Eigenfunction concentration in polyhedra

Mihajlo Cekić

Joint work with B. Georgiev and M. Mukherjee

Laboratoire Mathematiques d'Orsay, Université Paris Saclay

Paris, February 1, 2020

Summary

- Let (M, g) be a compact Riemannian manifold with boundary and denote by $-\Delta_g$ the Laplace-Beltrami operator. There are Dirichlet eigenfunctions $\{u_j\}_{j=1}^{\infty} \subset L^2(M)$ with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. satisfying: $-\Delta_g u_j = \lambda_j^2 u_j$, $u_j|_{\partial M} = 0$ and $\|u_j\|_{L^2} = 1$.

Question

Describe the set of weak limits of the probability distributions $|u_j|^2(x) d\,\mathrm{vol}_g$ as $j \to \infty$, i.e. measures ν such that for all $\varphi \in C_0^\infty(M^{\mathrm{int}})$, $\lim_{k\to\infty} \int_M \varphi |u_{j_k}|^2 d\,\mathrm{vol}_g = \int_M \varphi d\nu$ for a subsequence j_k.

- Physically, the u_j are pure quantum states (probabilities) and we want to describe the high energy behaviour of the u_j. Quantum-classical correspondence: properties of the geodesic flow are related to the quantum limits.

- Examples: on the square $[0,1]^2$ we have eigenfunctions $u_{jk} = 2\sin(j\pi x)\sin(k\pi y)$. Then $|u_{jj}|^2 dxdxy \to dxdy$, $|u_{j1}|^2 dxdy \to 2\sin^2(\pi y)dxdy$. On the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$: (normalised) restriction of $\Re(x+iy)^m$ concentrates on the geodesic $z=0$.
- The billiard flow Φ_t is defined as $\Phi_t(x,v) = (\gamma(t), \dot\gamma(t))$, where $\gamma(0) = x$, $\dot\gamma(t) = v$ is geodesic. At the boundary Φ_t respects the law of refection. If ∂M piecewise smooth, defined on a set of full (Liouville) measure of the phase space of unit vectors $SM = \{(x,v) \in TM : |v|_x = 1\}$.

Theorem (Quantum Ergodicity)

If the billiard flow is ergodic, then there a density one set $S \subset \mathbb{N}$ such that $\{u_j\}_{j\in S}$ converges weakly to the uniform measure (i.e. equidistributes).

Proved by: Gérard-Leichtnam (1993), Zelditch-Zworski (1995) in the boundary case; Shnirelman (1974), Colin de Verdière (1985) and Zelditch (1987) in the closed case.

- Quantum unique ergodicity conjecture (Rudnick-Sarnak): if (M, g) has negative curvature, then all eigenfunctions equidistribute. known for arithmetic surfaces (Lindenstrauss 2006). Recent significant progress by Dyatlov-Jin-Nonnenmacher: if (M, g) is a negatively curved surface, then weak limits have *full support*.
- Hassell-Hillairet (2010): the Bunimovich stadium (ergodic) admits a sequence of eigenfunctions that does not equidistribute, i.e. scarring happens.
- Mechanisms to obtain lower bounds on eigenfunctions: unique continuation (depends on the eigenvalue) and geometric control.

Assume $M = P \subset \mathbb{R}^n$ is a convex polyhedron. Denote by $\mathcal{S} \subset \partial P$ the set of singularities of ∂P, i.e. the $(n-2)$-skeleton of the boundary.

Theorem (C-Georgiev-Mukherjee 2020)

If $\mathcal{U} \subset P$ is an open set containing $\mathcal{S}$, then there is a constant $c(\mathcal{U}) > 0$ such that for all $j \in \mathbb{N}$

$$\int_{\mathcal{U}} |u_j|^2 dx \geq c(\mathcal{U}).$$

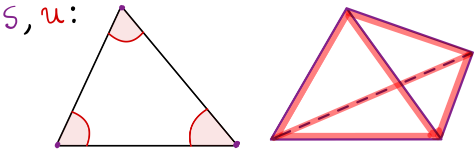


Figure: The singular set $\mathcal{S}$ and its neighbourhood $\mathcal{U}$.

- Remarks:
  - First result for $n > 2$. Works also for Neumann eigenfunctions.
  - Dependence of $c(\mathcal{U})$ on the size (width) of $\mathcal{U}$ not known.
  - $\mathcal{U}$ does not satisfy the geometric control condition.
  - Argument works also for billiards containing “periodic tubes”.
- Previous results:
  - Burq-Zworski (2004, 2005): there is no concentration in the interior rectangle of the Bunimovich stadium (and more).
  - Marzuola (2006): partially rectangular billiards.
  - Hillairet, Hasell, Marzuola (2008): prove the theorem for $n = 2$.
  - Marklof-Rudnick (2011): density 1 result for rational polygons.
- Plan for the rest of the talk: study closed orbits of the billiard flow on polyhedra and prove some control estimates.

Summary

- Consider (M, g) as before. Define $h_j := \lambda_j^{-1}$, so that $(-h_j^2 \Delta_g - 1)u_j = 0$.
- For $a \in C_0^\infty(T^* M^{\text{int}})$ compactly supported in the cotangent space of the interior of M, we may quantise it to $\text{Op}_h(a) = a(x, hD)$, where $D = -i\partial$ formally replaces ξ: $a(x, hD) = \mathcal{F}^{-1} a(x, h\xi)\mathcal{F}$.

Definition (Semiclassical Measure)

A measure μ on $T^* M^{\text{int}}$ is called a semiclassical measure if there is a subsequence $\{u_{j_k}\}_{k=1}^\infty$, such that for all $a \in C_0^\infty(T^* M^{\text{int}})$

$$\langle \text{Op}_{h_{j_k}}(a) u_{j_k}, u_{j_k} \rangle_{L^2} \to_{k \to \infty} \int_{T^* M} a \, d\mu.$$

- Microlocal limits or weak limits in phase space.

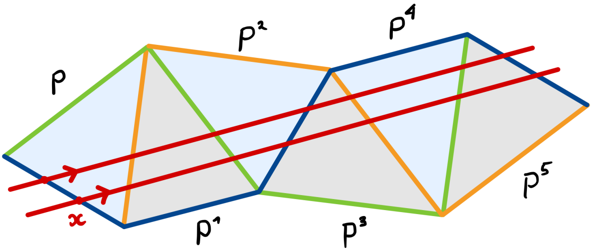
- Properties of semiclassical measures (at least when $\partial M = \emptyset$):
  1. Existence (compactness).
  2. $\operatorname{supp} \mu \subset S^*M = \{(x, \xi) \in T^*M : |\xi| = 1\}$ (ellipticity).
  3. μ is invariant under the geodesic/billiard flow (propagation).
- If $a(x, \xi) = a(x)$, then we get $\int_M a |u_{j_k}|^2 d\,\mathrm{vol}_g \to \int_M a \pi_* d\mu$.
- More precise versions of QE available in terms of semiclassical measures: describe the set of all semiclassical measures associated to $\{u_j\}_{j=1}^{\infty}$.

- Let $P \subset \mathbb{R}^n$ be a convex polyhedron. Set $\Gamma := \partial P$, $T\Gamma :=$ all inward pointing unit tangent vectors and

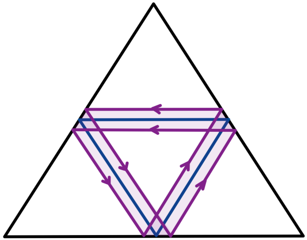
$$T\Gamma_1 := \{x \in T\Gamma : \text{ the forward orbit of } x \text{ never hits } \mathcal{S}\}.$$

  Then $T\Gamma_1 \subset T\Gamma$ has full measure and the first return map $f : T\Gamma_1 \to T\Gamma_1$ is well-defined.
- Enumerate faces of P by $1, \dots, \ell$. Obtain symbolic dynamics: $x \in T\Gamma_1 \implies$ string $w(x)$ in $\Sigma_\ell^+ = \{1, \dots, \ell\}^{\mathbb{N}}$. Set $\Sigma_P^+ :=$ all possible strings; $X(w) := \{x \in T\Gamma_1 : w(x) = w\}$.
- Properties:
  - $w(x) = w(y) \implies x$ and y are parallel.
  - $X(w)$ is convex and $y \in \partial X(w)$ comes arbitrarily close to $\mathcal{S}$.

- A billiard trajectory γ defines an unfolding or corridor: $P^\infty = P^0, P^1, \dots, P^m, \dots$ obtained by reflecting in faces along γ.



- $\sigma :=$ reflection in $\mathbb{R}^n$. Define the double $D := P \sqcup \sigma P / \sim$ identifying the points on ∂P. $D_0 := D \setminus \mathcal{S}$ has a Euclidean structure.
- Let $U \subset \mathbb{R}^n$ and $F : U \times \mathbb{R} \to D_0$ a local isometry. A tube is the image $F(U \times \mathbb{R})$ and U is its cross-section. A periodic tube satisfies $F(x, t + L) = F(\mathcal{R}x, t)$ for all $x \in U$, some $\mathcal{R} \in O(n-1)$ and $L > 0$.



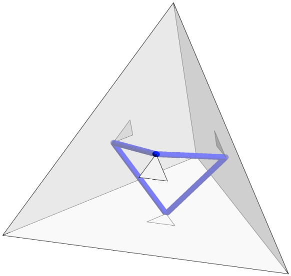


Figure: Left: equilateral triangle and the periodic tube around the central orbit. Right: Conway loop in a regular tetrahedron (central triangle has side length 1/10).

- Periodic orbits can be "thickened" to periodic tubes.
- The Conway loop corresponds to an irrational rotation and is stable under perturbation (by the work of N. Bedaride).

Theorem (Galperin-Krüger-Troubetzkoy)

Let $w \in \Sigma_P^+$. Then:

1. *If w is periodic with minimal period k, there exists $x(w) \in X(w)$ so that $x(w)$ is periodic with minimal period k. The set $X(w)$ generates a periodic tube with a convex cross-section $\Omega \subset \mathbb{R}^{n-1}$ and an associated isometry $\mathcal{R}_0 \in O(n-1)$ fixing Ω.*
2. *If $w \in \Sigma_P^+$ is non-periodic, then the closure of a trajectory generated by $x \in X(w)$ intersects the singular set $\mathcal{S}$.*

- For the first part: look at the unfolding and apply the Brower's fixed point theorem to the convex set $X(w)$.
- Second part: more subtle, argue by contradiction.

- Extend u_j to an eigenfunction on D_0 by setting $u_j := -u_j \circ \sigma$ on σP.
- Contradiction argument: assume (up to subsequence)

$$\lim_{j \to \infty} \int_{\mathcal{U}} |u_j|^2 dx = 0.$$

  Then, $\exists$ a semiclassical measure μ on $T^* D_0$ (item 1), such that $\operatorname{supp} \mu \subset S^* D_0$ (item 2) and μ invariant under geodesic flow (item 3).
- It can be shown that: $\mu = 0$ on $S^* \mathcal{U}_0$, where $\mathcal{U}_0 := \mathcal{U} \setminus \mathcal{S}$ and μ is a probability measure (by ellipticity). Then by flow invariance

$$\mu = 0 \quad \text{on} \quad \cup_{t \in \mathbb{R}} \Phi_t(S^* \mathcal{U}_0).$$

- GKT theorem $\implies$ if $(x, \xi) \in \operatorname{supp}(\mu)$, then the symbol generated by (x, ξ) is periodic and its trajectory belongs to a periodic tube.

Summary

- Let $F : \Omega \times \mathbb{R} \to D_0$ be a periodic tube of length L and isometry $\mathcal{R} \in O(n)$. Claim: singular points are uniformly recurrent:

Lemma

Denote by $\pi_1 : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$ the projection. For every $\varepsilon > 0$, there is an $L(\varepsilon) > 0$ depending on T, such that for every $t \in \mathbb{R}$

$$\pi_1\Big(F^{-1}(\mathcal{S}) \cap \partial\Omega \times [t, t + L(\varepsilon)]\Big)$$

is ε-dense in $\partial\Omega$.

- Step 1: show that $\overline{\pi_1\big(F^{-1}(\mathcal{S})\big)} = \partial\Omega$. Compactness + observation about maximal tubes.
- Step 2: use that $\mathcal{R}$ is an isometry and $F(\Omega \times \mathbb{R})$ periodic.

- We say that $\mathcal{U}$ satisfies the finite tube condition, if there exist periodic tubes $T_1, \dots, T_N$ such that any orbit avoiding $\mathcal{U}$ belongs to some T_i.

Theorem (C-Georgiev-Mukherjee)

Any neighbourhood $\mathcal{U}$ of $\mathcal{S}$ satisfies the finite tube condition.

- Proof: if not, take distinct tubes $T_1, T_2, \dots$ generated by $x_1, x_2, \dots$. Extract a limiting periodic tube T generated by x, $x_i \to x$.
- By the previous lemma, singular points occur uniformly often. This means that the unfolded tube T_i^∞ will contain some of these points, contradiction.
- Possible to get a quantitative estimate relating lengths, rotations and the number of these periodic tubes.

- To obtain a contradiction, need two estimates.
- Say a subset $A \subset (M, g)$ satisfies satisfies the geometric control condition (GCC) if every geodesic in M hits A in finite time. If a neighbourhood ω of ∂M satisfies the GCC, then there is a $C = C(g_x, \omega) > 0$, such that for any $s \in \mathbb{R}$ and any v satisfying

$$(-\Delta_g - s)v = g, \quad v|_{\partial M} = 0,$$

with $v \in H_0^1(M)$ and $g \in H^{-1}(M)$, we have the apriori estimate

$$\|v\|_{L^2(M)} \leq C(\|g\|_{H^{-1}(M)} + \|v|_\omega\|_{L^2(\omega)}).$$

- Proof of this fact: for bounded $s < -\varepsilon$ integrate by parts; for s bounded use the unique continuation principle and elliptic estimates (i.e. the case $\omega = M$). For large s argue by contradiction using semiclassical measures.

Theorem (C-Georgiev-Mukherjee)

Let $\varphi : M \to M$ be an isometry. Assume that $u \in H^1_{\mathrm{loc}}(M \times \mathbb{R}) \cap C(\mathbb{R}, H^1(M))$, such that $u(x, t + L) = u(\varphi(x), t)$ for all $(x, t) \in M \times \mathbb{R}$, for some $L > 0$. Define $\mathcal{C}_\varphi := M \times [0, L]/(x, L) \sim (\varphi(x), 0)$ to be the mapping torus determined by φ, with the inherited Riemannian metric from $M \times \mathbb{R}$. Assume u satisfies, for some $s \in \mathbb{R}$

$$(-\Delta_g - \partial_t^2 - s)u = f \text{ on } M \times \mathbb{R}, \quad u|_{\partial M \times \mathbb{R}} = 0,$$

where $f \in H^{-1}_{\mathrm{loc}}(M \times \mathbb{R}) \cap C(\mathbb{R}, H^{-1}(M))$. Let $\omega \subset M$ be an open neighbourhood of the boundary satisfying (GCC) and assume ω invariant under φ. Denote the mapping torus over ω by ω_φ. Then there exists a constant $C = C(M, g, \omega) > 0$, such that the following observability estimate holds:

$$\|u\|_{L^2(\mathcal{C}_\varphi)} \leq C(\|f\|_{H_x^{-1} L_t^2(\mathcal{C}_\varphi)} + \|u|_{\omega_\varphi}\|_{L^2(\omega_\varphi)}).$$

We prove this in the easy case first: $\varphi = \mathrm{id}$. Then $\mathcal{C}_\varphi = M \times \mathbb{S}^1$. Using Fourier expansion in the circle $e_k(t) = \frac{1}{\sqrt{2\pi}} e^{itk}$:

$$u(x,t) = \sum_{k \in \mathbb{Z}} u_k(x) e_k(t), \quad f(x,t) = \sum_{k \in \mathbb{Z}} f_k(x) e_k(t).$$

Then the equation becomes, for $k \in \mathbb{Z}$:

$$(-\Delta_g - (s - k^2)) u_k(x) = f_k(x).$$

Use Parseval's identity and apply the control estimate for each k:

$$\begin{aligned}\|u\|^2_{L^2(M \times \mathbb{S}^1)} = \sum_{k \in \mathbb{Z}} \|u_k\|^2_{L^2(M)} &\leq C \Big(\sum_{k \in \mathbb{Z}} \|f_k\|^2_{H^{-1}(M)} + \sum_{k \in \mathbb{Z}} \|u_k|_\omega\|^2_{L^2(\omega)} \Big) \\ &= C \big(\|f\|^2_{H_x^{-1} L_t^2(M)} + \|u|_{\omega \times \mathbb{S}^1}\|^2_{L^2(\omega \times \mathbb{S}^1)} \big).\end{aligned}$$

- For the general case, use the same idea and some theory of almost periodic functions. Studied by H. Bohr (1920). These are functions $f : \mathbb{R} \to X$ to a Banach space, uniformly approximated by trigonometric polynomials. If $\mathcal{M}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt$, the the Fourier-Bohr transform is defined as:

$$a(\lambda; f) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)e^{-i\lambda t}dt = \mathcal{M}\{f(t)e^{-i\lambda t}\}.$$

  Then there are countably many λ_k with $a_k = a(\lambda_k; f) \neq 0$, and

$$f(t) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}.$$

  A modified Parseval's identity holds: $\mathcal{M}(\|f\|^2) = \sum |a_k|^2$.
- Using that φ is an isometry, possible to show that $\mathbb{R} \ni t \mapsto u(t, \bullet) \in H_0^1(M) \subset L^2(M)$ is almost periodic.

- Assume $\mathcal{U}$ is the ε-neighbourhood of the singular set $\mathcal{S}$. Recall: μ a semiclassical (probability) measure on S^*D_0, $\mu = 0$ on $S^*\mathcal{U}_0$, and μ supported on periodic directions. Let $(x_0, \xi_0) \in \operatorname{supp} \mu$ generate a periodic tube $F : \Omega \times \mathbb{R} \to D_0$, of length L and rotation $\mathcal{R}$.
- Pullback u_n and μ to $T = \Omega \times \mathbb{R}$; these are invariant under $(x, t) \mapsto (\mathcal{R}x, t + L)$. Define $\Omega_r :=$ the complement of r-neighbourhood of $\partial\Omega$.
- By the Lemma, $\exists$ a symbol $a(\xi)$ such that: $a(\xi)$ supported in a small cone $\Upsilon \subset \mathbb{R}^n \setminus 0$ around dt; all lines in the direction of Υ with basepoint $x \in \Omega_{\varepsilon/2-\eta} \setminus \Omega_{\varepsilon/2+\eta}$ hit $F^{-1}(\mathcal{U}_\varepsilon)$ in finite time; $a = 1$ near dt and $a \circ \mathcal{R} = a$. Set $\Phi_n := \mathrm{Op}_{h_n}(a)$.
- Set $w_n := \Phi_n(\chi u_n) \in C^\infty(\mathcal{C}_\mathcal{R})$. Apply the main estimate to:

$$(-\Delta_\Omega - \partial_t^2 - \lambda_n^2)w_n = -\Phi_n((\Delta\chi)u_n) - 2\Phi_n(\nabla_x\chi \cdot \nabla_x u_n).$$

- $\implies a^2\chi^2\mu = 0 \implies \mu \equiv 0$ near (x_0, ξ_0), contradiction.

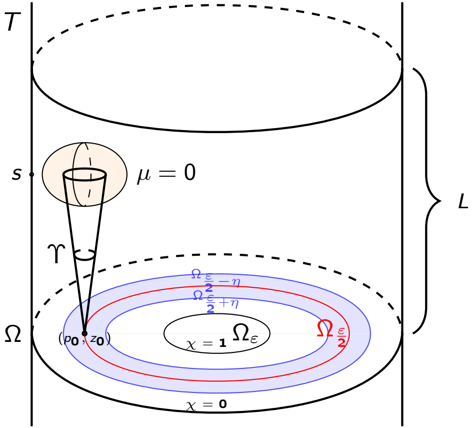


Figure: The periodic tube T of length L with disc cross-section Ω, point $(p_0, z_0) \in \partial\Omega_{\frac{\varepsilon}{2}}$, corresponding regions $\Omega_{\frac{\varepsilon}{2}\pm\eta}$, singular point $s \in \mathcal{S}$ and cone Υ. In orange is the set where $\mu = 0$.

Summary

- Which estimates are available on $c(\mathcal{U})$ depending on $\mathcal{U}$? Eg. if $\mathcal{U}$ is an ε-neighbourhood of $\mathcal{S}$, is it a function of ε?
- What if P is non-convex?

Thank you for your attention!