L^p-convolutions, multipliers and lacunarity for compact quantum groups

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Harmonic analysis on (classical or noncommutative) L^p-spaces is a useful tool in many fields of mathematics.

- Classical maths: PDEs, analysis, ergodic theory...
- Mathematical physics: spectral gap of generators of particle systems, existence of ground states, estimation of entropy, uncertainty principle...
- Operator algebra, operator spaces: approximation properties, solidity of group factors, operator space embedding problems

Among the above studies, a typical and important framework is the L^p-spaces associated to locally compact groups or group von Neumann algebras.

- How about L^p-spaces and harmonic analysis on quantum groups?

Framework and conventions

- $\mathbb{G} = (C(\mathbb{G}), \Delta)$: Woronowicz's compact quantum group
- $L^\infty(\mathbb{G})$, $\mathrm{Irr}(\mathbb{G})$
- h : Haar state
- $\mathrm{Pol}(\mathbb{G})$: dense Hopf $*$-algebra generated by matrix elements of irr representations
- S: antipode; R: unitary antipode; τ: scaling group
- $\hat{\mathbb{G}}$, $c_0(\hat{\mathbb{G}})$, $\ell^\infty(\hat{\mathbb{G}})$, $c_c(\hat{\mathbb{G}})$

Framework and conventions: L^p-spaces

$$\begin{aligned} \text{measure space on } G &\longleftrightarrow \text{von Neumann algebra } L^\infty(\mathbb{G}) \\ \text{Haar measure } dm \text{ on } G &\longleftrightarrow \text{Haar state } h \text{ on } L^\infty(\mathbb{G}) \\ \|f\|_p = (\int_G |f|^p dm)^{1/p} &\longleftrightarrow \|x\|_p = (h(|x|^p))^{1/p} \text{ (if } h \text{ tracial)} \end{aligned}$$

Briefly, the L^p-spaces $L^p((\mathbb{G}), 1 < p < \infty$ is an interpolation family of Banach spaces between $L^\infty(\mathbb{G})$ and $L^\infty(\mathbb{G})_*$.

- Define $\|x\|_1 = \|h(\cdot x)\|_{L^\infty(\mathbb{G})^*}$ for $x \in \mathrm{Pol}(\mathbb{G})$, and let $L^1(\mathbb{G})$ be the completion of $(\mathrm{Pol}(\mathbb{G}), \|\|_1)$. Define $L^p(\mathbb{G})$ to be the complex interpolation space

$$L^p(\mathbb{G}) = (L^\infty(\mathbb{G}), L^1(\mathbb{G}))_{1/p}, \quad 1 \le p \le \infty.$$

- Define $\ell^p(\hat{\mathbb{G}})$ on $c_c(\hat{\mathbb{G}})$ similarly.

L^p-Fourier series

We may define the Fourier transform

$$\mathcal{F} : \mathrm{Pol}(\mathbb{G}) \to c_c(\hat{\mathbb{G}}), \quad x \mapsto \hat{x},$$

where

$$\hat{x}(\pi) = h(\cdot x) \otimes \mathrm{id}((u^{(\pi)})^*) = [h(u_{ji}^{(\pi)^*} x)]_{i,j=1}^{n_\pi}, \quad \pi \in \mathrm{Irr}(\mathbb{G}).$$

The map can be extended to L^p-spaces.

- (Hausdorff-Young inequality) We have the extension

$$\mathcal{F} : L^p(\mathbb{G}) \to \ell^q(\hat{\mathbb{G}}) \text{ contraction}, \quad 1/p + 1/q = 1.$$

- (Plancherel theorem) $\mathcal{F} : L^2(\mathbb{G}) \to \ell^2(\hat{\mathbb{G}})$ unitary.

L^p-Convolutions

For a CQG $\mathbb{G}$ and $\varphi_1, \varphi_2 \in L^\infty(\mathbb{G})^*$, we may define

$$\varphi_1 \star \varphi_2 = (\varphi_1 \otimes \varphi_2) \circ \Delta.$$

This also induces the convolution $x_1 \star x_2$ for $x_1, x_2 \in L^1(\mathbb{G})$:

$$h(\cdot x_1 \star x_2) = h(\cdot x_1) \star h(\cdot x_2).$$

Young's inequality (Liu-W.-Wu, 15): for $x, y \in \mathrm{Pol}(\mathbb{G})$ we have

$$\|\tau_{\frac{\mathrm{i}}{p'}}(y) \star x\|_r \leq \|x\|_p \|y\|_q, \quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}, \frac{1}{p} + \frac{1}{p'} = 1$$

where $1 \leq p, q, r \leq \infty$ if h is tracial, and $1 \leq p, q, r \leq 2$ for general cases.

L^p-Convolutions and uncertainty principle

(Liu-W.-Wu 15; Liu-Wu 16)
Let $\mathbb{G}$ be of Kac type.

- A differentiation argument on the Hausdorff-Young inequality with respect to p gives the Hirschman-Beckner uncertainty principle for $\mathbb{G}$: let $H(x) = -h(x \ln x)$ denote the von Neumann entropy, then for all $x \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$,

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geq -4\|x\|_2^2 \ln \|x\|_2.$$

- The minimizers of the above inequality comes from the minimizers of Hausdorff-Young inequality and Young's inequality, which can be characterized via group-like projections or biprojections from the subfactor theory.
- The results further deduces the Hardy uncertainty principle on quantum groups.

More specific topics: (L^p, L^q)-estimates

A classical question: contruction of convolution operators from L^p to L^q: for eg, find positive Borel probablity measures μ so that for a given $p > 1$,

$$\exists p < q, \quad \forall f \in L_p(\mathbb{T}, \frac{dm}{2\pi}), \ \|\mu \star f\|_q \leq \|f\|_p.$$

Oberlin (1982): the Cantor-Lebesgue measure supported by the usual middle-third Cantor set satisfies the above property.
In fact it suffices to show

$$\exists p < 2, \quad \|\mu \star f\|_2 \leq \|f\|_p, \quad f \in L_p(\mathbb{Z}/3\mathbb{Z}, P)$$

where P is the normalized counting measure, $\mu(\{0\}) = \mu(\{2\}) = \frac{1}{2}$.

More specific topics: (L^p, L^q)-estimates

Motivated by Oberlin, Ritter (1984) showed that:

Theorem If G is an arbitrary finite group and $T_\mu : f \mapsto \mu \star f$ is the convolution operator associated to a probability measure μ on G, then

$$\big(\exists\, p < 2,\ \|T_\mu : L^p(G) \to L^2(G)\| = 1\big) \ \Leftrightarrow\ G = \langle ij^{-1} : i, j \in \operatorname{supp}\mu\rangle.$$

- The operator T with $\|T : L^p \to L^2\| = 1$ for a $p < 2$ will be said to be L^p-improving throughout the presentation.
- Question: similar constructions on (finite/compact) quantum groups?

Interpretation in the quantum language

Recall the classical result: for $T_\mu : f \mapsto \mu \star f,\ f \in C(G)$

$\big(\exists\, p < 2,\ \|T_\mu : L^p(G) \to L^2(G)\| = 1\big) \Leftrightarrow (*)\ G = \langle ij^{-1} : i, j \in \mathrm{supp}\,\mu\rangle.$

Take $\nu = \mu(\cdot^{-1}) \star \mu$. It is a simple observation that

$$(*) \Leftrightarrow\ \forall f > 0,\ \exists n \geq 1,\ \int f d\nu^{\star n} > 0.$$

This corresponds to an important notion in the study of random walks on quantum groups.

Definition (Kalantar,Neufang,Ruan 14) A state ψ on $C(\mathbb{G})$ is **non-degenerate** if

$$\forall 0 \neq x \in C(\mathbb{G})_+, \quad \exists n \geq 1, \quad \psi^{\star n}(x) > 0.$$

Characterizations in the quantum group setting

Theorem (W.) Let $\mathbb{G}$ be a finite quantum group and φ be a state on $C(\mathbb{G})$. Denote $\psi = (\varphi \circ S) \star \varphi$. Then

$$\exists 1 < p < 2,\ \forall\, x \in C(\mathbb{G}),\ \|\varphi \star x\|_2 \leq \|x\|_p$$

iff ψ is **non-degenerate**: $\forall 0 \neq x \in C(\mathbb{G})_+, \exists n \geq 1, \psi^{\star n}(x) > 0$.

We may also show that this L^p-improving property is stable under free product.

A key ingredient: non-degenerate states, applications

Proposition (W.) If φ is a non-degenerate state on $C(\mathbb{G})$, then

$$w^*\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\varphi^{\star k}=h.$$

Recall the Hopf image construction in P. Joziak's talk. Each $*$-representation π of $C(\mathbb{G})$ associates a quantum subgroup $\mathbb{G}_\pi\subset\mathbb{G}$. If G is a group and $X\subset G$ a closed subset with the restriction map $\pi: C(G)\to C(X)$, then $G_\pi=\langle X\rangle$.

Corollary (Banica,Franz,Skalski 12; W.) Let A be a unital C*-algebra with a unital $*$-homomorphism $\pi: C(\mathbb{G})\to A$, and $q: C(\mathbb{G})\to C(\mathbb{G}_\pi)$ the quotient map. Then given any faithful state φ on A,

$$h_{\mathbb{G}_\pi}\circ q=w^*\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(\varphi\circ\pi)^{\star k}.$$

A key ingredient: non-degenerate states, applications

This result is recently applied in a series of works of Banica on matrix models of quantum groups. For instance,

- (Banica 16) If $\mathbb{G} \subset O_n^*$, then we have a matrix model $C(\mathbb{G}) \subset \mathbb{M}_2(C(H))$ (H a compact Lie group) such that the Haar state of $\mathbb{G}$ coincides with $tr \otimes \int$.
- (Banica 16) We have a Weyl matrix model $\pi : C(S_N^+) \to \mathbb{M}_N(C(H))$ ($H \subset U_n$) such that the image of π is a Hopf algebra and the associated Haar state coincides with $tr \otimes \int$.
- (Banica 16; Banica-Nechita 16) For the deformed Fourier matrix model $\pi : C(S_{MN}^+) \to \mathbb{M}_{MN}(C(\mathbb{T}^{MN}))$, the law of the main character of the Hopf image is asymptotically free Poisson.

Back to (L^p, L^q)-estimates: $\Lambda(p)$-sets

The estimate of L^p-improving convolutions is often related to the study of Khintchine inequalities for some $E \subset \mathrm{Irr}(\mathbb{G})$: understand const C_p for $2 < p \leq \infty$ s.t. for all $x \in \mathrm{span}\{u_{ij}^{(\pi)} : \pi \in E, 1 \leq i, j \leq n_\pi\}$,

$$\|x\|_p \leq C_p \|x\|_2.$$

We call the subset E with the above property a $\Lambda(p)$-set with constant C_p.

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The notion has been widely studied in harmonic analysis.

- (Pisier 78&16) Let G be a compact abelian group. A subset $E \subset \hat{G}$ is a Sidon set, i.e.,

$$\exists C > 0, \quad \forall f \in \mathrm{span} E \subset C(G), \quad \sum_{\gamma \in E} |\hat{f}(\gamma)| \leq C \|f\|_\infty$$

  iff E is a $\Lambda(p)$-set with $C_p = O(\sqrt{p})$ for all $2 < p < \infty$.
- This means that these sets shares similar behaviors as i.i.d Gaussian variables, and provide a very weak notion of "independence" compared to freeness and tensor independence.

Sidon sets: quantum group setting

We may extend the notion to the quantum group setting.

Definition Let $\mathbb{G}$ be a compact quantum group. A subset $E \subset \mathrm{Irr}(\mathbb{G})$ is called a Sidon set if there is a constant $C > 0$ such that for all $x \in \mathrm{span}\{u_{ij}^{(\pi)} : \pi \in E, 1 \leq i, j \leq n_\pi\}$,

$$\|\hat{x}\|_{\ell^1(\hat{\mathbb{G}})} \leq C\|x\|_\infty.$$

The Sidon set has various characterizations: interpolation of bounded measures, multipliers, unconditional bases...

Sidon sets $\Rightarrow \Lambda(p)$-sets

It is well-known that if $\mathbb{G}$ is a compact group or the compact dual of a discrete group, then any Sidon set for $\mathbb{G}$ is a $\Lambda(p)$-set for $1 < p < \infty$.

- Case for compact groups: Figà-Talamanca, Marcus-Pisier;
- Case for (dual of) discrete groups: Picardello, Harcharras...
- Case for compact quantum groups: more delicate – Blendek-Michaliček 13': For Kac type CQG, Helgason-Sidon $\Rightarrow$ $\Lambda(2)$ for central functions.

Theorem (W.) Any Sidon set for $\mathbb{G}$ is a $\Lambda(p)$-set for $1 < p < \infty$.

Sidon sets $\Rightarrow \Lambda(p)$-sets, Fourier multipliers

L^p-Fourier multipliers play a key role in the argument. For each $a \in \ell^\infty(\hat{\mathbb{G}})$ we associate a “multiplier map”

$$m_a : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G}), \quad m_a x = \mathcal{F}^{-1}(a\hat{x}).$$

$M(L^p(\mathbb{G})) = \{a : m_a \text{ bdd on } L^p(\mathbb{G})\} \subset B(L^p(\mathbb{G}))$.

Theorem (W.) Let $2 < p < \infty$ and $E \subset \mathrm{Irr}(\mathbb{G})$. Then E is a $\Lambda(p)$-set iff

$$\forall\, a \in \oplus_{\pi \in E} \mathbb{M}_{n_\pi}, \quad \exists\, b \in M(L^p(\mathbb{G})), \quad b|_E = a.$$

Observations on non-unimodularity

Recall that we have a "modular element" $F = (F_\pi)_{\pi \in \mathrm{Irr}(\mathbb{G})}$ for $\hat{\mathbb{G}}$. This is Woronowicz's F-matrix with $F_\pi \in Mor(u^\pi, S^2(u^\pi))$.

Theorem Let $E \subset \mathrm{Irr}(\mathbb{G})$ a $\Lambda(p)$-set for $2 < p < \infty$. Then

$$\sup_{\pi \in E} \|F_\pi\| < +\infty.$$

Corollary $\mathrm{SU}(2)_q$ does not admit infinite $\Lambda(p)$-set for any $2 < p < \infty$.

General existence of $\Lambda(p)$-sets

Theorem (Bożejko; W.) Let (M, φ) be a vNa equipped with a normal faithful state φ. Let $B = \{x_i \in M : i \geq 1\}$ be an orthogonal system wrt φ s.t. $\sup_i \|x_i\|_\infty < \infty$. Then for each $1 \leq p < \infty$, there exists an infinite subset $\{x_{i_k} : k \geq 1\} \subset B$

$$\forall (c_k) \subset \mathbb{C}, \quad \|\sum_{k\geq 1} c_k x_{i_k}\|_{L^p(M)} \sim \Big(\sum_{k\geq 1} |c_k|^2\Big)^{\frac{1}{2}}.$$

Corollary Let $\mathbb{G}$ be a CQG. Let $E \subset \mathrm{Irr}(\mathbb{G})$ be an infinite subset with $\sup_{\pi \in E} d_\pi < \infty$. Then for each $1 \leq p < \infty$, there exists an infinite subset $E' \subset E$ which is $\Lambda(p)$-set.

Lacunarity for product quantum group

- $\prod_k \mathbb{G}_k$ with $O_{N_k} \subset \mathbb{G}_k \subset U_{N_k}^+$: Let $u^{(k)}$ be the fundamental rep of $\mathbb{G}_k$. Then $(u^{(k)})_k \subset \mathrm{Irr}(\prod_k \mathbb{G}_k)$ is a $\Lambda(p)$-set for all $2 \leq p < \infty$. This in particular yields a dimensional free estimate of L^p-norms: we have a universal constant C s.t. for all $c_{ij}^{(k)}, k \geq 1, 1 \leq i,j \leq N_k$,

$$\Big\| \sum_{k,i,j} c_{ij}^{(k)} u_{ij}^{(k)} \Big\|_{L^p(\mathbb{G}_k)} \leq Cp \Big\| \sum_{k,i,j} c_{ij}^{(k)} u_{ij}^{(k)} \Big\|_{L^2(\mathbb{G}_k)}.$$

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- $C(\prod_k \mathrm{SU}_{q_k}(2)) = \otimes_k C(\mathrm{SU}_{q_k}(2))$: Let $u^{(k)}$ be the fundamental rep of $\mathrm{SU}_{q_k}(2)$. Then the above type of Khintchine inequality holds iff

$$\inf_k q_k > 0.$$

Lacunarity for product quantum group

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- How about $\prod_k \mathrm{SU}_{q_k}(N_k)$???

Central lacunarity for G_q and O_N^+

- Lemma: compact quantum groups with the same fusion rules and the same dimension functions have identical central Sidon sets (i.e. $E \subset \mathrm{Irr}(\mathbb{G})$ s.t. for all $(c_\pi)_{\pi \in E} \subset \mathbb{C}$,

$$\| \sum_{\pi \in E} c_\pi \chi_\pi \| \sim \sum_{\pi \in E} n_\pi |c_\pi|,$$

and any Sidon set is central Sidon.)

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- Example on $\prod_k \mathrm{SU}_{q_k}(N_K)$: Let $u^{(k)}$ be the fundamental rep of $\mathrm{SU}_{q_k}(N_k)$. Then $(u^{(k)})_{k \geq 1} \subset \mathrm{Irr}(\prod_k \mathrm{SU}_{q_k}(N_K))$ central Sidon set.

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- Drinfeld-Jimbo deformation: a compact semi-simple Lie group $G \rightsquigarrow$ a compact quantum group G_q $(0<q<1)$. Then for any simply connected compact semi-simple Lie group G, G_q does not admit infinite (central) Sidon set.

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- fusion rule of $O_N^+ \sim$ fusion rule of $\mathrm{SU}(2)$. So O_N^+ does not admit infinite (central) Sidon set.

Lacunarity for $\mathrm{SU}_q(2)(0 < q < 1)$

- Recall $\mathrm{Irr}(\mathrm{SU}_q(2))$ is indexed by $\mathbb{N} \cup \{0\}$ and the fusion rule is same as that of O_N^+. But the subset subject to the following condition

$$E = (u^{(n_k)})_{k \geq 1} \subset \mathrm{Irr}(\mathrm{SU}_q(2)), \quad n_k \geq n_{k-1} + k$$

  satisfies the central $\Lambda(4)$ condition

$$\forall (c_n) \subset \mathbb{C}, \quad \| \sum_{n \in E} c_n \chi_n \|_4 \sim_q \| \sum_{n \in E} c_n \chi_n \|_2 = (\sum_n |c_n|^2)^{1/2}.$$

- There exists no infinite $\Lambda(p)$-sets for any $2 \leq p \leq \infty$ for $\mathrm{SU}_q(2)$.
- S.-G. Youn recently communicated to us that our methods on $\mathrm{SU}_q(2)$ applies to provide a first counterexample for Marcinkiewicz interpolation theorem on von Neumann algebras with non-tracial states. (Junge-Xu, JAMS 07: establish the Marcinkiewicz interpolation theorem for traces, i.e., weak (p_1, p_1) & strong (p_2, p_2) boundedness yields strong (p, p) for all $p_1 < p < p_2$)

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Thank you very much!