Individual ergodic theorems on von Neumann algebras

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Groups, Operators, and Banach Algebras Webinar, July 2020

Recall: Birkhoff ergodic theorem

Theorem Let (Ω, μ) be a measure space and let $T : \Omega \to \Omega$ be a measure-preserving transformation. Then for every $f \in L_p(\Omega)\,(1 \le p < \infty)$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k\omega)$$

exists for almost everywhere $\omega \in \Omega$. Moreover, if $\mu(\Omega) = 1$ and T is ergodic (i.e., $T(E) = E$ implies $\mu(E) = 0$ or 1), then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k\omega) = \mathbb{E}_\mu f(\omega), \quad \text{a.e. } \omega \in \Omega.$$

(“ time average = space average ”)

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Today’s topic: extensions to the quantum setting

Noncommutative measure theory

In order to establish a mathematical framework for quantum physics, von Neumann developed the theory of operator algebras. In his philosophy, in the quantum setting:

- functions $\to$ operators;
- topology $\to$ C*-algebras;
- bounded functions on measurable spaces $\to$ von Neumann algebras (unital $*$-subalgebra $\mathcal{M} \subset B(H)$ closed under the strong operator topology).
- (indicator function $\mathbb{1}_E$ on a) measurable set E $\to$ projection $e \in \mathcal{M}$, $e = e^* = e^2$.
- general (unbounded) measurable functions $\to$ closed densely defined operators on H affiliated with $\mathcal{M}$..

Noncommutative measure theory

- (integral against a) measure $\to$ normal semifinite faithful tracial weight $\tau : \mathcal{M}_+ \to [0, \infty]$.

$$\text{for } \mathcal{M} = L_\infty(\Omega, \mu) \hookrightarrow B(L_2(\Omega)), \quad \tau(f) = \int_\Omega f d\mu.$$

- p-integrable functions $\to$ noncommutative L_p-spaces

$$L_p(\mathcal{M}) := \| \ \|_p\text{-completion of } \{x \in \mathcal{M} : \|x\|_p < \infty\} \ (1 \leq p < \infty),$$

$$\text{where} \quad \|x\|_p = [\tau(|x|^p)]^{1/p}.$$

$$L_\infty(\mathcal{M}) := \mathcal{M}.$$

- measure-preserving transformation $\to$ $*$-automorphism $T : \mathcal{M} \to \mathcal{M}$ with $\tau \circ T = \tau$

Noncommutative measure theory: a.e. convergence

- Lance (76') introduced an analogue for the noncommutative setting of a.e. convergence:
  $(x_n)_{n\geq 1} \subset \mathcal{M}$ (or $L_p(\mathcal{M})$) is said to converge almost uniformly (a.u. in short) to x if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that
  $$\tau(e^{\perp}) < \varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|(x_n - x)e\|_\infty = 0.$$
- Egorov: for the classical probability space, a.u. convergence $\Leftrightarrow$ a.e. convergence.

Noncommutative Birkhoff ergodic theorem: case $p = \infty$

Theorem (Lance 76) Assume that $\tau(1) < \infty$.[1] Let T be a $*$-automorphism of $\mathcal{M}$ s.t. $\tau = \tau \circ T$. Then for $x \in \mathcal{M}$

$$\frac{1}{N} \sum_{k=0}^{N} T^k x \to Px \text{ a.u.}, \quad \text{as } N \to \infty,$$

where P is the projection onto $Fix(T) \subset \mathcal{M}$ (which exists).

[1] The classical Birkhoff ergodic theorem does not work for $p = \infty$ if $\mu(\Omega) = \infty$ (in which case $L_\infty \not\subseteq L_p$).

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As the classical setting, the theorem admits various extensions. For e.g.

- Conze, Dang-Ngoc 78': case for multi-operators/flows
- Kümmerer 78': case for general Markov operators
- Yeadon 77': case for $\tau(1) = \infty$ with $p = 1$

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Open for many other cases until 2007 (in particular the case $\tau(1) = \infty$ for $1 < p < \infty$ was open).

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Why was it difficult to get more?

In the classical setting, the pointwise convergence of (f_n) is usually equivalent to the L_p-estimate of the maximal function $\sup_n |f_n|$:

- Banach principle: For $T_n : L_p \to L_p$, (up to some easy assumption)

$$\| \sup_n |T_n f| \|_{p,\infty} \lesssim \|f\|_p, \forall f \Rightarrow T_n f \to f \text{ a.e.}, \forall f$$

- Stein maximal principle: converse is true in many cases.

However in the noncommutative setting, the notation $\sup_n |f_n|$ does not even make sense: we can find three positive matrices $A, B, C \in \mathbb{M}_2$ so that there exists no matrix $X \in \mathbb{M}_2$ with

$$\langle X\xi, \xi\rangle = \max\{\langle A\xi, \xi\rangle, \langle B\xi, \xi\rangle, \langle C\xi, \xi\rangle\}, \quad \forall \xi \in \mathbb{C}^2.$$

The aforementioned Lance et al's work indeed cooked a very technical and baby version of noncommutative analogues of above maximal inequalities for $p = 1$.

Formulation of noncommutative maximal norms

The appropriate version of noncommutative maximal inequalities appeared in Junge 02', derived from Pisier's theory of noncommutative vector-valued L_p-spaces.

Idea Although we cannot formulate the object $\sup_n |f_n|$ in the noncommutative setting, we may always formulate $\| \sup_n |f_n| \|_p$:

$$\| \sup_n |f_n| \|_{L_p(\Omega)} = \|(f_n)_n\|_{L_p(\Omega;\ell_\infty)}.$$

The RHS has a right noncommutative analogue:

- Pisier introduced the Banach space $L_p(\mathcal{M}; E)$ with $\mathcal{M} = \mathbb{M}_n$ for any operator space $E \subset B(H)$, based on the operator space theory.
- For $E = \ell_\infty$, the theory was generalized by Junge for general $\mathcal{M}$. An equivalent comprehensive definition: a positive sequence $(x_n) \subset L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M}; \ell_\infty)$ iff

$$\exists a \in L_p(\mathcal{M}) \text{ s.t. } x_n \le a, \ \forall n \in \mathbb{N}.$$

And $\|(x_n)\|_{L_p(\mathcal{M};\ell_\infty)} = \inf_a \|a\|_p$.

From maximal norms to almost uniform convergence

The space $L_p(\mathcal{M}; \ell_\infty)$ is very efficient for studying almost uniform convergence on vNAs.

- Defant, Junge, Xu: $\lim_m \|(x_n)_{n\geq m}\|_{L_p(\mathcal{M};\ell_\infty)} = 0$ with $p \geq 2$ implies $x_n \to 0$ a.u. (similarly for $p < 2$ up to technical adaptation)
- Noncommutative Banach principle (Chilin, Litvinov, Skalski,..): For a sequence of positive maps $T_n : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ and a dense subspace $E \subset L_p(\mathcal{M})$, if $T_n x$ converges a.u. for all $x \in E$ and

$$\|(T_n x)\|_{L_p(\mathcal{M};\ell_\infty)} \lesssim \|x\|_p, \quad \forall x \in L_p(\mathcal{M}), \qquad (*)$$

  then $T_n x$ converges a.u. for all $x \in L_p(\mathcal{M})$.

  As the classical setting, we say that (T_n) satisfies strong type (p, p) maximal inequality if it satisfies $(*)$.

Noncommutative Birkhoff ergodic theorem: general cases

The first strong (p, p) maximal inequality in the noncommutative setting is established by Junge-Xu:

Theorem (Junge-Xu 07) Let $1 < p < \infty$ and T be a $*$-automorphism of $\mathcal{M}$ s.t. $\tau = \tau \circ T$. Then

$$\|(\frac{1}{N}\sum_{k=0}^{N} T^k x)_N\|_{L_p(\mathcal{M};\ell_\infty)} \lesssim \|x\|_p, \quad \forall x \in L_p(\mathcal{M}).$$

In particular,

$$\frac{1}{N}\sum_{k=0}^{N} T^k x \to Px \text{ a.u. as } N \to \infty, \ \forall x \in L_p(\mathcal{M}),$$

where P is the projection onto $Fix(T) \subset L_p(\mathcal{M})$ (which exists).

- This is indeed a corollary of a deep theorem generalizing the famous Marcinkiewicz interpolation theorem to the noncommutative setting.
- Various generalizations: multiple operators/flows, type III setting..

Generalizations

Junge-Xu's method allows to obtain many more ergodic theorems on von Neumann algebras (even new for the case $\tau(1) < \infty$ and $p = \infty$):

Noncommutative Stein ergodic theorem:

Theorem (Junge-Xu 07) Let $1 < p < \infty$ and $T : \mathcal{M} \to \mathcal{M}$ be a positive contraction s.t. $\tau = \tau \circ T$ and $\tau((Tx)y) = \tau(x(Ty))$ for all x, y. Then (T^N) satisfies the strong type (p, p) inequality and

$$T^N x \to Px \text{ a.u. as } N \to \infty, \ \forall x \in L_p(\mathcal{M}),$$

where P is the projection onto $Fix(T) \subset L_p(\mathcal{M})$ (which exists).

Remark Also true for general symmetric Markov operators and quantum Markov semigroups, as well as for multiple operators/semigroups.

Generalizations

Noncommutative Nevo-Stein ergodic theorem for actions by free groups $\mathbb{F}_d (d < \infty)$:

Theorem (Anantharaman 06, Hu 08) Let $1 < p < \infty$ and $\alpha : \mathbb{F}_d \to Aut(\mathcal{M})$ be an action of $\mathbb{F}_d$ on $\mathcal{M}$ s.t. $\tau = \tau \circ \alpha$. Let $S_k = \{g \in \mathbb{F}_d : |g| = k\}$. Then

$$\frac{1}{|S_{2k}|} \sum_{g \in S_{2k}} \alpha_g x \to P_2 x \text{ a.u. as } k \to \infty, \ \forall x \in L_p(\mathcal{M}),$$

where P_2 is the projection onto $Fix(\alpha_g : g \in S_2) \subset L_p(\mathcal{M})$. And

$$\frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{|S_k|} \sum_{g \in S_k} \alpha_g x \to P_1 x \text{ a.u. as } N \to \infty, \ \forall x \in L_p(\mathcal{M}),$$

where P_1 is the projection onto $Fix(\alpha_g : g \in S_1) \subset L_p(\mathcal{M})$.

Generalizations

Noncommutative Calderon ergodic theorem for actions by groups of polynomial growth:

Theorem (Hong-Liao-W.) Let $1 < p < \infty$ and $\alpha : G \to Aut(\mathcal{M})$ be an action of a group G of polynomial growth on $\mathcal{M}$ s.t. $\tau = \tau \circ \alpha$. Let $B_k = \{g \in G : |g| \leq k\}$. Then

$$\frac{1}{|B_k|} \sum_{g \in B_k} \alpha_g x \to Px \text{ a.u. as } k \to \infty, \ \forall x \in L_p(\mathcal{M}),$$

where P is the projection onto $Fix(\alpha) \subset L_p(\mathcal{M})$.

- Calderon's classical work 53' on this ergodic theorem play a fundamental role in the modern real harmonic analysis, where he developed his celebrated transference principle, doubling condition and covering lemma in order to prove the ergodic theorem.

Ergodic theorem for positive contractions on L_p-spaces

Instead of transformations of measure spaces, we may consider more general actions on L_p-spaces.

Theorem (Ackoglu 75) Fix $1 < p < \infty$ and $T : L_p(\Omega) \to L_p(\Omega)$ be a positive contraction. Then the sequence of maps $(N^{-1} \sum_{k=0}^{N-1} T^k)_N$ satisfy the strong (p, p) maximal inequality and

$$\frac{1}{N} \sum_{k=0}^{N} T^k f \to Pf \quad \text{a.e. as } N \to \infty, \ \forall f \in L_p(\Omega),$$

where P is the projection onto $Fix(T) \subset L_p(\Omega)$.

Towards a noncommutative Ackoglu ergodic theorem

- The noncommutative analogue of Ackoglu's ergodic theorem remains open.
- Ackoglu's proof is based on a deep dilation result: $\exists$ another L_p-space $L_p(\tilde{\Omega}) \supset L_p(\Omega)$ and positive isometry $S : L_p(\tilde{\Omega}) \to L_p(\tilde{\Omega})$

$$T^n = QS^nJ, \quad \forall n \in \mathbb{N},$$

  where $J : L_p(\Omega) \to L_p(\tilde{\Omega})$ positive embedding and $Q : L_p(\tilde{\Omega}) \to L_p(\Omega)$ a positive projection.
- Junge-Le Merdy 07: Ackoglu's dilation does not work for noncommutative L_p-spaces any more. A counterexample can be found already for finite dimensional noncommutative L_p-spaces.
- Hong-Liao-W.: maximal inequalities and individual ergodic theorem for positive invertible isometries (or more general uniformly bounded actions by groups of polynomial growth) on noncommutative L_p-spaces.

Towards a noncommutative Ackoglu ergodic theorem

Recently we tried to answer the question for some subclasses of L_p-contractions.

Lamperti, Kan:

- We say that a map $T : L_p(\Omega) \to L_p(\Omega)$ separate supports if $fg = 0$ implies $(Tf)(Tg) = 0$.
- all such maps are induced by non-singular transformations of Ω and a multiple of measurable functions "Radon-Nikodym cocycle".

We may consider similar objects for noncommutative L_p-spaces:

Definition We say that a map $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ separate supports if for any τ-finite projection $e_1, e_2 \in \mathcal{M}$,

$$e_1 e_2 = 0 \Rightarrow (Te_1)^*(Te_2) = (Te_1)(Te_2)^* = 0.$$

- analogue of order-zero maps in the theory of C*-algebras
- all such maps are induced by (Jordan) automorphisms of $\mathcal{M}$ and a multiple of some affiliated unbounded elements. (independently studied by Le Merdy-Zadeh)

Towards a noncommutative Ackoglu ergodic theorem

Theorem (Hong-Ray-W.) Let $2 \le p < \infty$. Assume that $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ belongs to

$\overline{\text{conv}}^{sot}\{S : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ positive contractions seperating supports$\}$,

Then the sequence of maps $(N^{-1}\sum_{k=0}^{N-1} T^k)_N$ satisfy the strong (p,p) maximal inequality and

$$\frac{1}{N}\sum_{k=0}^{N} T^k x \to Px \ \text{ a.u. as } N \to \infty, \ \forall x \in L_p(\mathcal{M}),$$

where P is the projection onto $Fix(T) \subset L_p(\mathcal{M})$. (Similar results hold for $1 < p < 2$ up to standard adaptations.)

Remark The subclass seems to be quite large in some cases

$$\begin{aligned} &\{S : L_p([0,1]) \to L_p([0,1]) \text{ positive contractions}\} \\ = &\overline{\text{conv}}^{sot}\{S : L_p([0,1]) \to L_p([0,1]) \text{ positive Lamperti contractions}\}, \end{aligned}$$

which does recover the classical Akcoglu's ergodic theorem on $L_p([0,1])$.

Towards a noncommutative Ackoglu ergodic theorem

To obtain a complete solution to the noncommutative Ackoglu theorem, we need to understand the subclass (denoted by $\mathcal{C}$)

$\overline{\text{conv}}^{sot}\{S : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ positive contractions seperating supports$\}$,

- Questions: how to determine if a positive contraction $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ belongs to $\mathcal{C}$? Any characterizations or necessary/sufficient conditions? Even for finite dimensional nc L_p-spaces?
- how to determine if a completely positive contraction T on a C*-algebra belongs to the strongly closed convex hull of order-zero maps? Even for finite dimensional C*-algebras?

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Thank you very much!