L^p-multipliers and lacunarity for compact quantum groups

Simeng Wang

Université de Franche-Comté (Besançon) / IM PAN (Warsaw)

Saarbrücken, April 2016

Classical examples

- (Khintchine inequality) The i.i.d. Radamacher random variables $(\varepsilon_k)_{k\geq 1}$ on a probability space (Ω, P) satisfies

$$\forall (c_k) \subset \mathbb{C}, \quad \Big\| \sum_k c_k \varepsilon_k \Big\|_{L^p(\Omega)} \sim_p \Big(\sum_k |c_k|^2 \Big)^{1/2}, \quad 1 \leq p < \infty.$$

  Note: $(\varepsilon_k)_{k\geq 1}$ = generators of the Cantor group $\{0,1\}^{\mathbb{N}}$.

Classical examples

- (Khintchine inequality) The i.i.d. Radamacher random variables $(\varepsilon_k)_{k\geq 1}$ on a probability space (Ω, P) satisfies

$$\forall (c_k) \subset \mathbb{C}, \quad \Big\| \sum_k c_k \varepsilon_k \Big\|_{L^p(\Omega)} \sim_p \Big(\sum_k |c_k|^2 \Big)^{1/2}, \quad 1 \leq p < \infty.$$

  Note: $(\varepsilon_k)_{k\geq 1}$ = generators of the Cantor group $\{0,1\}^{\mathbb{N}}$.

- (Random matrix version) Let $(u^{(k)})_{k\geq 1}$ be independent $n_k \times n_k$ random unitary(/orthogonal/Gaussian) matrices. Then for $(A^{(k)})_{k\geq 1} \subset \oplus_k M_{n_k}(\mathbb{C})$ and $1 \leq p < \infty$,

$$\Big\| \sum_k \sum_{i,j} A_{ij}^{(k)} u_{ij}^{(k)} \Big\|_{L^p(\prod_N U_N)} \sim_p \Big\| \sum_k \sum_{i,j} A_{ij}^{(k)} u_{ij}^{(k)} \Big\|_{L^2(\prod_N U_N)}$$

$$= \Big(\sum_k n_k^{-1} Tr(|A^{(k)}|^2) \Big)^{1/2}.$$

  Note: view $(u^{(k)})_{k\geq 1}$ subset of irr representations of $\prod_N U_N$.

Classical examples

- (Haagerup-Khintchine inequality) The freely independent generators $(\lambda(\gamma_k))_{k\geq 1} \subset VN(\mathbb{F})$ ($\mathbb{F}$ the free group) satisfies

$$\forall (c_k) \subset \mathbb{C}, \quad \Big\| \sum_k c_k \lambda(\gamma_k) \Big\|_{L^p(\hat{\mathbb{F}})} \sim_p \Big(\sum_k |c_k|^2 \Big)^{1/2}, \quad 1 \leq p \leq \infty.$$

Classical examples

- (Haagerup-Khintchine inequality) The freely independent generators $(\lambda(\gamma_k))_{k\geq 1} \subset VN(\mathbb{F})$ ($\mathbb{F}$ the free group) satisfies

$$\forall (c_k) \subset \mathbb{C}, \quad \Big\| \sum_k c_k \lambda(\gamma_k) \Big\|_{L^p(\hat{\mathbb{F}})} \sim_p \Big(\sum_k |c_k|^2 \Big)^{1/2}, \quad 1 \leq p \leq \infty.$$

- More examples:
  - subset with large gaps in $\mathbb{Z}$,
  - generators of S_∞ (or more general Coxeter groups);
  - general construction in arbitrary discrete groups by Bożejko 75'.

$\Lambda(p)$-sets

- Let Γ be a discrete group, $2 \le p \le \infty$. $E \subset \Gamma$ is called a $\Lambda(p)$-set if
$$\forall (c_\gamma)_{\gamma \in E} \subset \mathbb{C}, \quad \Big\| \sum_{\gamma \in E} c_\gamma \lambda(\gamma) \Big\|_{L^p(\hat{\Gamma})} \sim_p \Big\| \sum_{\gamma \in E} c_\gamma \lambda(\gamma) \Big\|_{L^2(\hat{\Gamma})}.$$
- Let G be a compact group, $2 \le p \le \infty$. $E \subset \mathrm{Irr}(G)$ is called a $\Lambda(p)$-set if for all $\{c_{ij}^{(\pi)} : \pi \in E, 1 \le i,j \le n_\pi\}$,
$$\Big\| \sum_{\pi \in E} \sum_{i,j} c_{ij}^{(\pi)} u_{ij}^{(\pi)} \Big\|_{L^p(G)} \sim_p \Big\| \sum_{\pi \in E} \sum_{i,j} c_{ij}^{(\pi)} u_{ij}^{(\pi)} \Big\|_{L^2(G)}.$$

$\Lambda(p)$-sets

- Let Γ be a discrete group, $2 \leq p \leq \infty$. $E \subset \Gamma$ is called a $\Lambda(p)$-set if

$$\forall (c_\gamma)_{\gamma \in E} \subset \mathbb{C}, \quad \Big\| \sum_{\gamma \in E} c_\gamma \lambda(\gamma) \Big\|_{L^p(\hat{\Gamma})} \sim_p \Big\| \sum_{\gamma \in E} c_\gamma \lambda(\gamma) \Big\|_{L^2(\hat{\Gamma})}.$$

- Let G be a compact group, $2 \leq p \leq \infty$. $E \subset \mathrm{Irr}(G)$ is called a $\Lambda(p)$-set if for all $\{c_{ij}^{(\pi)} : \pi \in E, 1 \leq i,j \leq n_\pi\}$,

$$\Big\| \sum_{\pi \in E} \sum_{i,j} c_{ij}^{(\pi)} u_{ij}^{(\pi)} \Big\|_{L^p(G)} \sim_p \Big\| \sum_{\pi \in E} \sum_{i,j} c_{ij}^{(\pi)} u_{ij}^{(\pi)} \Big\|_{L^2(G)}.$$

- Note that in the framework of quantum groups $\mathrm{Irr}(\hat{\Gamma}) = \Gamma$, and the above two notions meet together:
  Let $\mathbb{G}$ be a compact quantum group, $2 \leq p \leq \infty$. $E \subset \mathrm{Irr}(\mathbb{G})$ is called a $\Lambda(p)$-set if for all $\{c_{ij}^{(\pi)} : \pi \in E, 1 \leq i,j \leq n_\pi\}$,

$$\Big\| \sum_{\pi \in E} \sum_{i,j} c_{ij}^{(\pi)} u_{ij}^{(\pi)} \Big\|_{L^p(\mathbb{G})} \sim_p \Big\| \sum_{\pi \in E} \sum_{i,j} c_{ij}^{(\pi)} u_{ij}^{(\pi)} \Big\|_{L^2(\mathbb{G})}.$$

Outline

1. Introduction to L^p-Fourier analysis on CQGs
2. $\Lambda(p)$-sets and lacunarity for L^p-multipliers
3. Examples and existence

Outline

1. Introduction to L^p-Fourier analysis on CQGs

2. $\Lambda(p)$-sets and lacunarity for L^p-multipliers

3. Examples and existence

Recall on compact quantum groups

- A compact quantum group is a pair $\mathbb{G} = (A, \Delta)$, where:

  A : a unital C*-algebra, $\Delta : A \to A \otimes A$ a *-homomorphism s.t.

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta,$$

$$\overline{\mathrm{span}}((1 \otimes A)\Delta(A)) = \overline{\mathrm{span}}((A \otimes 1)\Delta(A)) = A \otimes A.$$

  $A := C(\mathbb{G})$ is called the algebra of "continuous functions" on $\mathbb{G}$.

Recall on compact quantum groups

- A compact quantum group is a pair $\mathbb{G} = (A, \Delta)$, where:

  A : a unital C*-algebra, $\Delta : A \to A \otimes A$ a *-homomorphism s.t.

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta,$$

$$\overline{\mathrm{span}}((1 \otimes A)\Delta(A)) = \overline{\mathrm{span}}((A \otimes 1)\Delta(A)) = A \otimes A.$$

  $A := C(\mathbb{G})$ is called the algebra of "continuous functions" on $\mathbb{G}$.

- **Classical examples**:
  - $(C(G), \Delta_G)$ with G a compact group, $\Delta_G(f)(s,t) = f(st)$, $f \in C(G)$, $s, t \in G$.
  - $\hat{\Gamma} = (C_r^*(\Gamma), \Delta_{C_r^*(\Gamma)})$ for a discrete group Γ, $\Delta_{C_r^*(\Gamma)}(\lambda(\gamma)) = \lambda(\gamma) \otimes \lambda(\gamma)$, $\gamma \in \Gamma$.

Recall on compact quantum groups

- A compact quantum group is a pair $\mathbb{G} = (A, \Delta)$, where:

  A : a unital C*-algebra, $\Delta : A \to A \otimes A$ a *-homomorphism s.t.

  $$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta,$$

  $$\overline{\mathrm{span}}((1 \otimes A)\Delta(A)) = \overline{\mathrm{span}}((A \otimes 1)\Delta(A)) = A \otimes A.$$

  $A := C(\mathbb{G})$ is called the algebra of "continuous functions" on $\mathbb{G}$.

- **Classical examples**:
  - $(C(G), \Delta_G)$ with G a compact group, $\Delta_G(f)(s,t) = f(st)$, $f \in C(G)$, $s, t \in G$.
  - $\hat{\Gamma} = (C_r^*(\Gamma), \Delta_{C_r^*(\Gamma)})$ for a discrete group Γ, $\Delta_{C_r^*(\Gamma)}(\lambda(\gamma)) = \lambda(\gamma) \otimes \lambda(\gamma)$, $\gamma \in \Gamma$.
- There exists a Haar state h on $C(\mathbb{G})$ which is "translate invariant".

Class of unitary representations

- Unitary representation of $\mathbb{G}$: $u = [u_{ij}]_{i,j=1}^{n} \in \mathbb{M}_n(C(\mathbb{G}))$ unitary s.t.

$$\forall 1 \leq j, k \leq n,\ \Delta(u_{jk}) = \sum_{p=1}^{n} u_{jp} \otimes u_{pk}.$$

  $\mathrm{Irr}(\mathbb{G})$: equivalent class of all such irreducible representations u. For each $\pi \in \mathrm{Irr}(\mathbb{G})$ choose a representative $u^{(\pi)}$.

Class of unitary representations

- Unitary representation of $\mathbb{G}$: $u = [u_{ij}]_{i,j=1}^{n} \in \mathbb{M}_n(C(\mathbb{G}))$ unitary s.t.

$$\forall 1 \leq j, k \leq n,\ \Delta(u_{jk}) = \sum_{p=1}^{n} u_{jp} \otimes u_{pk}.$$

  $\mathrm{Irr}(\mathbb{G})$: equivalent class of all such irreducible representations u. For each $\pi \in \mathrm{Irr}(\mathbb{G})$ choose a representative $u^{(\pi)}$.
- All such matrix coefficients $u_{ij}^{(\pi)}$ $(\pi \in \mathrm{Irr}(\mathbb{G}))$ spans a **dense** algebra of "polynomials" $\mathrm{Pol}(\mathbb{G}) \subset C(\mathbb{G})$.

Class of unitary representations

- Unitary representation of $\mathbb{G}$: $u = [u_{ij}]_{i,j=1}^n \in \mathbb{M}_n(C(\mathbb{G}))$ unitary s.t.

$$\forall 1 \leq j, k \leq n,\ \Delta(u_{jk}) = \sum_{p=1}^n u_{jp} \otimes u_{pk}.$$

  $\mathrm{Irr}(\mathbb{G})$: equivalent class of all such irreducible representations u. For each $\pi \in \mathrm{Irr}(\mathbb{G})$ choose a representative $u^{(\pi)}$.
- All such matrix coefficients $u_{ij}^{(\pi)}$ $(\pi \in \mathrm{Irr}(\mathbb{G}))$ spans a **dense** algebra of "polynomials" $\mathrm{Pol}(\mathbb{G}) \subset C(\mathbb{G})$.
- The completions of $\mathrm{Pol}(\mathbb{G})$ wrt different topologies give rise to various typical "function" spaces on $\mathbb{G}$:

$$L^2(\mathbb{G}), \quad C_r(\mathbb{G})(\subset B(L^2(\mathbb{G}))), \quad L^\infty(\mathbb{G})(\subset B(L^2(\mathbb{G}))).$$

  In particular, if $\mathbb{G} = \hat{\Gamma} = (C_r^*(\Gamma), \Delta_{C_r^*(\Gamma)})$, then

$$\mathrm{Irr}(\mathbb{G}) = \Gamma, \mathrm{Pol}(\mathbb{G}) = \mathbb{C}\Gamma, C_r(\mathbb{G}) = C_r^*(\Gamma), L^\infty(\mathbb{G}) = VN(\Gamma).$$

dual discrete quantum group and non-unimodularity

In the framework of locally compact quantum groups, there is a "dual" discrete quantum group, denoted $\hat{\mathbb{G}}$, subject to the following $*$-algebra

$$c_c(\hat{\mathbb{G}}) := \oplus_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathbb{M}_{n_\pi}(\mathbb{C}).$$

dual discrete quantum group and non-unimodularity

In the framework of locally compact quantum groups, there is a "dual" discrete quantum group, denoted $\hat{\mathbb{G}}$, subject to the following $*$-algebra

$$c_c(\hat{\mathbb{G}}) := \oplus_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathbb{M}_{n_\pi}(\mathbb{C}).$$

Recall: a discrete group is always unimodular. But the discrete quantum group $\hat{\mathbb{G}}$ can be non-unimodular (i.e., $\mathbb{G}$ not of Kac type).

- We have a modular element

$$F = (F_\pi)_{\pi \in \mathrm{Irr}(\mathbb{G})}, \quad F_\pi \in Mor(u^{(\pi)}, S^2(u^{(\pi)}))\ (F_\pi \in \mathbb{M}_{n_\pi}(\mathbb{C})).$$

  It is possible $\|F\| = \sup_\pi \|F_\pi\| = +\infty$.
- Left Haar functional on $c_c(\hat{\mathbb{G}})$: $\hat{h} = \oplus_\pi Tr(F_\pi) Tr(\cdot F_\pi)$.
- F is trivial iff h on $L^\infty(\mathbb{G})$ is tracial. Woronowicz used this F to implement the modular property of non-tracial Haar state h on $\mathbb{G}$.

Fourier series

It is well-known that there is a bijective correspondence

$$\mathrm{Pol}(\mathbb{G}) \to c_c(\hat{\mathbb{G}}) = \oplus_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathbb{M}_{n_\pi}(\mathbb{C}), \quad u_{ij}^{(\pi)} \mapsto e_{ij}^{(\pi)} \in \mathbb{M}_{n_\pi}(\mathbb{C}).$$

On the Hilbert space level, we may modify and define

$$\mathcal{F} : \mathrm{Pol}(\mathbb{G}) \to c_c(\hat{\mathbb{G}}), \quad u_{ij}^{(\pi)} \mapsto d_\pi^{-1} e_{ji}^{(\pi)} F_\pi^{-1},$$

then $\mathcal{F}$ is a bijection on $\mathrm{Pol}(\mathbb{G})$ and extends to a unitary on the L^2-level

$$h(x^* x) = \hat{h}((\mathcal{F}x)^*(\mathcal{F}x)), \quad x \in \mathrm{Pol}(\mathbb{G}).$$

Moreover $\mathcal{F}$ can be extended to a contraction on L^p-spaces.

Noncommutative L^p-spaces

Define $\|x\|_1 = \|h(\cdot x)\|_{L^\infty(\mathbb{G})^*}$ for $x \in \mathrm{Pol}(\mathbb{G})$, and let $L^1(\mathbb{G})$ be the completion of $(\mathrm{Pol}(\mathbb{G}), \|\|_1)$. (Note: h may be non-tracial!)

Noncommutative L^p-spaces

Define $\|x\|_1 = \|h(\cdot x)\|_{L^\infty(\mathbb{G})^*}$ for $x \in \mathrm{Pol}(\mathbb{G})$, and let $L^1(\mathbb{G})$ be the completion of $(\mathrm{Pol}(\mathbb{G}), \|\|_1)$. (Note: h may be non-tracial!)
Define $L^p(\mathbb{G})$ to be the complex interpolation space

$$L^p(\mathbb{G}) = (L^\infty(\mathbb{G}), L^1(\mathbb{G}))_{1/p}, \quad 1 \leq p \leq \infty.$$

Define $\ell^p(\hat{\mathbb{G}})$ on $c_c(\hat{\mathbb{G}})$ similarly.

Noncommutative L^p-spaces

Define $\|x\|_1 = \|h(\cdot x)\|_{L^\infty(\mathbb{G})^*}$ for $x \in \mathrm{Pol}(\mathbb{G})$, and let $L^1(\mathbb{G})$ be the completion of $(\mathrm{Pol}(\mathbb{G}), \|\|_1)$. (Note: h may be non-tracial!)
Define $L^p(\mathbb{G})$ to be the complex interpolation space

$$L^p(\mathbb{G}) = (L^\infty(\mathbb{G}), L^1(\mathbb{G}))_{1/p}, \quad 1 \leq p \leq \infty.$$

Define $\ell^p(\hat{\mathbb{G}})$ on $c_c(\hat{\mathbb{G}})$ similarly.

- (Hausdorff-Young inequality) We have the extension

$$\mathcal{F} : L^p(\mathbb{G}) \to \ell^q(\hat{\mathbb{G}}) \text{ contraction}, \quad 1 \leq p \leq 2, 1/p + 1/q = 1.$$

Noncommutative L^p-spaces

Define $\|x\|_1 = \|h(\cdot x)\|_{L^\infty(\mathbb{G})^*}$ for $x \in \mathrm{Pol}(\mathbb{G})$, and let $L^1(\mathbb{G})$ be the completion of $(\mathrm{Pol}(\mathbb{G}), \|\|_1)$. (Note: h may be non-tracial!)
Define $L^p(\mathbb{G})$ to be the complex interpolation space

$$L^p(\mathbb{G}) = (L^\infty(\mathbb{G}), L^1(\mathbb{G}))_{1/p}, \quad 1 \leq p \leq \infty.$$

Define $\ell^p(\hat{\mathbb{G}})$ on $c_c(\hat{\mathbb{G}})$ similarly.

- (Hausdorff-Young inequality) We have the extension

$$\mathcal{F} : L^p(\mathbb{G}) \to \ell^q(\hat{\mathbb{G}}) \text{ contraction}, \quad 1 \leq p \leq 2, 1/p + 1/q = 1.$$

- (Plancherel theorem) $\mathcal{F} : L^2(\mathbb{G}) \to \ell^2(\hat{\mathbb{G}})$ unitary.

L^p-Fourier multipliers

- **Multipliers**: Each $a = (a_\pi)_{\pi \in \mathrm{Irr}(\mathbb{G})} \in \prod_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathbb{M}_{n_\pi}$ induces a map

$$m_a : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G}), \quad m_a x = \mathcal{F}^{-1}(a\mathcal{F}x).$$

We say a is a left bounded multiplier on $L^p(\mathbb{G})$ if m_a extends to a bdd map on $L^p(\mathbb{G})$. (similar def. for right multipliers)
$M(L^p(\mathbb{G})) = \{$left & right bdd multipliers on $L^p(\mathbb{G})\} \subset B(L^p(\mathbb{G}))$.

L^p-Fourier multipliers

- **Multipliers**: Each $a = (a_\pi)_{\pi \in \mathrm{Irr}(\mathbb{G})} \in \prod_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathbb{M}_{n_\pi}$ induces a map

$$m_a : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G}), \quad m_a x = \mathcal{F}^{-1}(a\mathcal{F}x).$$

  We say a is a left bounded multiplier on $L^p(\mathbb{G})$ if m_a extends to a bdd map on $L^p(\mathbb{G})$. (similar def. for right multipliers) $M(L^p(\mathbb{G})) =$\{left & right bdd multipliers on $L^p(\mathbb{G})$\} $\subset B(L^p(\mathbb{G}))$.
- Daws, Neufang, Junge, Ruan (09'-12'): study completely bounded multiplier on $L^\infty(\mathbb{G})$. However, the theory of bounded but non completely bounded multipliers on $L^\infty(\mathbb{G})$ is quite unclear; it is easy to establish

$$\|a\|_{\ell^\infty(\hat{\mathbb{G}})} \leq \|a\|_{M_{cb}(L^\infty(\mathbb{G}))}.$$

  But if a is bounded multiplier but not completely bounded, it is even not clear if $a \in \ell^\infty(\hat{\mathbb{G}})$ (Daws 10').

L^p-Fourier multipliers

Proposition (W.) For $1 \leq p \leq \infty$, a is a bounded left multiplier on $L^p(\mathbb{G})$ iff $F^{-1/2}a^*F^{1/2}$ is a bounded right multiplier on $L^p(\mathbb{G})$ (with same norm).

Proposition (W.; partially communicated by M. Junge)

$$\|F^{\frac{1}{4}-\frac{1}{2p}}aF^{-\frac{1}{4}+\frac{1}{2p}}\|_{\ell^\infty(\hat{\mathbb{G}})} \leq \|a\|_{M(L^p(\mathbb{G}))}.$$

In particular, if h is tracial,

$$\|a\|_{\ell^\infty(\hat{\mathbb{G}})} \leq \|a\|_{M(L^\infty(\mathbb{G}))}.$$

Outline

$\Lambda(p)$-sets

- Let Γ be a discrete group, $2 \le p \le \infty$. $E \subset \Gamma$ is called a $\Lambda(p)$-set if

$$\forall (c_\gamma)_{\gamma \in E} \subset \mathbb{C}, \quad \Big\| \sum_{\gamma \in E} c_\gamma \lambda(\gamma) \Big\|_{L^p(\hat{\Gamma})} \sim_p \Big\| \sum_{\gamma \in E} c_\gamma \lambda(\gamma) \Big\|_{L^2(\hat{\Gamma})}.$$

- Let G be a compact group, $2 \le p \le \infty$. $E \subset \mathrm{Irr}(G)$ is called a $\Lambda(p)$-set if for all $\{c_{ij}^{(\pi)} : \pi \in E, 1 \le i, j \le n_\pi\}$,

$$\Big\| \sum_{\pi \in E} \sum_{i,j} c_{ij}^{(\pi)} u_{ij}^{(\pi)} \Big\|_{L^p(G)} \sim_p \Big\| \sum_{\pi \in E} \sum_{i,j} c_{ij}^{(\pi)} u_{ij}^{(\pi)} \Big\|_{L^2(G)}.$$

- Note that in the framework of quantum groups $\mathrm{Irr}(\hat{\Gamma}) = \Gamma$, and the above two notions meet together:
  Let $\mathbb{G}$ be a compact quantum group, $2 \le p \le \infty$. $E \subset \mathrm{Irr}(\mathbb{G})$ is called a $\Lambda(p)$-set if for all $\{c_{ij}^{(\pi)} : \pi \in E, 1 \le i, j \le n_\pi\}$,

$$\Big\| \sum_{\pi \in E} \sum_{i,j} c_{ij}^{(\pi)} u_{ij}^{(\pi)} \Big\|_{L^p(\mathbb{G})} \sim_p \Big\| \sum_{\pi \in E} \sum_{i,j} c_{ij}^{(\pi)} u_{ij}^{(\pi)} \Big\|_{L^2(\mathbb{G})}.$$

Characterization via L^p-multipliers

Theorem (W.) Let $2 < p < \infty$ and $E \subset \mathrm{Irr}(\mathbb{G})$. Then:

$$E \text{ is a } \Lambda(p)\text{-set}$$

if and only if

$$\forall\, a \in \oplus_{\pi \in E} \mathbb{M}_{n_\pi}, \quad \exists\, b \in M(L^p(\mathbb{G})), \quad b|_E = a.$$

Remark: The theorem for classical cases was discussed by Harcharras 99', Figa-Talamanca 70s', Pisier 81'.
The completely bdd version of above thm is not established yet.
(For $\mathbb{G} = \hat{\Gamma}$ dual of discrete group: Harcharras 99';
For $\mathbb{G} = G$ compact group: Hare-Mohanty 15'.)

Application: Sidon sets $\Rightarrow \Lambda(p)$-sets

Definition Let $\mathbb{G}$ be a compact quantum group. $E \subset \mathrm{Irr}(\mathbb{G})$ is a Sidon set if for any sequence of matrices $c^{(\pi)} = [c_{ij}^{(\pi)}]_{1 \leq i,j \leq n_\pi}, \pi \in E$,

$$\Big\| \sum_{\pi \in E} \sum_{i,j} c_{ij}^{(\pi)} u_{ij}^{(\pi)} \Big\|_{L^\infty(\mathbb{G})} \sim \sum_{\pi \in E} \mathrm{Tr}(|c^{(\pi)}|).$$

- Rudin first studied Sidon sets in $\mathbb{Z}$ motivated by harmonic analysis and number theory.
- In classical cases, the Sidonicity leads to various equivalent characterizations via multipliers, $\Lambda(p)$-conditions, random variables/matrices, which also saw strong links with Banach space theory, entropy and probability theory (Drury, Rider, Bourgain, Pisier).
- In our recent work we extend some of these classical results, which in particular answers an open question of Picardello on the equivalence of Sidon and strong Sidon sets for non-amenable groups.

Application: Sidon sets $\Rightarrow \Lambda(p)$-sets

For compact groups or (dual of) discrete groups, it is well-known that any Sidon set is a $\Lambda(p)$-set (Figa-Talamanca - Rider, Marcus - Pisier, Picardello, Harcharras).

Application: Sidon sets $\Rightarrow \Lambda(p)$-sets

For compact groups or (dual of) discrete groups, it is well-known that any Sidon set is a $\Lambda(p)$-set (Figa-Talamanca - Rider, Marcus - Pisier, Picardello, Harcharras).

For quantum groups: more delicate –

Theorem (Blendek-Michaliček 13') Let $\mathbb{G}$ be a CQG s.t. the Haar state h is tracial on $C(\mathbb{G})$. Denote

$$\chi_\pi = \sum_{i=1}^{n_\pi} u_{ii}^{(\pi)}, \quad \pi \in \mathrm{Irr}(\mathbb{G}).$$

If $E \subset \mathrm{Irr}(\mathbb{G})$ is a Helgason-Sidon set ($\subsetneq$ Sidon set), then for all $(c_\pi)_{\pi \in E} \subset \mathbb{C}$ and $x = \sum_{\pi \in E} c_\pi \chi_\pi$,

$$\|x\|_2 \sim \|x\|_1.$$

Application: Sidon sets $\Rightarrow \Lambda(p)$-sets

Recall

Theorem (W.) Let $2 < p < \infty$ and $E \subset \mathrm{Irr}(\mathbb{G})$. Then:

$$E \text{ is a } \Lambda(p)\text{-set}$$

if and only if

$$\forall\, a \in \oplus_{\pi \in E} \mathbb{M}_{n_\pi}, \quad \exists\, b \in M(L^p(\mathbb{G})), \quad b|_E = a.$$

Corollary Any Sidon set for $\mathbb{G}$ is a $\Lambda(p)$-set for $1 < p < \infty$.

Proof.

$E \subset \mathrm{Irr}(\mathbb{G})$ is a Sidon set

$\Rightarrow \forall\, a \in \oplus_{\pi \in E} \mathbb{M}_{n_\pi}, \quad \exists\, b \in M(L^\infty(\mathbb{G})), \quad b|_E = a.$

$\Rightarrow \forall\, a \in \oplus_{\pi \in E} \mathbb{M}_{n_\pi}, \quad \exists\, b \in M(L^p(\mathbb{G})), \quad b|_E = a.$

$\Rightarrow E$ is a $\Lambda(p)$-set. □

Outline

Lacunarity for free orthogonal groups O_N^+

- The algebra of functions $C(O_N^+)$ is the universal C*-algebra generated by elements $u_{ij} = u_{ij}^*, 1 \le i, j \le N$ such that the matrix $u = [u_{ij}]_{i,j}$ is unitary.

Lacunarity for free orthogonal groups O_N^+

- The algebra of functions $C(O_N^+)$ is the universal C*-algebra generated by elements $u_{ij} = u_{ij}^*, 1 \leq i, j \leq N$ such that the matrix $u = [u_{ij}]_{i,j}$ is unitary.
- There exists no infinite $\Lambda(p)$-sets for any $4 \leq p \leq \infty$ for O_N^+:

Lacunarity for free orthogonal groups O_N^+

- The algebra of functions $C(O_N^+)$ is the universal C*-algebra generated by elements $u_{ij} = u_{ij}^*, 1 \leq i,j \leq N$ such that the matrix $u = [u_{ij}]_{i,j}$ is unitary.
- There exists no infinite $\Lambda(p)$-sets for any $4 \leq p \leq \infty$ for O_N^+:

  Recall $\mathrm{Irr}(O_N^+)$ is indexed by $\mathbb{N} \cup \{0\}$ and the fusion rule is

$$u^{(n)} \otimes u^{(m)} \cong u^{(|n-m|)} \oplus u^{(|n-m|+2)} \oplus \cdots \oplus u^{(n+m)}.$$

  Consider $\chi_n = \sum u_{ii}^{(n)}$, then

$$\|\chi_n\|_4^4 = h((\chi_n^2)^2) = h((\chi_0 + \chi_2 + \cdots + \chi_{2n})^2) = n+1,$$

  but $\|\chi_n\|_2 = 1$.

Lacunarity for free orthogonal groups O_N^+

- The algebra of functions $C(O_N^+)$ is the universal C*-algebra generated by elements $u_{ij} = u_{ij}^*, 1 \leq i, j \leq N$ such that the matrix $u = [u_{ij}]_{i,j}$ is unitary.
- There exists no infinite $\Lambda(p)$-sets for any $4 \leq p \leq \infty$ for O_N^+:

  Recall $\mathrm{Irr}(O_N^+)$ is indexed by $\mathbb{N} \cup \{0\}$ and the fusion rule is

  $$u^{(n)} \otimes u^{(m)} \cong u^{(|n-m|)} \oplus u^{(|n-m|+2)} \oplus \cdots \oplus u^{(n+m)}.$$

  Consider $\chi_n = \sum u_{ii}^{(n)}$, then

  $$\|\chi_n\|_4^4 = h((\chi_n^2)^2) = h((\chi_0 + \chi_2 + \cdots + \chi_{2n})^2) = n + 1,$$

  but $\|\chi_n\|_2 = 1$.
- There are more questions on the optimal $\Lambda(p)$ constant wrt a fixed length function on $O_N^+ \rightsquigarrow$ optimal (log-)Sobolov inequalities and rapid decay on O_N^+. (Recent studies by Brannan, Vergnioux; Franz, Wang)

Lacunarity for $\mathrm{SU}_q(2)(0 < q < 1)$

- The algebra of functions $C(\mathrm{SU}_q(2))$ is the universal C*-algebra generated by elements α, γ such that the matrix below is unitary:

$$\begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}.$$

Lacunarity for $\mathrm{SU}_q(2)(0 < q < 1)$

- The algebra of functions $C(\mathrm{SU}_q(2))$ is the universal C*-algebra generated by elements α, γ such that the matrix below is unitary:

$$\begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}.$$

- Recall $\mathrm{Irr}(\mathrm{SU}_q(2))$ is indexed by $\mathbb{N} \cup \{0\}$ and the fusion rule is same as that of O_N^+. But the subset subject to the following condition

$$E = (u^{(n_k)})_{k \geq 1} \subset \mathrm{Irr}(\mathrm{SU}_q(2)), \quad n_k \geq n_{k-1} + k$$

satisfies the central $\Lambda(4)$ condition

$$\forall (c_n) \subset \mathbb{C}, \quad \| \sum_{n \in E} c_n \chi_n \|_4 \sim_q \| \sum_{n \in E} c_n \chi_n \|_2 = (\sum_n |c_n|^2)^{1/2}.$$

Lacunarity for $\mathrm{SU}_q(2)(0 < q < 1)$

- The algebra of functions $C(\mathrm{SU}_q(2))$ is the universal C*-algebra generated by elements α, γ such that the matrix below is unitary:

$$\begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}.$$

- Recall $\mathrm{Irr}(\mathrm{SU}_q(2))$ is indexed by $\mathbb{N} \cup \{0\}$ and the fusion rule is same as that of O_N^+. But the subset subject to the following condition

$$E = (u^{(n_k)})_{k \geq 1} \subset \mathrm{Irr}(\mathrm{SU}_q(2)), \quad n_k \geq n_{k-1} + k$$

satisfies the central $\Lambda(4)$ condition

$$\forall (c_n) \subset \mathbb{C}, \quad \| \sum_{n \in E} c_n \chi_n \|_4 \sim_q \| \sum_{n \in E} c_n \chi_n \|_2 = (\sum_n |c_n|^2)^{1/2}.$$

- There exists no infinite $\Lambda(p)$-sets for any $2 \leq p \leq \infty$ for $\mathrm{SU}_q(2)$.

Lacunarity for tensor product quantum group

- $C(\prod_k O_{N_k}^+) = \otimes_k C(O_{N_k}^+)$: Let $u^{(k)}$ be the fundamental rep of $O_{N_k}^+$. Then $(u^{(k)})_k \subset \mathrm{Irr}(\prod_k O_{N_k}^+)$ is a $\Lambda(p)$-set for all $2 \leq p < \infty$. We obtain a Khintchine type inequality for these "n.c. random matrices": for all $\{c_{ij}^{(k)} : k \geq 1, 1 \leq i,j \leq N_k\}$,

$$\Big\| \sum_{k,i,j} c_{ij}^{(k)} u_{ij}^{(k)} \Big\|_{L^p(\prod_k O_{N_k}^+)} \sim_p \Big\| \sum_{k,i,j} c_{ij}^{(k)} u_{ij}^{(k)} \Big\|_{L^2(\prod_k O_{N_k}^+)}.$$

  Similarly for U_N^+, O_N^*, etc.

Lacunarity for tensor product quantum group

- $C(\prod_k O^+_{N_k}) = \otimes_k C(O^+_{N_k})$: Let $u^{(k)}$ be the fundamental rep of $O^+_{N_k}$. Then $(u^{(k)})_k \subset \mathrm{Irr}(\prod_k O^+_{N_k})$ is a $\Lambda(p)$-set for all $2 \leq p < \infty$. We obtain a Khintchine type inequality for these "n.c. random matrices": for all $\{c^{(k)}_{ij} : k \geq 1, 1 \leq i,j \leq N_k\}$,

$$\Big\| \sum_{k,i,j} c^{(k)}_{ij} u^{(k)}_{ij} \Big\|_{L^p(\prod_k O^+_{N_k})} \sim_p \Big\| \sum_{k,i,j} c^{(k)}_{ij} u^{(k)}_{ij} \Big\|_{L^2(\prod_k O^+_{N_k})}.$$

  Similarly for U^+_N, O^*_N, etc.
- $C(\prod_k \mathrm{SU}_{q_k}(2)) = \otimes_k C(\mathrm{SU}_{q_k}(2))$: Let $u^{(k)}$ be the fundamental rep of $\mathrm{SU}_{q_k}(2)$. Then the above type of Khintchine inequality holds iff

$$\inf_k q_k > 0.$$

Lacunarity for tensor product quantum group

- $C(\prod_k O_{N_k}^+) = \otimes_k C(O_{N_k}^+)$: Let $u^{(k)}$ be the fundamental rep of $O_{N_k}^+$. Then $(u^{(k)})_k \subset \mathrm{Irr}(\prod_k O_{N_k}^+)$ is a $\Lambda(p)$-set for all $2 \leq p < \infty$. We obtain a Khintchine type inequality for these "n.c. random matrices": for all $\{c_{ij}^{(k)} : k \geq 1, 1 \leq i,j \leq N_k\}$,

$$\Big\| \sum_{k,i,j} c_{ij}^{(k)} u_{ij}^{(k)} \Big\|_{L^p(\prod_k O_{N_k}^+)} \sim_p \Big\| \sum_{k,i,j} c_{ij}^{(k)} u_{ij}^{(k)} \Big\|_{L^2(\prod_k O_{N_k}^+)}.$$

  Similarly for U_N^+, O_N^*, etc.
- $C(\prod_k \mathrm{SU}_{q_k}(2)) = \otimes_k C(\mathrm{SU}_{q_k}(2))$: Let $u^{(k)}$ be the fundamental rep of $\mathrm{SU}_{q_k}(2)$. Then the above type of Khintchine inequality holds iff

$$\inf_k q_k > 0.$$

- How about $\prod_k \mathrm{SU}_{q_k}(N_k)$???

Central lacunarity for G_q and O_N^+

- Lemma: compact quantum groups with the same fusion rules and the same dimension functions have identical central Sidon sets (i.e. $E \subset \mathrm{Irr}(\mathbb{G})$ s.t. for all $(c_\pi)_{\pi \in E} \subset \mathbb{C}$,

$$\| \sum_{\pi \in E} c_\pi \chi_\pi \| \sim \sum_{\pi \in E} n_\pi |c_\pi|,$$

and any Sidon set is central Sidon.)

Central lacunarity for G_q and O_N^+

- Lemma: compact quantum groups with the same fusion rules and the same dimension functions have identical central Sidon sets (i.e. $E \subset \mathrm{Irr}(\mathbb{G})$ s.t. for all $(c_\pi)_{\pi \in E} \subset \mathbb{C}$,

$$\| \sum_{\pi \in E} c_\pi \chi_\pi \| \sim \sum_{\pi \in E} n_\pi |c_\pi|,$$

  and any Sidon set is central Sidon.)
- Example on $\prod_k \mathrm{SU}_{q_k}(N_K)$: Let $u^{(k)}$ be the fundamental rep of $\mathrm{SU}_{q_k}(N_k)$. Then $(u^{(k)})_{k \geq 1} \subset \mathrm{Irr}(\prod_k \mathrm{SU}_{q_k}(N_K))$ central Sidon set.

Central lacunarity for G_q and O_N^+

- Lemma: compact quantum groups with the same fusion rules and the same dimension functions have identical central Sidon sets (i.e. $E \subset \mathrm{Irr}(\mathbb{G})$ s.t. for all $(c_\pi)_{\pi \in E} \subset \mathbb{C}$,

$$\|\sum_{\pi \in E} c_\pi \chi_\pi\| \sim \sum_{\pi \in E} n_\pi |c_\pi|,$$

and any Sidon set is central Sidon.)
- Example on $\prod_k \mathrm{SU}_{q_k}(N_K)$: Let $u^{(k)}$ be the fundamental rep of $\mathrm{SU}_{q_k}(N_k)$. Then $(u^{(k)})_{k \geq 1} \subset \mathrm{Irr}(\prod_k \mathrm{SU}_{q_k}(N_K))$ central Sidon set.
- Drinfeld-Jimbo deformation: a compact semi-simple Lie group $G \rightsquigarrow$ a compact quantum group G_q $(0 < q < 1)$. Then for any simply connected compact semi-simple Lie group G, G_q does not admit infinite (central) Sidon set.

Central lacunarity for G_q and O_N^+

- Lemma: compact quantum groups with the same fusion rules and the same dimension functions have identical central Sidon sets (i.e. $E \subset \mathrm{Irr}(\mathbb{G})$ s.t. for all $(c_\pi)_{\pi \in E} \subset \mathbb{C}$,

$$\| \sum_{\pi \in E} c_\pi \chi_\pi \| \sim \sum_{\pi \in E} n_\pi |c_\pi|,$$

  and any Sidon set is central Sidon.)
- Example on $\prod_k \mathrm{SU}_{q_k}(N_K)$: Let $u^{(k)}$ be the fundamental rep of $\mathrm{SU}_{q_k}(N_k)$. Then $(u^{(k)})_{k \geq 1} \subset \mathrm{Irr}(\prod_k \mathrm{SU}_{q_k}(N_K))$ central Sidon set.
- Drinfeld-Jimbo deformation: a compact semi-simple Lie group $G \rightsquigarrow$ a compact quantum group G_q $(0 < q < 1)$. Then for any simply connected compact semi-simple Lie group G, G_q does not admit infinite (central) Sidon set.
- fusion rule of $O_N^+ \sim$ fusion rule of $\mathrm{SU}(2)$. So O_N^+ does not admit infinite (central) Sidon set.

General existence of $\Lambda(p)$-sets

Theorem (Bożejko; W.) Let (M, φ) be a vNa equipped with a normal faithful state φ. Let $B = \{x_i \in M : i \geq 1\}$ be an orthogonal system wrt φ s.t. $\sup_i \|x_i\|_\infty < \infty$. Then for each $1 \leq p < \infty$, there exists an infinite subset $\{x_{i_k} : k \geq 1\} \subset B$

$$\forall (c_k) \subset \mathbb{C}, \quad \|\sum_{k \geq 1} c_k x_{i_k}\|_{L^p(M)} \sim \Big(\sum_{k \geq 1} |c_k|^2\Big)^{\frac{1}{2}}.$$

Corollary Let $\mathbb{G}$ be a CQG. Let $E \subset \mathrm{Irr}(\mathbb{G})$ be an infinite subset with $\sup_{\pi \in E} d_\pi < \infty$. Then for each $1 \leq p < \infty$, there exists an infinite subset $E' \subset E$ which is $\Lambda(p)$-set.

Thank you very much!