Pointwise convergence of noncommutative Fourier series

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joint work with Guixiang Hong (Wuhan) & Xumin Wang (Paris)

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Background

Given a $f \in L_p(\mathbb{T})$, the Fourier series is

$$f \sim \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k \cdot \theta} \qquad 0 \leq \theta \leq 1.$$

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N.B.: summation method $\longleftrightarrow$ (family of) suitable Fourier multipliers

Fourier series on group von Neumann algebra

Let Γ be a discrete group.
We have the left regular representation $\lambda : G \to B(\ell_2(\Gamma)),\ \lambda(g)\delta_h = \delta_{gh}$
& group von Neumann algebra $VN(\Gamma) = \lambda(\Gamma)'' \subset B(\ell_2(\Gamma))$.
Note: $VN(\mathbb{Z}) = L_\infty(\mathbb{T})$
Each $x \in VN(\Gamma)$ admits a formal Fourier series

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Giving a symbol $m : \Gamma \to \mathbb{C}$, the associated Fourier multiplier is defined by

$$T_m : x \mapsto \sum_{g\in\Gamma} m(g)\hat{x}(g)\lambda(g).$$

Mean convergence - approximation properties of groups

mean convergence of different summation methods
$T_m x = \sum_g m(g)\hat{x}(g)\lambda(g) \leftrightsquigarrow$ **approximation properties** of groups

- amenability: $\exists$ finitely supported functions $(m_N)_{N\in\mathbb{N}}$ on Γ s.t. $T_{m_N} : VN(\Gamma) \to VN(\Gamma)$ unital completely positive & $T_{m_N} x \xrightarrow{w^*} x, \forall x$. (e.g., Fejér means on $L_\infty(\mathbb{T})$)

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- Haagerup property: $\exists$ c_0-functions $(m_N)_{N\in\mathbb{N}}$ on Γ s.t. $T_{m_N} : VN(\Gamma) \to VN(\Gamma)$ unital completely positive $\& T_{m_N}x \xrightarrow{w^*} x, \forall x.$ (e.g., Poisson means on $L_\infty(\mathbb{T})$)
- weak amenability: $\exists$ finitely supported-functions $(m_N)_{N\in\mathbb{N}}$ on Γ s.t. $T_{m_N} : VN(\Gamma) \to VN(\Gamma)$ completely bounded, $\sup_N \|T_{m_N}\|_{cb} < \infty$ $\& T_{m_N}x \xrightarrow{w^*} x, \forall x.$ (e.g., Bochner-Riesz means on $L_\infty(\mathbb{T})$)

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$\rightsquigarrow$ Cowling-Haagerup's work on W*-rigidity of Lie groups of real rank 1; Popa's deformation/rigidity theory; strong solidity, Cartan subalgebras; spectral truncation in noncomutative geometry

Pointwise convergence - nc ergodic theory and Fourier analysis on quantum tori

- Junge-Xu JAMS 07':
  Pointwise convergence of quantum Markov semigroups on $VN(\Gamma)$ (corollary of nc maximal ergodic inequality)
- Chen-Xu-Yin, CMP 13':
  Pointwise convergence of Fejér means, Poisson means and Bochner-Riesz means on **Quantum Tori**
  (using transference method to semi-commutative setting studied in [Mei, Memoirs AMS 07']).

The problem of pointwise approximations of nc L_p space

How about pointwise convergence of (finitely supported) summation methods of noncommutative Fourier series associated to non-abelian groups?

Aim: find a suitable sequence $(m_N)_N$ (finitely supported) such that $T_{m_N}x \to x$ "pointwise" (to define later).

Noncommutative L_p-spaces

- Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal faithful trace τ and $S_{\mathcal{M}}$ be the linear span of $S_{\mathcal{M}}^{+}$ which is the set of all positive x in $\mathcal{M}$ such that $\tau(suppx) < \infty$.
- Let $1 \leq p < \infty$. We define

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}}, \qquad \forall x \in S_{\mathcal{M}}$$

where $|x| = (x^*x)^{\frac{1}{2}}$ is the modulus of x. We denote the completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$ as $L_p(\mathcal{M})$ which is called the non-commutative L_p space associated with $(\mathcal{M}, \tau)$. For convenience, we usually set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$.

Noncommutative analogue of pointwise convergence

The noncommutative analogue of pointwise convergence dates back to [Segal, Ann. Math. 53'] and [Padmanabhan, Trans. AMS 67'].

Definition (Lance, Invent 76') Let $x_n, x \in L_p(\mathcal{M})$. $(x_n)_{n\geq 0}$ is said to converge almost uniformly (a.u. in short) to x if for any $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\lim_{n\to\infty} \|(x_n - x)e\|_\infty = 0 \quad \text{and} \quad \tau(e^\perp) < \varepsilon.$$

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Remark In commutative probability space:

$$\text{a.u. convergence} \xLeftrightarrow{\text{Egorov's Theorem}} \text{a.e. convergence}$$

Commutative case:

The study of pointwise convergence relies on that of maximal inequalities (i.e., L_p-estimates of the maximal function $\sup_N |f_N|$).

$$\text{maximal inequalities} \xRightarrow{\text{Banach Principle}} \text{a.e. convergence}$$

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Remark: $\sup_N |f_n|$ does not necessarily make sense in the noncommutative setting.

Noncommutative analogue of $\|\sup_N |f_n|\|_p = \|(f_n)\|_{L_p(\Omega;\ell_\infty)}$

Pisier, Asterique 98'; Junge, Crelle 02':

$L_p(\mathcal{M};\ell_\infty) :=$ space of all sequences $(x_n)_n \subset L_p(\mathcal{M})$ with factorization

$$x_n = a y_n b, \qquad n \geq 0,\ a, b \in L_{2p}(\mathcal{M}), y = (y_n) \subset \mathcal{M} \text{ bounded.}$$

The norm of x in $L_p(\mathcal{M};\ell_\infty)$ is given by

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\{\|a\|_{2p}\|y\|_\infty\|b\|_{2p}\}$$

where the infimum runs over all factorizations of x as above.

Convention: $\|\sup_n^+ x_n\|_p \triangleq \|(x_n)\|_{L_p(\mathcal{M};\ell_\infty)}$.

An equivalent and more intuitive description:
if $(x_n)_{n\geq 0}$ is a sequence of self-adjoint operators in $L_p(\mathcal{M})$, we have

$$\|\sup_{n\geq 0}{}^+ x_n\|_p = \inf\{\|a\|_p : a \in L_p(\mathcal{M})_+, -a \leq x_n \leq a, \forall n \geq 0\}.$$

Noncommutative maximal inequalities

Let $1 \leq p \leq \infty$. Consider a family of maps $\Phi_n : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ for $n \in \mathbb{N}$. We say $(\Phi_n)_{n\geq 0}$ is of strong type (p, p) with constant C if

$$\|\sup_{n\geq 0}{}^{+}\Phi_n(x)\|_p \leq C\|x\|_p, \qquad x \in L_p(\mathcal{M}).$$

A standard method (Banach principle, etc; Chilin-Litvinov 16',20'): Maximal inequality + some additional conditions (easy to verify) $\Rightarrow$ a.u. convergence on L_p

Junge-Xu JAMS 07': Markov semigroups on noncommutative L_p-spaces satisfies the strong (p, p) maximal inequalities and a.u. convergences for $p > 1$.

How to prove the maximal inequalities for nc Fourier multipliers?

- **Commutative (or semi-commutative) setting**: compare the kernel of multipliers with the one of some known maximal operators (e.g. Hardy-Littlewood, etc).
- **Quantum tori** (Chen-Xu-Yin, CMP 13'): transfer into the semi-commutative setting based on [Mei, Memoirs of AMS 07'].
- **Our setting for nc L_p on group algebras**: the "kernel" is only a formal element in a nc L_1-space; no natural transference into semi-commutative L_p-spaces. New ideas are required.

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- Our methods: estimate

$$\|\sup_N{}^+(T_{m_N}x - P_{t_N}x)\|_p$$

in terms of the symbols m_N, where $(P_t)_t$ an (abstract) suitable quantum Markov semigroup.

Main criteria: smooth case

$\ell : \Gamma \to \mathbb{R}_+$ conditionally negative definite function.
$T_{m_t} : VN(\Gamma) \to VN(\Gamma),\ t > 0$, unital positive Fourier multipliers.

Theorem Assume there exist $\alpha > 0$ and $\eta \in \mathbb{N}_+$ such that

- for all $g \in \Gamma$ and almost all $t > 0$,

$$|1 - m_t(g)| \lesssim \frac{\ell(g)^\alpha}{t}, \quad |m_t(g)| \lesssim \frac{t}{\ell(g)^\alpha},$$

- for all $g \in \Gamma$, $t \mapsto m_t(g)$ is cont and piecewise η-differentiable, and for all $1 \leq k \leq \eta$,

$$\left|\frac{d^k m_t(g)}{dt^k}\right| \lesssim \frac{1}{t^k},$$

then for all $1 + \frac{1}{2\eta} < p < \infty$ and all $x \in L_p(VN(\Gamma))$,

$$\|\sup_t{}^+ T_{m_t} x\|_p \lesssim \|x\|_p \qquad \text{and} \qquad T_{m_t} x \to x \text{ a.u. as } t \to \infty.$$

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Remark 2 Special case for $\eta = 1$ and $p > 3/2$:

$$|1 - m_N(g)| \lesssim \frac{\ell(g)^\alpha}{N}, \quad |m_N(g)| \lesssim \frac{N}{\ell(g)^\alpha}, \quad |m_{N+1}(g) - m_N(g)| \lesssim \frac{1}{N}.$$

Main criteria: case of exponential convergence/lacunarity

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Remark The positivity assumption on (T_{m_N}) can be removed for $p \geq 2$.

Remark The two theorems also hold for general vNa with Fourier-like expansions. For instance,

- $L_\infty(G)$, where G is a compact group or a compact quantum group. (will be discussed it later)
- Operator-valued functions space $L_\infty(\mathbb{R}^d; \mathcal{N})$.
- Twisted group von Neumann algebra, such as quantum torus.
- [Popa-Vaes] central multipliers on rigid C^*-tensor category .
- [Bozejko-Speicher] q-deformed von Neumann algebra $\Gamma_q(H)$.
- ....

(Very) few words on the strategy of proof

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Recall that for the Markov semigroup $P_t : \lambda(g) \mapsto e^{-t\ell(g)}\lambda(g)$, we have $\|{\sup_t}^+ P_t x\|_p \lesssim \|x\|_p$ (proved by Junge-Xu). We need to show that

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- A side remark: sharper estimate of Junge-Le Merdy-Xu's nc square function estimate for $p \to 1$: $\| \, \|_{H_p(\mathcal{M};(P_t))} \lesssim (p-1)^{-6} \| \, \|_{L_p(\mathcal{M})}$.

Back to discrete groups and approximate properties

Recall: Γ is amenable if $\exists$ finitely supported functions $(m_N)_{N\in\mathbb{N}}$ on Γ s.t. $T_{m_N} : VN(\Gamma) \to VN(\Gamma)$ unital completely positive & $T_{m_N}x \xrightarrow{w^*} x, \forall x$.

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- YES (at least up to a subsequence independent of x)
- To study this, we will apply our previous theorem. So we need a suitable Markov semigroup "close" to (T_{m_N}). Our study is based on a typical philosophy of abstract harmonic analysis:

Existence of "nice" approximate units of Fourier multipliers
$\Leftrightarrow$ Existence of "nice" Markov semigroups

Back to discrete groups and approximate properties

Recall: Γ is amenable if $\exists$ finitely supported functions $(m_N)_{N\in\mathbb{N}}$ on Γ s.t. $T_{m_N} : VN(\Gamma) \to VN(\Gamma)$ unital completely positive & $T_{m_N}x \xrightarrow{w^*} x, \forall x$.

Recall our initial question: does $T_{m_N}x \to x$ a.u.?

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  - Schoenberg: positive definite function ($\leftrightarrow$ completely positive Fourier multiplier) $\leftrightarrow$ conditionally negative definition function
  - Cipriani-Sauvageot 17': a vNa is amenable iff it admits a quantum Markov semigroup with Dirichlet form of sub-exp spectral growth
  - Jolissaint-Martin 04', Caspers-Skalski 15': a vNa has Haagerup property iff it admits a L_2-compact quantum Markov semigroup.

Back to discrete groups and approximate properties

Based on similar spirits, for any such (m_N) (up to possible passage to a subsequence), we may find conditionally negative definite function $\ell : \Gamma \to \mathbb{R}_+$ satisfying the assumption of the previous theorem, more precisely,

$$|1 - m_N(g)| \lesssim \frac{\ell(g)^2}{2^N}, \quad |m_N(g)| \lesssim \frac{2^N}{\ell(g)^2}.$$

In particular we get

Theorem If Γ is amenable, then there exists finitely supported functions $(m_N)_{N\in\mathbb{N}}$ on Γ s.t. $T_{m_N} : VN(\Gamma) \to VN(\Gamma)$ unital completely positive and $T_{m_N}x \to x$ a.u. for all $x \in L_p(VN(\Gamma))$ and all $p > 1$.

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Remark similar results holds for $p \geq 2$ for more general groups with the almost completely positive approximation property (ACPAP) (De Commer-Freslon-Yamashita 14'; intuitively with Haagerup property and weak amenability at the same time; e.g. free groups, $SL(2;\mathbb{Z})$ etc)

Remark similar results for quantum groups.

A few words on analogues of Dirichlet means

The case $m_N = \mathbb{1}_{K_N}$ for some increasing subsets $(K_N) \subset \Gamma$ is much more difficult. i.e., do we have

$$\sum_{g \in K_N} \hat{x}(g)\lambda(g) \to x \text{ a.u.}, \quad x \in L_p(VN(\Gamma))\,?$$

- Junge-Nielsen-Ruan-Xu, Adv. Math. 04': for the free group $\Gamma = \mathbb{F}_d$, $L_p(VN(\mathbb{F}_d))$ has a Schauder basis for $1 < p < \infty$.
- Bozejko-Fendler, Banach Center Publ 06': $\sum_{|g| \leq N} \hat{x}(g)\lambda(g) \not\to x$ in $L_p(VN(\mathbb{F}_d))$ for $p > 3$ or $1 \leq p < 3/2$.

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Theorem If Γ has ACPAP, then there exists increasing subsets $(K_N) \subset \Gamma$ s.t.

$$\sum_{g \in K_N} \hat{x}(g)\lambda(g) \to x \text{ a.u.}, \quad x \in L_2(VN(\Gamma)).$$

Example 1: generalized Féjer means on $\mathbb{R}^d$

Recall that classical Féjer multipliers on $\mathbb{R}^d$:

$$m_t(\xi) = \prod_{i=1}^{d} \left(1 - \frac{|\xi_i|}{t}\right) \mathbb{1}_{[-t,t]}(\xi_i)$$

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where $Q_t = [-\frac{t}{2}, \frac{t}{2}]^d$, μ is the Lebesgue measure.
Let B be a symmetric convex body in $\mathbb{R}^d$ such that the interior contains 0. We define the functions φ_t associated to B as

$$\varphi_t(\xi) = \frac{\mu(B_t \cap (\xi + B_t))}{\mu(B_t)}, \quad \text{where } B_t = \{\xi \in \mathbb{R}^d : \xi/t \in B\}.$$

Easy to check that T_{φ_t} is a u.c.p. map.
(1) for any $1 < p \leq \infty$ and for any $f \in L_p(\mathbb{R})$ with $1 < p < \infty$, $T_{\varphi_{2^j}}(f) \to f$ a.e. as $j \to \infty$.
(2) for any $f \in L_p(\mathbb{R})$ with $3/2 < p < \infty$, $T_{\varphi_t}(f) \to f$ a.e. as $t \to \infty$.
$\rightsquigarrow$ More pb on $p = 1$ case and dim-free estimates (cf. end of file)

Example 2: amenable groups, nilpotent groups

Γ amenable $\rightsquigarrow$ Følner sequence $(K_N)_N \subset \Gamma$ with

$$m_N(g) = \frac{|K_N \cap gK_N|}{|K_N|} \to 1, \quad N \to \infty$$

where $|\ |$ is the counting measure. $T_{m_N} : VN(\Gamma) \to VN(\Gamma)$ unital completely positive.

- $\exists$ subsequence $(N_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ s.t. $T_{m_{N_j}}(x) \to x$ a.u. as $j \to \infty$ or all $x \in L_p(VN(\Gamma))$ with $p > 1$.
- For $\Gamma = \langle S \rangle$ 2-step nilpotent, we may take $K_N = S^N$ and $N_j = 2^j$; moreover $T_{m_N}(x) \to x$ a.u. or all $x \in L_p(VN(\Gamma))$ with $p > 3/2$.
- higher step cases: similar results depending on the geometric estimates $|S^{N+1} \setminus S^N|$ (cf. Stoll, Breuillard, etc)

Example 3: compact group

Let G be a compact metrizable group. Let $\mathrm{Irr}(G)$ be the collection of equivalence classes of irreducible representations of G. For $\pi \in \mathrm{Irr}(G)$,

$$\hat{f}(\pi) = \int_G f(x)\pi(x^{-1})dx \in B(H_\pi), f \in L_p(G).$$

Corollary $\exists$ finitely supported functions $m_N : \mathrm{Irr}(G) \to \mathbb{R}$ s.t.

$$(T_{m_N} f)(x) := \sum_{\pi \in \mathrm{Irr}(G)} m_N(\pi) \dim(\pi) \mathrm{Tr}(\hat{f}(\pi)\pi(x)), \quad x \in G, f \in L_p(G)$$

defines unital positive operators on $L_p(G)$ and

$$\lim_{N\to\infty} T_{m_N} f = f \text{ a.e.}, \quad f \in L_p(G), 1 < p < \infty.$$

Remark m_N can be explicitly constructed from the representation theory of G. ("Folner sets" on the dual quantum group $\hat{G}$)

Example 4: hyperbolic groups

Let Γ be a hyperbolic group so that the word length function $|\ |$ is conditionally negative (e.g., free group or hyperbolic Coxeter group). The following Bochner-Riesz means are studied in [Mei-de la Salle TAMS 18']: for a fixed $\delta > 1$ we take

$$B_N^\delta f = \sum_{g \in \Gamma : |g| \leq N} \left(1 - \frac{|g|^2}{N^2}\right)^\delta \hat{f}(g)\lambda(g), \quad f \in L_p(VN(\Gamma)).$$

Corollary Let $2 \leq p \leq \infty$. Then

$$\|\sup_{N \in \mathbb{N}}{}^+ B_N^\delta x\|_p \lesssim \|x\|_p, \qquad x \in L_p(VN(\Gamma)),$$

and for any $x \in L_p(VN(\Gamma))$, $B_N^\delta(x)$ converges a.u. to x as $N \to \infty$.

Remark Similar results hold for $p = 2$ and $\delta > 0$. Open for $\delta = 0$.

Example 5: dimension free bounds of noncommutative Hardy-Littlewood maximal operators on $L_p(\mathbb{R}^d; L_p(\mathcal{N}))$

- Bourgain 86-87', Carbery 86' The classical dimension free bounds of Hardy-Littlewood maximal of convex body
- Mei, MAMS 07': noncommutative analogue of Hardy-Littlewood maximal inequalities for balls respect to Euclidean metrics.
- Hong-Liao-W., Duke Math J. 20': noncommutative analogue of Hardy-Littlewood maximal inequalities for balls in general doubling spaces.
- Hong, Illinois J Math 18' The dimension free bounds in this noncommutative setting only for Euclidean balls.

Example 5: dimension free bounds of noncommutative Hardy-Littlewood maximal operators on $L_p(\mathbb{R}^d; L_p(\mathcal{N}))$

Corollary Let B be a symmtric convex body in $\mathbb{R}^d$ and $\mathcal{N}$ a semifinite von Neumman algebra. Define $\Phi_r : L_p(\mathbb{R}^d; L_p(\mathcal{N})) \to L_p(\mathbb{R}^d; L_p(\mathcal{N}))$

$$\Phi_r(f)(x) = \frac{1}{\mu(B)} \int_B f(x - ry)dy.$$

There exists $C_p > 0$ independent of the dimension d such that

(1) $\|{\sup_{j\in\mathbb{Z}}}^+ \Phi_{2^j}(f)\|_p \leq C_p \|f\|_p, \qquad f \in L_p(\mathbb{R}^d; L_p(\mathcal{N})), 1 < p \leq \infty.$

(2) $\|{\sup_{r\geq 0}}^+ \Phi_r(f)\|_p \leq C_p \|f\|_p, \qquad f \in L_p(\mathbb{R}^d; L_p(\mathcal{N})), \frac{3}{2} < p \leq \infty.$

(3) If B is the ℓ_q-ball $\{(x_i)_{i=1}^d : \sum_{i=1}^d |x_i|^q \leq 1\}$ with $q \in 2\mathbb{N}$,

$$\|{\sup_{r\geq 0}}^+ \Phi_r(f)\|_p \leq C_p \|f\|_p, \qquad f \in L_p(\mathbb{R}^d; L_p(\mathcal{N})), 1 < p \leq \infty.$$

Two problems on $\mathbb{R}^d$

Let B be any fixed (symmtric) convex body in $\mathbb{R}^d$. Recall that the Fourier multiplier T_{φ_t} is defined associated to B, with symbols φ_t defined as

$$\varphi_t(\xi) = \frac{\mu(B_t \cap (\xi + B_t))}{\mu(B_t)}.$$

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Problem 2 Can we take $C_{p,d}$ to be independent of d?

Comments on Problem 1

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Yes for cube (cf. Grafakos' book);
For general symmetric convex bodies, the best estimate is due to Brandolini-Hofmann-Iosevich GAFA 03', for large r

$$\int_{S^{d-1}} |\hat{\chi}_B(rx')|^2 dx' \lesssim r^{-(d+1)}$$

which has important applications to the distribution of lattice points in convex domains and the Falconer distance problem etc.

Thank you very much!