On the convergence of random walks on quantum groups: cutoff phenonmenon and its limit profiles

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Background

Consider a “random” process on a set S: at each step/time, for each $x \in S$, the probability of transiting from x to a $y \in S$ is $p(x, y) \in [0, 1]$ so that $\sum_{y \in S} p(x, y) = 1$.
(In particular each step corresponds to a distribution on S)

Amongst the central topics in probability theory:

- Does it converge after a long time? In particular does it converge to the uniform distribution?
- How does it converge? e.g., speed of convergence?

Background: convergence of random walks

The theory was initiated by Poincaré (1912): when we are playing cards, after a sufficiently long time, all the permutations of cards appear with equal probabilities.

CHAPITRE XVI.

QUESTIONS DIVERSES.

225. Battage des cartes. — Je me suis occupé dans l'introduction des problèmes relatifs au joueur qui bat un jeu de cartes. Pourquoi, quand le jeu a été battu assez longtemps, admettons-nous que toutes les permutations des cartes, c'est-à-dire tous les ordres dans lesquels ces cartes peuvent être rangées, doivent être également probables? C'est ce que nous allons examiner de plus près.

A warm-up example: Diaconis-Shahshahani's card shuffle

The first precise (and surprising) computation of sharp convergences is given by Diaconis-Shahshahani (1981) ("random transpositions"). The intuition is as follows:

- Take a deck of N cards and spread them on a table. (N very large)
- Randomly select one card uniformly and then select another one in the same way.
- If the same card is chosen twice: do nothing
- Otherwise: swap the two cards.

Then the convergence to uniform happens in a short window of time around $N \ln(N)/2$:

- Before $N \ln(N)/2$ steps, the distribution stays far from uniform;
- After $N \ln(N)/2$ steps, the distribution stays close to uniform.

Rigorous model for the random transposition

Intepretation of the previous process:

- Consider the random process on the permutation group of N points S_N: for each step, the probability of transitioning between $\sigma, \gamma \in S_N$ is $p(\sigma, \gamma) = \mu_N(\sigma^{-1}\gamma)$ with

$$\mu_N = \frac{N-1}{N}\mu_{\mathrm{tran}} + \frac{1}{N}\delta_{\mathrm{id}},$$

  where $\mu_{\mathrm{tran}} =$ uniform measure on the set of transpositions.
- The distribution at the k-th step is then given by the convolution:

$$\mu_N^{*k}(\sigma) := \sum_{\substack{\sigma_1,\dots,\sigma_k \in S_N \\ \sigma_1 \cdots \sigma_k = \sigma}} \mu_N(\sigma_1) \cdots \mu_N(\sigma_k), \quad \sigma \in S_N.$$

  We are interested in the (total variation) distance between μ_N^{*k} and the uniform distribution on S_N (namely the translate-invariant Haar measure on S_N).

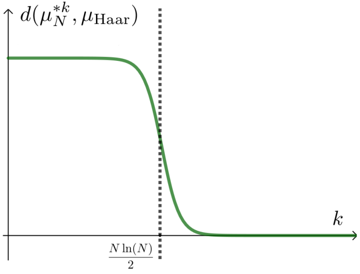
Cutoff phenomenon for random transpositions

Denote μ_{Haar} = Haar measure on S_N. The total variation distance

$$d(\mu_N^{*k}, \mu_{\mathrm{Haar}}) := \sup_{A \subset S_N} |\mu_N^{*k}(A) - \mu_{\mathrm{Haar}}(A)| = \frac{1}{2} \|\mu_N^{*k} - \mu_{\mathrm{Haar}}\|_{\mathrm{TV}}$$

Theorem (Diaconis-Shahshahani 81') For $\epsilon > 0$, as $N \to \infty$

$$d(\mu_N^{*(1-\epsilon)N\ln(N)/2}, \mu_{\mathrm{Haar}}) \to 1, \quad d(\mu_N^{*(1+\epsilon)N\ln(N)/2}, \mu_{\mathrm{Haar}}) \to 0.$$



To summarize,

- In mathematics, "random walks" on groups are described by convolutions: for a compact group G and a measure μ on G,

$$\int_G f d\mu^{*k} := \int_{G^k} f(\sigma_1 \cdots \sigma_k) d\mu(\sigma_1) \cdots d\mu(\sigma_k).$$

  We care about the convergence of $(\mu^{*k})_{k\in\mathbb{N}}$, for e.g., to the Haar measure μ_{Haar} on G.
- The cutoff at time $(t_N)_{N\in\mathbb{N}}$ for the measures $(\mu_N)_{N\in\mathbb{N}}$ on a family of groups $(G_N)_{N\in\mathbb{N}}$ refers to the convergence of TV distance

$$d(\mu_N^{*(1-\epsilon)t_N}, \mu_{\mathrm{Haar}}) \to 1, \quad d(\mu_N^{*(1+\epsilon)t_N}, \mu_{\mathrm{Haar}}) \to 0.$$

- After D-S, plenty of similar cutoff phenomena for other random walks have been discovered. Still a competitive field in today's proba theory.
- Similar questions for continuous processes (Brown motion etc), a priori for convergence of continuous convolution semigroups of measures.

Cutoff profiles

We may try to get a better understanding on how the "fall" occurs in the short window. Recall that the cutoff at time $(t_N)_{N\in\mathbb{N}}$ for the measures $(\mu_N)_{N\in\mathbb{N}}$ refers to the convergence of TV distance

$$d(\mu_N^{*(1-\epsilon)t_N}, \mu_{\mathrm{Haar}}) \to 1, \quad d(\mu_N^{*(1+\epsilon)t_N}, \mu_{\mathrm{Haar}}) \to 0.$$

We may look for "higher order terms" $(w_N)_{N\in\mathbb{N}}$ so that

$$f(c) := \lim_{N\to\infty} d(\mu_N^{*(t_N + c w_N)}, \mu_{\mathrm{Haar}}), \quad \forall c \in \mathbb{R}$$

defines a cont function decreasing from 1 to 0, called the cutoff profile.

Cutoff profiles

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We may look for "higher order terms" $(w_N)_{N\in\mathbb{N}}$ so that

$$f(c) := \lim_{N\to\infty} d(\mu_N^{*(t_N+cw_N)}, \mu_{\mathrm{Haar}}), \quad \forall c \in \mathbb{R}$$

defines a cont function decreasing from 1 to 0, called the cutoff profile.

- Various works since 90s: In many cases, the profile can be described using Gaussian distributions (e.g. Ehrenfest Urn) or Poisson laws (e.g. dovetail shuffle, simple exclusion process on the circle,..), etc.
- Teyssier 20': for Diaconis-Shahshahani's random transpositions,

$$f(c) := \lim_{N\to\infty} d(\mu_N^{*\frac{1}{2}(N\ln(N)+cN)}, \mu_{\mathrm{Haar}}) = d(\mathrm{Poiss}(1+e^{-c}), \mathrm{Poiss}(1)).$$

Topics for today's talk:

- Cutoff phenomenon for "quantum random transpositions"
- Describe its cutoff profile using distributions from free probability theory.
- Similar results for some continuous processes on quantum groups, notably Brownian motions or heat semigroups on free orthogonal groups. (Novelty even in the classical setting)

Quantum permutations

- Recall that a permutation matrix $C = [c_{ij}]_{1 \leq i,j \leq N} \in S_N$ is such that

$$c_{ij} \in \{0,1\}, \quad CC^t = C^t C = I.$$

  The algebra $C(S_N)$ of functions on S_N is generated by the functions $C \mapsto c_{ij}$.
- The idea for "quantum" is simply to replace scalars c_{ij} by operators (for instance in the recent study of non-local games in QIT): (Shuzhou Wang) We consider the universal C*-algebra generated by N^2 elements $(u_{ij})_{1 \leq i,j \leq N}$ s.t. for the matrix $U = (u_{ij})_{1 \leq i,j \leq N}$,

$$u_{ij} = u_{ij}^* = u_{ij}^2, \quad UU^t = U^t U = I.$$

  As a intuitive notation, we denote this alg by $C(S_N^+)$ and call S_N^+ the quantum permutation group.

Convolution and random walks on quantum groups

- Analogue of group multiplications: $*$-homomorphism

$$\Delta : C(S_N^+) \to C(S_N^+) \otimes C(S_N^+), \quad u_{ij} \mapsto \sum_{k=1}^{N} u_{ik} \otimes u_{kj}.$$

- Analogue of convolutions: for two states $\varphi_1, \varphi_2 \in C(S_N^+)^*$,

$$\varphi_1 * \varphi_2 \coloneqq (\varphi_1 \otimes \varphi_2) \circ \Delta.$$

- Analogue of Haar measure: $\exists$ unique state $h \in C(S_N^+)^*$ s.t. for all state $\varphi \in C(S_N^+)^*$, $\varphi * h = h * \varphi = h$, called the Haar state. (Denote the vNa $L^\infty(S_N^+)$ = GNS construction of $C(S_N^+)$ wrt h)
- related previous works on convergence of φ^{*k} to h: convergence of ergodic average (Woronowicz); case of other (finite) quantum groups: convergence for central states (Baraquin), ergodic thm (JP McCarthy), speed of convergence (Baraquin, JP McCarthy), ..

Analogue of total variation distance on quantum groups

Noncommutative analogues of Banach algebras of measures:

- Fourier algebra $L^\infty(S_N^+)_*$ wrt convolution product (in the classical setting: all abs cont measures wrt Haar measure)
- Fourier-Stieltjes algebra $C(S_N^+)^*$ wrt convolution product (in the classical setting: all bounded measures)
- $L^\infty(S_N^+)_*$ embeds isometrically as an ideal of $C(S_N^+)^*$.
- Recall that for a unital commutative C*-algebra $C(K)$ and a functional $\varphi \in C(K)^*$ given by $\varphi = \int \cdot d\nu$, then $\|\varphi\|_{C(K)^*} = \|\nu\|_{\mathrm{TV}}$.

So we may regard the distance in $C(S_N^+)^*$ as an analogue of total variation distance : for two states φ_1, φ_2,

$$d(\varphi_1, \varphi_2) := \frac{1}{2}\|\varphi_1 - \varphi_2\|_{C(S_N^+)^*} \Big(= \sup_{p=p^*=p^2 \in C(S_N^+)^{**}} |\varphi_1(p) - \varphi_2(p)| \Big).$$

Freslon 19': First cutoff phenomenon on genuine quantum groups. But the case of quantum random transpositions was left open.

Quantum random transpositions

Recall: classical random transpositions given by $\mu_N = \frac{N-1}{N}\mu_{\mathrm{tran}} + \frac{1}{N}\delta_{\mathrm{id}}$

- μ_{tran} is unif distribution on $C := \{\mathit{transpositions}\}$. Note that C is a conjugacy class, so for $\mathbb{E} = |S_N|^{-1}\int \mathrm{ad}(\sigma)d\sigma$,

$$\int_{S_N} f d\mu_{\mathrm{tran}} = \int_{S_N} (\mathbb{E}f) d\mu_{\mathrm{tran}} = (\mathbb{E}f)((12)).$$

- ad-invariant elements of $C(S_N^+)$ are generated by $\sum_i u_{ii}$; $\exists$ cond expectation $\mathbb{E} : C(S_N^+) \to C^*(\sum_i u_{ii})$. We consider analogously

$$\varphi_{\mathrm{tran}}(f) = (\pi \circ \mathbb{E}f)((12)), \quad f \in C(S_N^+),$$

  where $\pi : C(S_N^+) \to C(S_N)$ denotes the abelianization. (Intuitively unif distribution on the quantum conjugacy class of transpositions)
- counit $\varepsilon : C(S_N^+) \to \mathbb{C}$, unique state s.t. $\varepsilon * \varphi = \varphi$, $\forall \varphi \in C(S_N^+)^*$.
- Problem: cutoff for $\varphi_N := \frac{N-1}{N}\varphi_{\mathrm{tran}} + \frac{1}{N}\varepsilon$?

Cutoff for quantum random transpositions

$$\varphi_N : C(S_N^+) \to \mathbb{C}, \quad \varphi_N = \frac{N-1}{N}\varphi_{\mathrm{tran}} + \frac{1}{N}\varepsilon$$

Theorem (Freslon-Teyssier-W) For $\epsilon > 0$, as $N \to \infty$,

$$d(\varphi_N^{*(1-\epsilon)\frac{N\ln(N)}{2}}, h) \to 1, \quad d(\varphi_N^{*(1+\epsilon)\frac{N\ln(N)}{2}}, h) \to 0.$$

Moreover we have the cutoff profile: for $c \in \mathbb{R}$, as $N \to \infty$,

$$d(\varphi_N^{*\frac{1}{2}(N\ln(N)+cN)}, h)$$

$$\to d\left(D_{\sqrt{1+e^{-c}}}\left(\mathrm{Meix}^+\left(\frac{1-e^{-c}}{\sqrt{1+e^{-c}}}, \frac{-e^{-c}}{1+e^{-c}}\right)\right) * \delta_{e^{-c}}, \mathrm{Meix}^+(1,0)\right)$$

where: - $D_r(\mu)$ the r-dilation of μ (i.e. $rX \sim D_r(\mu)$ if $X \sim \mu$)
- Meix^+ denotes the free Meixner law.

Free Meixner (/Poisson/semicircular) law

Free Meixner laws are introduced by Bozejko, Bryc, Saitoh, Yoshida, as analogues of classical Meixner laws. The absolutely continuous part with parameters $a \in \mathbb{R}, b \geq 1$ is given by

$$d\,\mathrm{Meix}^{+}(a,b)(t) = \frac{\sqrt{4(1+b)-(t-a)^2}}{2\pi(bt^2+at+1)}dt.$$

- $b = 0$: free Poisson law for $\lambda > 1$

$$d\,\mathrm{Poiss}^{+}(\lambda,\alpha)(t) = \frac{1}{2\pi\alpha t}\sqrt{4\lambda\alpha^2-(t-\alpha(1+\lambda))^2}dt$$

- $a = b = 0$: free semicircular law $(2\pi)^{-1}\sqrt{4-t^2}dt$.

(Very typical distributions in free probaility theory, intuitively describing asymptotic distributions of large dim random matrices and probabilistic phenomena in von Neumann algebras.)

Case of continuous processes

We may also consider continuous random processes indexed by $t \in \mathbb{R}_+$.

- Méliot 14': Brownian motions on simple simply connected compact Lie groups satisfy the cutoff phenomenon at the time $\ln(N)$ where N is the rank of the group.
- More difficult for the cutoff profile. Very few is known even in the classical setting.
- We will establish precise cutoff profiles of Brownian motions on the quantum analogue of orthogonal groups!

Free quantum orthogonal groups

- Recall that an orthogonal matrix $C = [c_{ij}]_{1 \le i,j \le N} \in O_N$ is such that

$$c_{ij} \in \mathbb{R}, \quad CC^t = C^t C = I.$$

  The algebra $C(O_N)$ of functions on O_N is generated by the functions $C \mapsto c_{ij}$.

- As the case of S_N^+, we replace scalars c_{ij} by operators: (Shuzhou Wang) We consider the universal C*-algebra generated by N^2 elements $(u_{ij})_{1 \le i,j \le N}$ s.t. for the matrix $U = (u_{ij})_{1 \le i,j \le N}$,

$$u_{ij} = u_{ij}^*, \quad UU^t = U^t U = I.$$

  As a intuitive notation, we denote this alg by $C(O_N^+)$ and call O_N^+ the free orthogonal group.

- Similar theory on the multiplicative structures, convolutions, Haar state,..

Lévy process on O_N^+

A Lévy process on a classical group can be viewed as a càdlàg family of random variables $(X_t)_{t\in\mathbb{R}}$ on G so that the distributions μ_t of $X_t X_0^{-1}$ satisfying

- $\mu_0 = \delta_{\mathrm{Id}}$,
- $\mu_t * \mu_s = \mu_{t+s}$,
- $\lim_{t\to 0} \mu_t = \mu_0$ weakly.

As the discrete case, our random process on quantum groups is realized by convolutions of states. In particular, we consider the convolution semigroup of states, i.e. a family of states $(\psi_t)_{t\in\mathbb{R}_+}$ on $C(O_N^+)$ with

- $\psi_0 = \varepsilon : u_{ij} \mapsto \delta_{ij}$,
- $\psi_t * \psi_s = \psi_{t+s}$,
- $\lim_{t\to 0} \psi_t(x) = \psi_0(x)$ for all $x \in *alg\langle u_{ij}, 1 \le i, j \le N\rangle$.

This equivalently defines a so-called Lévy process on O_N^+ (Schürmann).

Lévy process on O_N^+

- Hunt: In classical probability theory, any Lévy process on a Lie group can be decomposed into a Brownian motion part and the other part (Lévy-Khinchin decomposition)
- Cipriani-Franz-Kula: Similar results on O_N^+. The analogue of Brownian motion part is determined by

$$\psi_t^{(N)}(\chi_n) = P_n(N) e^{-t \frac{P_n'(N)}{P_n(N)}}$$

  where χ_n is the character of the n-th irreducible representation and P_n is the Cheybyshev polynomials of 2nd kind: $P_n(2\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$.
- $C^*\langle \chi_n, n \in \mathbb{N} \rangle$ is isomorphic to $C([-N, N])$ via the map $\chi_n \mapsto P_n$, so we obtain a classical random process on $[-N, N]$ associated to the semigroup of measures $(m_t^{(N)})_{t \in \mathbb{R}}$ with $\int_{-N}^{N} P_n dm_t^{(N)} = P_n(N) e^{-t \frac{P_n'(N)}{P_n(N)}}$.

Proposition (Freslon-Teyssier-W)

- $\frac{P_n'(N)}{P_n(N)} = \frac{1}{\sqrt{N^2-4}} n + \frac{N-\sqrt{N^2-4}}{N^2-4} + (n+1)O(N^{-2n-1}),\ n \geq 1.$ (the remainder can also be explicitly expressed);
- If $t > N\ln(N)$, then $m_t^{(N)}$ is supported in $[-2,2]$ and has a L^2-density with respect to the semicircular measure $\sqrt{4-t^2}dt$.
- If $t < N\ln(N)$ and N large, then $m_t^{(N)} = \alpha_N(t)\delta_{\tilde{N}(t)} + \tilde{m}_t^{(N)}$, where $\alpha_N(t) \in \mathbb{R}$ and $\tilde{N}(t) \notin [-2,2]$, and $\tilde{m}_t^{(N)}$ is supported in $[-2,2]$ with L^2-density with respect to the semicircular measure.

Remark (1) Intuitively, this classical process walks from some $x \notin [-2,2]$ towards $[-2,2]$, and at some point jumps into it where it remains forever; in $[-2,2]$, asymptotically very similar to the Poisson semigroup on $SU(2)$ or the free Ornstein-Uhlenbeck process.
(2) Sharp contrast with classical Brownian motions: any classical non-degenerate Lévy process automatically has an L^2-density with respect to the Haar measure (M. Liao).

Cutoff of Brownian motions on O_N^+

Recall $d(\phi, \psi) = \|\phi - \psi\|_{C(O_N^+)^*}$.

Theorem (Freslon-Teyssier-W) For $\epsilon > 0$, as $N \to \infty$,

$$d(\psi^{(N)}_{(1-\epsilon)N\ln(N)}, h) \to 1, \quad d(\psi^{(N)}_{(1+\epsilon)N\ln(N)}, h) \to 0.$$

Moreover we have the cutoff profile: for $c \in \mathbb{R}$, as $N \to \infty$,

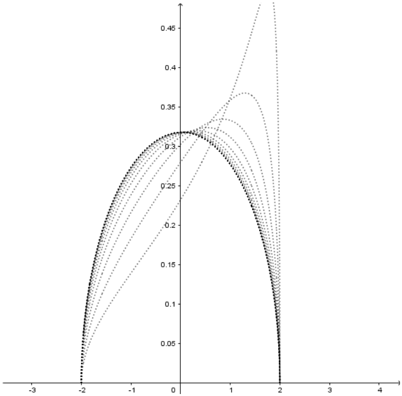
$$d(\varphi^{(N)}_{N\ln(N)+cN}, h) \to d\left(\mathrm{Poiss}^+\left(e^{2c}, -e^{-c}\right) * \delta_{e^c+e^{-c}}, \nu_{\mathrm{SC}}\right)$$
$$\left(= d\left(\mathrm{Meix}^+\left(-e^{-c}, 0\right) * \delta_{e^{-c}}, \mathrm{Meix}^+(0,0)\right)\right),$$

where: Poiss^+ denotes the free Poisson law, ν_{SC} the free semicircular law, Meix^+ the free Meixner law.

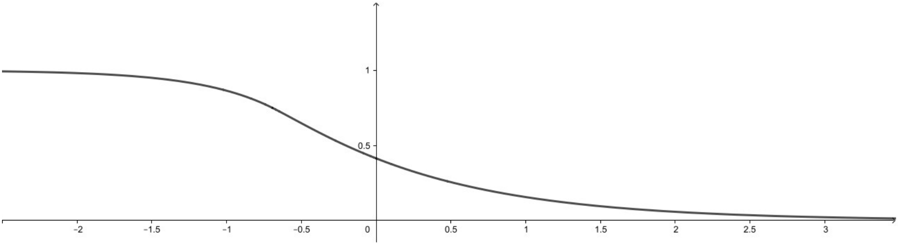
Remark Same cutoff time for L^p-distance $\|\varphi^{(N)}_{N\ln(N)+cN} - 1\|_p$ for all $1 \leq p \leq \infty$ and $c > 0$; but in the classical setting the time for $p = \infty$ is different.

$$d(\varphi^{(N)}_{N\ln(N)+cN}, h) \to d\left(\mathrm{Meix}^{+}\left(-e^{-c}, 0\right) * \delta_{e^{-c}}, \mathrm{Meix}^{+}(0,0)\right).$$

A plot of the density of $\mathrm{Meix}^{+}(-e^{-c}, 0) * \delta_{e^{-c}}$ for $0 < c < 5$:



The profile $f(c) = d\left(\text{Poiss}^{+}\left(e^{2c}, -e^{-c}\right) * \delta_{e^{c}+e^{-c}}, \nu_{\mathrm{SC}}\right)$:



Thank you very much!