Université de Franche-Comté - Polish Academy of Sciences

Some problems in harmonic analysis on quantum groups

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Outline

Introduction to abstract harmonic analysis on quantum groups

Convolutions of states and L^p-improving operators

Sidon sets and $\Lambda(p)$-sets

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Classical frameworks

Abstract harmonic analysis studies Fourier series, integrals and related problems on topological groups. There are two important frameworks studied in recent decades:

- Harmonic analysis on compact groups
- Harmonic analysis on group algebras of discrete groups

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- Harmonic analysis on compact groups
- Harmonic analysis on group algebras of discrete groups

Observation:

- G a compact group $\rightsquigarrow$ comultiplication on $C(G)$:
  $\Delta_G : C(G) \to C(G \times G), \quad \Delta_G(f)(s,t) = f(st), \quad s,t \in G.$
  $(C(G), \Delta_G)$ determines G.
- Γ a discrete group $\rightsquigarrow$ group C*-algebra $C_r^*(\Gamma)$ with comultiplication $\Delta_{C_r^*(\Gamma)} : C_r^*(\Gamma) \to C_r^*(\Gamma) \otimes C_r^*(\Gamma)$. $(C_r^*(\Gamma), \Delta_{C_r^*(\Gamma)})$ determines Γ.
  (If Γ is abelian, we have a compact Pontryagin dual group $\hat{\Gamma}$ with $(C(\hat{\Gamma}), \Delta_{\hat{\Gamma}}) \cong (C_r^*(\Gamma), \Delta_{C_r^*(\Gamma)})$.)

Woronowicz's Compact quantum groups

We may extend the previous arguments in a more general framework, that is, compact quantum groups.

- A compact quantum group is a pair $\mathbb{G} = (A, \Delta)$, where:

A : a unital C*-algebra, $\quad \Delta : A \to A \otimes A \quad$ a *-homomorphism s.t.

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta,$$

$$\overline{\mathrm{span}}((1 \otimes A)\Delta(A)) = \overline{\mathrm{span}}((A \otimes 1)\Delta(A)) = A \otimes A.$$

$A := C(\mathbb{G})$ is called the algebra of "continuous functions" on $\mathbb{G}$.

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  $A := C(\mathbb{G})$ is called the algebra of "continuous functions" on $\mathbb{G}$.

- There exists a Haar state h on $C(\mathbb{G})$ which is "translate invariant".

Towards Fourier analysis: the "dual group"

Recall the Fourier transform on a compact abelian group G,

$$\mathcal{F} : \text{functions on } G \to \text{functions on Pontryagin dual } \hat{G}.$$

- G non-abelian: replace $\hat{G}$ by $\mathrm{Irr}(G)$ (irreducible representations)

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- For a compact quantum group $\mathbb{G}$: $\mathrm{Irr}(\mathbb{G})$?

- Unitary representation of $\mathbb{G}$: $u = [u_{ij}]_{i,j=1}^{n} \in \mathbb{M}_n(C(\mathbb{G}))$ unitary s.t.

$$\forall 1 \leq j, k \leq n,\ \Delta(u_{jk}) = \sum_{p=1}^{n} u_{jp} \otimes u_{pk}.$$

  $\mathrm{Irr}(\mathbb{G})$: equivalent classes of all such irreducible representations u. For each $\pi \in \mathrm{Irr}(\mathbb{G})$ choose a representative $u^{(\pi)}$.

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  $\mathrm{Irr}(\mathbb{G})$: equivalent classes of all such irreducible representations u. For each $\pi \in \mathrm{Irr}(\mathbb{G})$ choose a representative $u^{(\pi)}$.
- All such matrix coefficients $u_{ij}^{(\pi)}$ ($\pi \in \mathrm{Irr}(\mathbb{G})$) span a **dense** algebra of "polynomials" $\mathrm{Pol}(\mathbb{G}) \subset C(\mathbb{G})$.

Towards Fourier analysis: the "dual group"

In the framework of locally compact quantum groups, there is a "dual" discrete quantum group, denoted $\hat{\mathbb{G}}$, subject to the following $*$-algebra

$$c_c(\hat{\mathbb{G}}) := \oplus_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathbb{M}_{n_\pi}(\mathbb{C}).$$

We may define the Fourier transform

$$\mathcal{F} : \mathrm{Pol}(\mathbb{G}) \to c_c(\hat{\mathbb{G}}), \quad x \mapsto \hat{x},$$

where

$$\hat{x}(\pi) = (h \otimes \mathrm{id})((u^{(\pi)})^*(x \otimes 1)) = [h(u_{ji}^{(\pi)^*} x)]_{i,j=1}^{n_\pi}, \quad \pi \in \mathrm{Irr}(\mathbb{G}).$$

(Recall: for a compact group G, $f \in C(G)$,

$$\hat{f}(\pi) = \int_G u^{(\pi)}(g)^* f(g) dm(g) = \left[\int_G u_{ji}^{(\pi)^*} f dm\right]_{i,j=1}^{n_\pi} .)$$

L^p-Fourier series

Briefly, we obtain the Fourier transform

$$\mathcal{F} : \mathrm{Pol}(\mathbb{G}) \to c_c(\hat{\mathbb{G}}) := \oplus_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathbb{M}_{n_\pi}(\mathbb{C}), \quad x \mapsto \hat{x}.$$

The map can be extended to L^p-spaces.

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- $(\pi_h, H_h) =$ GNS of $\mathrm{Pol}(\mathbb{G})$ wrt h, $L^\infty(\mathbb{G}) = \pi_h(\mathrm{Pol}(\mathbb{G}))'' \subset B(H_h)$.

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- $(\pi_h, H_h) =$ GNS of $\mathrm{Pol}(\mathbb{G})$ wrt h, $L^\infty(\mathbb{G}) = \pi_h(\mathrm{Pol}(\mathbb{G}))'' \subset B(H_h)$.
- Define $\|x\|_1 = \|h(\cdot x)\|_{L^\infty(\mathbb{G})^*}$ for $x \in \mathrm{Pol}(\mathbb{G})$, and let $L^1(\mathbb{G})$ be the completion of $(\mathrm{Pol}(\mathbb{G}), \|\|_1)$. Define $L^p(\mathbb{G})$ to be the complex interpolation space

$$L^p(\mathbb{G}) = (L^\infty(\mathbb{G}), L^1(\mathbb{G}))_{1/p}, \quad 1 \leq p \leq \infty.$$

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- Define $\ell^p(\hat{\mathbb{G}})$ on $c_c(\hat{\mathbb{G}})$ similarly.

L^p-Fourier series

Briefly, we obtain the Fourier transform

$$\mathcal{F} : \mathrm{Pol}(\mathbb{G}) \to c_c(\hat{\mathbb{G}}) := \oplus_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathbb{M}_{n_\pi}(\mathbb{C}), \quad x \mapsto \hat{x}.$$

The map can be extended to L^p-spaces.

- (Hausdorff-Young inequality) We have the extension

$$\mathcal{F} : L^p(\mathbb{G}) \to \ell^q(\hat{\mathbb{G}}) \text{ contraction}, \quad 1/p + 1/q = 1.$$

- (Plancherel theorem) $\mathcal{F} : L^2(\mathbb{G}) \to \ell^2(\hat{\mathbb{G}})$ unitary.

More remark on $\hat{\mathbb{G}}$: non-unimodularity

Recall: a discrete group is always unimodular.
But the discrete quantum group $\hat{\mathbb{G}}$ can be non-unimodular.

There are different "left/right invariant" Haar weights on $\ell^\infty(\hat{\mathbb{G}})$, and a modular element F linking them,

$$F = (F_\pi)_{\pi \in \mathrm{Irr}(\mathbb{G})}, \quad F_\pi \in \mathbb{M}_{n_\pi}(\mathbb{C}).$$

It is possible $\|F\| = \sup_\pi \|F_\pi\| = +\infty$.

In fact, F is trivial iff h on $L^\infty(\mathbb{G})$ is tracial. Woronowicz used this F to implement the modular property of non-tracial Haar state h on $\mathbb{G}$.

Convolutions

- Recall: For a compact group G and $\mu_1, \mu_2 \in M(G)$,

$$\forall f \in C(G), \quad \int_G f d(\mu_1 \star \mu_2) = \int_{G \times G} f(gh) d\mu_1(g) d\mu_2(h).$$

If $d\mu_1 = f_1 dm, d\mu_2 = f_2 dm$ with $f_1, f_2 \in L^1(G)$, then "define"

$$(f_1 \star f_2) dm = (f_1 dm) \star (f_2 dm).$$

- Similarly, for a CQG $\mathbb{G}$ and $\varphi_1, \varphi_2 \in L^\infty(\mathbb{G})^*$, we may define

$$\varphi_1 \star \varphi_2 = (\varphi_1 \otimes \varphi_2) \circ \Delta.$$

This also induces the convolution $x_1 \star x_2$ for $x_1, x_2 \in L^1(\mathbb{G})$:

$$h(\cdot x_1 \star x_2) = h(\cdot x_1) \star h(\cdot x_2).$$

Convolutions on L^p-spaces

The modular element F gives rise to a scaling group τ of $*$-automorphisms of $L^\infty(\mathbb{G})$ such that

$$\tau_t(u^{(\pi)}) = F_\pi^{\mathrm{i}t} u^{(\pi)} F_\pi^{\mathrm{i}t}, \quad \pi \in \mathrm{Irr}(\mathbb{G}).$$

Young's inequality (Liu-W.-Wu, 2015): for $x, y \in \mathrm{Pol}(\mathbb{G})$ we have

$$\|\tau_{\frac{\mathrm{i}}{p'}}(y) \star x\|_r \leq \|x\|_p \|y\|_q, \quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}, \frac{1}{p} + \frac{1}{p'} = 1$$

where $1 \leq p, q, r \leq \infty$ if h is tracial, and $1 \leq p, q, r \leq 2$ for general cases (counterexample existing for $r = \infty$).
Remark: similar for LCQGs

Fourier multipliers

- **Multipliers**: Each $a = (a_\pi)_{\pi \in \mathrm{Irr}(\mathbb{G})} \in \prod_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathbb{M}_{n_\pi}$ induces a map
$$m_a : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G}), \quad m_a x = \mathcal{F}^{-1}(a\hat{x}).$$
We say a is a left bounded multiplier on $L^p(\mathbb{G})$ if m_a extends to a bdd map on $L^p(\mathbb{G})$. (similar def. for right multipliers)
$M(L^p(\mathbb{G})) =$ {left & right bdd multipliers on $L^p(\mathbb{G})$} $\subset B(L^p(\mathbb{G}))$.

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- Daws, Neufang, Junge, Ruan (09'-12'): study completely bounded multiplier on $L^\infty(\mathbb{G})$. Easy to establish

$$\|a\|_{\ell^\infty(\hat{\mathbb{G}})} \leq \|a\|_{M_{cb}(L^\infty(\mathbb{G}))}.$$

Fourier multipliers

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Proposition (Partially communicated by M. Junge)

$$\|F^{\frac{1}{4}-\frac{1}{2p}} a F^{-\frac{1}{4}+\frac{1}{2p}}\|_{\ell^\infty(\hat{\mathbb{G}})} \leq \|a\|_{M(L^p(\mathbb{G}))}.$$

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Background

A classical question: contruction of positive Borel probablity measures μ so that for a given $p > 1$,

$$\exists p < q, \quad \forall f \in L_p(\mathbb{T}, \frac{dm}{2\pi}), \ \|\mu \star f\|_q \leq \|f\|_p.$$

Oberlin (1982): the Cantor-Lebesgue measure supported by the usual middle-third Cantor set satisfies the above property.
In fact it suffices to show

$$\exists p < 2, \quad \|\mu \star f\|_2 \leq \|f\|_p, \quad f \in L_p(\mathbb{Z}/3\mathbb{Z}, P)$$

where P is the normalized counting measure, $\mu(\{0\}) = \mu(\{2\}) = \frac{1}{2}$.

Background

Motivated by Oberlin, Ritter (1984) showed that:

Theorem

If G is an arbitrary finite group and $T_\mu : f \mapsto \mu \star f$ is the convolution operator associated to a probability measure μ on G, then

$$\left(\exists\, p < 2,\ \|T_\mu : L^p(G) \to L^2(G)\| = 1\right) \ \Leftrightarrow\ G = \langle ij^{-1} : i, j \in \operatorname{supp} \mu \rangle.$$

The operator T with $\|T : L^p \to L^2\| = 1$ for a $p < 2$ will be said to be L^p-improving throughout the presentation.

Motivating question

Similar constructions on (finite/compact) quantum groups?

Interpretation in the quantum language

Recall the classical result: for $T_\mu : f \mapsto \mu \star f, \ f \in C(G)$

$\big(\exists\, p < 2, \ \| T_\mu : L^p(G) \to L^2(G) \| = 1\big) \ \Leftrightarrow \ (*)\ G = \langle ij^{-1} : i, j \in \operatorname{supp} \mu \rangle.$

Take $\nu = \mu(\cdot^{-1}) \star \mu$. It is a simple observation that

$$(*) \Leftrightarrow \ \forall f > 0, \ \exists n \geq 1, \ \int f d\nu^{\star n} > 0.$$

This corresponds to an important notion in the study of random walks on quantum groups.

Definition (Vaes, Vergnioux 07; Kalantar, Neufang, Ruan 14)

A state ψ on $C(\mathbb{G})$ is **non-degenerate** if

$$\forall 0 \neq x \in C(\mathbb{G})_+, \quad \exists n \geq 1, \quad \psi^{\star n}(x) > 0.$$

Main result

Theorem (W.)

Let $\mathbb{G}$ *be a finite quantum group and* φ *be a state on* $C(\mathbb{G})$. *Denote* $\psi = (\varphi \circ S) \star \varphi$. *Then*

$$\exists 1 < p < 2, \ \forall \ x \in C(\mathbb{G}), \ \|\varphi \star x\|_2 \leq \|x\|_p$$

iff ψ *is* **non-degenerate**: $\forall 0 \neq x \in C(\mathbb{G})_+, \exists n \geq 1, \psi^{\star n}(x) > 0.$

Infinite case: a free product construction

Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two CQGs. There exist a CQG $\mathbb{G} := \mathbb{G}_1 \hat{*} \mathbb{G}_2$ s.t.

$$C(\mathbb{G}) = C(\mathbb{G}_1) * C(\mathbb{G}_2), \quad L^\infty(\mathbb{G}) = L^\infty(\mathbb{G}_1) \bar{*} L^\infty(\mathbb{G}_2).$$

Theorem (W.)

Let $\mathbb{G}_1, \ldots, \mathbb{G}_n$ be finite quantum groups and let each φ_i be a state on $C(\mathbb{G}_i)$, $i \in \{1, \ldots, n\}$. Take $\mathbb{G} = \mathbb{G}_1 \hat{*} \cdots \hat{*} \mathbb{G}_n$ and denote by $\varphi = \varphi_1 * \cdots * \varphi_n$ the conditionally free product. If for each i,

$$\exists 1 \leq p < 2 \text{ s.t. } \forall\, x \in C(\mathbb{G}),\ \|\varphi_i \star x\|_2 \leq \|x\|_p,$$

then the convolution operator $T : x \mapsto \varphi \star x$, $x \in C(\mathbb{G})$ satisfies

$$\exists 1 < p' < 2, \quad \|T : L^{p'}(\mathbb{G}) \to L^2(\mathbb{G})\| = 1.$$

Key ingredient I: L^p-improvement vs spectral gap

Theorem (W.)

Let A be a finite dimensional C-algebra equipped with a faithful tracial state τ, and $T : A \to A$ be a unital $2-$positive trace preserving map on A. Then*

$$\exists 1 < p < 2, \quad \|T : L^p(A) \to L^2(A)\| = 1$$

if and only if

$$\sup_{x \in A\setminus\{0\}, \tau(x)=0} \frac{\|Tx\|_2}{\|x\|_2} \left(= \sup_{\|x\|_2=1, \tau(x)=0} \langle |T|x, x\rangle^{1/2}_{L^2(A,\tau)} \right) < 1.$$

Also, the L^p-improving property is stable under free product.

Key ingredient II: non-degenerate states, applications

Proposition (W.)

If φ is a non-degenerate state on $C(\mathbb{G})$, then

$$w^*\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \varphi^{\star k} = h.$$

This proposition yields useful results for studying quantum subgroups ($\rightarrow$ next slides).

Key ingredient II: non-degenerate states, applications

Recently, Skalski-Sołtan discussed the quantum subgroups "generated by a quantum subset" via the Hopf image of Banica-Bichon. Each $*$-representation π of $C(\mathbb{G})$ associates a quantum subgroup $\mathbb{G}_\pi \subset \mathbb{G}$. If G is a group and $X \subset G$ a closed subset with the restriction map $\pi : C(G) \to C(X)$, then $G_\pi = \langle X \rangle$.

Corollary (Banica,Franz,Skalski 12; W.)

Let A be a unital C^*-algebra with a unital $*$-homomorphism $\pi : C(\mathbb{G}) \to A$, and $q : C(\mathbb{G}) \to C(\mathbb{G}_\pi)$ the quotient map. Then given any faithful state φ on A,

$$h_{\mathbb{G}_\pi} \circ q = w^*\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\varphi \circ \pi)^{\star k}.$$

Key ingredient II: non-degenerate states, applications

This result is recently applied in a series of works of Banica on matrix models of quantum groups. For instance,

- (Banica 16) If $\mathbb{G} \subset O_n^*$, then we have a matrix model $C(\mathbb{G}) \subset \mathbb{M}_2(C(H))$ (H a compact Lie group) such that the Haar state of $\mathbb{G}$ coincides with $tr \otimes \int$.
- (Banica 16) We have a Weyl matrix model $\pi : C(S_N^+) \to \mathbb{M}_N(C(H))$ ($H \subset U_n$) such that the image of π is a Hopf algebra and the associated Haar state coincides with $tr \otimes \int$.
- (Banica 16; Banica-Nechita 16) For the deformed Fourier matrix model $\pi : C(S_{MN}^+) \to \mathbb{M}_{MN}(C(\mathbb{T}^{MN}))$, the law of the main character of the Hopf image is asymptotically free Poisson.

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Sidon sets: classical settings

Definition

Let G be a compact abelian group and $\Gamma = \hat{G}$ be the dual discrete group. A subset $E \subset \Gamma$ is called a Sidon set if

$$\forall (\alpha_\gamma) \subset \mathbb{C}, \quad \sum_{\gamma \subset E} |\alpha_\gamma| \sim \| \sum_{\gamma \in E} \alpha_\gamma \gamma \|_{L^\infty(G)}.$$

- Various characterizations: interpolation of bounded measures, multipliers, $\Lambda(p)$-estimations, unconditional bases...
- Typical examples: Rademacher functions; lacunarity in $\mathbb{Z}$...

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Noncommutative generalizations:

- $G \rightsquigarrow$ compact non-abelian group, $\Gamma \rightsquigarrow \mathrm{Irr}(G)$;
- $\Gamma \rightsquigarrow$ discrete non-abelian group, $L^\infty(G) \rightsquigarrow L^\infty(\hat{\Gamma}) = VN(\Gamma)$ group vNa. ($\hat{\Gamma}$ the dual quantum group of Γ, $\mathrm{Irr}(\hat{\Gamma}) = \Gamma$)

Sidon sets: classical settings

Let G be a compact group.
For $f \in L^\infty(G)$ and $\pi \in \mathrm{Irr}(G)$, recall $\hat{f}(\pi) = \int_G f(g)u^{(\pi)}(g)^* dm(g)$,

$$\|\hat{f}\|_1 = \sum_{\pi \in \mathrm{Irr}(G)} d_\pi \mathrm{Tr}(|\hat{f}(\pi)|).$$

Theorem (Figà-Talamanca)

Consider $E \subset \mathrm{Irr}(G)$. TFAE:

1. *E is a Sidon set, i.e.,* $\mathrm{supp}(\hat{f}) \subset E \Rightarrow \|\hat{f}\|_1 \sim \|f\|_\infty$;
2. $\oplus_{\pi \in E} \mathbb{M}_{n_\pi} = \{\hat{\mu}|_E : \mu \in M(G)\}$;
3. $\oplus^{c_0}_{\pi \in E} \mathbb{M}_{n_\pi} = \{\hat{f}|_E : f \in L^1(G)\}$.

Sidon sets: quantum group setting

Let $\mathbb{G}$ be a compact quantum group. F the modular element for $\hat{\mathbb{G}}$. For $x \in L^\infty(\mathbb{G})$, $\pi \in \mathrm{Irr}(\mathbb{G})$, recall $\hat{x}(\pi) = (h \otimes \mathrm{id})((u^{(\pi)})^*(x \otimes 1))$,

$$\|\hat{x}\|_1 = \sum_{\pi \in \mathrm{Irr}(\mathbb{G})} d_\pi \mathrm{Tr}(|\hat{x}(\pi) F_\pi|). \quad (d_\pi = \mathrm{Tr}(F_\pi))$$

Theorem (W.)

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1. *E is a Sidon set, i.e., $\mathrm{supp}(\hat{x}) \subset E \Rightarrow \|\hat{x}\|_1 \sim \|x\|_\infty$,*
2. $\oplus_{\pi \in E} \mathbb{M}_{n_\pi} = \{\hat{\varphi}|_E : \varphi \in C_r(\mathbb{G})^*\}$;
3. $\oplus^{c_0}_{\pi \in E} \mathbb{M}_{n_\pi} = \{\hat{x}|_E : x \in L^1(\mathbb{G})\}$.

Sidon sets: classical settings revisited

The previous theorem improves a result of Picardello: (Picardello proved it in the special case that Γ is amenable)

Corollary

Let Γ be a discrete group (not necessarily amenable). TFAE:

1. *$E \subset \Gamma$ is a Sidon set, i.e.,*

$$\forall (\alpha_\gamma) \subset \mathbb{C}, \quad \sum_{\gamma \in E} |\alpha_\gamma| \sim \| \sum_{\gamma \in E} \alpha_\gamma \lambda(\gamma) \|_{VN(\Gamma)};$$

2. *$E \subset \Gamma$ is a strong Sidon set, i.e.,*

$$c_0(E) = \{ f|_E : f \in A(\Gamma) (\cong L^1(\hat{\Gamma})) \}.$$

Remark: various generalizations vs amenability

- $\mathbb{G}$ is called coamenable if $\epsilon : u_{ij}^{(\pi)} \mapsto \delta_{ij}$ is bdd on $L^\infty(\mathbb{G})$. Recall our convention: $\hat{\Gamma} = (C_r^*(\Gamma), \Delta_{C_r^*(\Gamma)})$ being a compact quantum group. $\hat{\Gamma}$ is coamenable iff Γ is amenable.
- If $\mathbb{G}$ is not coamenable, there are various (non-equivalent) analogue of Sidon sets: weak Sidon sets, interpolation sets of multipliers, unconditional Sidon sets, Leinert sets, etc. (For $\hat{\Gamma}$: Pisier 95')

Sidon sets $\Rightarrow \Lambda(p)$-sets

Definition

$E \subset \mathrm{Irr}(\mathbb{G})$ is a $\Lambda(p)$-set if

$$\|x\|_p \sim \|x\|_1, \quad \mathrm{supp}(\hat{x}) \subset E.$$

It is well-known that if $\mathbb{G}$ is a compact group or the compact dual of a discrete group, then any Sidon set for $\mathbb{G}$ is a $\Lambda(p)$-set for $1 < p < \infty$.

- Case for compact groups: Figà-Talamanca, Marcus-Pisier;
- Case for (dual of) discrete groups: Picardello, Harcharras...
- Case for compact quantum groups: more delicate – Blendek-Michaliček 13': Helgason-Sidon $\Rightarrow$ central $\Lambda(2)$ in Kac case.

Sidon sets $\Rightarrow \Lambda(p)$-sets, Fourier multipliers

Theorem (W.)

Let $2 < p < \infty$ and $E \subset \mathrm{Irr}(\mathbb{G})$. Then:

$$\|x\|_p \sim \|x\|_1, \quad \mathrm{supp}(\hat{x}) \subset E,$$

if and only if

$$\forall\, a \in \oplus_{\pi \in E} \mathbb{M}_{n_\pi}, \quad \exists\, b \in M(L^p(\mathbb{G})), \quad b|_E = a.$$

Remark: the completely bdd version of above thm is not established yet.
(For $\mathbb{G} = \hat{\Gamma}$ dual of discrete group: Harcharras 99';
For $\mathbb{G} = G$ compact group: Hare-Mohanty 15'.)

Sidon sets $\Rightarrow \Lambda(p)$-sets, Fourier multipliers

Theorem (W.)

Let $2 < p < \infty$ and $E \subset \mathrm{Irr}(\mathbb{G})$. Then:

$$\|x\|_p \sim \|x\|_1, \quad \mathrm{supp}(\hat{x}) \subset E,$$

if and only if

$$\forall\, a \in \oplus_{\pi \in E} \mathbb{M}_{n_\pi}, \quad \exists\, b \in M(L^p(\mathbb{G})), \quad b|_E = a.$$

Corollary

Any Sidon set for $\mathbb{G}$ is a $\Lambda(p)$-set for $1 < p < \infty$.

Observations on non-unimodularity I

Recall that we have a modular element F for $\hat{\mathbb{G}}$.

Theorem

Let $E \subset \mathrm{Irr}(\mathbb{G})$ a $\Lambda(p)$-set for $2 < p < \infty$. Then

$$\sup_{\pi \in E} \|F_\pi\| < +\infty.$$

Drinfeld-Jimbo deformation: a compact semi-simple Lie group $G \rightsquigarrow$ a compact quantum group G_q $(0 < q < 1)$, with $\hat{G}_q$ non-unimodular.

Corollary

$\mathrm{SU}(2)_q$ *does not admit infinite $\Lambda(p)$-set for any $2 < p < \infty$.*

Observations on non-unimodularity II

More words on noncommutative L^p:
Let h be non-tracial on $\mathbb{G}$ (equivalently $\hat{\mathbb{G}}$ non-unimodular)

- According to Kosaki 84': For each $0 \leq \theta \leq 1$, there is a complex interpolation scale $(L^p_{(\theta)}(\mathbb{G}))_{1\leq p\leq\infty}$ between $L^\infty(\mathbb{G})$ and $L^\infty(\mathbb{G})_*$, which are isometrically isomorphic but not equal.

  In the previous slides we have indeed taken $L^p(\mathbb{G}) = L^p_{(0)}(\mathbb{G})$.

- Casper 13': The boundedness of Fourier transform depends on θ.

Observations on non-modularity II

Proposition

Our definition of $\Lambda(p)$-sets is independent of θ.
That is, let $2 < p < \infty$, $0 \leq \theta, \theta' \leq 1$ and $E \subset \mathrm{Irr}(\mathbb{G})$. Then:

$$\|x\|_{L^p_{(\theta)}(\mathbb{G})} \sim \|x\|_{L^1_{(\theta)}(\mathbb{G})}, \quad \mathrm{supp}(\hat{x}) \subset E,$$

if and only if

$$\|x\|_{L^p_{(\theta')}(\mathbb{G})} \sim \|x\|_{L^1_{(\theta')}(\mathbb{G})}, \quad \mathrm{supp}(\hat{x}) \subset E.$$

Existence of $\Lambda(p)$-sets

Theorem (Bożejko; W.)

Let (M, φ) be a vNa equipped with a normal faithful state φ. Let $B = \{x_i \in M : i \geq 1\}$ be an orthogonal system wrt φ s.t. $\sup_i \|x_i\|_\infty < \infty$. Then for each $1 \leq p < \infty$, there exists an infinite subset $\{x_{i_k} : k \geq 1\} \subset B$

$$\forall (c_k) \subset \mathbb{C}, \quad \|\sum_{k\geq 1} c_k x_{i_k}\|_{L^p(M)} \sim \Big(\sum_{k\geq 1} |c_k|^2\Big)^{\frac{1}{2}}.$$

Existence of $\Lambda(p)$-sets

Corollary

Let $\mathbb{G}$ be a CQG. Let $E \subset \mathrm{Irr}(\mathbb{G})$ be an infinite subset with $\sup_{\pi \in E} d_\pi < \infty$. Then for each $1 \leq p < \infty$, there exists an infinite subset $E' \subset E$ s.t.

$$\|x\|_p \sim \|x\|_1, \quad \mathrm{supp}(\hat{x}) \subset E'.$$

Some examples

- $\prod_k \mathbb{G}_k$ with $O_{N_k} \subset \mathbb{G}_k \subset U_{N_k}^+$: the collection of fundamental representations of all $\mathbb{G}_k$ gives a (weak) Sidon set and a $\Lambda(p)$-set for $1 < p < \infty$.
- $\prod_k \mathrm{SU}_{q_k}(2)$: the collection of fundamental representations of all $\mathrm{SU}_{q_k}(2)$ gives a Sidon set and a $\Lambda(p)$-set for $1 < p < \infty$ iff $\inf_k q_k > 0$.
- O_N^+ admits no infinite central $\Lambda(p)$-set for $p > 4$; but we can construct a concrete central $\Lambda(4)$-set for $\mathrm{SU}_q(2)$.
- For any simply connected compact semi-simple Lie group G, the Drinfeld-Jimbo deformation G_q does not admit infinite (central) Sidon set.

Thank you very much!