Resampling-based estimation of the accuracy of satellite ephemerides

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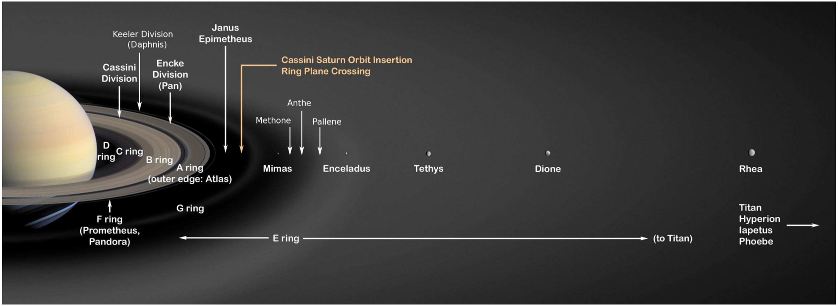
[6]Université de Lille

Precision of ephemerides of natural satellites of Saturn



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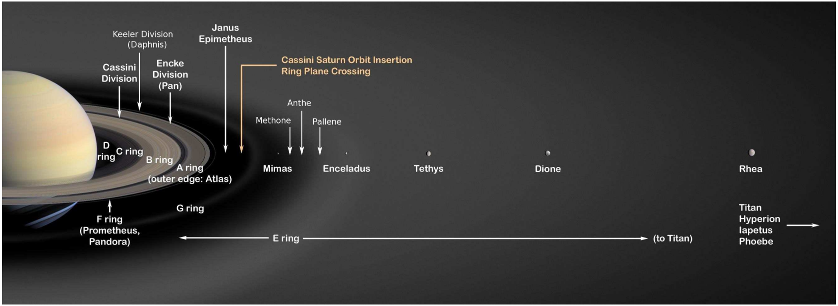

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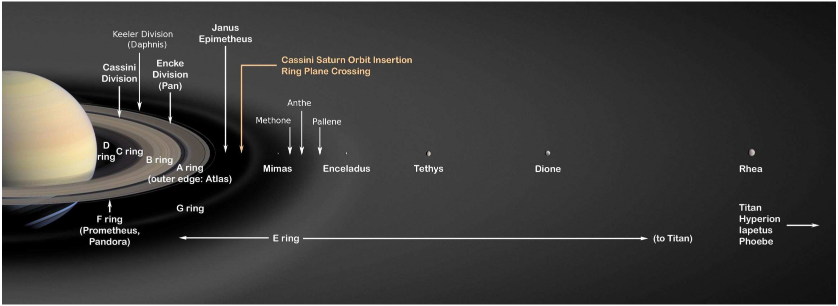

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- Problem: how accurate are the ephemerides, in particular **far from the observation period**?
- Two main examples: Mimas (revolution period: 0.942 days) and Titan (revolution period: 15.945 days)

The models: TASS & NUMINT

- TASS1.7 (Analytic Theory of Saturnian Satellites):
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$\Rightarrow$ Model: $x(t) = \varphi(c, t) \in \mathcal{X}$, where $c \in \mathcal{C} \subset \mathbb{R}^p$ parameter space
e.g., $x(t) = (\alpha(t), \delta(t))$

Internal error of the models

- True position $P(t) \neq \varphi(c, t)$ for every $c \in \mathcal{C}$

$\Rightarrow$ Internal error (or bias)

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- Neglected (or unknown physical effects) in TASS and NUMINT

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$$(t_1, X_1), \ldots, (t_N, X_N) \in \mathbb{R} \times \mathcal{X}$$

$$X_i = P(t_i) + \varepsilon_i \qquad \mathbb{E}\left[\varepsilon_i\right] = 0$$

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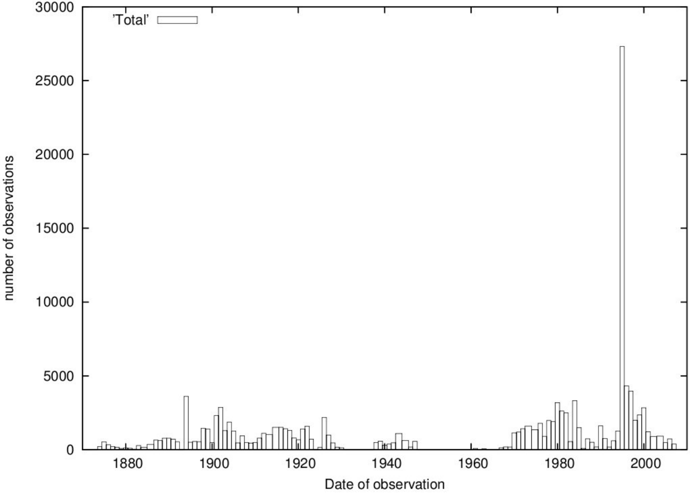
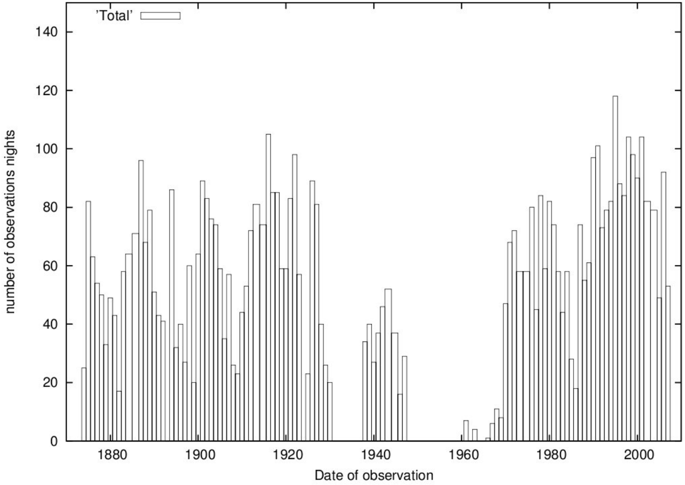
- uncertainty of observation time (especially for old observations)

Time distribution of observations



Time distribution of observation nights



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- Optimization method: start with $c = \overline{c}$ + linearization $\Rightarrow$ iterate until convergence

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- Observation errors
- Optimization error
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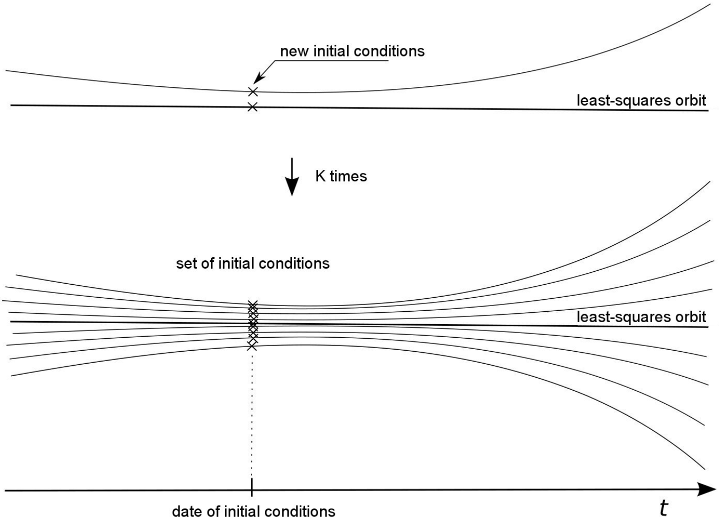

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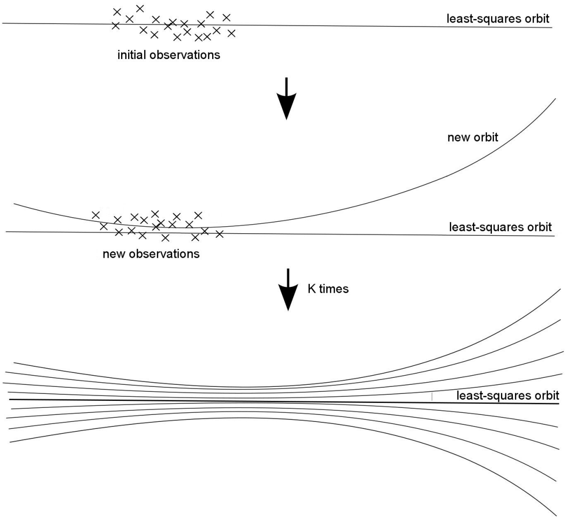

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- Many observations $\Rightarrow$ long-term ephemerides & complex models $\Rightarrow$ new methods needed

Resampling heuristics (bootstrap, Efron 1979)

Real world : $P \xrightarrow{\text{sampling}} P_n \Longrightarrow \widehat{c} = \widehat{c}(P_n)$

$$\text{precision} = F_t(P, P_n) = \left(\varphi(c^\star, t) - \varphi(\widehat{c}, t) \right)^2$$

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resampling $\quad P_n^W = \frac{1}{n} \sum_{i=1}^{n} W_i \delta_{(X_i, Y_i)} \quad \text{with } W \sim \mathcal{M}(n; n^{-1}, \ldots, n^{-1})$

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subsampling $\quad P_n^W = \dfrac{1}{\operatorname{Card}(I)} \sum_{i \in I} \delta_{(X_i, Y_i)}$ with $I \subset \{1, \dots, n\}$ random

The bootstrap for estimating the extrapolated error

$$\forall k \in \{1, \ldots, B\}, \qquad D_N^{(k)} = (t_{I_1^{(k)}}, X_{I_1^{(k)}}), \ldots, (t_{I_N^{(k)}}, X_{I_N^{(k)}})$$

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- Assumption: blocks are (almost) independent

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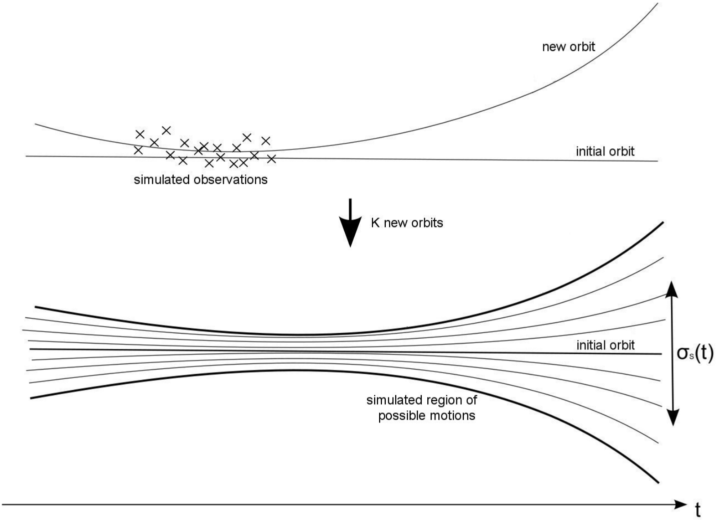
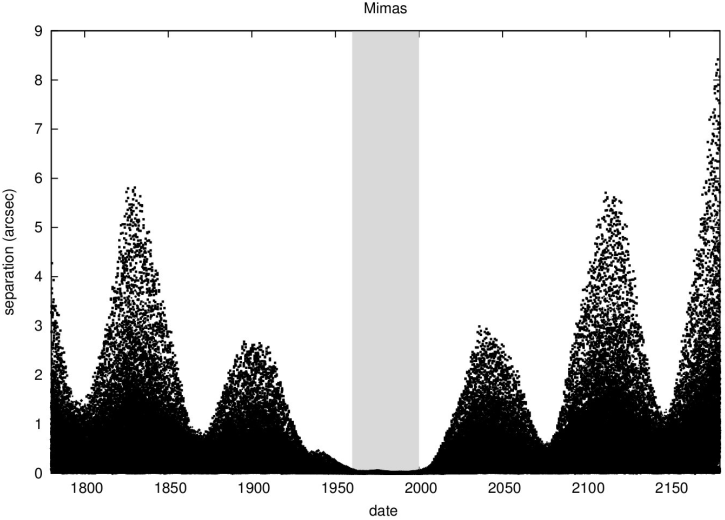
$\Rightarrow$ Dependent observations, of rather good quality

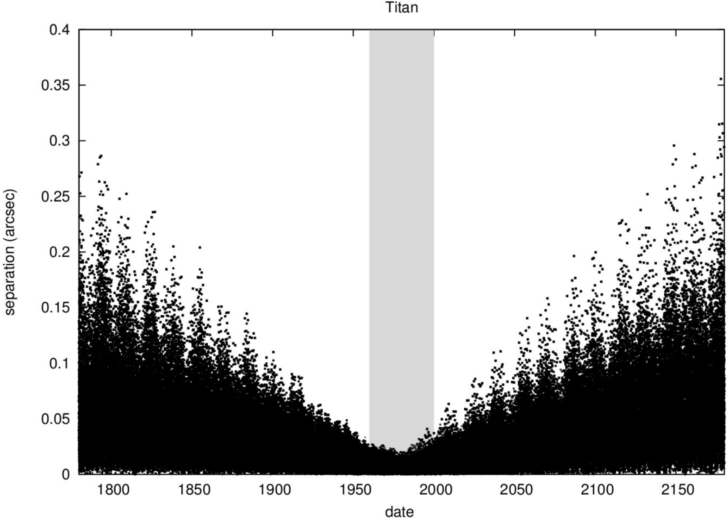
Region of possible motions ($K = 200$)



Region of possible motions: Mimas (TASS)



Region of possible motions: Titan (TASS)



Average size of the region of possible motions

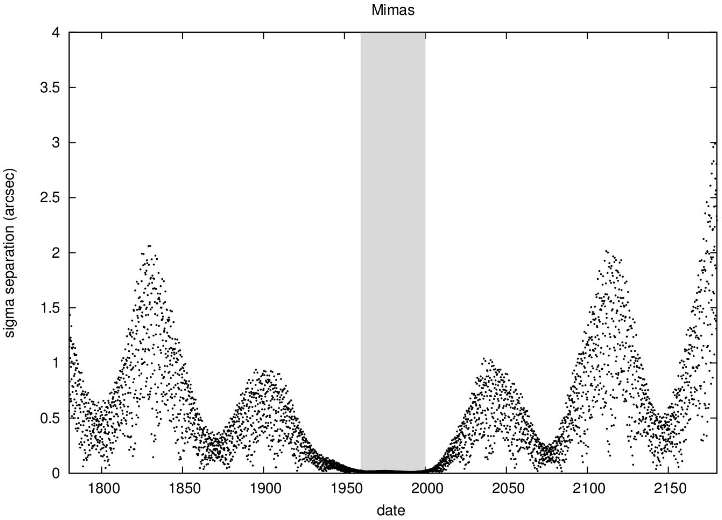
$$\sigma_S(t) = \sqrt{\frac{1}{K} \sum_{k=1}^{K} \left(s_k(t) \right)^2}$$

where

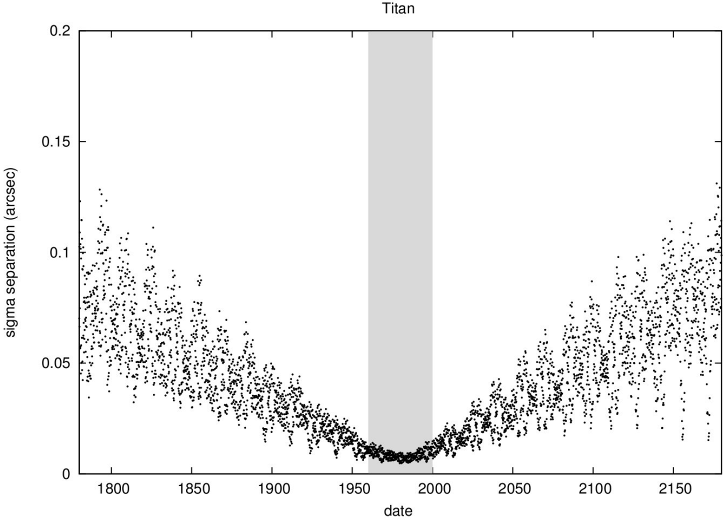
$$s_k(t) = \sqrt{\left(\left(\alpha^{(k)}(t) - \alpha^{(0)}(t) \right) \cos\left(\delta^{(0)}(t) \right) \right)^2 + \left(\delta^{(k)}(t) - \delta^{(0)}(t) \right)^2}$$

is the (angular) separation between the k-th orbit and the inital orbit at time t

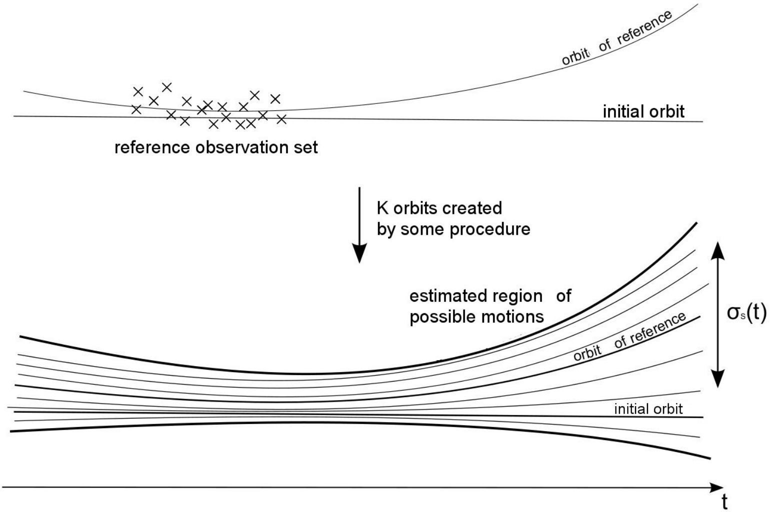
Size of the region of possible motions: Mimas (TASS)



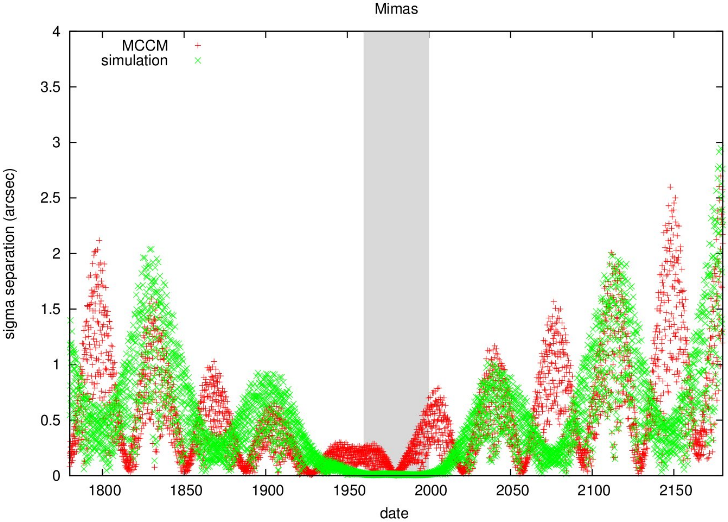
Size of the region of possible motions: Titan (TASS)



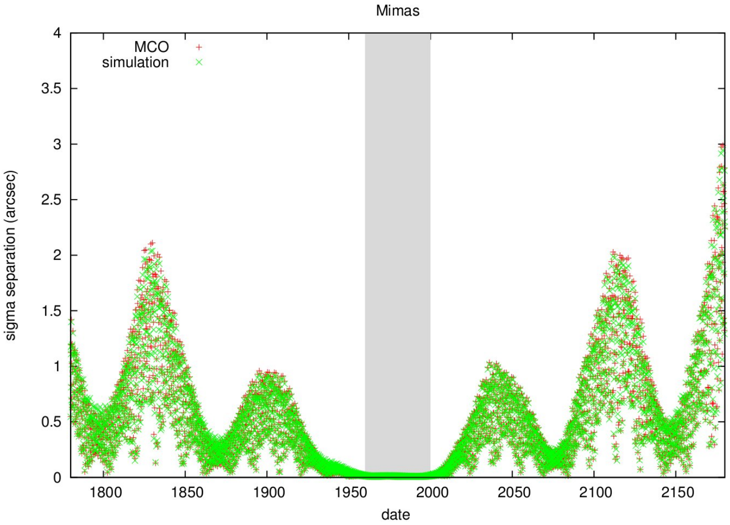
Principle of simulations



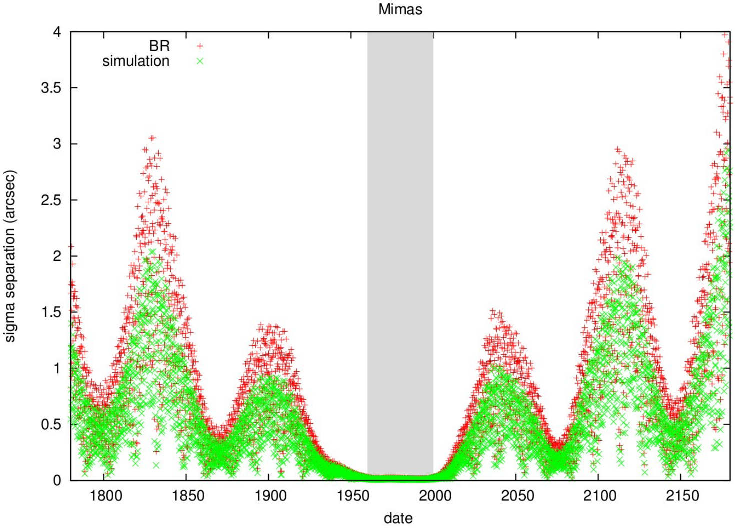
Performance of MCCM: Mimas ($B = 200$), TASS



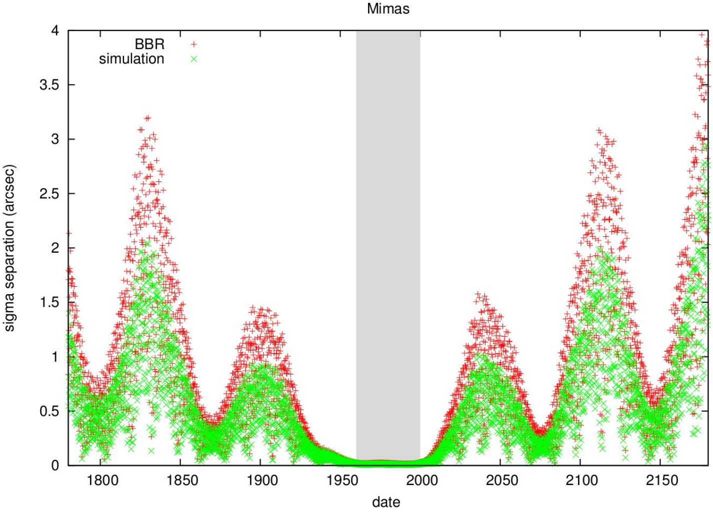
Performance of MCO: Mimas ($B = 200$), TASS



Performance of the Bootstrap: Mimas ($B = 200$), TASS



Performance of the Block Bootstrap: Mimas ($B = 200$)



Correlation coefficient and multiplying factor (TASS)

correlation coefficient $\rho_S = \text{corr}(\sigma_S^{\text{sim}}(t)\,,\,\sigma_S^{\text{estim}}(t))$

multiplying factor κ_S

	Mimas		Titan	
Method	ρ_S	κ_S	ρ_S	κ_S
MCCM	0.511	1.876	0.955	0.790
MCO	0.999	1.001	0.994	0.966
Bootstrap	1.000	1.458	0.999	1.456
Block Bootstrap	0.999	1.484	0.999	1.441

($B = 200$; only for one simulated reference orbit)

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- Problem of choosing the blocks:
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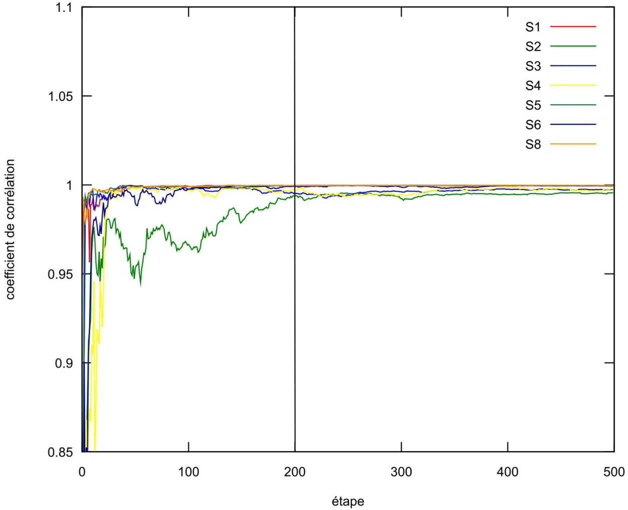
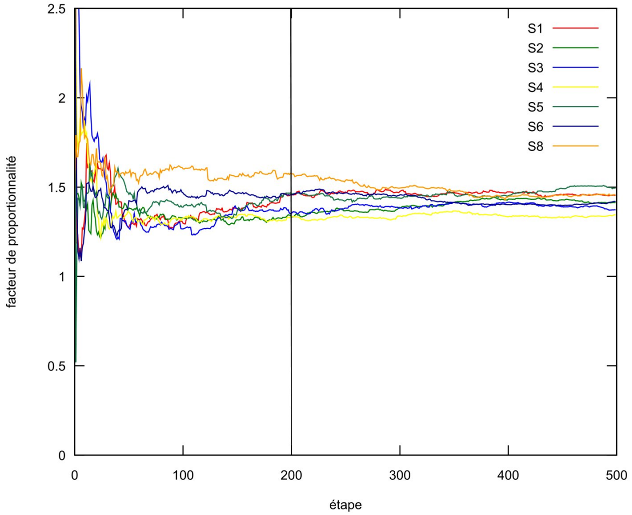
- Multiplying factor for the bootstrap $\in [1.4; 1.5]$: why? how general is this?

  κ_S seems much closer to 1 for NUMINT

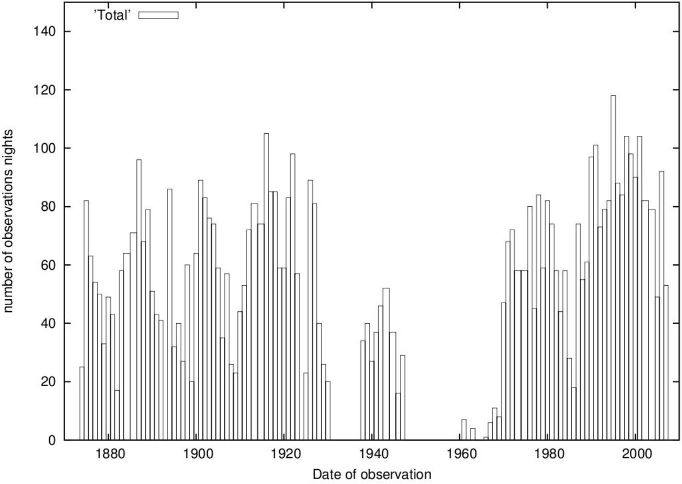
How many resamples do we need? ρ_S (TASS)



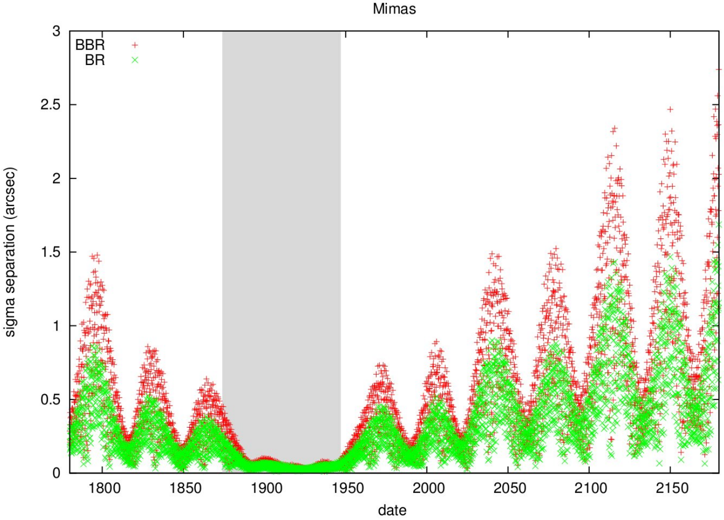
How many resamples do we need? m_S (TASS)



Application: old vs. recent observations



Precision of old observations: Mimas (TASS)



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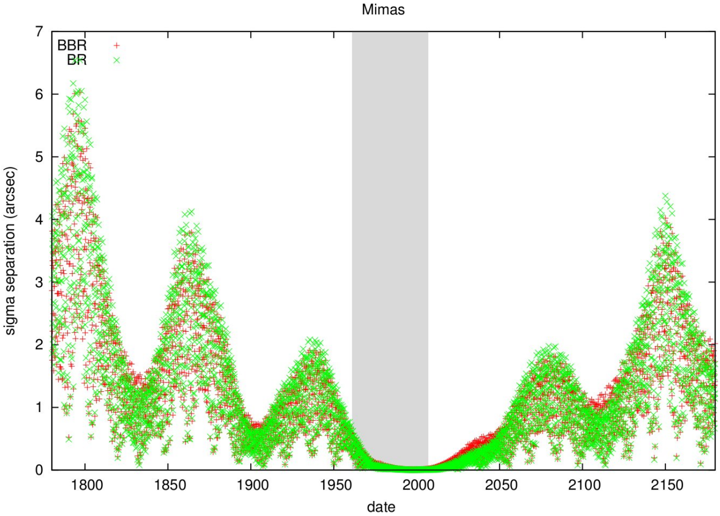
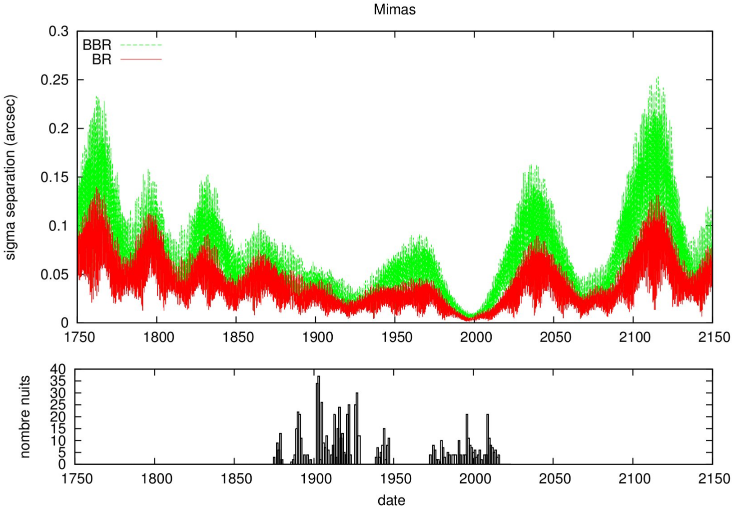
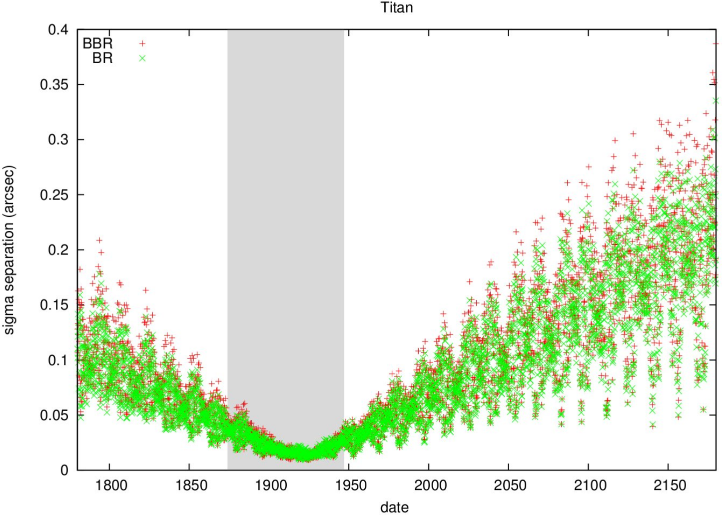


Precision when using all the observations: Mimas (TASS)



Precision of old observations: Titan (TASS)



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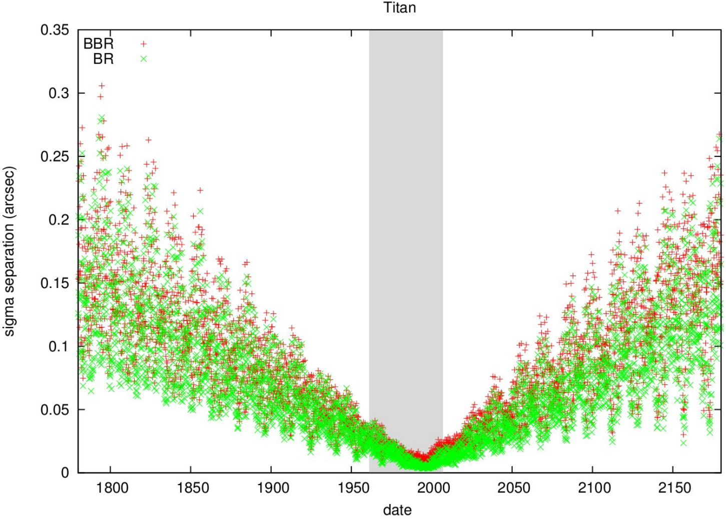

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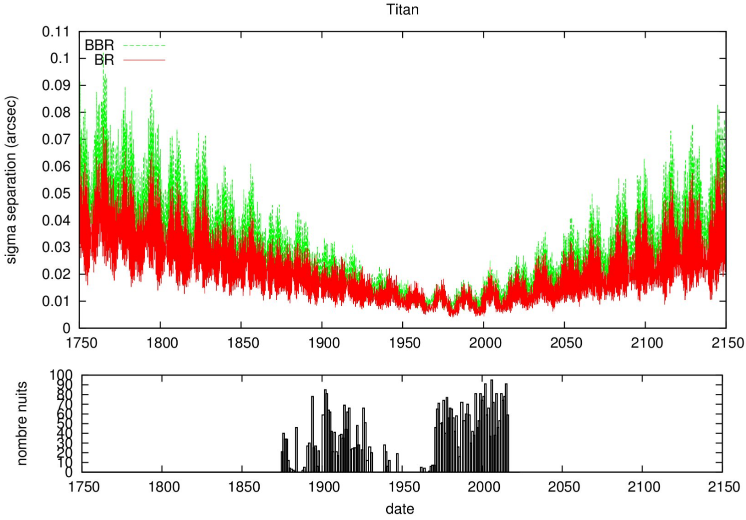
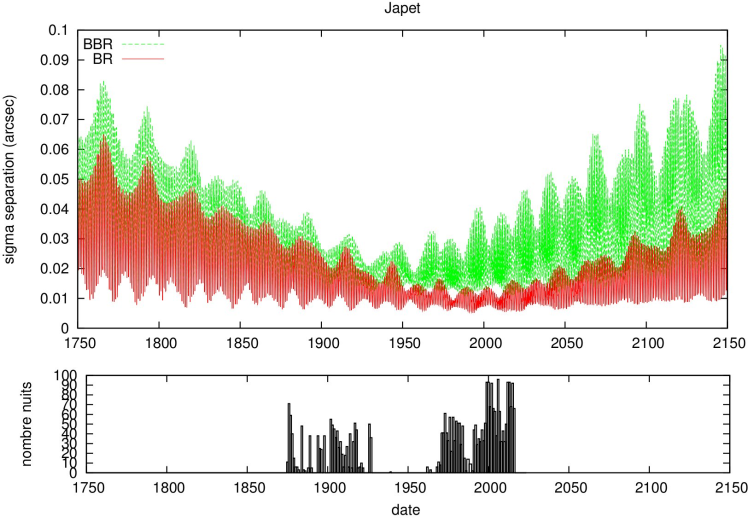


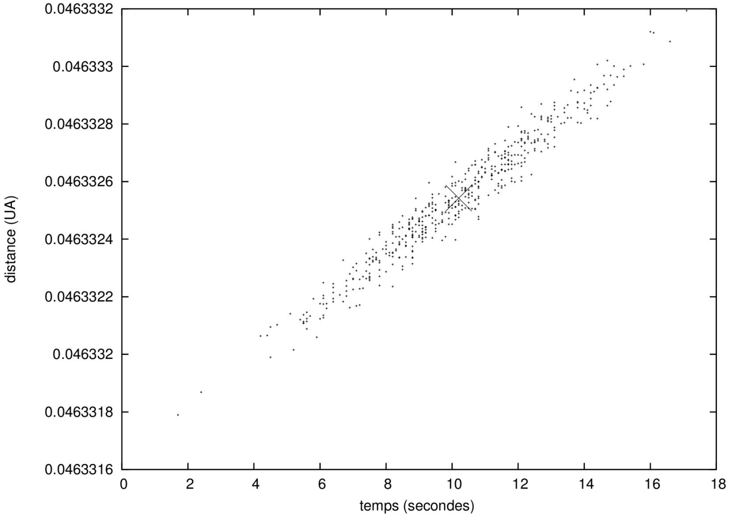
Precision when using all the observations: Japet (TASS)



Astronomical conclusions

- qualitative differences between satellites:
  fast motion (Mimas) / slow motion (Titan)
  main term of the mean longitude
- accurate observations on a short period can be less useful than noisy observations on a long period
  $\Rightarrow$ old observations indeed are useful
- Other applications (Desmars' Ph.D., 2009):
  - expected improvement of reducing errors: Gaia mission (a few observations very accurate + improvement of the accuracy of past observations)
  - asteroids: Toutatis (time-space accuracy of close approaches to Earth)

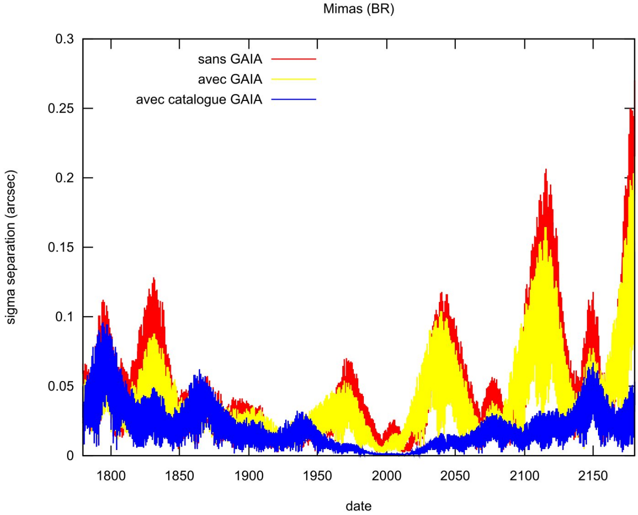
Toutatis: will December, 12th be the end of the world?



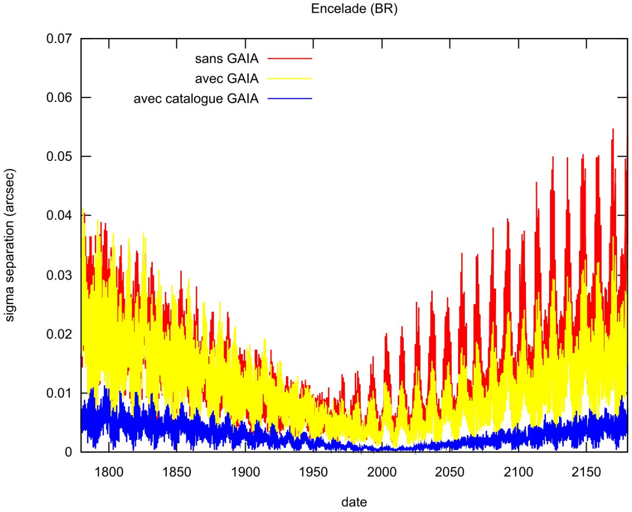
Mathematical conclusions

- Bootstrap: versatile and robust method for estimating the extrapolated error
- Building blocks ⇒ handling dependence between observations
- Open problems:
  - Multiplying factor κ_S
  - Formal proofs: known results in simpler statistical frameworks only
  - Theoretical link between sensitivity to initial conditions and resampling-based estimators of extrapolated error
  - What about other resampling methods (e.g., subsampling)?

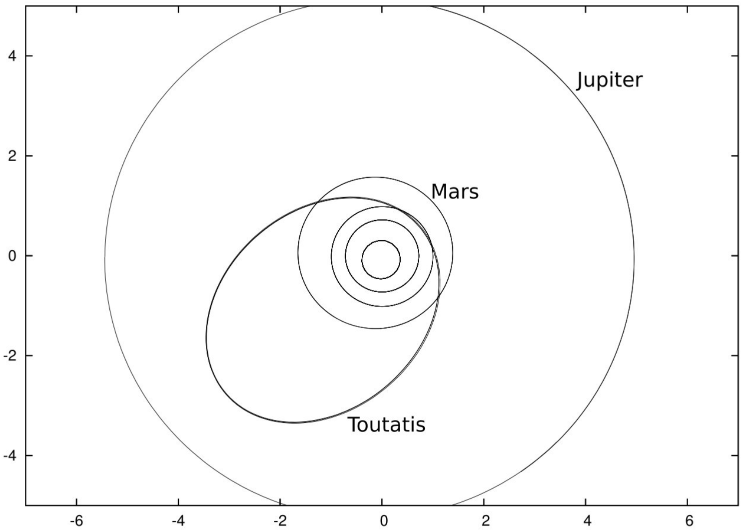
Expected improvement of precision thanks to Gaia results: Mimas



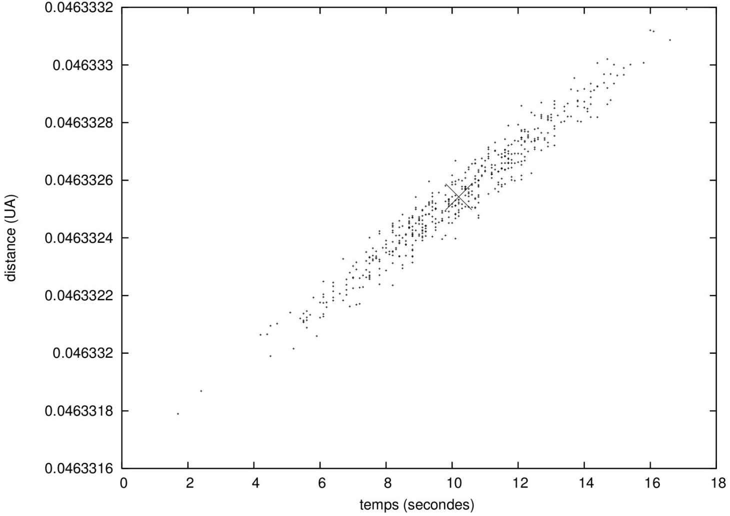
Expected improvement of precision thanks to Gaia results: Encelade



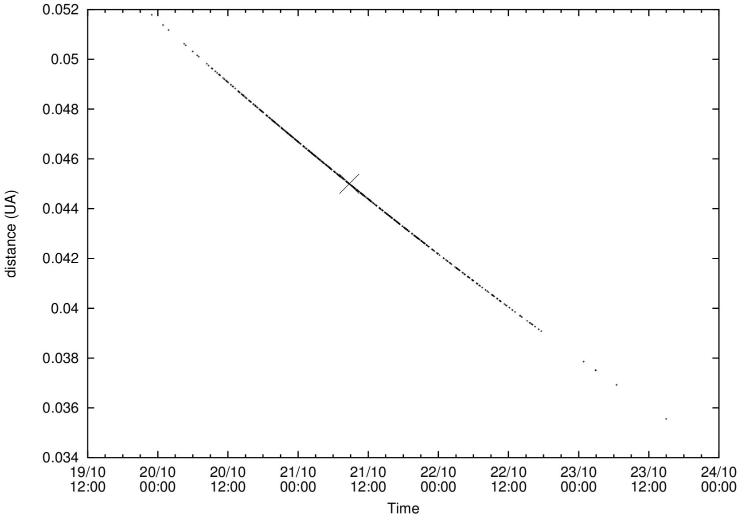
Toutatis orbit



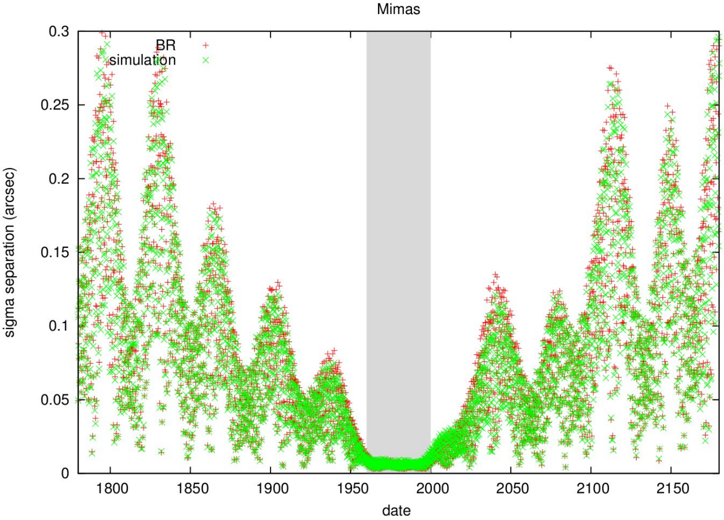
Toutatis: time-space precision of close approach to Earth on December, 12th 2012



Toutatis: time-space precision of close approach to Earth on October 2322



Results with NUMINT instead of TASS ($B = 30$ samples)



Results with NUMINT instead of TASS ($B = 30$ samples)

	Mimas		Titan	
Method	ρ_S	κ_S	ρ_S	κ_S
MCO	0.989	0.848	0.997	0.723
Bootstrap	0.999	1.041	0.997	0.832
Block Bootstrap	0.981	0.999	0.997	0.842