Cutoff profiles for quantum Lévy processes and quantum random transpositions

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Background

Consider a “random” process on a set S: at each step/time, for each $x \in S$, the probability of transiting from x to a $y \in S$ is $p(x, y) \in [0, 1]$ so that $\sum_{y \in S} p(x, y) = 1$.
(In particular each step corresponds to a distribution on S)

Amongst the central topics in probability theory:

- Does it converge after a long time? In particular does it converge to the uniform distribution?
- How does it converge? e.g., speed of convergence?

Background: convergence of random walks

The theory was initiated by Poincaré (1912): when we are playing cards, after a sufficiently long time, all the permutations of cards appear with equal probabilities.

CHAPITRE XVI.

QUESTIONS DIVERSES.

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225. Battage des cartes. — Je me suis occupé dans l'introduction des problèmes relatifs au joueur qui bat un jeu de cartes. Pourquoi, quand le jeu a été battu assez longtemps, admettons-nous que toutes les permutations des cartes, c'est-à-dire tous les ordres dans lesquels ces cartes peuvent être rangées, doivent être également probables? C'est ce que nous allons examiner de plus près.

Background: convergence of random walks and cutoff phenomenon

The first precise (and surprising) computation of sharp convergences is given by Diaconis-Shahshahani (1981) ("random transpositions").
The intuition is as follows:

- Take a deck of N cards and spread them on a table. (N very large)
- Randomly select one card uniformly and then select another one in the same way.
- If the same card is chosen twice: do nothing
- Otherwise: swap the two cards.

Then the convergence to uniform happens in a short window of time around $N \ln(N)/2$:

- Before $N \ln(N)/2$ steps, the distribution stays far from uniform;
- After $N \ln(N)/2$ steps, the distribution stays close to uniform.

Rigorous model for the random transposition

Intepretation of the previous process:

- Consider the random process on the permutation group of N points S_N: for each step, the probability of transitioning between $\sigma, \gamma \in S_N$ is $p(\sigma, \gamma) = \mu_N(\sigma^{-1}\gamma)$ with

$$\mu_N = \frac{N-1}{N}\mu_{\mathrm{tran}} + \frac{1}{N}\delta_{\mathrm{id}},$$

  where $\mu_{\mathrm{tran}} =$ uniform measure on the set of transpositions.
- The distribution at the k-th step is then given by the convolution:

$$\mu_N^{*k}(\sigma) := \sum_{\substack{\sigma_1,\ldots,\sigma_k \in S_N \\ \sigma_1\cdots\sigma_k = \sigma}} \mu_N(\sigma_1)\cdots\mu_N(\sigma_k), \quad \sigma \in S_N.$$

  We are interested in the (total variation) distance between μ_N^{*k} and the uniform distribution on S_N (namely the translate-invariant Haar measure on S_N).

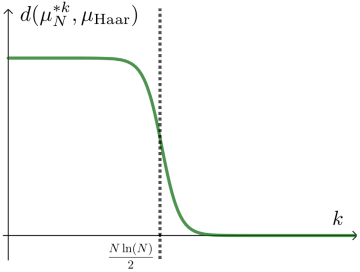
Cutoff phenomenon for random transpositions

Denote μ_{Haar} = Haar measure on S_N. The total variation distance

$$d(\mu_N^{*k}, \mu_{\mathrm{Haar}}) := \sup_{A \subset S_N} |\mu_N^{*k}(A) - \mu_{\mathrm{Haar}}(A)| = \frac{1}{2}\|\mu_N^{*k} - \mu_{\mathrm{Haar}}\|_{\mathrm{TV}}$$

Theorem (Diaconis-Shahshahani 81') For $\epsilon > 0$, as $N \to \infty$

$$d(\mu_N^{*(1-\epsilon)N\ln(N)/2}, \mu_{\mathrm{Haar}}) \to 1, \quad d(\mu_N^{*(1+\epsilon)N\ln(N)/2}, \mu_{\mathrm{Haar}}) \to 0.$$



After D-S, plenty of similar cutoff phenomena for other random walks have been discovered. Still a competitive field in today's proba theory.

Cutoff profiles

We may try to get a better understanding on how the "fall" occurs in the short window. Recall that the cutoff at time $(t_N)_{N\in\mathbb{N}}$ for the measures $(\mu_N)_{N\in\mathbb{N}}$ refers to the convergence of TV distance

$$d(\mu_N^{*(1-\epsilon)t_N}, \mu_{\mathrm{Haar}}) \to 1, \quad d(\mu_N^{*(1+\epsilon)t_N}, \mu_{\mathrm{Haar}}) \to 0.$$

We may look for "higher order terms" $(w_N)_{N\in\mathbb{N}}$ so that

$$f(c) := \lim_{N\to\infty} d(\mu_N^{*(t_N+cw_N)}, \mu_{\mathrm{Haar}}), \quad \forall c \in \mathbb{R}$$

defines a cont function decreasing from 1 to 0, called the cutoff profile.

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- Various works since 90s: In many cases, the profile can be described using Gaussian distributions (e.g. Ehrenfest Urn) or Poisson laws (e.g. dovetail shuffle, simple exclusion process on the circle,..), etc.
- Teyssier 20': for Diaconis-Shahshahani's random transpositions,

$$f(c) := \lim_{N\to\infty} d(\mu_N^{*\frac{1}{2}(N\ln(N)+cN)}, \mu_{\mathrm{Haar}}) = d(\mathrm{Poiss}(1+e^{-c}), \mathrm{Poiss}(1)).$$

Topics for today's talk:

- Cutoff phenomenon for "quantum random transpositions"
- Describe its cutoff profile using distributions from free probability theory.
- Similar results for some continuous processes on quantum groups. (Novelty even in the classical setting)

Quantum permutations

- Recall that a permutation matrix $C = [c_{ij}]_{1 \leq i,j \leq N} \in S_N$ is such that

$$c_{ij} \in \{0,1\}, \quad CC^t = C^t C = I.$$

  The algebra $C(S_N)$ of functions on S_N is generated by the functions $C \mapsto c_{ij}$.

- The idea for “quantum” is simply to replace scalars c_{ij} by operators (for instance in the recent study of non-local games in QIT): (Shuzhou Wang) We consider the universal C*-algebra generated by N^2 elements $(u_{ij})_{1 \leq i,j \leq N}$ s.t. for the matrix $U = (u_{ij})_{1 \leq i,j \leq N}$,

$$u_{ij} = u_{ij}^* = u_{ij}^2, \quad UU^t = U^t U = I.$$

  As a intuitive notation, we denote this alg by $C(S_N^+)$ and call S_N^+ the quantum permutation group.

Convolution and random walks on quantum groups

(Recall Isabelle's talk on Tuesday)

- Analogue of group multiplications: $*$-homomorphism

$$\Delta : C(S_N^+) \to C(S_N^+) \otimes C(S_N^+), \quad u_{ij} \mapsto \sum_{k=1}^{N} u_{ik} \otimes u_{kj}.$$

- Analogue of convolutions: for two states $\varphi_1, \varphi_2 \in C(S_N^+)^*$,

$$\varphi_1 * \varphi_2 := (\varphi_1 \otimes \varphi_2) \circ \Delta.$$

- Analogue of Haar measure: $\exists$ unique state $h \in C(S_N^+)^*$ s.t. for all state $\varphi \in C(S_N^+)^*$, $\varphi * h = h * \varphi = h$, called the Haar state. (Denote the vNa $L^\infty(S_N^+)$ = GNS construction of $C(S_N^+)$ wrt h)
- related previous works on convergence of φ^{*k} to h: convergence of ergodic average (Woronowicz); case of other (finite) quantum groups: convergence for central states (Baraquin), ergodic thm (JP McCarthy), speed of convergence (Baraquin, JP McCarthy), ..

Analogue of total variation distance on quantum groups

Noncommutative analogues of Banach algebras of measures:

- Fourier algebra $L^\infty(S_N^+)_*$ wrt convolution product (in the classical setting: all abs cont measures wrt Haar measure)
- Fourier-Stieltjes algebra $C(S_N^+)^*$ wrt convolution product (in the classical setting: all bounded measures)
- $L^\infty(S_N^+)_*$ embeds isometrically as an ideal of $C(S_N^+)^*$.
- Recall that for a unital commutative C*-algebra $C(K)$ and a functional $\varphi \in C(K)^*$ given by $\varphi = \int \cdot d\nu$, then $\|\varphi\|_{C(K)^*} = \|\nu\|_{\mathrm{TV}}$.

So we may regard the distance in $C(S_N^+)^*$ as an analogue of total variation distance : for two states φ_1, φ_2,

$$d(\varphi_1, \varphi_2) := \frac{1}{2}\|\varphi_1 - \varphi_2\|_{C(S_N^+)^*} \Big(= \sup_{p=p^*=p^2 \in C(S_N^+)^{**}} |\varphi_1(p) - \varphi_2(p)| \Big).$$

Freslon 19': First cutoff phenomenon on genuine quantum groups. But the case of quantum random transpositions was left open.

Quantum random transpositions

Recall: classical random transpositions given by $\mu_N = \frac{N-1}{N}\mu_{\mathrm{tran}} + \frac{1}{N}\delta_{\mathrm{id}}$

- μ_{tran} is unif distribution on $C := \{\textit{transpositions}\}$. Note that C is a conjugacy class, so for $\mathbb{E} = |S_N|^{-1} \int \mathrm{ad}(\sigma) d\sigma$,

$$\int_{S_N} f d\mu_{\mathrm{tran}} = \int_{S_N} (\mathbb{E}f) d\mu_{\mathrm{tran}} = (\mathbb{E}f)((12)).$$

- ad-invariant elements of $C(S_N^+)$ are generated by $\sum_i u_{ii}$; $\exists$ cond expectation $\mathbb{E} : C(S_N^+) \to C^*(\sum_i u_{ii})$. We consider analogously

$$\varphi_{\mathrm{tran}}(f) = (\pi \circ \mathbb{E}f)((12)), \quad f \in C(S_N^+),$$

  where $\pi : C(S_N^+) \to C(S_N)$ denotes the abelianization. (Intuitively unif distribution on the quantum conjugacy class of transpositions)
- counit $\varepsilon : C(S_N^+) \to \mathbb{C}$, unique state s.t. $\varepsilon * \varphi = \varphi$, $\forall \varphi \in C(S_N^+)^*$.
- Problem: cutoff for $\varphi_N := \frac{N-1}{N}\varphi_{\mathrm{tran}} + \frac{1}{N}\varepsilon$?

Cutoff for quantum random transpositions

$$\varphi_N : C(S_N^+) \to \mathbb{C}, \quad \varphi_N = \frac{N-1}{N}\varphi_{\mathrm{tran}} + \frac{1}{N}\varepsilon$$

Theorem (Freslon-Teyssier-W) For $\epsilon > 0$, as $N \to \infty$,

$$d(\varphi_N^{*(1-\epsilon)\frac{N\ln(N)}{2}}, h) \to 1, \quad d(\varphi_N^{*(1+\epsilon)\frac{N\ln(N)}{2}}, h) \to 0.$$

Moreover we have the (right) cutoff profile: for $c > 0$, as $N \to \infty$,

$$d(\varphi_N^{*\frac{1}{2}(N\ln(N)+cN)}, h)$$

$$\to d\left(D_{\sqrt{1+e^{-c}}}\left(\mathrm{Meix}^+\left(\frac{1-e^{-c}}{\sqrt{1+e^{-c}}}, \frac{-e^{-c}}{1+e^{-c}}\right)\right) * \delta_{e^{-c}}, \mathrm{Meix}^+(1,0)\right)$$

where: - $D_r(\mu)$ the r-dilation of μ (i.e. $rX \sim D_r(\mu)$ if $X \sim \mu$)
- Meix^+ denotes the free Meixner law.

Free Meixner (/Poisson/semicircular) law

Free Meixner laws are introduced by Bozejko, Bryc, Saitoh, Yoshida, as analogues of classical Meixner laws. The absolutely continuous part with parameters $a \in \mathbb{R}, b \geq 1$ is given by

$$d\,\mathrm{Meix}^{+}(a,b)(t) = \frac{\sqrt{4(1+b)-(t-a)^2}}{2\pi(bt^2+at+1)}dt.$$

- $b = 0$: free Poisson law for $\lambda > 1$

$$d\,\mathrm{Poiss}^{+}(\lambda,\alpha)(t) = \frac{1}{2\pi\alpha t}\sqrt{4\lambda\alpha^2-(t-\alpha(1+\lambda))^2}dt$$

- $a = b = 0$: free semicircular law $(2\pi)^{-1}\sqrt{4-t^2}dt$.

Comments on the non absolute-continuity

$$\varphi_N : C(S_N^+) \to \mathbb{C}, \quad \varphi_N = \frac{N-1}{N}\varphi_{\mathrm{tran}} + \frac{1}{N}\varepsilon$$

The usual method of studying cutoff phenomena is based on the L^2-estimates of the density functions and then on the noncommutative Fourier analysis, which breaks down in this setting, since ε is not bounded on $L^\infty(S_N^+)$!

One key step toward the previous main theorem is the following approximation:

Theorem We have

$$d(\varphi_N^{*\frac{1}{2}(N\ln(N)+cN)}, \varphi_{\mathrm{tran}}^{*\frac{1}{2}(N\ln(N)+cN)}) \to 0 \quad \text{as } N \to \infty.$$

The theorem reduced the problem to that of $\varphi_{\mathrm{tran}}^{*\frac{1}{2}(N\ln(N)+cN)}$, which is absolute cont wrt h and the Fourier analytic tools become available.

Case of continuous processes

We may also consider continuous random processes indexed by $t \in \mathbb{R}_+$.

- Méloit 14': Brownian motions on simple simply connected compact Lie groups satisfy the cutoff phenomenon at the time $\ln(N)$ where N is the rank of the group.
- More difficult for the cutoff profile. Very few is known even in the classical setting.
- We will establish precise cutoff profiles of Brownian motions on the quantum analogue of orthogonal groups!

Free quantum orthogonal groups

- Recall that an orthogonal matrix $C = [c_{ij}]_{1\le i,j\le N} \in O_N$ is such that

$$c_{ij} \in \mathbb{R}, \quad CC^t = C^tC = I.$$

  The algebra $C(O_N)$ of functions on O_N is generated by the functions $C \mapsto c_{ij}$.
- As the case of S_N^+, we replace scalars c_{ij} by operators: (Shuzhou Wang) We consider the universal C*-algebra generated by N^2 elements $(u_{ij})_{1\le i,j\le N}$ s.t. for the matrix $U = (u_{ij})_{1\le i,j\le N}$,

$$u_{ij} = u_{ij}^*, \quad UU^t = U^tU = I.$$

  As a intuitive notation, we denote this alg by $C(O_N^+)$ and call O_N^+ the free orthogonal group.
- Similar theory on the multiplicative structures, convolutions, Haar state,..

Lévy process on O_N^+

As the discrete case, our random process on quantum groups is realized by convolutions of states. In particular, we consider the convolution semigroup of states, i.e. a family of states $(\psi_t)_{t\in\mathbb{R}_+}$ on $C(O_N^+)$ with

- $\psi_0 = \varepsilon : u_{ij} \mapsto \delta_{ij}$,
- $\psi_t * \psi_s = \psi_{t+s}$,
- $\lim_{t\to 0} \psi_t(x) = \psi_0(x)$ for all $x \in *alg\langle u_{ij}, 1 \leq i,j \leq N\rangle$.

This equivalently defines a Lévy process on O_N^+ (Schürmann).

Cipriani-Franz-Kula: classify all adjoint-invariant Lévy processes on O_N^+. The Brownian motion part is determined by

$$\psi_t^{(N)}(\chi_n) = P_n(N)e^{-t\frac{P_n'(N)}{P_n(N)}}$$

where χ_n is the character of the n-th irreducible representation and P_n is the Cheybyshev polynomials of 2nd kind: $P_n(2\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$.

Comments on the non absolute-continuity

The Brownian motion on O_N^+ has many properties in sharp contrast with those of classical Brownian motions. Recall that any classical non-degenerate Lévy process automatically has an L^2-density with respect to the Haar measure (M. Liao). But in the quantum setting we have:

Proposition

- If $t < N\ln(N) + o(1)$, then $\psi_t^{(N)}$ is not absolutely continuous with respect to the Haar state,
- If $t > N\ln(N) + o(1)$, then $\psi_t^{(N)}$ is absolutely continuous with respect to the Haar state.

Cutoff of Brownian motions on O_N^+

Theorem (Freslon-Teyssier-W) For $\epsilon > 0$, as $N \to \infty$,

$$d(\psi^{(N)}_{(1-\epsilon)N\ln(N)}, h) \to 1, \quad d(\psi^{(N)}_{(1+\epsilon)N\ln(N)}, h) \to 0.$$

Moreover we have the (right) cutoff profile: for $c > 0$, as $N \to \infty$,

$$d(\varphi^{(N)}_{N\ln(N)+cN}, h) \to d\left(\mathrm{Poiss}^+\left(e^{2c}, -e^{-c}\right) * \delta_{e^c+e^{-c}}, \nu_{\mathrm{SC}}\right)$$
$$\left(= d\left(\mathrm{Meix}^+\left(-e^{-c}, 0\right) * \delta_{e^{-c}}, \mathrm{Meix}^+(0,0)\right)\right),$$

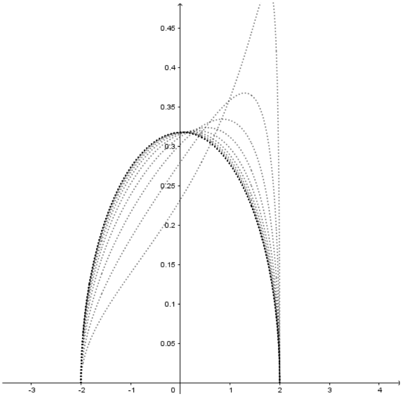
where: Poiss^+ denotes the free Poisson law, ν_{SC} the free semicircular law, Meix^+ the free Meixner law.

Remark Also a weaker result for $c < 0$: $\varphi^{(N)}_{N\ln(N)+cN} - h$ weakly converges to $\mathrm{Poiss}^+\left(e^{2c}, -e^{-c}\right) * \delta_{e^c+e^{-c}} - \nu_{\mathrm{SC}}$.

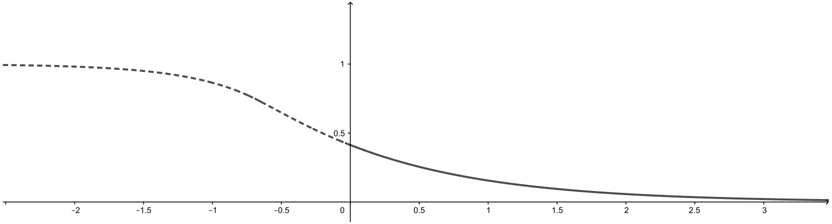
Remark Same cutoff time for L^p-distance $\|\varphi^{(N)}_{N\ln(N)+cN} - 1\|_p$ for all $1 \leq p \leq \infty$; but in the classical setting the time for $p = \infty$ is different.

$d(\varphi^{(N)}_{N\ln(N)+cN}, h) \to d\left(\mathrm{Meix}^{+}\left(-e^{-c}, 0\right) * \delta_{e^{-c}}, \mathrm{Meix}^{+}(0,0)\right).$

A plot of the density of $\mathrm{Meix}^{+}(-e^{-c}, 0) * \delta_{e^{-c}}$ for $0 < c < 5$:



The profile $f(c) = d\left(\text{Poiss}^{+}\left(e^{2c}, -e^{-c}\right) * \delta_{e^{c}+e^{-c}}, \nu_{\text{SC}}\right)$:



A few more examples

- Brownian motions on free real spheres introduced by Das-Franz-Xumin Wang: cutoff time at $\frac{1}{2}\ln(N)$
- Brownian motions on S_N^+ introduced by Franz-Kula-Skalski: cutoff time at $\frac{1}{2}N\ln(N)$
- Brownian motions on U_N^+ by free product construction: cutoff time at $N\ln(N)$

Thank you very much!