

# Noncommutative maximal ergodic inequalities for some group actions

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## Recall: Birkhoff ergodic theorem

Let  $(\Omega, P)$  be a probability space and let  $T : \Omega \rightarrow \Omega$  be a measure-preserving transformation. Define

$$A_N f = \frac{1}{N} \sum_{k=0}^{N-1} f \circ T^k, \quad f \in L_p(\Omega). \quad (1 \leq p \leq \infty)$$

**Theorem (Birkhoff)** (1)  $(A_N)_{N \geq 1}$  satisfies the weak type  $(1, 1)$  and strong type  $(p, p)$  inequalities, i.e.

$$\sup_{\lambda > 0} \lambda P(\{\sup_N |A_N f| \geq \lambda\}) \lesssim \|f\|_1, \quad f \in L_1(\Omega),$$

$$\|\sup_N |A_N f|\|_p \lesssim \|f\|_p, \quad f \in L_p(\Omega), \quad p > 1.$$

(2) Let  $I$  be the  $\sigma$ -subalgebra of  $T$ -invariant sets. For all  $f \in L_p(\Omega)$ ,

$$\lim_N A_N f = \mathbb{E}(f|I) \quad a.e.$$

## Generalizations to the noncommutative setting

Aim: analogues for actions on **von Neumann algebras** and noncommutative  $L_p$ -spaces

- ▶ Lance (76') introduced an analogue for the noncommutative setting of a.e. convergence:

Let  $\mathcal{M}$  be a vNA equipped with a normal faithful semifinite trace  $\varphi$ .  $(x_n)_{n \geq 1} \subset L_p(\mathcal{M})$  is said to converge **almost uniformly** (a.u. in short) to  $x$  if for every  $\varepsilon > 0$  there is a projection  $e \in \mathcal{M}$  such that

$$\varphi(e^\perp) < \varepsilon \quad \text{and} \quad \lim_{n \rightarrow \infty} \|(x_n - x)e\|_\infty = 0.$$

- ▶ Egorov: in the setting of classical probability spaces,

$$\text{a.u. convergence} \Leftrightarrow \text{a.e. convergence.}$$

- ▶ Pisier, Junge: noncommutative maximal norms via  $L_p(\mathcal{M}; \ell_\infty)$ .

## Junge-Xu's noncommutative ergodic theorem

Let  $T \in \text{Aut}(\mathcal{M})$  s.t.  $\varphi = \varphi \circ T$  (or a general Dunford-Schwartz operator). Let

$$A_N = \frac{1}{N} \sum_{k=0}^{N-1} T^k, \quad N \geq 1.$$

### Theorem (Lance 76; Yeadon 77; Junge-Xu 07)

(1) **weak type (1, 1) inequality**:  $\forall x \in L_1^+(\mathcal{M}), \lambda > 0, \exists e \in \text{Proj}(\mathcal{M})$  s.t.

$$\tau(1 - e) \lesssim \frac{\|x\|_1}{\lambda}, \quad eA_N(x)e \leq \lambda e, \quad N \geq 1.$$

**strong type (p, p) inequalities** ( $p > 1$ ):  $\forall x \in L_p^+(\mathcal{M}), \exists a \in L_p^+(\mathcal{M})$

$$A_N(x) \leq a, \quad \|a\|_p \lesssim \|x\|_p, \quad N \geq 1.$$

(2) For  $x \in L_p(\mathcal{M})$  ( $1 \leq p < \infty$ )

$$\lim_N A_N x = Px \quad \text{a.u.},$$

where  $P$  is the projection onto  $\text{Fix}(T) \subset L_p(\mathcal{M})$ .

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► **Problem**: more general actions beyond Junge-Xu's setting?

## Recall: Calderon ergodic theorem for group actions

$G$ : locally compact group,  $m$ : Haar measure,  $d$ : an invariant metric

Assume  $B_r := \{g \in G : d(g, e) \leq r\}$  satisfy

- ▶ **doubling condition**:  $m(B_{2r}) \leq Cm(B_r)$ ,  $r > 0$ .
- ▶ **asymptotically invariance (or Følner condition)**: for every  $g \in G$ ,

$$\lim_{r \rightarrow \infty} \frac{m((B_r g) \triangle B_r)}{m(B_r)} = 0.$$

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**Theorem** Assume that  $G$  acts on a probability space  $(\Omega, P)$  by measure-preserving transformations  $(T_g)_{g \in G}$ . Let  $\mathcal{I}$  be the  $\sigma$ -subalgebra of  $G$ -invariant sets. Then

$$A_r f := \frac{1}{m(B_r)} \int_{B_r} (f \circ T_g) dm(g), \quad f \in L_p(\Omega)$$

satisfies the weak type  $(1, 1)$  and strong type  $(p, p)$  inequalities, and for every  $f \in L_p(\Omega)$ , we have

$$\lim_{r \rightarrow \infty} A_r f = \mathbb{E}(f | \mathcal{I}) \quad \text{a.e.}$$

## Main result

$G$ : locally compact group,  $m$ : Haar measure,  $d$ : an invariant metric

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$$\lim_{r \rightarrow \infty} \frac{m((B_r g) \triangle B_r)}{m(B_r)} = 0.$$

**Theorem (Hong-Liao-W.)** Assume that  $\alpha : G \rightarrow \text{Aut}(\mathcal{M})$  is a continuous homomorphism such that  $\varphi = \varphi \circ \alpha_g$  for all  $g \in G$ . Then

$$A_r := \frac{1}{m(B_r)} \int_{B_r} \alpha_g dm(g), \quad r > 0,$$

satisfies the weak type  $(1, 1)$  and strong type  $(p, p)$  inequalities, and for  $x \in L_p(\mathcal{M})$  ( $1 \leq p < \infty$ ),

$$A_r x \rightarrow Px \text{ a.u., as } r \rightarrow \infty,$$

where  $P$  is the projection onto  $\bigcap_{g \in G} \text{Fix}(\alpha_g) \subset L_p(\mathcal{M})$ .



## Ingredients: group-theoretic tools

We may use the group theory to prove the theorem for partial cases:  $G$  discrete group of polynomial growth generated by a finite subset  $V = V^{-1}$ ,  $d$  word metric,  $p > 1$ :

- ▶ Gromov, Bass, Wolf,...:  $G$  contains a finitely generated nilpotent subgroup of finite index. We may view

$$G = F \times \mathbb{Z}^{d(G)} \text{ as a set (} F \text{ finite)}$$

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- ▶ Following this spirit, a careful study on the structure of  $G$  yields that

$$V^n \hookrightarrow F \times [0, cn^{j_1}] \times \cdots \times [0, cn^{j_{d(G)}}] \subset F \times \mathbb{Z}^{d(G)}.$$

- ▶ Then it is easy to deduce the maximal inequalities for  $A_n = \frac{1}{|V^n|} \sum_{g \in V^n} \alpha_g$  from Junge-Xu's maximal inequalities.

## Ingredients: noncomm. Calderon transference principle

For the general cases, we may first reduce the problem to translation actions. Assume  $G$  is amenable.

- ▶ **Calderon**: maximal inequalities for translation action  $G \times G \rightarrow G$   
 $\Rightarrow$  maximal inequalities for general action  $G \times (\Omega, P) \rightarrow (\Omega, P)$ .

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**Theorem (Hong-Liao-W.)** Let  $(\mu_n)_{n \geq 1}$  be a sequence of Radon probability measures on  $G$ . If

$$A'_n f = \int_G f(\cdot h) d\mu_n(h), \quad f \in L_p(G; L_p(\mathcal{M})), n \geq 1,$$

satisfies weak/strong  $(p, p)$  inequalities, then for a trace preserving action  $\alpha$  on  $\mathcal{M}$ ,

$$A_n x = \int_G \alpha_g x d\mu_n(g), \quad x \in L_p(\mathcal{M}), n \geq 1$$

satisfies the maximal inequalities of the same type.

**Remark:** For the strong type  $(p, p)$  inequalities, it suffices to assume that  $\alpha$  is an action on  $L_p(\mathcal{M})$  (for a **fixed**  $p$ ).

## Ingredients: probabilistic tools

- ▶ [Mei 07](#): Classical  $BMO(\mathbb{R}^n)$  is the intersection of  $n + 1$  copies of dyadic- $BMO$ . Doob's martingale inequality implies the Hardy-Littlewood maximal inequality on  $\mathbb{R}^n$ .
- ▶ [Naor-Tao 10](#), [Hytonen-Kaimera 12](#): filtrations of random partitions of doubling metric spaces

**Proposition** Let  $(X, d, \mu)$  be a doubling metric measure spaces. For some finite  $N$  and for each  $1 \leq i \leq N$ , one may construct a filtration  $\{\mathcal{I}_n^{(i)} : n \in \mathbb{Z}\}$  of  $\sigma$ -subalgebras on  $X$ . For each  $r$  and  $x \in X$ ,  $\exists n(r), i$  s.t. for  $f \in L_p(X; L_p(\mathcal{M}))_+$ ,

$$\frac{1}{\mu(B(x, r))} \int_{B(x, r)} f \leq c(\mathbb{E}(\cdot | \mathcal{I}_{n(r)}^{(i)}) \otimes Id_{L_p(\mathcal{M})})(f)(x).$$

## Ingredients: Operator-valued Hardy-Littlewood maximal inequalities on doubling spaces

So by the noncommutative Doob martingale inequality (Cuculescu 71, Junge 02, Junge-Xu 07, Jiao-Osekowski-Wu 18), we obtain

**Corollary** Let  $(X, d, \mu)$  be a doubling metric measure spaces. Then

$$M_r f(x) := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f, \quad x \in X, r > 0, f \in L_p(X; L_p(\mathcal{M}))$$

satisfies the weak  $(1, 1)$  and strong  $(p, p)$  inequalities.

In particular, taking  $(X, d, \mu) = (G, d, m)$  the group with doubling conditions, we obtain the desired maximal inequalities on  $G$ .

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**Remark** For  $p > 1$ , our previous transference methods also work for general positive actions on  $L_p$ -spaces (not necessarily arising from  $\text{Aut}(\mathcal{M})$ ). But for  $p = 1$  these do not hold. However in the case of word metrics, we may provide an alternative approach.  $\triangleright \triangleright$

## Ingredients: Domination via Markov operators (case of word metrics)

$G$ : group of polynomial growth wrt a symmetric cpt generating set  $V$ .  
 $d$ : word metric.

**Gaussian estimate in random walk theory:** Let  $f$  be a “nice” density function with  $\text{supp}(f) \supset V$ .  $\exists c > 0, \forall k$ ,

$$f^{*k}(g) \geq \frac{ce^{-d(e,g)^2/k}}{m(B_{\sqrt{k}})}, \quad g \in B_k.$$

**Proposition** Let  $T = \frac{1}{m(V)} \int_V \alpha_g dm(g)$ . Then there exists a constant  $c$  such that

$$\frac{1}{m(V^n)} \int_{V^n} \alpha_g x dm(g) \leq \frac{c}{n^2} \sum_{k=1}^{2n^2} T^k x, \quad x \in L_p^+(\mathcal{M}).$$



## Comments on individual ergodic theorems

- ▶ Junge, Xu, ...: One may deduce the a.u. convergence from maximal ergodic inequalities for Dunford-Schwartz operators (which relies on the boundedness on  $L_1 + L_\infty$ ).
- ▶ Hong-Liao-W: The same holds for general power bounded operators on a  $L_p$ -space for a **fixed**  $p$  (our new argument does not require boundedness on  $L_1 + L_\infty$ ).
- ▶ Therefore the main theorem can be generalized for any **fixed**  $1 < p < \infty$  and any action  $\alpha : G \rightarrow B(L_p(\mathcal{M}))$  satisfying:
  - Continuity
  - Uniform boundedness:  $\sup_{g \in G} \|\alpha_g : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})\| < \infty$ .
  - Positivity : for all  $g \in G$ ,  $\alpha_g x \geq 0$  if  $x \geq 0$  in  $L_p(\mathcal{M})$ .

## Comments on individual ergodic theorems

In particular, we have

**Corollary Fix**  $1 < p < \infty$ . Let  $T : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})$  be a positive invertible operator with positive inverse such that  $\sup_{n \in \mathbb{Z}} \|T^n\| < \infty$ . Then

$$A_n = \frac{1}{2n+1} \sum_{k=-n}^n T^k, \quad n \in \mathbb{N}$$

satisfies strong  $(p, p)$  inequalities. For all  $x \in L_p(\mathcal{M})$ ,  $(A_n x)_{n \geq 1}$  converges b.a.u if  $1 < p < 2$ , and converges a.u if  $p \geq 2$ .

**Remark** The result has been generalized for completely positive complete isometries or more general support-separating complete contractions by Hong-Wang-Ray (in progress).

Thank you very much!