hat{A}_h^T|\varphi_h\rangle = \langle\varphi_h|\hat{T}_{h\mathbf{k}'_h}|\varphi_h\rangle, \tag{3.3}$$

with the vector $\mathbf{k}'_h = A^{-T}\mathbf{k}$. From the definition of the Ehrenfest time, the "large" eigenvalue of A^{-T} is $(\pm e^{\lambda})^T = \pm C(h)\frac{1}{h}$, where the prefactor satisfies $e^{-\lambda} \leq C(h) \leq 1$. The decomposition of that vector in the eigenbasis of A then reads:

$$h\mathbf{k}'_h = \pm e^{\lambda T} h\mathbf{k}^{stable} \pm e^{-\lambda T} h\mathbf{k}^{unstable} = \pm C(h)\mathbf{k}^{stable} + \mathcal{O}(h^2). \tag{3.4}$$

The vector $h\mathbf{k}'_h$ thus has a finite length (bounded from above and from below uniformly in h), and is asymptotically parallel to the stable axis. Because the slope of that axis is irrational, the *distance* between $h\mathbf{k}'_h$ and the lattice $\frac{1}{d}\mathbb{Z}^2$ is bounded from below by a constant $c(\mathbf{k}) > 0$ uniformly w.r. to h.

To finish the proof, we write the above overlap as a coherent state integral:

$$|\langle\varphi_h|\hat{T}_{h\mathbf{k}'_h}|\varphi_h\rangle| = \left|\int_{\mathbb{T}^2} \frac{dx}{h}\ \langle\varphi_h|x\rangle\langle x|\hat{T}_{h\mathbf{k}'_h}|\varphi_h\rangle\right| \leq \int_{\mathbb{T}^2} \frac{dx}{h}\ |\langle x|\varphi_h\rangle|\,|\langle x - h\mathbf{k}'_h|\varphi_h\rangle|. \tag{3.5}$$

We then split the integral on the RHS between $D(\mathcal{S}, a_h)$ and its complement. The integral on $D(\mathcal{S}, a_h)$ is estimated through a Cauchy-Schwarz inequality:

$$\int_{D(\mathcal{S},a_h)} \frac{dx}{h}\ |\langle x|\varphi_h\rangle|\,|\langle x - h\mathbf{k}'_h|\varphi_h\rangle| \leq \sqrt{\mu_h\big(D(\mathcal{S}, a_h)\big)\ \mu_h\big(D(\mathcal{S}, a_h) - h\mathbf{k}'_h\big)}.$$

Due to above-mentioned property of $h\mathbf{k}'_h$ and the fact that $x_i - x_j \in \frac{1}{d}\mathbb{Z}^2$, for small enough a_h the set $D(\mathcal{S}, a_h) - h\mathbf{k}'_h$ has no intersection with $D(\mathcal{S}, a_h)$. As a consequence, using (3.1), the second factor under the square-root on the RHS is bounded above by a_h, and the full RHS by $\sqrt{a_h(1 + a_h)}$. The remaining integral over $\mathbb{T}^2\backslash D(\mathcal{S}, a_h)$ admits the same upper bound for similar reasons. □

Proposition 3.2. *Let $(\mathcal{S}, \alpha)$ be a finite weighted set of A-periodic points, and ν an A-invariant probability measure satisfying $\nu(\mathcal{S}) = 0$. Suppose that the sequence $\{|\varphi_{\mathcal{S},h}\rangle\}_{h\to 0}$ converges semiclassically to the measure $\delta_{\mathcal{S},\alpha}$, and that a second sequence $\{|\varphi_{\nu,h}\rangle\}_{h\to 0}$ converges to ν. Then the states $|\varphi'_{\mathcal{S},h}\rangle = \hat{A}_h^T|\varphi_{\mathcal{S},h}\rangle$, $|\varphi'_{\nu,h}\rangle = \hat{A}_h^T|\varphi_{\nu,h}\rangle$ satisfy:*

$$\forall\mathbf{k} \in \mathbb{Z}^2, \quad \langle\varphi'_{\mathcal{S},h}|\hat{T}_{h\mathbf{k}}|\varphi'_{\nu,h}\rangle \xrightarrow{h\to 0} 0.$$

Proof. We use similar methods as for the previous proposition. Namely, as in Eq. (3.3) we rewrite the overlap as

$$\langle\varphi'_{\mathcal{S},h}|\hat{T}_{h\mathbf{k}}|\varphi'_{\nu,h}\rangle = \langle\varphi_{\mathcal{S},h}|\hat{T}_{h\mathbf{k}'_h}|\varphi_{\nu,h}\rangle = \int_{\mathbb{T}^2} \frac{dx}{h}\ \langle\varphi_{\mathcal{S},h}|x\rangle\langle x|\hat{T}_{h\mathbf{k}'_h}|\varphi_{\nu,h}\rangle \tag{3.6}$$

with $h\mathbf{k}'_h$ given by Eq. (3.4). We want to cut this integral into two parts. Using the same notations as above, the assumptions of the proposition imply the existence of a decreasing sequence $a_h \xrightarrow{h\to 0} 0$ such that the Husimi measures of $|\varphi_{\mathcal{S},h}\rangle$ and $|\varphi_{\nu,h}\rangle$ satisfy

$$\mu_{\mathcal{S},h}\big(\mathbb{T}^2 \setminus D(\mathcal{S}, a_h)\big) \leq a_h, \quad \mu_{\mathcal{S},h}\big(\mathbb{T}^2\big) \leq 1 + a_h \quad \text{and} \quad \mu_{\nu,h}\big(\mathbb{T}^2\big) \leq 1 + a_h.$$

These inequalities also hold if we replace a_h by any decreasing sequence $b_h \geq a_h$, $b_h \to 0$. We can now bound the part of the integral (3.6) over the set $\mathbb{T}^2 \setminus D(\mathcal{S}, b_h)$:

$$\left| \int_{\mathbb{T}^2 \setminus D(\mathcal{S}, b_h)} \frac{dx}{h} \langle \varphi_{\mathcal{S},h} | x \rangle \langle x | \hat{T}_{h\mathbf{k}'_h} | \varphi_{\nu,h} \rangle \right| \leq \sqrt{\mu_{\mathcal{S},h}\big(\mathbb{T}^2 \setminus D(\mathcal{S}, b_h)\big) \, \mu_{\nu,h}\big(\mathbb{T}^2\big)}$$
$$\leq \sqrt{b_h (1 + b_h)} \xrightarrow{h \to 0} 0.$$

The second part of the integral is also estimated by Cauchy-Schwarz:

$$\left| \int_{D(\mathcal{S}, b_h)} \frac{dx}{h} \langle \varphi_{\mathcal{S},h} | x \rangle \langle x | \hat{T}_{h\mathbf{k}'_h} | \varphi_{\nu,h} \rangle \right| \leq \sqrt{\mu_{\mathcal{S},h}\big(\mathbb{T}^2\big) \, \mu_{\nu,h}\big(D(\mathcal{S}, b_h) - h\mathbf{k}'_h\big)}.$$

We now show that the sequence b_h can be chosen such that the second factor under the square root vanishes in the semiclassical limit, so that the full RHS vanishes as well. Indeed, using Eq. (3.4) and the bound on $C(h)$, one realizes that for h small enough, the set $D(\mathcal{S}, b_h) - h\mathbf{k}'_h$ is contained in the thin "tube"

$$T_{b_h} \stackrel{\text{def}}{=} D(\mathcal{S}, 2b_h) + \big([-1, -\mathrm{e}^{-\lambda}] \cup [e^{-\lambda}, 1]\big)\mathbf{k}^{stable}.$$

As $b_h \to 0$, this tube decreases to $T_0 = \mathcal{S} + \big([-1, -\mathrm{e}^{-\lambda}] \cup [e^{-\lambda}, 1]\big)\mathbf{k}^{stable}$, that is a union of segments of the stable manifold ($T_0 = \mathcal{S}$ if $k = 0$). The following simple lemma is proven in Appendix B.

Lemma 3.3. *For any invariant probability measure ν, any finite set of periodic points $\mathcal{S}$ and any $0 < a < b < \infty$, one has $\nu(\mathcal{S} + [a, b]\mathbf{e}^{stable}) = 0$, where $\mathbf{e}^{stable}$ is a unit vector in the stable direction.*

This lemma implies that $\nu(T_0) = 0$ in the case $k \neq 0$, the case $k = 0$ being trivial. Since $b_h \searrow 0$, the regularity of the Borel measure ν entails:

$$\lim_{b_h \to 0} \nu(T_{b_h}) = \nu(T_0) = 0.$$

Since by assumption the Husimi measures $\mu_{\nu,h}$ converge towards ν, one can by a diagonal argument choose the sequence $b_h \searrow 0$ (making sure that $b_h \geq a_h$) such that:

$$\mu_{\nu,h}\big(T_{b_h}\big) \xrightarrow{h \to 0} 0.$$

This finally implies that $\mu_{\nu,h}\big(D(\mathcal{S}, b_h) - h\mathbf{k}'_h\big) \xrightarrow{h \to 0} 0.$ □

4. Maximal Localization of the Semiclassical Measures

We call τ a periodic orbit of A of period T_τ, and denote its associated measure by

$$\delta_\tau = \frac{1}{T_\tau} \sum_{x \in \tau} \delta_x.$$

We consider a *finite* set $\mathcal{F}$ of periodic orbits, associate a weight $w_\tau > 0$ to each orbit ($\sum_{\tau \in \mathcal{F}} w_\tau = 1$), and construct the pure point invariant measure

$$\delta_{\mathcal{F},w} = \sum_{\tau \in \mathcal{F}} w_\tau \, \delta_\tau.$$

All periodic points of A have rational coordinates, so this measure is a particular instance of the measure $\delta_{\mathcal{S},\alpha}$ considered in Proposition 3.1. Indeed, grouping the periodic orbits of $\mathcal{F}$ together yields the set of rational points $\mathcal{S} = \mathcal{S}_{\mathcal{F}} = \cup_{\tau\in\mathcal{F}}\tau = \{x_1, \ldots, x_n\}$, and the weight allocated to each point x_i reads $\alpha_i = \frac{w_\tau}{T_\tau}$. From now on, the two notations $\delta_{\mathcal{F},w}$ and $\delta_{\mathcal{S},\alpha}$ will refer to the same invariant measure. We now state the central result of this article.

Theorem 4.1. *Let μ be a normalized A-invariant Borel measure of A, and $(\mathcal{F}, w)$ a finite weighted set of periodic orbits. μ may be decomposed into $\mu = \beta\delta_{\mathcal{F},w} + (1-\beta)\nu$, where ν is a normalized invariant measure satisfying $\nu(\mathcal{S}_{\mathcal{F}}) = 0$. If μ is a semiclassical measure of A, then its Lebesgue component has a weight $\geq \beta$, which in turn implies $\beta \leq 1/2$.*

Any invariant Borel measure $\mu \in \mathfrak{M}_A$ can obviously be decomposed in the above way, with $0 \leq \beta \leq 1$. We do not assume the measure ν to be continuous, but allow it to contain a pure point component localized on a (possibly countable) set of periodic orbits disjoint with $\mathcal{F}$. The statement of the theorem is of course stronger if we include in $(\mathcal{F}, w)$ as many orbits as possible. By a simple limit argument, one may eventually take for ν the continuous component of μ, allowing $(\mathcal{F}, w)$ to be a *countable* weighted set of orbits (still taking $\sum_{\tau\in\mathcal{F}} w_\tau = 1$): one then obtains Theorem 1.1 as a simple corollary of the one above.

Proof of Theorem 4.1. There are two steps in the proof. Let $\{|\psi_h\rangle\}_{h\to 0}$ be a sequence of eigenstates of $\hat{A}_h$ admitting μ as a semiclassical measure. Our first objective is to decompose the state $|\psi_h\rangle$ into $|\psi_{\mathcal{S},h}\rangle + |\psi_{\nu,h}\rangle$, such that the sequence $\{|\psi_{\mathcal{S},h}\rangle\}$ (resp. $\{|\psi_{\nu,h}\rangle\}$) converges to the measure $\beta\delta_{\mathcal{S},\alpha}$ (resp. the measure $(1-\beta)\nu$). This decomposition will be obtained by "projecting" $|\psi_h\rangle$ on appropriate (h-dependent) small neighborhoods of $\mathcal{S}$.

In the second part we will be guided by the following simple idea. Because $|\psi_h\rangle$ is an eigenstate of $\hat{A}_h$, $|\psi_h\rangle \propto \hat{A}_h^t|\psi_h\rangle = \hat{A}_h^t|\psi_{\mathcal{S},h}\rangle + \hat{A}_h^t|\psi_{\nu,h}\rangle$ for any $t \in \mathbb{Z}$, in particular for the Ehrenfest time $t = T$. From Proposition 3.1, the sequence of states $\{\hat{A}_h^T|\psi_{\mathcal{S},h}\rangle\}$ is equidistributed; together with Proposition 3.2, that implies that the semiclassical measure of $\hat{A}_h^T|\psi_h\rangle$ (that is, μ) contains a Lebesgue part of weight $\geq \beta$.

First part. Splitting the eigenstates. To "project" the eigenstates in a small neighbourhood of $\mathcal{S}$, we will use a function $\vartheta \in C^\infty(\mathbb{R}^2)$ satisfying $0 \leq \vartheta(x) \leq 1$ on $\mathbb{R}^2$, $\vartheta(x) = 0$ outside the disk $D(0, 2)$ and $\vartheta(x) = 1$ inside the disk $D(0, 1)$.

For any (small) $r > 0$, and any point $x_i \in \mathcal{S}$, let $\vartheta_{i,r}(x) = \sum_{\mathbf{n}\in\mathbb{Z}^2} \vartheta\left(\frac{x-x_i-n}{r}\right)$, which can be seen as a smooth function on $\mathbb{T}^2$, localized around x_i. We will also need the function $\vartheta_r = \sum_{i=1}^n \vartheta_{i,r}$ to take all points of $\mathcal{S}$ into account: this function is supported in the neighborhood $D(\mathcal{S}, 2r)$ of $\mathcal{S}$. There is an $r_0 > 0$ such that the disks $D(x_i, 2r_0)$ do not overlap each other, which implies $\vartheta_{i,r}\vartheta_{j,r} \equiv 0$ if $i \neq j$.

Fixing $0 < r \leq r_0$, we consider the anti-Wick quantization of these functions, $\hat{\vartheta}_{i,r}^{AW}$ and use them to decompose $|\psi_h\rangle$. We first observe that for any $x_i \in \mathcal{S}$,

$$\langle\psi_h|\hat{\vartheta}_{i,r}^{AW}|\psi_h\rangle \xrightarrow{h\to 0} \beta\delta_{\mathcal{S},\alpha}(\vartheta_{i,r}) + (1-\beta)\nu(\vartheta_{i,r}) = \beta\alpha_i + (1-\beta)\nu(\vartheta_{i,r}).$$

Furthermore, one has for any pair $x_i,\ x_j \in \mathcal{S}$,

$$\begin{aligned}\langle\psi_h|\hat{\vartheta}_{i,r}^{AW}\hat{\vartheta}_{j,r}^{AW}|\psi_h\rangle &\xrightarrow{h\to 0} \beta\delta_{\mathcal{S},\alpha}(\vartheta_{i,r}\vartheta_{j,r}) + (1-\beta)\nu(\vartheta_{i,r}\vartheta_{j,r})\\ &= \delta_{ij}\left(\beta\alpha_i + (1-\beta)\nu(\vartheta_{i,r}^2)\right).\end{aligned}$$

On the other hand, the regularity of the (Borel) measure ν entails:

$$\forall x_i \in \mathcal{S}, \qquad \nu(\vartheta_{i,r}) \xrightarrow{r\to 0} \nu(\{x_i\}) = 0 \qquad \text{and similarly} \qquad \nu(\vartheta_{i,r}^2) \xrightarrow{r\to 0} 0.$$

From the above limits, one can construct by a diagonal argument a decreasing sequence of radii $r(h) \xrightarrow{h\to 0} 0$ such that

$$\forall x_i,\ x_j \in \mathcal{S}, \quad \langle\psi_h|\hat{\vartheta}_{i,r(h)}^{AW}|\psi_h\rangle \xrightarrow{h\to 0} \beta\alpha_i \quad \text{and} \quad \langle\psi_h|\hat{\vartheta}_{i,r(h)}^{AW}\hat{\vartheta}_{j,r(h)}^{AW}|\psi_h\rangle \xrightarrow{h\to 0} \delta_{ij}\beta\alpha_i. \tag{4.1}$$

We now show that the two sequences of vectors

$$|\psi_{\mathcal{S},h}\rangle \stackrel{\text{def}}{=} \hat{\vartheta}_{r(h)}^{AW}|\psi_h\rangle, \qquad |\psi_{\nu,h}\rangle \stackrel{\text{def}}{=} \big(\mathrm{Id} - \hat{\vartheta}_{r(h)}^{AW}\big)|\psi_h\rangle \tag{4.2}$$

provide the desired decomposition of $|\psi_h\rangle$, that is the corresponding sequences respectively converge to the measures $\beta\delta_{\mathcal{S},\alpha}$ and $(1-\beta)\nu$. The first statement $\mu_{\mathcal{S},h} \xrightarrow{h\to 0} \beta\delta_{\mathcal{S},\alpha}$ seems quite natural: the operator $\hat{\vartheta}_{r(h)}^{AW}$ acts as a "microlocal projector" onto the set $\mathcal{S}$. This property is precisely expressed in the following lemma, which we prove in Appendix A.

Lemma 4.2. *For any $f \in C^\infty(\mathbb{T}^2)$, $x_i \in \mathcal{S}$ and any decreasing sequence $r(h) \xrightarrow{h\to 0} 0$, one has*

$$\|\hat{f}^{AW}\,\hat{\vartheta}_{i,r(h)}^{AW} - f(x_i)\,\hat{\vartheta}_{i,r(h)}^{AW}\| \xrightarrow{h\to 0} 0,$$

where $\|.\|$ is the operator norm on $\mathcal{H}_h$.

This lemma together with the properties (4.1) immediately yield the required limits:

$$\forall f \in C^\infty(\mathbb{T}^2), \quad \langle\psi_{\mathcal{S},h}|\hat{f}^{AW}|\psi_{\mathcal{S},h}\rangle \xrightarrow{h\to 0} \sum_i \beta\alpha_i f(x_i) = \beta\delta_{\mathcal{S},\alpha}(f), \tag{4.3}$$

$$\langle\psi_{\nu,h}|\hat{f}^{AW}|\psi_{\nu,h}\rangle \xrightarrow{h\to 0} \mu(f) - \beta\delta_{\mathcal{S},\alpha}(f) = (1-\beta)\nu(f). \tag{4.4}$$

Second part. Playing with time evolution. We will follow a strategy close to the one used to prove Proposition 3.1. We consider a fixed $\mathbf{k} \in \mathbb{Z}^2$, and focus on the overlap $\langle\psi_h|\hat{T}_{h\mathbf{k}}|\psi_h\rangle$. From the semiclassical assumption on $\{|\psi_h\rangle\}$, one has

$$\langle\psi_h|\hat{T}_{h\mathbf{k}}|\psi_h\rangle \xrightarrow{h\to 0} \mu(F_\mathbf{k}) = \beta\delta_{\mathcal{S},\alpha}(F_\mathbf{k}) + (1-\beta)\nu(F_\mathbf{k}). \tag{4.5}$$

On the other hand, since $|\psi_h\rangle = |\psi_{\mathcal{S},h}\rangle + |\psi_{\nu,h}\rangle$ is an eigenstate of $\hat{A}_h$, one may rewrite

$$\langle\psi_h|\hat{T}_{h\mathbf{k}}|\psi_h\rangle = \langle\psi_h|\hat{A}_h^{-T}\,\hat{T}_{h\mathbf{k}}\,\hat{A}_h^T|\psi_h\rangle \tag{4.6}$$

$$= \langle\psi'_{\mathcal{S},h}|\hat{T}_{h\mathbf{k}}|\psi'_{\mathcal{S},h}\rangle + 2\Re\Big(\langle\psi'_{\nu,h}|\hat{T}_{h\mathbf{k}}|\psi'_{\mathcal{S},h}\rangle\Big) + \langle\psi'_{\nu,h}|\hat{T}_{h\mathbf{k}}|\psi'_{\nu,h}\rangle, \tag{4.7}$$

with the notation $|\psi'_{\mathcal{S},h}\rangle = \hat{A}_h^T|\psi_{\mathcal{S},h}\rangle$, $|\psi'_{\nu,h}\rangle = \hat{A}_h^T|\psi_{\nu,h}\rangle$ and T is the Ehrenfest time. We are now in position to collect the dynamical results of Sect. 3:

- From the asymptotic localization (4.3) of $|\psi_{\mathcal{S},h}\rangle$ and Proposition 3.1, the first term in (4.7) converges to $\beta dx(F_\mathbf{k}) = \beta\delta_{\mathbf{k},\mathbf{0}}$ as $h \to 0$.

- From the properties (4.3–4.4) and Proposition 3.2, the cross-terms in (4.7) vanish in the semiclassical limit: $\langle \psi'_{\nu,h} | \hat{T}_{h\mathbf{k}} | \psi'_{\mathcal{S},h} \rangle \xrightarrow{h \to 0} 0$.

Using Eq. (4.5), the last term in (4.7) has therefore the semiclassical behaviour

$$\langle \psi'_{\nu,h} | Op_h(F_{\mathbf{k}}) | \psi'_{\nu,h} \rangle \xrightarrow{h \to 0} \mu(F_{\mathbf{k}}) - \beta dx(F_{\mathbf{k}}).$$

Since this limit holds for any $\mathbf{k} \in \mathbb{Z}^2$, it means that the sequence $\{|\psi'_{\nu,h}\rangle\}_{h\to 0}$ admits the semiclassical measure $\mu - \beta dx$. Because semiclassical measures are limits of Husimi measures, they cannot contain any negative part. Therefore, $\mu - \beta dx$ must be a positive measure, which implies that the Lebesgue component of μ has a weight $\geq \beta$. This component being contained in $(1-\beta)\nu$, one has finally $(1-\beta) \geq \beta \Leftrightarrow \beta \leq 1/2$. □

5. Remarks and Comments

5.1. On the set of semiclassical measures. Theorem 1.1 constrains the set of semiclassical measures $\mathfrak{M}_{A,\mathrm{SC}}$ to be a *proper subset* of the set $\mathfrak{M}_A$ of invariant Borel measures. One can easily show that $\mathfrak{M}_{A,\mathrm{SC}}$ is a *closed set* of $\mathfrak{M}_A$. Indeed, if for any $n > 0$ the sequence $S_n = \{|\psi_{h,n}\rangle\}_{h\to 0}$ converges towards a normalized semiclassical measure μ_n, and that in turn the measures μ_n weak-$*$ converge to a measure μ, then one can extract a function $n(h) \xrightarrow{h\to 0} \infty$ such that $\{|\psi_{h,n(h)}\rangle\}_{h\to 0}$ converges to μ.

Every open neighbourhood of $\mathfrak{M}_A$ contains a pure point measure of type δ_τ (τ a periodic orbit) [19, 14], therefore Theorem 1.1 implies that the set $\mathfrak{M}_{A,\mathrm{SC}}$ is *nowhere dense* in $\mathfrak{M}_A$ (*i.e.* its interior is empty).

On the other hand, the results of [7] show that $\mathfrak{M}_{A,\mathrm{SC}}$ contains all measures of the type $\frac{\delta_\tau + dx}{2}$. Since the measures $\{\delta_\tau\}$ are dense in $\mathfrak{M}_A$ and the set $\mathfrak{M}_{A,\mathrm{SC}}$ is closed, this implies

$$\forall \nu \in \mathfrak{M}_A, \qquad \frac{\nu + dx}{2} \in \mathfrak{M}_{A,\mathrm{SC}}.$$

This inclusion together with the constraint imposed by Thm. 1.1 do not suffice to fully identify the set $\mathfrak{M}_{A,\mathrm{SC}}$. We do not know if a singular continuous invariant measure can be a semiclassical measure. The set of invariant continuous measures is dense in $\mathfrak{M}_A$ [19], so in any case $\mathfrak{M}_{A,\mathrm{SC}}$ cannot contain all continuous invariant measures.

5.2. About the Ehrenfest time. Proposition 3.1 means that **any** sequence of localized states $\{|\varphi_h\rangle\}_{h\to 0}$ evolves towards a sequence of equidistributed states at the Ehrenfest time $T = \frac{|\log h|}{\lambda} + \mathcal{O}(1)$. To achieve this goal, the prefactor $1/\lambda$ defining T is crucial, and cannot be modified without stronger assumptions on the localization of $|\varphi_h\rangle$. Indeed, for any $\epsilon > 0$, one can construct a sequence of states $|\varphi_h\rangle$ semiclassically localized at the origin, such that the evolved states $|\psi_h\rangle = \hat{A}^{(1-\epsilon)T}|\varphi_h\rangle$ are still localized at the same point. Explicitly, consider the coherent state at the origin $|0\rangle = |0\rangle_h$, and take the sequences

$$|\varphi_h\rangle \stackrel{\mathrm{def}}{=} \hat{A}_h^{-(1-\epsilon)T/2}|0\rangle, \quad |\psi_h\rangle = \hat{A}_h^{(1-\epsilon)T}|\varphi_h\rangle = \hat{A}_h^{(1-\epsilon)T/2}|0\rangle.$$

At the "microscopic scale", the states $|\varphi_h\rangle$ and $|\psi_h\rangle$ are very different: the former is stretched along the stable axis, the latter along the unstable axis. However, the "length" of both states is of the order $h^{\epsilon/2}$, so this difference of shape is invisible at the measure-theoretic level, and both sequences admit the semiclassical measure δ_0.

On the other hand, there exist an infinite sequence of Planck's constants $h_k^{-1} \in \mathbb{N}$, $h_k \to 0$ such that starting from the states $\{|\psi_{h_k}\rangle\}$ defined above (localized at the origin), the evolved states $\{\hat{A}^{(1+\epsilon)T}|\psi_{h_k}\rangle\}$ are localized at the origin as well. These special values of h correspond to "short quantum periods" of the quantized cat map [1]. Let us remind that for any $h^{-1} \in \mathbb{N}$, the quantum cat map $\hat{A}_h$ is periodic, meaning that there exists $P_h \in \mathbb{N}$ and $\theta_h \in [0, 2\pi]$ such that $\hat{A}_h^{P_h} = \mathrm{e}^{\mathrm{i}\theta_h}\mathrm{Id}_h$ [8]. Besides, there exists an infinite subsequence $h_k \to 0$ for which these periods are as short as $P_{h_k} = 2T_{h_k} + \mathcal{O}(1)$. As a result, one has

$$\hat{A}_{h_k}^{(1+\epsilon)T}|\psi_{h_k}\rangle = \hat{A}_{h_k}^{3(1+\epsilon)T/2}|0\rangle = \mathrm{e}^{\mathrm{i}\theta_{h_k}}\hat{A}_{h_k}^{3(1+\epsilon)T/2-P_k}|0\rangle = \mathrm{e}^{\mathrm{i}\theta_{h_k}}\hat{A}_{h_k}^{(-1+\epsilon)T/2+\mathcal{O}(1)}|0\rangle.$$

The state on the RHS is close to $|\varphi_{h_k}\rangle$ at the microscopic level, and therefore admits δ_0 for semiclassical measure. In conclusion, the time $(1-\epsilon)T$ can be too short, and $(1+\epsilon)T$ too long to produce the transition localized $\to$ equidistributed described in Prop. 3.1.

5.3 More general maps? Proposition 3.1 can be extended to a broad class of quantized automorphisms on tori of dimension $2d$ with $d > 1$ [2, Thm. 5.1]. Let e^{λ} be the maximal modulus of the eigenvalues of A, E_λ the direct sum of the generalized eigenspaces corresponding to eigenvalues of modulus e^{λ}, and $E_{<\lambda}$ the direct sum of the remaining eigenspaces. The precise conditions on the automorphism A for the proposition to be satisfied are the following:

(1) A is ergodic, meaning that $\lambda > 0$ and that none of its eigenvalues is a root of unity. A is then automatically mixing, but need not be hyperbolic (it may have eigenvalues on the unit circle).
(2) the subspace $E_{<\lambda}$ has a trivial intersection with $\mathbb{Z}^{2d}$.
(3) A restricted to E_λ is diagonalizable.

The two last conditions are always satisfied for ergodic automorphisms in the case $d = 1$, but not necessarily in higher dimension. They are needed to obtain the crucial estimate (5.37) of [2]. One easily checks that Proposition 3.2 (and Lemma 3.3 which it depends on) also holds for higher-dimensional automorphisms satisfying the above conditions.

As opposed to the proof of [2, Thm. 1.2], our Thm. 4.1 crucially relies on the two dynamical propositions of Sect. 3, while the remaining ingredient in the proof of the theorem (namely, the splitting of eigenstates into two parts) can be straightforwardly transposed to higher dimension. Therefore, Theorems 4.1 and 1.1 also hold in higher dimension for the class of ergodic automorphisms described above. We do not know if the upper bound $1/2$ is sharp in dimension $d > 1$; in fact, the periods P_h of the quantized automorphisms then satisfy $\frac{P_h}{|\log h|} \xrightarrow{h\to 0} \infty$ ([5′]), which makes the construction of [7] impossible to generalize.

Back to the 2-dimensional torus, a natural extension of the above results would concern the perturbations of the linear map A of the form $M = \mathrm{e}^{-tX_H} \circ A$, where X_H is the vector field generated by a Hamiltonian $H \in C^\infty(\mathbb{T}^2)$. For t sufficiently small, this map is still Anosov. The challenge consists in generalizing Propositions 3.1 and 3.2 to those maps, with an appropriate definition of the Ehrenfest time. The trick (3.3) used in the linear case to prove these propositions cannot be used for a nonlinear perturbation, the problem starting from the poor control of Egorov's property (2.3) for times of order T.

Finally, one may also try to prove a similar property for chaotic flows, *e.g.* the geodesic flow on a compact Riemann surface of negative curvature. In such a setting, some interesting results have been recently obtained by R. Schubert, pertaining to the long-time evolution of Lagrangian states, which is a first step towards the proof of Proposition 3.1 in this setting.

Appendix A. Proof of Lemma 4.2

We start by showing that for x outside a small disk around x_i, the state $\hat{\vartheta}^{AW}_{i,r(h)}|x\rangle$ is asymptotically small ($|x\rangle$ is a torus coherent state at the point x). More precisely, there exists a sequence of radii $R(h) \searrow 0$ such that, for any $x_i \in \mathcal{S}$ and any sequence of points $\{x_h \in \mathbb{T}^2\}$ satisfying $x_h \notin D(x_i, R(h))$, then $\|\hat{\vartheta}^{AW}_{i,r(h)}|x_h\rangle\| \leq h^2$ for sufficiently small h.

First of all, we recall a couple of estimates on torus coherent states, valid for small enough h.

- For any $x \in \mathbb{T}^2$, $\||x\rangle\| \leq 2$.
- For any $x, y \in \mathbb{T}^2$, one has $|\langle x|y\rangle| \leq 5\exp\{\pi|x-y|^2/2h\}$, where $|x-y|$ denotes the torus distance between the points x, y.

Now, the operator $\hat{\vartheta}^{AW}_{i,r(h)}$ is a combination of projectors $|x\rangle\langle x|$ for points x in the disk $D(x_i, 2r(h))$. Therefore, taking $R(h) = 2r(h) + \sqrt{2h|\log h|}$ will do the job: any $x_h \notin D(x_i, R(h))$ and any $x \in D(x_i, 2r(h))$ satisfy $|x - x_h| \geq \sqrt{2h|\log h|}$, which implies for h small enough $|\langle x|x_h\rangle| \leq 5\exp(-2\pi h|\log h|/2h) \leq h^3$. One finally gets

$$\|\hat{\vartheta}^{AW}_{i,r(h)}|x_h\rangle\| \leq \int_{D(x_i,2r(h))} \frac{dx}{h}\ \||x\rangle\|\ |\langle x|x_h\rangle| \leq 2h^2\ Vol(D(x_i, 2r(h))) \leq h^2.$$

We are now in position to estimate the operator product

$$\hat{f}^{AW}\ \hat{\vartheta}^{AW}_{i,r(h)} = \int_{\mathbb{T}^2} \frac{dy}{h}\ |y\rangle f(y)\langle y|\,\hat{\vartheta}^{AW}_{i,r(h)}$$

by separating the integral into two parts. On the one hand, the integral over $\mathbb{T}^2 \setminus D(x_i, R(h))$ is bounded above by $2h\|f\|_\infty$ from the above results. On the other hand, on $D(x_i, R(h))$ the function $f(y)$ is equal to the function $f(x_i) + g_h(y)$, where $g_h(y) \stackrel{\text{def}}{=} (f(y) - f(x_i))\vartheta_{i,R(h)}(y)$. Since $g_h(y)$ is uniformly bounded on $\mathbb{T}^2$, the same arguments as above yield

$$\hat{f}^{AW}\ \hat{\vartheta}^{AW}_{i,r(h)} = f(x_i)\hat{\vartheta}^{AW}_{i,r(h)} + \hat{g}^{AW}_h\ \hat{\vartheta}^{AW}_{i,r(h)} + \mathcal{O}(h).$$

The function g_h actually decreases uniformly with h due to the smoothness of f:

$$\|g_h\|_\infty \leq \sup_{|y-x_i|\leq 2R(h)} \big(|f(y) - f(x_i)|\big) \leq 2\,\|df\|_\infty R(h).$$

This upper bound also applies to the anti-Wick quantization of g_h, so that
$\|\hat{g}^{AW}_h \hat{\vartheta}^{AW}_{i,r(h)}\| \leq 2\|df\|_\infty R(h) \stackrel{h\to 0}{\longrightarrow} 0.$ □

Appendix B. Proof of Lemma 3.3

We first replace $\mathcal{S}$ by the *finite* invariant set it generates, $\mathcal{S}' = \cup_{n\in\mathbb{Z}} A^n(\mathcal{S})$. We then want to prove that if $I_0 \stackrel{\text{def}}{=} \mathcal{S}' + [a,b]\mathbf{e}^{stable}$ with $0 < a < b$, then $\nu(I_0) = 0$ if ν is an invariant probability measure. Let n_0 be an integer such that $a\mathrm{e}^{\lambda n_0} > b$. Then, the sets

$$I_j \stackrel{\text{def}}{=} A^{jn_0}(I_0) = \mathcal{S}' + [a\mathrm{e}^{\lambda j n_0}, b\mathrm{e}^{\lambda j n_0}]\mathbf{e}^{stable}, \quad j \in \mathbb{Z}$$

are pairwise disjoint. The invariant measure ν will satisfy for any $J \geq 0$:

$$\nu\Big(\bigcup_{j=-J}^{J} I_j\Big) = \sum_{j=-J}^{J} \nu(I_j) = (2J+1)\nu(I_0).$$

Since $\nu(\mathbb{T}^2) = 1$, one must therefore have $\nu(I_0) = 0$. □

This lemma can be easily generalized to the case of the higher-dimensional ergodic toral automorphisms satisfying the conditions stated in Sect. 5.3. It can also be extended to any Anosov map M on $\mathbb{T}^2$, the straight segments making up I_0 being replaced by segments of the stable manifolds of a set of periodic points.

Acknowledgements. We have benefitted from insightful discussions with R. Schubert, and Y. Colin de Verdière, whose remarks also motivated this work. We are grateful to M. Dimassi for pointing out to us a gap in the proof of Prop. 3.2. Both authors acknowledge the support of the Mathematical Sciences Research Institute (Berkeley) where this work was completed.

References

1. Bonechi, F., De Bièvre, S.: Exponential mixing and $\ln\hbar$ timescales in quantized hyperbolic maps on the torus. Commun. Math. Phys. **211**, 659–686 (2000)
2. Bonechi, F., De Bièvre, S.:Controlling strong scarring for quantized ergodic toral automorphisms. Duke Math J. **117**, 571–587 (2003); Section 5 appears only in the preprint mp-arc/02-81 (2002)
3. Bouzouina, A., De Bièvre, S.: Equipartition of the eigenfunctions of quantized ergodic maps on the torus. Commun. Math. Phys. **178**, 83–105 (1996)
4. Chang, C.-H., Krüger, T., Schubert, R.: *Quantizations of piecewise affine maps on the torus and their quantum limits*. In preparation
5. Colin de Verdière, Y.: Ergodicité et fonctions propres du Laplacien. Commun. Math. Phys. **102**, 497–502 (1985)

5′. Corvaja, P., Rudnick, Z., Zannier, U.: A lower bound for periods of matrices. Preprint math. NT/0309215 (2003)

6. Degli Esposti, M.: Quantization of the orientation preserving automorphisms of the torus. Ann. Inst. Henri Poincaré **58**, 323–341 (1993)
7. Faure, F., Nonnenmacher, S., De Bièvre, S.: Scarred eigenstates for quantum cat maps of minimal periods, Commun. Math. Phys. **239**, 449–492 (2003)
8. Hannay, J.H., Berry, M.V.: Quantization of linear maps-Fresnel diffraction by a periodic grating. Physica D **1**, 267–290 (1980)
9. Heller, E.J.: Bound-state eigenfunctions of classically chaotic Hamiltonian systems: Scars of periodic orbits. Phys. Rev. Lett. **53**, 1515–1518 (1984)
10. Koyama, S.: Quantum ergodicity of Eisenstein series for arithmetic 3-manifolds. Commun. Math. Phys. **215**, 477–486 (2000)
11. Kurlberg, P., Rudnick, Z.: Hecke theory and equidistribution for the quantization of linear maps of the torus. Duke Math. J. **103**, 47–77 (2000)
12. Lindenstrauss, E.: *Invariant measures and arithmetic quantum unique ergodicity*. Preprint (2003)
13. Luo, W., Sarnak, P.: Quantum ergodicity of eigenfunctions on $PSL_2(\mathbb{Z}) \setminus H^2$. Publ. Math. IHES **81**, 207–237 (1995)
14. Marcus, B.: A note on periodic points of toral automorphisms. Monatsh. Math. **89**, 121–129 (1980)

15. Marklof, J., Rudnick, Z.: Quantum unique ergodicity for parabolic maps. Geom. Funct. Anal. **10**, 1554–1578 (2000)
16. Petridis, Y., Sarnak, P.: Quantum unique ergodicity for $SL_2(\mathcal{O}) \setminus H^3$ and estimates for L-functions. J. Evol. Equ. **1**, 277–290 (2001)
17. Rudnick, Z., Sarnak, P.: The behaviour of eigenstates of arithmetic hyperbolic manifolds. Commun. Math. Phys. **161**, 195–231 (1994)
18. Schnirelman, A.: Ergodic properties of eigenfunctions. Usp. Math. Nauk **29**, 181–182 (1974)
19. Sigmund, K.: Generic Properties of Invariant Measures for Axiom A-Diffeormorphisms. Invent. Math. **11**, 99–109 (1970)
20. Wolpert, S. A.: The modulus of continuity for $\Gamma_0(m)\backslash\mathbb{H}$ semi-classical limits. Commun. Math. Phys. **216**, 313–323 (2001)
21. Zelditch, S.: Uniform distribution of the eigenfunctions on compact hyperbolic surfaces. Duke Math. J **55**, 919–941 (1987); Index and dynamics of quantized contact transformations, Ann. Inst. Fourier **47**, 673–682 (1996)

Communicated by P. Sarnak