Eigenvalue spacings for 1D singular Schrödinger operators

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A general question

Consider a (self-adjoint) 1D semiclassical Schrödinger operator

$$P_h u = -h^2 u'' + V(x)u, \quad \text{and} \quad E \in \operatorname{spec} P_h.$$

$$d_h(E) \stackrel{\text{def}}{=} \inf\{|E - \tilde{E}|, \ \tilde{E} \in \operatorname{spec} P_h, \tilde{E} \neq E\}.$$

Q : Lower bounds on $d_h(E)$? uniform within a subset $\Omega \subset (0, +\infty) \times \mathbb{R}$ of pairs (h, E) ?

Old question, some regimes are well-known and can be extracted from the literature on Sturm-Liouville problems and then semiclassical analysis (Titchmarsch, Olver, Hörmander, Maslov .... Helffer-Robert, Dimassi-Sjöstrand, Zworski ....).

Non-critical energies

On the half-line $[0, +\infty)$, Dir./Neu. b.c. at 0.
We set $\Omega =]0, h_0] \times [a, b],\ 0 < a_0 < a < b < b_0$ and make the following assumptions on V.

1. The potential V is smooth, non-negative, $V(0) = 0$, $\liminf_{+\infty} V(x) > b_0$.
2. $\exists \delta' > 0,\ \forall x > 0,\ V(x) \in [a_0, b_0] \implies V'(x) \geqslant \delta'$.

Spacing at non-critical energies

There exists $h_0, \mathrm{c} > 0$ such that

$$\forall h \leqslant h_0, \quad \forall E \in \operatorname{spec} P_h$$
$$(h, E) \in \Omega \implies d_h(E) \geqslant \mathrm{c} \cdot h.$$

- Energy surface is connected (no tunnel effect).
- Q : uniformity of the constant c with respect to V ?

Bottom of the well

On the half-line $[0, +\infty)$, Dir./Neu. b.c. at 0. Assume

1. There exist a smooth W and $\gamma > 0$ such that $V(x) = x^{\gamma} W(x), \ x > 0$.
2. W is positive on $[0, +\infty)$ and $\liminf_{+\infty} V(x) > 0$.

Fix $M > 0$ and set

$$\Omega \stackrel{\text{def}}{=} \left\{ \left(h, E\right), \ h \leqslant h_0, \ E \leqslant M \cdot h^{\frac{2\gamma}{\gamma+2}} \right\}.$$

Then there exists $\mathrm{c}, h_0 > 0$ such that

$$\forall h \leqslant h_0, \quad \forall E \in \operatorname{spec} P_h$$
$$(h, E) \in \Omega \implies d_h(E) \geqslant \mathrm{c} \cdot h^{\frac{2\gamma}{\gamma+2}}.$$

Idea of proof : use a h dependent scaling.
(See Friedlander-Solomyak for a related result)

Some questions

- What about non-critical energies for singular potentials ?
- What about the intermediate regime ? Estimate $d_h(E_h)$ if

$$E_h \to 0, \quad h^{-\frac{2\gamma}{\gamma+2}} E_h \to +\infty.$$

- What about potentials of the form $x \mapsto x^\gamma W(x)$ on the half-line $[-B, +\infty)$? This question is related to the adiabatic Ansatz in a stadium billiard.

Uniform spacing

Theorem

Assume that $\gamma > 0$ and W is smooth and positive on $[0, +\infty)$. Let $V = x^\gamma W$ and P_h the Dirichlet or Neumann realization of $-h^2 u'' + V$ on $[0, +\infty)$. If $\liminf_{x \to +\infty} V(x) > 0$, there exist $\varepsilon, h_0, \mathrm{c} > 0$ such that

1. For all $h \leqslant h_0$, $\operatorname{spec} P_h \cap [0, \varepsilon]$ is purely discrete,
2. For any $h \leqslant h_0$ and any E in $\operatorname{spec} P_h \cap [0, \varepsilon]$,

$$d_h(E) \geqslant \mathrm{c}h \cdot E^{\frac{\gamma-2}{2\gamma}}.$$

The proof follows from

- the spacing of order h at non-critical energies, uniformly w.r.t. W,
- The bottom of the well computation,
- A contradiction argument.

Comments

$$\text{Uniform spacing}: \; d_h(E) \geqslant ch \cdot E^{\frac{\gamma-2}{2\gamma}}.$$

- The estimate is coherent with the non-critical energies and the bottom of the well.

$$E_h \text{ of order } 1 \implies d_h(E_h) \text{ of order } \; h,$$
$$E_h \text{ of order } h^{\frac{2\gamma}{\gamma+2}} \implies d_h(E_h) \text{ of order } \; h^{1+\frac{\gamma-2}{\gamma+2}} = h^{\frac{2\gamma}{\gamma+2}}.$$

- If $\gamma = 2$ then the spacing is everywhere of order h. This is coherent with the harmonic oscillator computation.

Comments

- Compared to h the spacing between low lying eigenvalues is large if $\gamma < 2$ and small if $\gamma > 2$.
- We also prove Bohr-Sommerfeld rules for singular potentials on a half-line (will be needed for the gluing case).

Other related works in the semiclassical literature :

- Semi-excited states (using Birkhoff Normal form techniques, starting with Sjöstrand).
- Diffraction by conormal potential (Gannot-Wunsch).
- Anharmonic oscillator (Voros)

Dealing with the intermediate regime

Choose a sequence $(E_h, u_h)_{h\geqslant 0}$ in the intermediate regime :

$$E_h \to 0, \quad h^{-\frac{2\gamma}{\gamma+2}} E_h \to +\infty.$$

and perform a E-dependent scaling : $\tilde{v}_h(\cdot) = \tilde{u}_h(E^{\frac{1}{\gamma}}\cdot)$.

$$-h^2 \tilde{u}_h'' + (x^\gamma W(x) - \tilde{E})\tilde{u}_h = 0 \qquad \Longleftrightarrow$$

$$-h^2 E_h^{-1-\frac{2}{\gamma}} \tilde{v}_h'' + \left(z^\gamma W(E_h^{\frac{1}{\gamma}} z) - \frac{\tilde{E}_h}{E_h}\right)\tilde{v}_h = 0.$$

- $W(E_h^{\frac{1}{\gamma}}\cdot)$ converges to the constant function $W(0)$ (uniformly on every compact set),
- $\bar{h} \overset{\mathrm{def}}{=} hE_h^{-\frac{2+\gamma}{2\gamma}}$ tends to 0. New semiclassical parameter.

$$-\bar{h}^2 \tilde{v}_h'' + (z^\gamma \tilde{W}_h(z) - \frac{\tilde{E}_h}{E_h})\tilde{v}_h = 0.$$

$$\frac{\tilde{E}_h}{E_h} \longrightarrow 1.$$

We now work near the energy 1 which is non-critical. Spacing of order h at non-critical energies + uniformity w.r.t. potential will imply

$$\left|\frac{\tilde{E}_h}{E_h} - 1\right| \geqslant c\bar{h} \implies |\tilde{E}_h - E_h| \geqslant ch \cdot E^{\frac{\gamma-2}{2\gamma}}$$

Spacing of order h : a general strategy

For any $h > 0$, there is a unique function $G_h(\cdot\,;\,E)$ such that

$$(P_h - E)G_h(\cdot\,;\,E) = 0$$
$$\int_0^{+\infty} |G_h(x\,;\,E)|^2\,dx = 1,$$
$$\forall x \geqslant c,\ G_h(x\,;\,E) > 0.$$

The spacing is obtained by showing that if $|E_h - \tilde{E}_h| = o(h)$ then $G_h(\cdot\,;\,E_h)$ and $G_h(\cdot\,;\,\tilde{E}_h)$ cannot be orthogonal.

Spacing of order h : classical tools

- (Non-)concentration estimates, semiclassical measures.
- Exponential estimates in the classically not-allowed region.
- WKB expansions in classically allowed region.
- Dealing with the turning point : Maslov Ansatz, Airy approximations.

Revisit these techniques to gain uniformity w.r.t. the energy and the potential.

Global assumptions

We fix $\gamma > 0$, $0 < b < c < d$, $\mathcal{K}$ a compact set in $C^\infty([0,d])$ equipped with its Fréchet topology and K a compact set in $(0,+\infty)$. We denote by $\mathcal{V}$ the set of potentials such that

- There exists W smooth and positive on $[0,\infty)$ such that $\forall x > 0,\ V(x) = x^\gamma W(x)$.
- $\forall x \geqslant d,\ V(x) \geqslant V(d)$.
- $\forall x \in (0,d],\ V'(x) > 0$.
- The restriction of W to $[0,d]$ belongs to $\mathcal{K}$.
- The following estimates hold

$$\forall (V,E) \in \mathcal{V} \times K,\ \forall x \in [0,b],\ E - V(x) > 0.$$
$$\forall (V,E) \in \mathcal{V} \times K,\ \forall x \in [c,d],\ V(x) - E > 0.$$

Uniform bounds

For any $(V, E) \in \mathcal{V} \times K$, the assumptions imply

- There is a unique solution x_E to the equation $V(x_E) = E$ (the turning point).
- $[0, b]$ is in the classically allowed region and $(E - V)$ is uniformly bounded below on it.
- $[c, +\infty)$ is in the classically not allowed region.
- The turning point x_E always belong to $[b, c]$. Since, on $[b, c]$, V' is uniformly bounded below, it is non-degenerate.
- Finally, for any ℓ, $W^{(\ell)}$ is, uniformly on $[0, d]$, bounded above by some C_ℓ.
- If γ is an integer, $W^{(\ell)}$ can be replaced by $V^{(\ell)}$ in the latter statement.

Semiclassical measures for families of potential

Let $(V_h, E_h)_{h \geqslant 0}$ be a family in $\mathcal{V} \times K$, then up to extracting a subsequence, there exists (V_0, E_0) and a measure μ such that for any smooth function with compact support in $(0, d) \times \mathbb{R}$,

$$\langle \mathrm{Op}_h(a) G_h, G_h \rangle \to \int a(x, \xi)\, d\mu.$$

Then the semiclassical measure is supported by $\{\xi^2 + V_0(x) = E_0\}$ and is invariant under the hamiltonian flow.

Moral : The estimates that are obtained using semiclassical measures and the standard contradiction argument are uniform when the potential varies in a compact set.

Sketch of Proof of the invariance

We first extract a subsequence so that W_h converges in $C^\infty([0,d])$ to W_0, and E_h to E_0.

$$\begin{aligned}\frac{h}{i}\langle \mathrm{Op}_h(\{p_0,a\})G_h, G_h\rangle &= \langle [P_0, \mathrm{Op}_h(a)]G_h, G_h\rangle \\ &= \langle [P_h, \mathrm{Op}_h(a)]G_h, G_h\rangle \\ &\quad + \langle [V_h - V_0, \mathrm{Op}_h(a)]G_h, G_h\rangle \\ &= \frac{h}{i}\langle \mathrm{Op}_h(\{V_h - V_0, a\})G_h, G_h\rangle.\end{aligned}$$

We now use the fact that the norm of a pseudodifferential operator on L^2 depends on the uniform norm of a finite number of derivatives of the symbol and that $\{V_h - V_0, a\}$ and all its derivatives converge uniformly to 0 on the support of a.

- In dimension 1, μ is thus determined up to a factor. This gives a way to address the turning point.

Dealing with the turning point

Lemma

$$\forall \varepsilon > 0,\ \exists \eta, h_0 > 0\ \forall (V, E) \in \mathcal{V} \times K,\ \forall h \leqslant h_0,$$
$$\int_{x_E-\eta}^{x_E+\eta} |G_h(x\,;\,E)|^2\,dx \leqslant \varepsilon \int_0^b |G_h(x\,;\,E)|^2\,dx.$$

Sketch of proof : take a cutoff that is localized near the turning point and use the invariance to relate it to a cutoff in the classically allowed region.

- It can also be proved using Airy approximation near the turning point (e.g. Yafaev '11).
- Using this estimate, we are able to compute $\langle G_h(\cdot,\ \ E), G_h(\cdot, \tilde{E})\rangle$ by looking only in the classically allowed region where we have standard WKB expansions.

Uniform spacing for smooth potential

- Perform WKB expansion on $[0, x_E - \eta]$.
- Prove that any solution admits such a WKB expansion, and thus also $G_h(\cdot\,;\,E)$. (still 2 degrees of freedom).
- Reduce to 1 degree of freedom by using the Maslov Ansatz and a Wronskian argument to pass above the turning point.

$$\exists c_h \neq 0,\ \forall x \in [0, x_E - \eta],$$
$$G_h(x\,;\,E) =$$
$$c_h(E - V(x))^{-\frac{1}{4}} \cos\left[\frac{1}{h}\int_x^{+\infty} (E - V(y))_+^{\frac{1}{2}} dy - \frac{\pi}{4}\right] + O(h)$$

- This yields the estimate

$$c_h^{-1} G(x\,;\,h) - \tilde{c}_h^{-1} G(x\,;\,\tilde{E}) \,=\, O(\frac{E-\tilde{E}}{h}) + O(h).$$

- Estimate c_h by the mass and compute the needed scalar product
- It follows that this scalar product cannot be 0 when $\frac{E-\tilde{E}}{h}$ is too small.

For non-integer γ

We need to make sure that all the mass does not concentrate at the origin. We will match the WKB expansion to a *boundary layer* near 0.

Recall the WKB Ansatz on $I_h = [a_h, b]$:

$$u_h(x) \sim \exp(\frac{i}{h} S(x)) \sum_{k \geqslant 0} h^k A_k(x).$$

The method leads to

- The eikonal equation :

$$\forall x \in I_h,\ S'(x)^2 \ = \ E - V(x).$$

- The homogeneous transport equation :

$$\forall x \in I_h,\ 2S'(x)A_0'(x) \ + \ S''(x)A_0(x) \ = \ 0.$$

- The inhomogeneous transport equations :

$$\forall k \geqslant 0,\ 2S' A_{k+1}' \ + \ S'' A_{k+1} \ = \ iA_k''.$$

Estimates for WKB

The eikonal and the transport equations can be solved on I_h because $E - V$ is positive on I_h. We choose the following solution :

$$\forall x \in I_h,$$
$$S'(x) = \sqrt{E - V(x)},$$
$$A_0(x) = [S'(x)]^{-\frac{1}{2}} = [E - V(x)]^{-\frac{1}{4}},$$
$$\forall k \geqslant 0,\ A_{k+1}(x) = -\frac{i}{2} A_0(x) \cdot \int_x^b A_k''(y) A_0(y)\, dy.$$

We see that for large k, A_k will involve high order derivatives of A_0 that blow-up when x goes to 0. We need to track this behaviour in the construction to determine α such that WKB approximation holds on $[h^\alpha, b]$

Generalized Taylor expansions

Let $\mathcal{A}$ be a discrete set of exponents. We consider the smooth functions on $(0, b]$ that admits, near 0 an asymptotic expansion

$$\sum_{\alpha \in \mathcal{A}} a_\alpha x^\alpha$$

We prove that for any k, ℓ, $A_k^{(\ell)}$ admits such an expansion with

$$\mathcal{A}_{k,0} = \{m\gamma + n - k,\ m \geqslant 1, n \geqslant 0\} \cup \{0\}$$
$$\mathcal{A}_{k,\ell} = \{m\gamma + n - k - \ell,\ m \geqslant 1, n \geqslant 0\}, \quad \ell \geqslant 1.$$

It follows that WKB expansions hold on $[h^\alpha, b]$ for any $\alpha < 1$.

The matching region

On $[0, h^\beta]$, we solve

$$-h^2 u_h'' + (x^\gamma W(x) - E)u_h = 0,$$

by treating the term $x^\gamma W(x) u_h$ as a inhomogeneous term. Applying the variation of constants leads to a system of equations that can be solved by a Banach-Picard iteration scheme provided that

$$\beta > \frac{1}{\gamma + 1}.$$

- If $\beta > \frac{1}{\gamma+1}$ and $\beta < \alpha < 1$ then the intervals $[0, h^\beta]$ and $[h^\alpha, b]$ overlap.

Performing the matching

We can implement the Banach-Picard iteration scheme and understand how the Cauchy data at 0 and at h^α are related. Using WKB, we understand how the Cauchy data at h^α and at b are related.
The Cauchy data at b is well understood using the Maslov Ansatz that allows to go beyond the turning point.
In the end, we are able to write an asymptotic expansion for the Cauchy data at 0 of $G_h(\cdot\,; E)$.

- We can then estimate the mass near 0 and obtain the spacing by the same method as above.
- We can also write down Bohr-Sommerfeld rules for singular potentials on the half-line $[0, +\infty)$.

The Gluing problem

- When the halfline is $[-B, +\infty)$ then we have to write a collinearity equation for the Cauchy data at 0 of the function $G(\cdot\,; E)$ that we have constructed on $[0, +\infty)$ and the explicit solution on $[-B, 0]$.
- New regimes appear. E.g. for the potential x_+^2 there are eigenvalues of order h^2, whereas on $[0, +\infty)$, the lowest energy is of order h.