The Newhouse phenomenon*

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We build an open set of non-hyperbolic C^2-diffeomorphisms of a surface:

Theorem 1 (Newhouse [N1])**.** *Let M be a compact surface. There exists a non-empty open set $\mathscr{U} \subset \mathrm{Diff}^2(M)$ of non-hyperbolic diffeomorphisms. Moreover any diffeomorphism in a dense G_δ-subset of $\mathscr{U}$ has infinitely many sinks.*

In contrast with hyperbolic diffeomorphisms, the generic systems in $\mathscr{U}$ have infinitely many chain-recurrent classes.

Recall that in dimension $d=1$, the Morse-Smale diffeomorphisms are dense in $\mathrm{Diff}^r(M)$ for any $r \geq 1$. In dimension $d \geq 3$, there exists a non-empty open set $\mathscr{U} \subset \mathrm{Diff}^1(M)$ of non-hyperbolic diffeomorphisms [N2]. One question remains about the density of hyperbolicity:

On a surface M, is the set of hyperbolic diffeomorphisms dense in $\mathrm{Diff}^1(M)$?

Remark 1. The proof of Newhouse's theorem we present in this chapter is due to N. Gourmelon and myself. It shows the following stronger version on any compact surface M:

For any $\alpha \in (0,1)$ and for $C>0$ large enough, let $\mathrm{Diff}^{1+\alpha}_C(M)$ be the set of $C^{1+\alpha}$ diffeomorphisms g such that the α-Hölder norms of Dg, Dg^{-1} are bounded by $C>0$, endowed with the C^1-topology. It contains a non-empty open set $\mathscr{U}$ of non-hyperbolic diffeomorphisms. Moreover any diffeomorphism in a dense G_δ-subset of $\mathscr{U}$ has infinitely many sinks.

1 Dynamics of the horseshoe

We consider the classical construction of the horseshoe.

a) The horseshoe. We consider a $C^{1+\alpha}$-diffeomorphism f of a surface M and a rectangle R diffeomorphic to $[0,1]^2$ and we assume that the cone field criterion is satisfied: there exist $\lambda, \beta \in (0,1)$ such that for any $x \in R \cap f^{-1}(R)$, we have:

- $Df_x(\mathbb{R}^2 \setminus \mathscr{C}^s) \subset \mathscr{C}^u$,
- for any $v \in \mathscr{C}^u_x$, $\|Df_x.v\| \geq \lambda^{-1}\|v\|$,
- for any $v \in \mathscr{C}^s_{f(x)}$, $\|Df^{-1}_{f(x)}.v\| \geq \lambda^{-1}\|v\|$,

where

$$\mathscr{C}^s_x = \{v = v_1 + v_2 \in \mathbb{R}^2,\ \beta\|v_2\| > \|v_1\|\},$$
$$\mathscr{C}^u_x = \{v = v_1 + v_2 \in \mathbb{R}^2,\ \beta\|v_1\| > \|v_2\|\}.$$

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Let us consider two vertical strips $P_1, P_2 \subset R$, i.e. two disjoint rectangles whose horizontal boundaries are contained in the two horizontal boundaries of R and whose vertical boundaries are graphs of the form $\{(\psi(t), t), t \in [0,1]\}$, disjoint from the vertical boundaries of R. (We take P_1 to the left of P_2.)

Similarly we consider two disjoint horizontal strips $Q_1, Q_2 \subset R$ (with Q_1 below Q_2) and we suppose:

$$f^{-1}(R) \cap R = P_1 \cup P_2,$$

$$f(P_1) = Q_1, \quad f(P_2) = Q_2.$$

The maximal invariant set K in R is contained in $(P_1 \cup P_2) \cup (Q_1 \cup Q_2)$ and is hyperbolic. (See figure 1.)

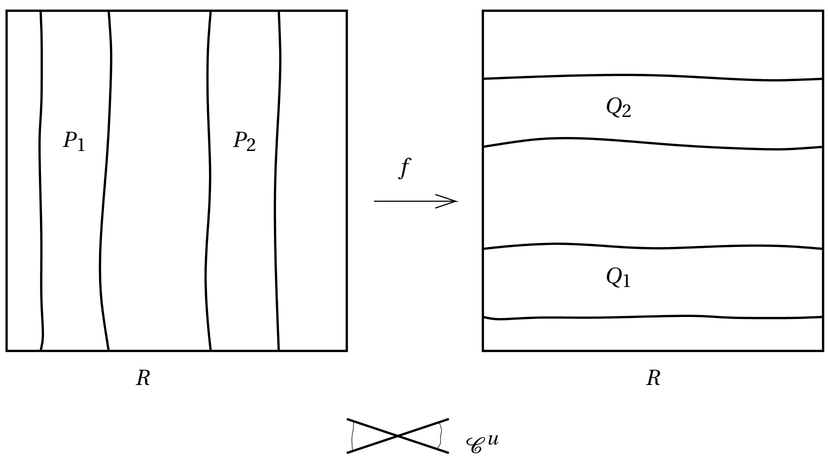

Figure 1: The horseshoe map.

b) Local invariant manifolds. For any $x \in K$, the connected component of $W^u(x) \cap R$ containing x is a graph of the form $\{(s, \varphi(s)), s \in [0,1]\}$, tangent to $\mathscr{C}^u$ (the function φ is C^1 and β-Lipschitz). We denote it by $W^u_{loc}(x)$.

Similarly, the connected component of $W^s(x) \cap R$ containing x is a graph of the form $\{(\psi(t), t), t \in [0,1]\}$, tangent to $\mathscr{C}^s$ (the function ψ is C^1 and β-Lipschitz). We denote it by $W^s_{loc}(x)$.

For any $x, y \in K$, we have the following properties:

- $f(W^s_{loc}(x)) \subset W^s_{loc}(f(x))$ and $f^{-1}(W^u_{loc}(x)) \subset W^s_{loc}(f^{-1}(x))$.
- $W^s_{loc}(x), W^s_{loc}(y)$ (resp. $W^u_{loc}(x), W^u_{loc}(y)$) are either disjoint or equal.
- $W^s_{loc}(x), W^u_{loc}(y)$ intersect in a unique point. The intersection is transverse and contained in K.
- In particular, K is transitive, hence it is a basic set.

c) Folding region. We introduce two open sets $L_s \subset (P_1 \cup P_2) \setminus (Q_1 \cup Q_2)$ and $L_u \subset (Q_1 \cup Q_2) \setminus (P_1 \cup P_2)$ and an integer $N \geq 2$ such that $f^N(L_s) = L_u$. (See figure 2.) We are aimed to describe the dynamics inside

$$R \cup f(L_s) \cup \cdots \cup f^{N-1}(L_s).$$

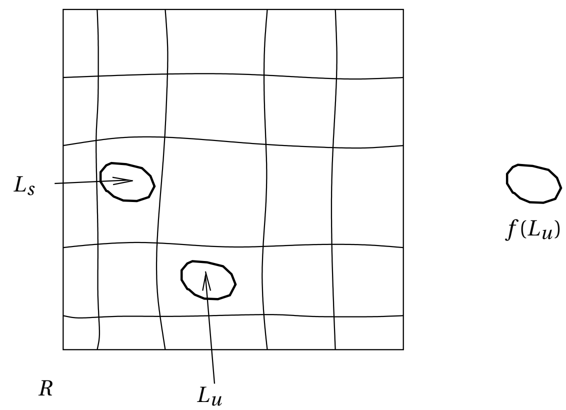

Figure 2: The folding region.

d) Construction. One can build such a dynamics by deformation of the classical hyperbolic diffeomorphism of the sphere S^2 built by Smale [S, pages 770–773], as it is picture in figure 3.

These properties are C^1-robust.

Lemma 1. *Any diffeomorphism g that is C^1-close to f satisfies the same properties, when one replaces P_1, P_2 by the new connected components of $R \cap g^{-1}(R)$ and replace Q_1, Q_2, L_u by the new images $g(P_1), g(P_2), g^N(L_s)$.*

2 Transverse combinatorial structure

Let us denote by $\mathscr{W}^u$ the union of the local unstable manifolds $W^u_{lot}(x)$ with $x \in K$. It is a closed subset of R. We describe the transverse structure of $\mathscr{W}^u$. One can note (but we will not use it) that $\mathscr{W}^u$ intersect each vertical line of R as a Cantor set.

The connected components of $\text{Int}(R) \setminus \mathscr{W}^u$ are open rectangles G bounded by two local unstable manifolds: G^+ (above) and G^- (below).

The connected component G which contains the region between Q_1 and Q_2 is called *generation* 0 *component* and denoted by G_0. The union of $\mathscr{W}^u \cap Q_1$ with the components G contained in Q_1 and the union of $\mathscr{W}^u \cap Q_2$ with the components G contained in Q_2 are two closed rectangles denoted as B_1 and B_2 respectively. (See figure 4.)

We immediately get:

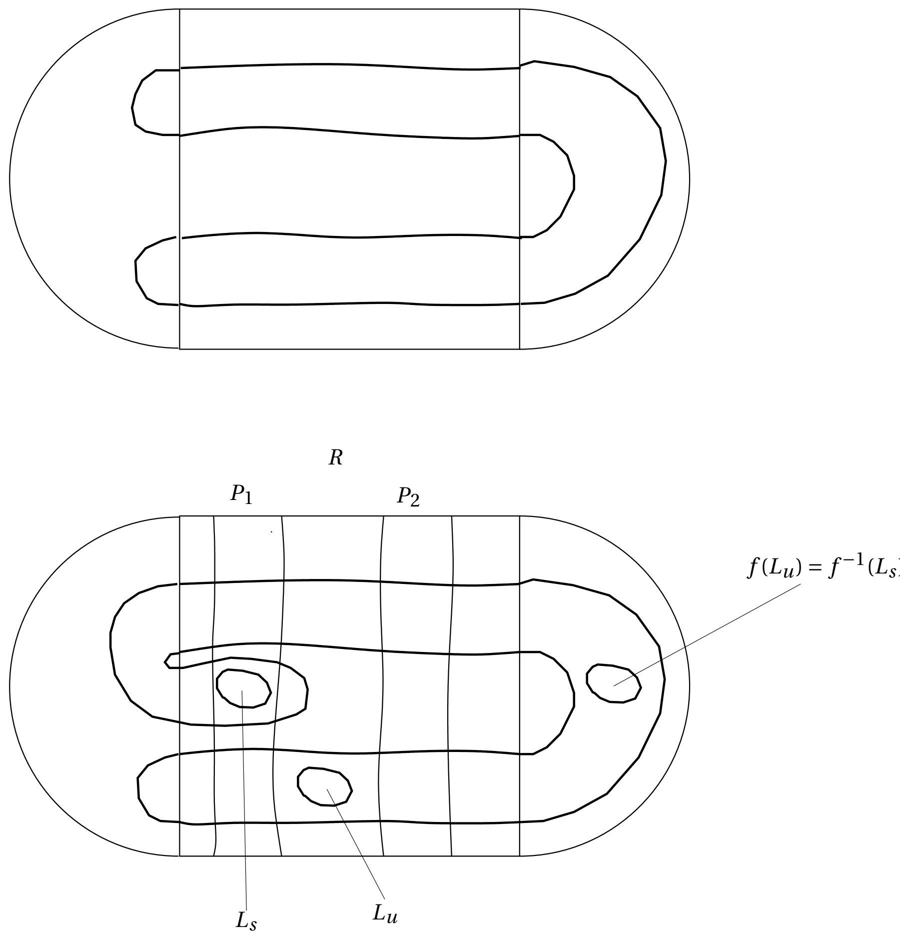

Figure 3: The standard horseshoe (above). The realization of the horseshoe map with a folding (below).

Lemma 2. *For any component G different from G_0, the pre image $f^{-1}(G)$ is contained in a component G'. There exists an integer $n \geq 1$ such that $f^{-n}(G) \subset G_0$, which is called the* generation of G.

To any such component G we associate the rectangles B^+, B^-, which are the connected components of $f^n(B_1)$ and $f^n(B_2)$ that are adjacent to G. (The rectangle B^+ is above and B^- is below.) In the following we sometimes denotes B_0^+ the rectangle B_1 or B_2 which contains $f^{-n}(B^+)$; the other one is denoted B_0^-.

The same structure holds for the union $\mathscr{W}^s$ of the local stable manifolds $W^s_{lot}(x)$.

3 Stable and unstable holonomies

The stable and unstable manifolds are as smooth as the diffeomorphism, but in general they do not belong to a smooth foliation. For a surface diffeomorphism and a $C^{1+\alpha}$-diffeomorphism, the transverse smoothness, that is the smoothness of the holonomy of the local stable and unstable manifolds, is however Lipschitz.

Definition 1. Let γ, γ' be two transverse arcs to a local unstable manifold $W^u_{loc}(x)$. The *holon-*

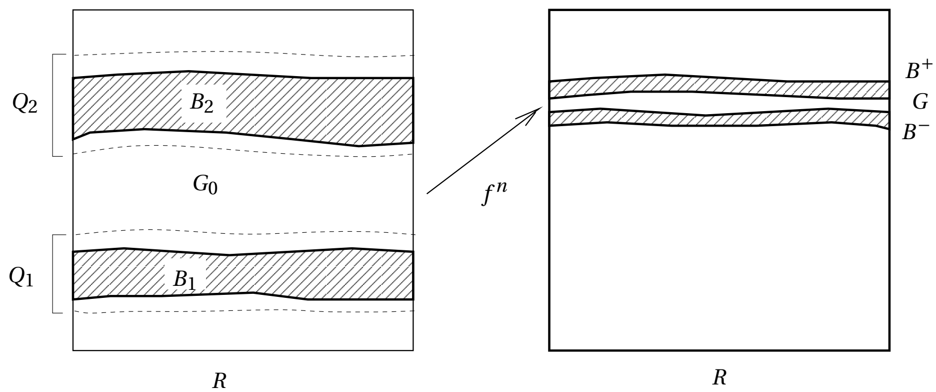

Figure 4: Combinatorics of components.

omy between γ,γ' is the map which associates to any intersection z between γ and $W^u_{loc}(x')$ close to $W^u_{loc}(x)$ the unique point $z' \in \gamma' \cap W^u_{loc}(x')$.

For any $x, y \in K$ we estimate the distance between the unstable leaves $W^u_{loc}(x)$ and $W^u_{loc}(y)$ by considering vertical graphs $\gamma = \{(\psi(t), t), t \in [0,1]\}$ associated to a β-Lipschitz function ψ and the distance between $W^u_{loc}(x) \cap \gamma$ and $W^u_{loc}(y) \cap \gamma$ along γ. It is denoted $d_\gamma(W^u_{loc}(x), W^u_{loc}(y))$.

We then define

$$d^+(W^u_{loc}(x), W^u_{loc}(y)) = \sup_\gamma d_\gamma(W^u_{loc}(x), W^u_{loc}(y)),$$

$$d^-(W^u_{loc}(x), W^u_{loc}(y)) = \inf_\gamma d_\gamma(W^u_{loc}(x), W^u_{loc}(y)).$$

Proposition 1. *Let us assume that f is a $C^{1+\alpha}$-diffeomorphism and consider some $C > 0$. Then, there exists $\Delta > 1$ and a C^1-neighborhood $\mathscr{U}$ of f with the following property.*

For any $C^{1+\alpha}$-diffeomorphism $g \in \mathscr{U}$ such that the α-Hölder norms of Dg, Dg^{-1} are bounded by $C > 0$, for any x, y in the hyperbolic continuation K_g of K we have:

$$d^+(W^u_{loc}(x), W^u_{loc}(y)) \leq \Delta . d^-(W^u_{loc}(x), W^u_{loc}(y)).$$

Proof. Let us consider two local unstable manifolds $W^u_{loc}(x)$ and $W^u_{loc}(y)$, two transverse graphs γ,γ' and the subintervals I, I' inside γ,γ' which connect $W^u_{loc}(x)$ and $W^u_{loc}(y)$. We have to show that $|I| \leq \Delta |I'|$, where $\Delta > 0$ is a uniform constant and $|I|$ is the length of the arc I.

Note that the backward iterates of the endpoints of I, I' are contained in R and tangent to the cone $\mathscr{C}^s$. The length of $f^{-k}(I), f^{-k}(I')$ increases exponentially as n growths, whereas they are tangent to the thiner cone fields $Df^{-n}.\mathscr{C}^s$.

Let J_1, J_2 be the arcs of $W^u_{loc}(x)$ and $W^u_{loc}(y)$ which connect γ,γ'. Their backward iterates are all contained in R and their length decreases exponentially. One deduces that there exists $N \geq 0$ such that $f^{-N}(I)$, $f^{-N}(I')$, $f^{-N}(J_1)$, $f^{-N}(J_2)$ have the same length (up to a multiplicative constant depending on β, λ). See figure 5.

Lemma 3. *There exists a foliation of the subrectangle of R bounded by $W^u_{loc}(f^{-N}(x))$ and $W^u_{loc}(f^{-N}(y))$, whose leaves are tangent to $\mathscr{C}^u$. Its holonomy defines a homeomorphism Π_N between the transverse arcs $f^{-N}(I)$ and $f^{-N}(I')$, which is bi-Lipschitz with a uniform constant $Lip(\Pi_N) > 0$.*

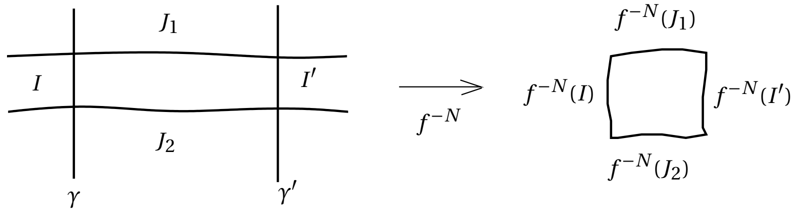

Figure 5: Image of the unstable strip by the iterate f^{-N}.

Proof. The two leaves $W^u_{loc}(f^{-N}(x))$ and $W^u_{loc}(f^{-N}(y))$ are the graphs of two β-Lipschitz functions φ_1, φ_2. For $u \in [1,2]$, the functions

$$\varphi_u := (2-u)\varphi_1 + (u-1)\varphi_2$$

are β-Lipschitz and their graphs define the leaves of the foliation. We denote by Π_N the holonomy map between $f^{-N}(I)$ and $f^{-N}(I')$.

Let $z \in f^{-N}(I)$ and $z' = \Pi_N(z)$. Let V, V' be two vertical segments which connect the local unstable manifolds of $f^{-n}(x), f^{-n}(y)$ and which contain z and z' respectively. The holonomy $\Pi_{V,V'}$ between V and V' is linear since the leaves φ_t have been obtained as barycenters. As a consequence, $\Pi_{V,V'}$ is a Lipschitz map whose constant $\frac{|V'|}{|V|}$ is bounded since $f^{-N}(I)$, $f^{-N}(I')$, $f^{-N}(J_1)$, $f^{-N}(J_2)$ have comparable lengths.

The holonomy map Π_N may be decomposed as

$$\Pi_N = \Pi_{V', f^{-N}(I')} \circ \Pi_{V,V'} \circ \Pi_{f^{-N}(I),V}.$$

The holonomy map $\Pi_{f^{-N}(I),V}$ fixes z. In a small neighborhood of z, the slopes of the leaves of the foliation are close to a constant with norm smaller than β and the slope of $f^{-N}(I)$ above the second coordinate is close to a constant with norm smaller than β. Hence the Lipschitz constant of $\Pi_{f^{-N}(I),V}$ at z is uniformly bounded. One argues similarly for $\Pi_{V',f^{-N}(I')}$. This gives the conclusion of the lemma. $\square$

From the previous lemma we obtain a bi-lipschitz homeomorphism $\Pi_{I,I'}$ between I and I' defined by

$$\Pi_{I,I'} = f^N \circ \Pi_N \circ f^{-N}.$$

Its Lipschitz constant at $\zeta \in I$ is bounded by

$$\|Df^N_{|f^{-N}(I')}(\Pi_N \circ f^{-N}(\zeta))\| \ \mathrm{Lip}(\Pi_N) \ \|Df^{-N}_{|I}(\zeta)\|.$$

We thus have to bound the following quantity for $z \in f^{-N}(I)$:

$$\frac{\|Df^N_{|f^{-N}(I')}(\Pi_N(z))\|}{\|Df^N_{|f^{-N}(I)}(z)\|} = \prod_{i=0}^{N-1} \frac{\|Df_{|f^{i-N}(I')}(f^i \circ \Pi_N(z))\|}{\|Df_{|f^{i-N}(I)}(f^i(z))\|}. \tag{1}$$

Lemma 4. *There exists $C' > 0$ and $\lambda' \in (0,1)$ such that if $p_i, p'_i \in (-\beta, \beta)$ denote the slope of $f^{i-N}(I), f^{i-N}(I')$ (above the vertical) at $f^i(z)$ and $f^i(\Pi_n(z))$, then*

$$|p'_i - p_i| \le C' {\lambda'}^{N-i}.$$

Proof. Let $L > 0$ bounds the length of local unstable manifolds. Since z and $\Pi_N(z)$ belong to a same leaf tangent to $\mathscr{C}^u$, the cone field criterion gives:

$$d(f^i(\Pi_N(z)), f^i(z)) \le \lambda^{N-i} d(f^N(\Pi_N(z)), f^N(z)) \le \lambda^{N-i}.L.$$

By the cone field criterion, if we consider two vectors at a same point x with slopes p, p', then their images Df^{-1} will have slopes $\bar{p}, \bar{p}'$ satisfying

$$|\bar{p}' - \bar{p}| \le \lambda^2 |p' - p|.$$

This gives

$$\begin{aligned}|p'_{i-1} - p_{i-1}| &\le |p'_{i-1} - \bar{p}| + |\bar{p} - p_{i-1}| \\ &\le \lambda^2 |p'_i - p_i| + C.d(f^i(z), f^i(\Pi_N(z)))^\alpha \\ &\le \lambda^2 |p'_i - p_i| + C.L^\alpha.\lambda^{(N-i)\alpha},\end{aligned}$$

where $\bar{p}$ is the slope of the image of the vectors with slope p_i by $Df^{-1}(f^i(\Pi_N(z)))$ (rather than by $Df^{-1}(f^i(z))$).

Since $|p'_N - p_N| \le 2\beta$, we thus obtain:

$$|p'_i - p_i| \le C' \lambda^{\alpha(N-i)},$$

where $C' > 0$ is a uniform constant. □

One deduces that there exists $C'' > 0$ uniform such that:

$$\log \frac{\|Df_{|f^{i-N}(I')}(f^i \circ \Pi_N(z))\|}{\|Df_{|f^{i-N}(I)}(f^i(z))\|} \le C'' d(f^i \circ \Pi_N(z), f^i(z)) + C'' \|p_i - p'_i\|,$$

which is exponentially small in $N - i$. One deduces that (1) is uniformly bounded by a constant $\Delta > 0$.

Any $C^{1+\alpha}$-diffeomorphism g which is C^1-close to f and such that the $C^{1+\alpha}$-norm of Dg, Dg^{-1} is bounded by C satisfies the same estimates. □

Remark 2. When the lengths of I, I', J_1, J_2 are small and the slopes of γ, γ' are close to a same constant, we get $|I'| \le \Delta |I|$ for a constant Δ arbitrarily close to 1.

4 Thickness

Let us consider a local stable manifold $W^s_{loc}(x)$ and an open connected component U of $W^s_{loc}(x) \setminus K$: it is the intersection of a component G with $W^s_{loc}(x)$. We are aimed to compare U with the sizes along $W^s_{loc}(x)$ of the adjacent rectangles B^+, B^- to G.

Definition 2. The *stable thickness* of K at U is

$$\tau(K, U) = \frac{\min(|B^- \cap W^s_{loc}(x)|, |B^+ \cap W^s_{loc}(x)|)}{|U|}.$$

The *stable thickness of* K is

$$\tau^s(K) = \inf_U \tau(K, U).$$

Let z be a point on an unstable leaf $W^u(x)$, $x \in K$, and let γ be a C^1-arc at z transverse to $W^u(x)$.

Lemma 5. *There exists $N \geq 1$ such that $f^{-N}(z)$ belongs to $W^u_{loc}(f^{-N}(x))$ and $T_{f^{-N}(z)} f^{-N}(\gamma)$ is contained in $\mathscr{C}^s$.*

Proof. Let us consider N_0 such that $f^{-N_0}(z)$ belongs to $W^u_{loc}(f^{-N_0}(x))$. For $N_1 \geq 0$ large enough, the complement of the cone $Df^{N_1}(f^{-(N_1+N_0)}(z)).\mathscr{C}^s$ is arbitrarily thin and contains $T_{f^{-N_0}(x)} W^u_{loc}(f^{-N_0}(x))$. It is thus transverse to $f^{N_0}(\gamma)$. One deduces that $Df^{-N}(\gamma)$ is tangent to $\mathscr{C}^s$ at $f^{-N}(z)$. □

Definition 3. The *local thickness at z* is:

$$\tau^s(z) = \liminf_{U_z \to z} \frac{\min(|B_z^+ \cap \gamma|, |B_z^- \cap \gamma|)}{|U_z|},$$

where U_z is the image by f^N of a connected component of $f^{-N}(\gamma) \setminus \mathscr{W}^u$ (i.e. there exists a component G such that $U_z = \gamma \cap f^N(G)$) and $B_z^\pm$ are the image by f^N of the rectangles $B^\pm$ that are adjacent to G.

The following shows that the definition does not depend from N or γ.

Proposition 2. *a) $\tau^s(K) > 0$.*
b) $\tau^s(z) = \tau^s(K)$.
c) The stable thickness $\tau^s(K_g)$ of the hyperbolic continuation K_g depends continuously on g for the C^1-topology on the space of $C^{1+\alpha}$ diffeomorphisms such that the α-norm of Dg, Dg^{-1} is bounded by $C > 0$.

We will use the following lemma.

Lemma 6. *For any $\Delta > 1$ and $C > 0$, there exists $\varepsilon > 0$ such that any diffeomorphism g that is C^1-close to f and such that the α-norm of Dg, Dg^{-1} is bounded by $C > 0$ has the following property.*

For any C^1-graph $\sigma = \{(\psi(t), t)\}$ which intersects $\mathscr{W}^u$, any $n \geq 0$ such that:

- *the slopes $\{D\psi(t)\}$ are contained in an interval smaller than ε and included in $(-\beta, \beta)$,*
- *the length of $f^{-n}(\sigma)$ is smaller than ε,*

and for any $y_1, y_2 \in \sigma$, we have

$$\Delta^{-1} < \frac{\|Df^n(y_1)_{|\sigma}\|}{\|Df^n(y_2)_{|\sigma}\|} < \Delta.$$

Proof. The proof is similar than for lemma 4. We denote p_i, p'_i the slope of $f^i(\sigma)$ at $f^i(y_1)$ and $f^i(y_2)$. The distance $d(f^i(y_1), f^i(y_2))$ is smaller than $\varepsilon.\lambda^i$

This gives

$$|p'_{i-1} - p_{i-1}| \leq \lambda^2 |p'_i - p_i| + C.\varepsilon^\alpha.\lambda^{i\alpha}.$$

We thus obtain:

$$|p_i' - p_i| \le \begin{cases} C'\varepsilon^\alpha\lambda^i & \text{if } \varepsilon^\alpha\lambda^i \ge \varepsilon\lambda^{2(N-i)}, \\ C'\varepsilon\lambda^{2(N-i)} & \text{otherwise,} \end{cases}$$

where $C' > 0$ is a uniform constant.

One deduces that

$$\log \frac{\|Df_{|f^{-i}(\gamma)}(f^i(y_1))\|}{\|Df_{|f^{-i}(\gamma)}(f^i(y_2))\|}$$

is smaller than $C''\varepsilon^\alpha(\lambda'')^{\min(i,N-i)}$, where $C'' > 0$ and $\lambda'' \in (0,1)$ are uniform constants. One concludes as for lemma 4. □

Proof of proposition 2. Let us consider x, U, $B^\pm$ as in the definition of $\tau(K,U)$. Note that one can assume that the point x belongs to the boundary of U. The diffeomorphism f^n sends $G_0 \cap W^s_{loc}(f^{-n}(x))$, $B_0^\pm \cap W^s_{loc}(f^{-n}(x))$ on U and $B^\pm \cap W^s_{loc}(x)$. This shows that there exists $y_1 \in B_0^+ \cap W^s_{loc}(f^{-n}(x))$ and $y_2 \in G_0 \cap W^s_{loc}(f^{-n}(x))$ such that

$$\frac{|B^+ \cap W^s_{loc}(x)|}{|U|} = \frac{|B_0^+ \cap W^s_{loc}(f^{-n}(x))|}{|G_0 \cap W^s_{loc}(f^{-n}(x))|} \cdot \frac{\|Df^n_{|W^s_{loc}(x)}(y_1)\|}{\|Df^n_{|W^s_{loc}(x)}(y_2)\|}.$$

From the previous lemma, the right hand is uniformly bounded from below. A similar property holds for $\frac{|B^- \cap W^s_{loc}(x)|}{|U|}$. This gives the property a).

Let us consider z, N, γ as in the definition of local thickness. Since $\|Df^{-N}_{|\gamma}(\zeta)\|$ is arbitrarily close to $\|Df^{-N}_{|\gamma}(z)\|$ as ζ is close to z, the local thickness at z (along γ) and at $f^{-N}(z)$ (along $f^{-N}(\gamma)$) coincide. One can thus assume in the following that $N = 0$. Let us fix $\delta > 0$. There exists ℓ such that for any interval U of generation larger than ℓ in definition 2, we have $\tau(U,K) > \tau^s(K) - \delta$.

Let us consider the ratio $|B_z^+ \cap \gamma|/|U_z|$ which appears in the definition of the local thickness. As before there exists $n \ge 0$ such that $f^{-n}(U_z)$ and $f^{-n}(B_z^+)$ coincide with the intersections of $f^{-n}(\gamma)$ with G of generation ℓ and B^+ adjacent to G. One deduces that there exists $y_1 \in B^+ \cap f^{-n}(\gamma)$ and $y_2 \in f^{-n}(U_z)$ such that denoting $\sigma = f^{-n}(\gamma)$ we have:

$$\frac{|B_z^+ \cap \gamma|}{|U_z|} = \frac{|B^+ \cap \sigma|}{|G \cap \sigma|} \cdot \frac{\|Df^n_{|\gamma}(y_1)\|}{\|Df^n_{|\gamma}(y_2)\|} \ge \Delta^{-1}(\tau^s(K) - \delta).$$

Note that Δ is arbitrarily close to 1 as U_z converges to z. Arguing similarly with $|B_z^- \cap \gamma|/|U_z|$ one deduces $\tau^s(z) \ge \tau^s(K)$.

Let us prove the other inequality. One considers $\bar{x} \in K$, a component $\bar{U} = G \cap W^s_{loc}(\bar{x})$ of $W^s_{loc}(\bar{x}) \setminus \mathscr{W}^u$ which is small, and the interval $B^\pm \cap W^s_{loc}(\bar{x})$ adjacent. One can assume that $\bar{x}$ is a boundary point of $\bar{U}$. Since K is transitive, there exists $y \in K$ close to x and negative iterates $f^{-n}(y)$ arbitrarily close to $\bar{x}$. One can replace $\bar{x}$ by the intersection point between $W^u_{loc}(\bar{x})$ and $W^s_{loc}(f^{-n}(y))$: this changes the ratios $|B^\pm \cap W^s_{loc}(\bar{x})|/\bar{U}$ only a little. We are now reduced to the case there exist iterates $f^n(\bar{x})$ arbitrarily close to x with ℓ arbitrarily large. One deduces that there exists $B_n^\pm$ and G_n close to $W^u_{loc}(f^n(\bar{x}))$ which are mapped by f^{-n} inside $B^\pm$ and G. As before we have

$$\frac{|B_n^\pm \cap \gamma|}{|G_n \cap \gamma|} \le \Delta \frac{|B^\pm \cap \sigma|}{|G \cap f^{-n}(\sigma)|},$$

where σ is the connected component of $f^{-n}(\gamma) \cap R$ containing $f^{-n}(z)$. Arguing as in the proof of the inclination lemma, we deduces that σ and $W^s_{loc}(\bar{x})$ are arbitrarily close for the C^1-topology when n is large. One deduces also that Δ is arbitrarily close to 1. This proves $\tau^s(z) \leq \tau^s(K)$ and concludes the proof of the b).

The lemma 6 (and the arguments above) show that

$$\tau^s_n(K) := \inf_{U \text{ of generation } n} \tau(K, U)$$

for n large is close to $\tau^s(K)$, uniformly in the diffeomorphism g that is C^1-close to f and such that the C^α-norm of Dg , Dg^{-1} is bounded by $C > 0$. Since $\tau^s_n(K_g)$ depends continuously on g for the C^1-topology, one gets the property c). □

5 Robust tangencies

Definition 4. K has a *homoclinic tangency* if there exists a periodic orbit $O \subset K$ such that $W^s(O)$ and $W^u(O)$ have a non-transverse intersection.

K has a *generalized homoclinic tangency* if there exist $x, y \in K$ such that $W^s(x)$ and $W^u(y)$ have a non-transverse intersection.

K has a C^r*-robust generalized homoclinic tangency* if there exists a neighborhood $\mathcal{U}$ of f in $\text{Diff}^r(M)$ such that any $g \in \mathcal{U}$ has a generalized homoclinic tangency associated to the hyperbolic continuation K_g of K.

Theorem 2. *For any C^2 diffeomorphism f with a horseshoe K exhibiting a homoclinic tangency and satisfying $\tau^s(K) \cdot \tau^u(K) > 1$, then there exists a diffeomorphism g close to f in* $\text{Diff}^2(M)$ *which exhibits a C^2-robust generalized homoclinic tangency.*

6 Proof of Theorem 2

a) Preparation. Let us consider $x, y \in K$ and $z \in W^s(x) \cap W^u(y)$ a non-transverse intersection outside R.

We choose local coordinates (s, t) near the point z such that:

- $W^s(x)$ coincides locally with the graph $\{t = 0\}$,
- $W^u(x)$ coincides locally with the graph of a function $\phi \geq 0$ which take the values 0 only at 0.

This last property is obtained after a small C^k-perturbation in a neighborhood of $f^{-1}(z)$ (so that K, $W^s_{loc}(K)$, $W^u_{loc}(K)$ are not modified).

Let $\gamma = \{s = 0\}$ be a small vertical transversal through z. By lemma 5, there exists $N \geq 1$ such that $f^N(\gamma)$ is a graph $\{(s, \varphi(s)\}$ and $f^{-N}(\gamma)$ is a graph $\{(\psi(t), t)\}$ where φ, ψ are β-Lipschitz.

We will study the transverse structure of $f^N(\mathcal{W}^u)$ and $f^{-N}(\mathcal{W}^s)$ near z, using the results proved in the previous sections:

– If one chooses two small vertical transversals $\{s = s_1\}$ and $\{s = s_2\}$, close to z, the holonomy of the local stable and local unstable laminations $f^N(\mathcal{W}^u)$ and $f^{-N}(\mathcal{W}^s)$ are Lipschitzian with a constant $\Delta > 1$ close to 1.

– Reducing γ if necessary, the norm of the derivatives $\|Df^N(\zeta)_{|\gamma}\|$ and $\|Df^{-N}(\zeta)_{|\gamma}\|$ are almost constant for $\zeta \in \gamma$.

– If $\varepsilon > 0$ has been fix and γ is small enough, one deduces that for any component G^u of $\mathrm{Int}(R) \setminus \mathcal{W}^u$ and any adjacent rectangle R^u, one has:

$$\frac{|\gamma \cap f^N(B^u)|}{|\gamma \cap f^N(G^u)|} \geq (1-\varepsilon).\tau^u.$$

– A similar estimate for the components G^s of $\mathrm{Int}(R) \setminus \mathcal{W}^s$. By the Lipschitz control of the holonomies, these estimates are still valid for verticals $\{s = s_0\}$ close to γ.

b) Robust overlapping. We modify f again near $f^{-1}(z)$, in order to modify $f^N(\mathcal{W}^u)$ in a neighborhood of z without modifying $f^{-N}(\mathcal{W}^s)$.

The two tangent leaves $W^s(x)$ and $W^u(y)$ are limit in a neighborhood of z of leaves $G^{s,\pm}$ and $G^{u,\pm}$ respectively: G^s is the image by f^{-N} of a components of $\mathrm{Int}(R) \setminus \mathcal{W}^s$ and $G^{s,\pm}$ are its boundary leaves; similarly G^u is the image by f^N of a components of $\mathrm{Int}(R) \setminus \mathcal{W}^u$ and $G^{u,\pm}$ are its boundary leaves. See figure 6.

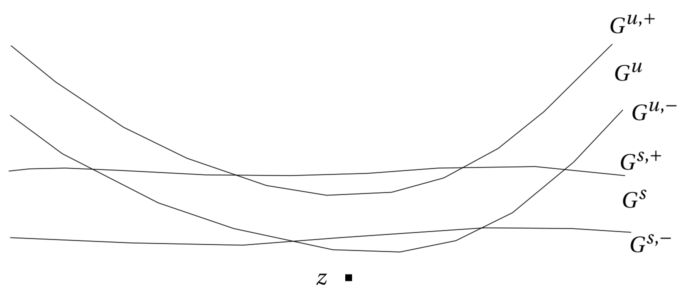

Figure 6: The overlapping between the stable and unstable laminations.

The perturbation is done in order that the following assumptions are satisfied in a neighborhood $(-\varepsilon,\varepsilon)^2$ of z, for two components G^u, G^s and for any diffeomorphism g in an open set $\mathcal{U} \subset \mathrm{Diff}^k(M)$ close to the initial diffeomorphism f:

(O1) $G^{u,-}$ meets $G^{s,-}$,

(O2) $G^{u,+}$ meets $G^{s,+}$,

(O3) $G^{u,+}$ does not meet $G^{s,-}$.

(And necessarly, $G^{u,-}$ meets $G^{s,+}$.)

We have to prove that any diffeomorphism $g \in \mathcal{U}$ has a generalized homoclinic tangency.

c) The induction.

Lemma 7. *Let us assume that the components G^s, G^u satisfy the assumptions (O1-3) of the previous section for a diffeomorphism g. Then either g has a generalized homoclinic tangency, or there exists components $\widehat{G}^s, \widehat{G}^u$ satisfying also assumptions (O1-3) such that:*

- *either $G^s = \widehat{G}^s$ and the generation of $\widehat{G}^u$ is larger than the one of G^u,*
- *or $G^u = \widehat{G}^u$ and the generation of $\widehat{G}^s$ is larger than the one of G^s.*

Proof. Let us introduce two rectangles B^s, B^u adjacent to G^s, G^u and such that $B^{s,+} = G^{s,-}$ and $B^{u,-} = G^{u,+}$ (see figure 7).

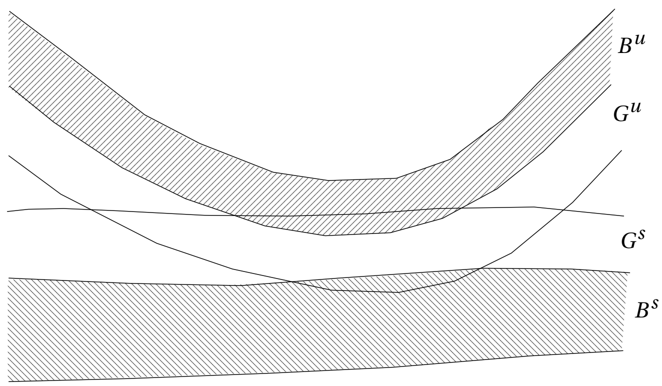

Figure 7: Overlapping between components and rectangles (1).

Claim 1. One of the following properties holds:

(a) $B^{s,-} < G^{u,-}$,

(b) $B^{u,+} > G^{s,+}$.

Proof. If one supposes by contradiction that these properties are not satisfied, there exists a vertical γ_1 close to z such that

$$(\gamma_1 \cap G^u) \supset (\gamma_1 \cap B^s),$$

and there exists a vertical γ_2 close to z such that

$$(\gamma_2 \cap G^s) \supset (\gamma_2 \cap B^u).$$

One deduces:

$$|\gamma_2 \cap G^u| \simeq |\gamma_1 \cap G^u| \geq |\gamma_1 \cap B^s| \simeq |\gamma_2 \cap B^s| \geq (1-\varepsilon)\tau^s(K)|\gamma_2 \cap G^s|.$$

Similarly:

$$|\gamma_2 \cap G^s| \geq |\gamma_2 \cap B^u| \geq (1-\varepsilon)\tau^u(K)|\gamma_2 \cap G^u|.$$

This gives

$$|\gamma_2 \cap G^u| \geq (1-\varepsilon)^2\tau^s(K)\tau^u(K)|\gamma_2 \cap G^u|,$$

which is a contradiction since $\tau^s(K)\tau^u(K) > 1$. □

Let us assume for instance that the first case of the claim $B^{s,-} < G^{u,-}$ holds. Among the leaves Σ of $f^{-N}(\mathscr{W}^s)$ which satisfy $\Sigma \leq G^{u,-}$, we choose the last one, i.e. minimizing the distance to $G^{u,-}$. Two cases are possible:

- either $\Sigma \leq G^{u,-}$ intersect each other: we have found a generalized homoclinic tangency,
- or $\Sigma < G^{u,-}$: in this case $\Sigma = \widetilde{G}^{s,-}$ where $\widetilde{G}^s$ is the image by f^{-N} of a new component of $\mathrm{Int}(R) \setminus \mathscr{W}^s$; by definition of Σ, the leaf $\widetilde{G}^{s,+}$ meets $G^{u,-}$. We also define $\widetilde{G}^u = G^u$.

If there is no generalized homoclinic tangency for g, we have found a new pair of components $\widetilde{G}^s, \widetilde{G}^u$ satisfying (see figure 8):

($\widetilde{O}1$) $\widetilde{G}^{u,+}$ does not meet $\widetilde{G}^{s,+}$ (and $\widetilde{G}^{s,-}$),

($\widetilde{O}2$) $\widetilde{G}^{u,-}$ meets $\widetilde{G}^{s,+}$,

($\widetilde{O}3$) $\widetilde{G}^{u,-}$ does not meet $\widetilde{G}^{s,-}$.

Note that $\widetilde{G}^s$ is contained in B^s, hence has a generation larger than the generation of G^s.

If the second case $B^{u,+} > G^{s,+}$ of the claim holds, the same argument gives the same conclusion (but this time $\widetilde{G}^s = G^s$ and the generation of $\widetilde{G}^u$ is larger than the generation of G^s).

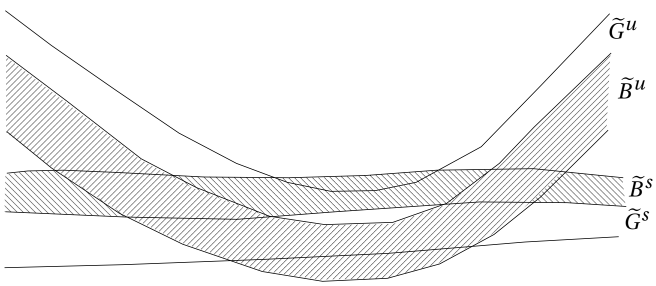

Figure 8: Overlapping between components and rectangles (2).

Let us consider the rectangles $\widetilde{B}^s, \widetilde{B}^u$ adjacent to $\widetilde{G}^s, \widetilde{G}^u$ such that $\widetilde{B}^{s,-} = \widetilde{U}^{s,+}$ and $\widetilde{B}^{u,+} = \widetilde{U}^{u,-}$. As before we prove:

Claim 2. One of the following properties holds:

(a) $\widetilde{B}^{s,+}$ meets $\widetilde{G}^{u,+}$,

(b) $\widetilde{B}^{u,-}$ meets $\widetilde{G}^{s,-}$.

We assume for instance that $\widetilde{B}^{u,-}$ meets $\widetilde{G}^{s,-}$ as in the first case of the claim. Among the leaves $\widetilde{\Sigma}$ of $f^N(\mathscr{W}^u)$ which meet $\widetilde{G}^{s,-}$, we choose as before the last one. Two cases are possible:

- either $\widetilde{\Sigma} \leq \widetilde{G}^{s,-}$: we have found a generalized homoclinic tangency,
- or $\widetilde{\Sigma} = \widehat{G}^{u,-}$ where $\widehat{G}^u$ is the image by f^N of a new component of $\mathrm{Int}(R) \setminus \mathscr{W}^u$; by definition of $\widetilde{\Sigma}$, the leaf $\widehat{G}^{u,+}$ is disjoint from $\widetilde{G}^{s,-}$. We also define $\widehat{G}^s = \widetilde{G}^s$.

If there is no generalized homoclinic tangency for g, we have found a new pair of components $\widehat{G}^s, \widehat{G}^u$ satisfying (O1-3). If $\widetilde{B}^{s,+}$ meets $\widetilde{G}^{u,+}$ as in the second case of the claim, we obtain the same conclusion by the same argument.

By construction,

$$\text{generation}(G^s) \leq \text{generation}(\widehat{G}^s),$$

$$\text{generation}(G^u) \leq \text{generation}(\widehat{G}^u),$$

and at least one of these inequality is strict. This ends the proof of the lemma 7. □

d) Robust tangency. From lemma 7 we obtain a sequence of components G^s_n, G^u_n such that:

- $\max(\text{generation}(G^s_n), \text{generation}(G^u_n)) \underset{n\to\infty}{\longrightarrow} \infty$,
- $G^s_n \cap G^u_n \neq \emptyset$,
- $G^{s,-}_n \cap G^{u,+} = \emptyset$.

If one assumes for instance that $\text{generation}(G^s_n) \underset{n\to\infty}{\longrightarrow} \infty$, one deduces that $G^{s,-}_n$ and $G^{s,+}_n$ converge toward a same leaf Σ^s of $f^{-N}(\mathscr{W}^s)$. One can also assume that $G^{u,+}_n$ converge toward a leaf Σ^u of $f^N(\mathscr{W}^u)$.

We obtain:

- Since $G^{u,+}_n \geq G^{s,-}_n$ we have $\Sigma^u \geq \Sigma^s$.
- Since $G^{u,+}_n$ meets $G^{s,+}_n$, we have that Σ^u meets Σ^s.

One deduces that Σ^u and Σ^s have a non-transverse intersection, hence g has a generalized homoclinic tangency. This ends the proof of the theorem 2.

7 Infinitely many sinks or sources

The following proposition ends the proof of the Newhouse's theorem 1 stated at the beginning of the chapter.

Proposition 3. *Let us consider an open set $\mathscr{U} \subset \mathrm{Diff}^k(M)$ and a transit if hyperbolic set which admits a hyperbolic continuation K_g for each $g \in \mathscr{U}$. Let us assume furthermore that K_g has a generalized homoclinic tangency for each $g \in \mathscr{U}$. Then, there exists a dense G_δ-subset $\mathscr{G} \subset \mathscr{U}$ such that any $g \in \mathscr{G}$ has infinitely many sinks or infinitely many sources.*

Proof. For $N \geq 1$, we define

$$\mathscr{G}_N = \{g \in \mathscr{U},\ g \text{ has at least } N \text{ sinks or sources}\}.$$

It is an open set. If it is dense for each $N \geq 1$, then the set $\mathscr{G} = \cap \mathscr{G}_N$ is a dense G_δ-set as announced. The proof of the proposition is thus a consequence of the following. □

Proposition 4. *For any diffeomorphism $f \in \mathrm{Diff}^k(M)$, for any transitive hyperbolic set K with a generalized homoclinic tangency and for any neighborhood U of K, there exists g arbitrarily close to f in $\mathrm{Diff}^k(M)$ with a sink or a source whose orbit meets U.*

The next lemma shows that one can assume that the diffeomorphism f in the previous proposition has a homoclinic tangency.

Lemma 8. *For any diffeomorphism $f \in \mathrm{Diff}^k(M)$, for any transitive hyperbolic set K with a generalized homoclinic tangency and for any neighborhood U of K, there exists g arbitrarily close to f which exhibits a homoclinic tangency associated to a periodic orbit contained in U.*

Proof. Let $x, y \in K$ and $N \geq 1$ such that $f^{-N}(W^s_{loc}(x))$ and $f^{-N}(W^u_{loc}(y))$ have a non-transverse intersection z. Since K is transitive, there exists a hyperbolic periodic orbit $O \subset U$ with two points p, q close to x and y respectively. Consequently, the leaves $f^{-N}(W^s_{loc}(p))$ and $f^{-N}(W^s_{loc}(x))$ are close to each other, the leaves $f^N(W^u_{loc}(q))$ and $f^N(W^s_{loc}(y))$ are close to each other.

A perturbation of f in a small neighborhood of $f^{-1}(z)$ will produce a tangency between $f^{-N}(W^s_{loc}(p))$ and $f^{-N}(W^u_{loc}(q))$. □

The end of the section is devoted to the proof of the second proposition.

Proof of proposition 4. Let O be a periodic orbit. In order to simplify the presentation, one will assume that O is a fixed point p. We choose some coordinates (s,t) near $p = (0,0)$ such that:

- $W^s_{loc}(p) = \{(0,t), t \in [-2,2]\}$ and $W^u_{loc}(p) = \{(s,0), s \in [-2,2]\}$,
- there exists $N \geq 1$ such that $f^N(1,0) = (0,1)$ and $f^N(W^u_{loc}(p))$ is tangent to $W^s_{loc}(p)$ at $(0,1)$ (i.e. $Df^N(1,0)$ sends the horizontal direction on the vertical one),
- $Df^N(1,0)$ sends the vertical direction on the horizontal one.

Let us consider a small vertical segment γ through $(1,0)$, a horizontal segment σ through $(0,1)$, and a large integer $m \geq 1$. The segments $f^{-m}(\gamma)$ and σ intersect at a point q. See figure 9.

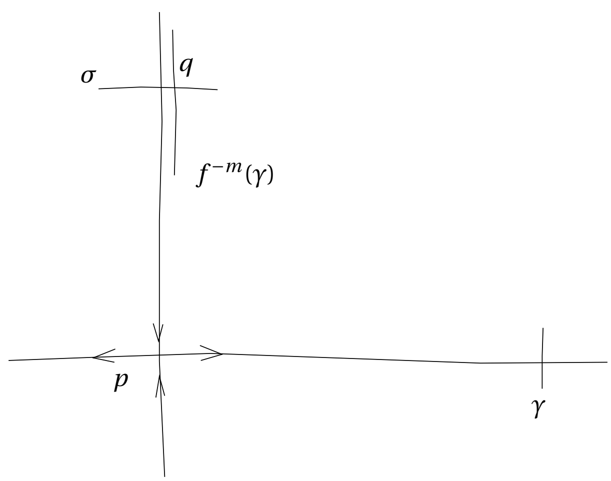

Figure 9: The homoclinic return giving birth to sinks or sources.

From the inclination lemma, $Df^m(f^{-m}(q))$ sends the horizontal direction to a direction close to the vertical and sends a direction close to the vertical to the vertical direction. Consequently, there exists a C^k-perturbation g of f supported near q and $f^{-1}(q)$ such that

– q is $m+N$-periodic for g,

– $Dg^{m+N}(q)$ exchange horizontal and vertical directions, and its determinant is different from ± 1: its eigenvalues have the same modulus, which is different from 1.

This proves that q is a sink or a source (depending if the determinant has modulus larger or smaller than 1). □

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