Noncommutative individual ergodic theorems for groups with polynomial growth

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Recall: Birkhoff ergodic theorem

Theorem Let (Ω, P) be a probability space and let $T : \Omega \to \Omega$ be a measure-preserving transformation. Let I be the σ-subalgebra of T-invariant sets. Then for every $f \in L_p(\Omega)\,(1 \leq p < \infty)$, we have

$$\frac{1}{N}\sum_{k=0}^{N-1} f \circ T^k \to \mathbb{E}(f|I), \quad \text{as } n \to \infty,$$

for a.e. $x \in \Omega$.

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Remark One may generalize the result for actions by groups with "nice" averages.

- Følner averages of actions by amenable groups: Calderon, Tempelman, Breuillard, Tessera, Nevo, Lindenstrauss...
- spherical averages of actions by free groups : Nevo, Stein.

Recall: Calderon ergodic theorem for group actions

G: locally compact group, m: Haar measure, d: an invariant metric

Assume $B_r := \{g \in G : d(g, e) \leq r\}$ satisfy

- doubling condition: $m(B_{2r}) \leq Cm(B_r), \quad r > 0.$
- asymptotically invariance (or Følner condition): for every $g \in G$,

$$\lim_{r \to \infty} \frac{m((B_r g) \triangle B_r)}{m(B_r)} = 0.$$

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Theorem Assume that G acts on a probability space (Ω, P) by measure-preserving transformations $(T_g)_{g\in G}$. Let $\mathcal{I}$ be the σ-subalgebra of G-invariant sets. Then for every $f \in L_p(\Omega)$, we have

$$\frac{1}{m(B_r)} \int_{B_r} (f \circ T_g) dm(g) \to \mathbb{E}(f|\mathcal{I}), \quad \text{as } r \to \infty,$$

for a.e. $x \in \Omega$.

Recall: Calderon ergodic theorem for group actions

Example: (Tessera 07; Breuillard 07)

- If G is generated by a symmetric compact subset V, and if G is of polynomial growth wrt the word metric:

$$m(V^n) \leq Cn^q, \quad n \in \mathbb{N}$$

  for some C and q, then $(V^n)_{n\in\mathbb{N}}$ are asymptotically invariant, and therefore

$$\frac{1}{m(V^n)} \int_{V^n} (f \circ T_g) dm(g) \to \mathbb{E}(f|I), \quad \text{as } n \to \infty,$$

  for a.e. $x \in \Omega$ and in $\|\|_p$.

- more general invariant metrics quasi-isometry to word metric, etc.

Generalizations to the noncommutative setting

Aim: analogues for group actions on von Neumann algebras

- Lance (76') introduced an analogue for the noncommutative setting of a.e. convergence:
  Let $\mathcal{M}$ be a vNA equipped with a normal faithful state φ.
  $(x_n)_{n\geq 1} \subset \mathcal{M}$ is said to converge almost uniformly (a.u. in short) to x if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\varphi(e^{\perp}) < \varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|(x_n - x)e\|_\infty = 0.$$

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- Egorov: in the classical case, a.u. convergence $\Leftrightarrow$ a.e. convergence.
- Noncommutative L_p-space: (assume φ tracial) Let $1 \leq p < \infty$.

$$L_p(\mathcal{M}) = \text{ completion of } (\mathcal{M}, \|\|_p), \quad \|x\|_p = [\varphi(|x|^p)]^{1/p}.$$

  One may define a.u. convergence for $L_p(\mathcal{M})$ in the same way.

Noncommutative individual ergodic theorem: $\mathbb{Z}$-actions

Theorem (Lance 76; Yeadon 77; Junge-Xu 07) Let T be an automorphism of $\mathcal{M}$ s.t. $\varphi = \varphi \circ T$. Then for $x \in L_p(\mathcal{M})$ $(1 \leq p < \infty)$

$$\frac{1}{N} \sum_{k=0}^{N} T^k x \to Px \ \text{ a.u.,} \quad \text{as } N \to \infty,$$

where P is the projection onto $Fix(T) \subset L_p(\mathcal{M})$.

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- Anantharaman 06, Hu 08: individual ergodic theorem for free group actions on von Neumann algebras
- How about actions by amenable groups?

Main result

G: locally compact group, m: Haar measure, d: an invariant metric

Assume $B_r := \{g \in G : d(g,e) \le r\}$ satisfy

- doubling condition: $m(B_{2r}) \le Cm(B_r), \quad r > 0.$
- asymptotically invariance (or Følner condition): for every $g \in G$,

$$\lim_{r\to\infty} \frac{m((B_r g) \triangle B_r)}{m(B_r)} = 0.$$

Theorem (Hong-Liao-W.) Assume that $\alpha : G \to Aut(\mathcal{M})$ is a w*-continuous homomorphism such that $\varphi = \varphi \circ \alpha_g$ for all $g \in G$. Then for $x \in L_p(\mathcal{M})$ $(1 \le p < \infty)$,

$$\frac{1}{m(B_r)} \int_{B_r} \alpha_g x dm(g) \to Px \ \text{ a.u.}, \quad \text{as } r \to \infty,$$

where P is the projection onto $\cap_{g\in G} Fix(\alpha_g) \subset L_p(\mathcal{M})$.

A standard scheme for studying a.u. convergence

In general, there is a standard scheme ("Banach principle") for the proof of a.u. convergence:

If we have:

- convergence in Banach space norm (well-known in our setting)
- a.u. convergence for a dense subspace in $L_p(\mathcal{M})$ (sometimes easy)
- "noncommutative maximal inequalities" (difficult)

Then we may deduce the a.u. convergence on $L_p(\mathcal{M})$. (Chilin, Litvinov, Skalski 05'-)

Maximal inequalities: Classically, for a family of positive maps $A_i : L_p \to L_p$, $i \in I$, we would like to estimate the L_p-bound of the maximal operator $f \mapsto \sup_i A_i f$. In this talk, we always refer to

- weak $(1, 1)$ inequality: $\forall \lambda,\ P(\{\omega \in \Omega : \sup_i A_i f > \lambda\}) \leq C \frac{\|f\|_1}{\lambda}$.
- strong (p, p) inequality: $\|\sup_i A_i f\|_p \leq C\|f\|_p,\ p > 1$.

Noncommutative maximal inequalities

There is a highly non-trivial generalized notion of noncommutative maximal inequalities (Pisier's work on $L_p(\mathcal{M};\ell_\infty)$), for which we will only cite the following observation in this talk:

Fact If $\forall x \in L_p(\mathcal{M})_+,\ 0 \leq A_r x \leq \widetilde{A}_r x$, then:
maximal inequalities of $(\widetilde{A}_r) \Rightarrow$ maximal inequalities of (A_r)
($\Rightarrow$ ergodic theorem in our setting)

Theorem (Hong-Liao-W.) Let G, α be as before. We have

$$A_r := \frac{1}{m(B_r)} \int_{B_r} \alpha_g dm(g), \quad r > 0,$$

satisfies the maximal inequalities.

Idea: Find nice control $A_r \leq \widetilde{A}_r$ for which the maximal inequalities of $(\widetilde{A}_r)$ are known.

- For $\mathbb{Z}_n$-actions : Brunel 73
- For free group actions : Nevo-Stein 94, Anantharaman 06

Domination via Markov operators (case of word metrics)

G: group of polynomial growth wrt a symmetric cpt generating set V.
d: word metric.

Gaussian estimate in random walk theory: Let f be a "nice" density function with $supp(f) \supset V$. $\exists c > 0$, $\forall k$,

$$f^{\star k}(g) \geq \frac{ce^{-d(e,g)^2/k}}{m(B_{\sqrt{k}})}, \quad g \in B_k.$$

Proposition For any integer n,

$$\frac{\chi_{V^n}}{m(V^n)} \leq \frac{c}{2n^2} \sum_{k=1}^{2n^2} f^{\star k}.$$

Let $T = \frac{1}{m(V)} \int_V \alpha_g dm(g)$. Then there exists a constant c such that

$$\frac{1}{m(V^n)} \int_{V^n} \alpha_g x dm(g) \leq \frac{c}{n^2} \sum_{k=1}^{2n^2} T^k x, \quad x \in L_p(\mathcal{M})_+.$$

Yeadon;Junge-Xu: RHS satisfies maximal inequalities. ($\Rightarrow$ LHS as well)

Domination via martingales (case of general metrics)

- Mei 07: Classical $BMO(\mathbb{R}^n)$ is the intersection of $n+1$ copies of dyadic-BMO. Doob's martingale inequality implies the Hardy-Littlewood maximal inequality on $\mathbb{R}^n$.
- Naor-Tao 10, Hytonen-Kaimera 12: filtrations of random partitions of doubling metric spaces

Proposition Let (X, d, μ) be a doubling metric measure spaces. For some finite N and for each $1 \leq i \leq N$, one may construct a filtration $\{\mathcal{I}_n^{(i)} : n \in_{\mathbb{Z}}\}$ of σ-subalgebras on X. For each r and $x \in X$, $\exists n(r), i$ s.t. for $f \in L_p(X; L_p(\mathcal{M}))_+$,

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \leq c\big(\mathbb{E}(\cdot|\mathcal{I}_{n(r)}^{(i)}) \otimes Id_{L_p(\mathcal{M})}\big)(f)(x).$$

Cuculescu,Junge,Xu: RHS ("noncommutative martingale" on $L_\infty(X)\bar{\otimes}\mathcal{M}$) satisfies maximal inequalities. (Hence LHS as well)

Operator-valued Hardy-Littlewood maximal inequalities

Corollary Let (X, d, μ) be a doubling metric measure spaces. Then

$$M_r f(x) := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f, \quad x \in X, r > 0, f \in L_p(X; L_p(\mathcal{M}))$$

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Taking $(X, d, \mu) = (G, d, m)$ the group with doubling conditions, we obtain the maximal inequalities for the translation action α on vector-valued functions on G:

Corollary We have

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satisfies the maximal inequalities.

- Then we can deduce the results for arbitrary actions by transference.

A noncommutative Calderon transference principle

Assume G is amenable.

- Calderon: maximal inequalities for translation action $G \times G \to G$
  $\Rightarrow$ maximal inequalities for general action $G \times (\Omega, P) \to (\Omega, P)$.

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- Calderon: maximal inequalities for translation action $G \times G \to G$
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Theorem (Hong-Liao-W.) Let $(\mu_n)_{n\geq 1}$ be a sequence of Radon probability measures on G. If

$$A'_n f = \int_G f(\cdot h) d\mu_n(h), \quad f \in L_p(G; L_p(\mathcal{M})), n \geq 1,$$

satisfies maximal inequalities, then for an action of G on $L_p(\mathcal{M})$,

$$A_n x = \int_G \alpha_g x d\mu_n(g), \quad x \in L_p(\mathcal{M}), n \geq 1$$

satisfies the maximal inequalities of the same type.

Thank you very much!