Last passage percolation on complete directed acyclic graphs: a study of the time constant

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Introduction

Consider the complete directed graph with vertex set $V = \{1, \dots, n\}$ and edge set

$$E = \{ (i, j) \mid 1 \le i < j \le n \}.$$

To each edge $(i, j) \in E$, assign a weight $X_{i,j}$ in $\{-\infty\} \cup \mathbb{R}$. Define the weight of a path $\pi = (i_1, \dots, i_k) \in \{1, \dots, n\}$ as the sum of the edges weights:

$$w_\pi := X_{i_1,i_2} + \cdots + X_{i_{k-1},i_k}.$$

The quantity of interest for the problem of last passage percolation is the **greatest weight of a directed path starting at** 1 **and ending at** n. More precisely, set

$$W_n := \max\{ w_{(i_1,\dots,i_k)} \mid k \in \mathbb{N},\ 1 = i_1 < i_2 < \cdots < i_k = n \}$$

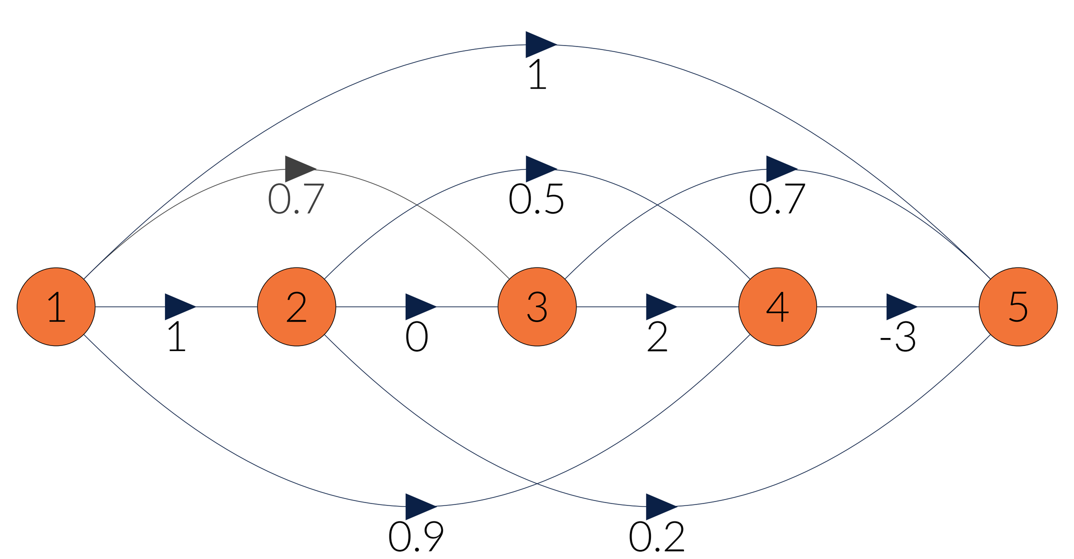

Figure 1. Illustration of the last passage percolation problem on the complete graph with 5 vertices. With the weights given below each edge of this graph, $W_1 = 0$, $W_2 = W_3 = 1$, $W_4 = 3$, and $W_5 = 1.7$. A heaviest path from 1 to 5 is $\pi = (1, 2, 3, 5)$.

Consider ν a probability distribution on $\{-\infty\} \cup \mathbb{R}$ and assume that $(X_{i,j})_{1 \le i < j \le n}$ are i.i.d with distribution ν. The following sub-additive property holds for all $1 \le m \le n$:

$$W_n \ge W_m + W_{m,n}.$$

Therefore, by Kingman's sub-additive theorem there exists a constant $C(\nu) \in \mathbb{R} \cup \{\pm\infty\}$, such that

$$\frac{W_n}{n} \xrightarrow[n\to\infty]{a.s.} C(\nu).$$

The constant $C(\nu)$ is called the **time constant**.

Basic properties of the time constant

We focus on the case where the support of ν is upper-bounded by some constant M. In this case,

$$C(\nu) \le M.$$

When $M \le 0$, it is easy to show that $C(\nu) = 0$.

Rescaling property

Assume that X has distribution ν and consider $\lambda > 0$. Let ν_λ denote the distribution of the random variable λX. It holds that

$$C(\nu_\lambda) = \lambda \cdot C(\nu).$$

The particular case of Barak-Erdős graphs

In terms of heaviest paths, $X_{i,j} = -\infty$ is equivalent to removing the edge (i, j) from the graph.

Therefore, if ν is supported by 1 and $-\infty$, the problem consists in studying the length of a longest path in a directed acyclic version of an Erdős-Rényi random graph: a Barak-Erdős graph.

In [3], Mallein and Ramassamy proved that the time constant for Barak-Erdős graphs is an analytic function in the probability of an edge to be in the graph. The two results on the analyticity of the time constant presented here extend that result:

Main results in the general case

Theorem (T. 23', Analyticity in the essential supremum's probability)

Consider $M > 0$ and a probability distribution μ on $[-\infty, M)$. Denote by δ_M the Dirac distribution at M. Then, the following map is **analytic** on $(0, 1]$:

$$p \mapsto C((1-p)\mu + p\delta_M)$$

Theorem (T. 23', Analyticity in several variables)

For $N \in \mathbb{N}$ and $a_1 > a_2 > \cdots > a_N \ge -\infty$, the map

$$(p_2, \dots, p_N) \mapsto C\left(\sum_{i=1}^{N} p_i \delta_{a_i}\right)$$

is **analytic** on $\{ (p_2, \dots, p_N) \in [0,1]^N \mid 0 \le p_2 + \cdots + p_N < 1 \}$, where $p_1 = 1 - (p_2 + \cdots + p_N)$.

Theorem (T. 23', Strict monotonicity)

The map $\nu \mapsto C(\nu)$ is **strictly increasing** for the stochastic order on the set of measures ν such that $\nu([-\infty, M]) = 1$ and $\nu(\{M\}) > 0$.

Theorem (T. 23', Continuity)

Consider d_{LP} the Lévy-Prohkorov metric on the set of probability measures on $\mathbb{R} \cup \{-\infty\}$. For any probability measure ν, set M_ν to be the smallest real number such that $\nu([-\infty, M_\nu]) = 1$. Consider the metric d defined by

$$d(\nu, \nu') = \max(d_{LP}(\nu, \nu'), |M_\nu - M_{\nu'}|).$$

Then, $\nu \mapsto C(\nu)$ is **continuous** for the metric d on the set of probability measures ν such that $M_\nu < +\infty$.

Interpretation of those results

- The strict monotonicity tells us something about the geometry of the heaviest paths: because the time constant is strictly monotonic, edge weights with arbitrarily large negative values in the support of ν may contribute to a heaviest path.
- In statistical physics, a breach of analyticity or continuity usually pinpoints a phase transition in the model. Here, we proved that there is no phase transition for the measures we considered.

The max growth system (MGS)

Coupling last passage percolation on the complete graph and the MGS

There exists a coupling between last passage percolation on the complete graph and an interacting particle system called the **max growth system (MGS)**. It was introduced in [1] by Foss, Konstantopoulos, Mallein and Ramassamy.

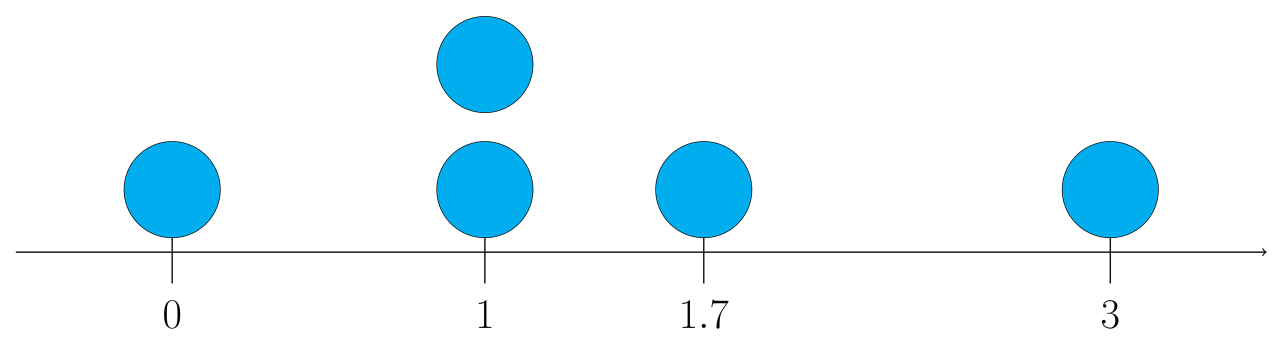

Figure 2. Illustration of the coupling between the MGS and last-passage percolation (see Figure 1 for the corresponding graph). The MGS is constructed as follows: the MGS at time n consists in n unlabeled particles. The position of the particles at time n are given by $(W_i)_{i \in \{1,\dots,n\}}$. The weight of a heaviest path W_n has the same asymptotic as the maximal position of a particle in this system at time n.

Renovation events

A crucial property of the MGS consists in the existence of "renovation events": A **renovation event** is a time t at which the positions of the particles added at a time $t' \ge t$ no longer depends on the positions of all the particles previously added, but only on the position of the particle at maximal position at time t.

The limiting density of renovation events is positive : $\prod_{k \ge 1}(1 - q^k)) > 0$, where $q = 1 - \nu(\{M\})$.

Result for measures supported by two positive atoms

Theorem (T. 23', Rationality)

For all $0 < m < M$, the following map is a rational function on $[0, 1]$:

$$p \mapsto C((1-p)\delta_m + p\delta_M).$$

Elements of proof for the case of two positive atoms

- The max growth system becomes a Markov chain on a finite state space when $Mm^{-1} \in \mathbb{N}$. In this case, a heaviest path contains no edge (i, j) with $j - i > \lfloor Mm^{-1} \rfloor$.
- Using a result from Foss, Konstantopoulos and Pyatkin [2], extend this result to all $0 < m \le M$.

References

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[3] Bastien Mallein and Sanjay Ramassamy. Two-sided infinite-bin models and analyticity for Barak-Erdős graphs. *Bernoulli*, 25(4B):3479–3495, 2019.

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