Review of syntomic cohomology

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Introduction: This paper reviews the recent foundations of syntomic cohomology. Syntomic cohomology is a p-adic cohomology theory developed by Bhatt, Morrow and Scholze in [BMS19]. Thanks to its deep relations with several important subjects in p-adic arithmetic geometry (integral p-adic Hodge theory [BMS18], [BMS19], the prismatic theory of [BS22], and motivic cohomology [BMS19], [AMMN20], [CMM21]), this will surely play a major role in the future developments of the field. While presenting syntomic cohomology, we encounter some of the main objects of p-adic Hodge theory, such as Fontaine's period rings, perfectoid rings, and p-adic cohomology theories. Finally, we present fundational results concerning syntomic cohomology, and motivate its expected (p-adic étale) motivic nature.

Chapter 1

Introduction

We fix a prime number p for the rest of the text.

1.1 General introduction

p-adic Hodge theory began officially in 1967, with Tate's foundational article [Tat67] on p-divisible groups. Since then, the variety of techniques involved has grown up as independent branches of mathematics, but Tate's original results still provide the main sources of inspiration for the new developments in the theory.

Let K be a complete discrete valuation field of characteristic 0, with perfect residue field k of characteristic p (for instance, think about a finite extension of $\mathbb{Q}_p$). Let $\mathcal{O}_K$ be the ring of integers of K, and C the p-completion of an algebraic closure $\overline{K}$ of K. As an algebraic replacement for singular cohomology, Grothendieck defined, for proper smooth varieties X over $\mathcal{O}_K$, the étale cohomology $R\Gamma_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$ (which is a cochain complex in the derived category $\mathcal{D}(\mathbb{Z}_l)$). The étale cohomology groups $\mathrm{H}^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell)$ are $\mathbb{Z}_\ell$-modules, and behave as singular cohomology groups do for analytic varieties over $\mathbb{C}$; in particular they compare to the algebraic de Rham cohomology groups of X. However, étale cohomology does not satisfy similar good behaviour when applied to a variety of characteristic p (such as X_k, the reduction of X modulo p), and when the prime number ℓ chosen for the coefficient ring is equal to p. That is, étale cohomology is not the good replacement for singular cohomology in the "p-adic context" (as opposed to the "ℓ-adic context", where $\ell \neq p$). To make up for this flaw, Grothendieck defined crystalline cohomology $R\Gamma_{\text{crys}}(X_k/W(k))$, which is defined over the ring of Witt vectors $W(k)$. The subject of p-adic Hodge theory can be described, as we explain now, as the comparison of these three p-adic cohomologies: étale cohomology $\mathrm{H}^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$, crystalline cohomology $\mathrm{H}^*_{\text{crys}}(X_k/W(k))$ and de Rham cohomology $\mathrm{H}^*_{\text{dR}}(X/\mathcal{O}_K)$.

For A an abelian variety defined over the valuation ring $\mathcal{O}_K$, Tate proved that for each integer $k \leqslant 2\dim(A)$, there exists a natural Galois-equivariant isomorphism

$$\mathrm{H}^k_{\text{ét}}(A_{\overline{K}}, C) \cong \bigoplus_{i+j=k} \mathrm{H}^j(A, \Omega^i_A) \otimes_K C(-i).$$

By Galois-equivariant, we mean that the absolute Galois group G_K of K acts on the two sides of the isomorphism, and that the action is compatible with this isomorphism. More precisely, the action on the left-hand side is induced by the natural action on the base change $A_{\overline{K}} := A \times_K \overline{K}$[1],

[1] This action, defined by functoriality of the construction, is sometimes called "transport de structure".

and it is defined on the right-hand side as the trivial action on $\mathrm{H}^j(A, \Omega^i_A)$, and the natural action on $C(-i)$. Here $C(i)$ is defined, for every integer $i \in \mathbb{Z}$, as $C(i) := \mathbb{Q}_p(i) \otimes_{\mathbb{Q}_p} C$, where $\mathbb{Q}_p(i)$ is the i-th tensor power of the one-dimensional p-adic representation $\mathbb{Q}_p(1)$, on which G_K acts as the cyclotomic character. Tate conjectured that such an equivariant decomposition should exist for any smooth projective variety defined over K. As a comparison, we know in the archimedean context that a similar isomorphism exists for any smooth and proper algebraic variety X over $\mathbb{C}$ (the usual complex numbers), and compares the singular and de Rham cohomologies of X:

$$\mathrm{H}^*(X^{\mathrm{an}}, \mathbb{Z}) \otimes_Z \mathbb{C} \cong \mathrm{H}^*_{\mathrm{dR}}(X/\mathbb{C}). \tag{1.1}$$

The point of these two isomorphisms (either in the archimedean, or in non-archimedean context, respectively) is that both sides contribute complementary informations on X, and that one can not reduce one onto the other.

Tate's conjecture, known as the Hodge–Tate conjecture, was first solved in some cases by Fontaine, who introduced in [Fon82] the graded ring $B_{\mathrm{HT}} := \bigoplus_{i \in \mathbb{Z}} C(i)$. This ring B_{HT} (named after Hodge and Tate), later called a "period ring" due to its relation to algebraic periods, provides a way to rewrite Tate's decomposition as the following isomorphism of graded K-vector spaces

$$(\mathrm{H}^k_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{G_K} \cong \bigoplus_{i+j=k} \mathrm{H}^j(A, \Omega^i_A).$$

Remark that we do not ask anymore for the isomorphism to be Galois-equivariant, since we took the fixed points for the Galois action on the left-hand side. This new way of writing the Tate's decomposition isomorphism was in fact extremely fruitful, and led to the proof of Tate's conjecture in the general case by Faltings, in 1988. The proof of Faltings basically relies on the construction of an intermediate cohomology which takes values in C-vector spaces, and then comparing each of the two sides of the isomorphism to this intermediate cohomology.

More recently, and after quite some years of active research and developments, this idea was taken much further, when Bhatt, Morrow and Scholze defined in [BMS18] and [BMS19] a unifying cohomology theory $R\Gamma_{\mathfrak{S}}(X)$ which specialises in étale, de Rham, and crystalline cohomologies. This new p-adic cohomology theory is called Breuil-Kisin cohomology, and, by using its comparison results to the other usual p-adic cohomologies, allows one to prove directly the usual conjectures of p-adic Hodge theory, such as the Hodge-Tate conjecture. In fact, this new Breuil-Kisin cohomology not only finds back some already known results in p-adic Hodge theory, but provides completely new types of results, for instance interpreting geometrically some phenonema about the (p-)torsion classes of our cohomology groups. Remark that this is not anodyne at all: in the archimedean context, the isomorphism (1.1), comparing singular and de Rham cohomologies, is constructed by integrating some differential forms over $\mathbb{C}$, and thus does not detect any integral properties of the singular cohomology $\mathrm{H}^*(X^{\mathrm{an}}, \mathbb{Z})$. Here, the results of [BMS18] and [BMS19] imply, via the Grotendieck comparison isomorphism between algebraic and de Rham cohomologies, some precise results about the torsion classes of $\mathrm{H}^*(X^{\mathrm{an}}, \mathbb{Z})$.

This is where syntomic cohomology enters the scene. To define the Breuil-Kisin cohomology, a new Grothendieck topology, called the quasisyntomic topology, is defined in [BMS19]. It is inspired by the syntomic topology introduced by Fontaine and Messing in the 1980's (which in particular explains the name). On this topology (which is formally a site), one can define generalisations of the Fontaine's period rings A_{crys}, B_{crys}, B_{HT}, ... These become sheaves for this topology, and thus the comparison isomorphisms of p-adic Hodge theory can be interpreted not only at the level of the (étale, crystalline, or de Rham) cohomology groups, but in fact directly as equivalences of complexes in the derived category, where we take tensor products with these new "period sheaves". This suggests we can now bring things full circle. Indeed, historically, the fundamental objects Tate needed to even state his comparison results or conjectures were not the Fontaine's period rings, but rather the rings "$\mathbb{Q}_p(i)$", for $i \in \mathbb{Z}$. These objects $\mathbb{Q}_p(i)$ are now turned, just as the period rings, into sheaves of complexes on the quasisyntomic topology, and these are what we call syntomic cohomology. In fact, most of the theory of [BMS19] is integral, as opposed to rational (the rational definitions can be deduced from the integral ones)[2], so we prefer to define syntomic cohomology as the sheaves of complexes $\mathbb{Z}_p(i)$, indexed by integers i.

These "syntomic cohomology" sheaves $\mathbb{Z}_p(i)$ are thus defined naturally in the context of p-adic Hodge theory. But in fact, they are expected to provide way more than just a nice definition. Indeed, since the foundational article of Tate in 1967, people (such as Grothendieck, Beilinson, or Deligne) had time to figure out some relations between p-adic Hodge theory (and typically the objects $\mathbb{Q}_p(i)$, or $\mathbb{Z}_p(i)$) and the so-called motivic cohomology theory. This motivic cohomology is a central theory in algebraic geometry; it has been constructed by Bloch, Levine, Suslin, and Voevodsky for smooth schemes over fields and over Dedekind rings, and is still conjectural in more general contexts. Via these relations to motivic cohomology, syntomic cohomology $\mathbb{Z}_p(i)$ is then expected to become a new central object in p-adic arithmetic geometry.

1.2 Overview of the mémoire

The plan is as follows. The reader is near to the end of Chapter 1, which is an introduction. The second chapter, that is the first after the introduction, is mainly an excuse to prepare to the third one. This second chapter introduces the classical theory of Hochschild and cyclic homology, and contains some of the main ideas concerning topological Hochschild and cyclic homology (which is the subject of the third chapter). Some of the classical results of the theory are reviewed, such as the Hochschild-Kostant-Rosenberg filtration, and the calculation of $\mathrm{HH}_*(\mathbb{F}_p)$.

The third chapter is the continuation of the second one, and presents topological Hochschild homology (denoted THH) and topological cyclic homology (denoted TC). The main reason why Chapter 2 is of any use is that, although some of the main ideas concerning THH and TC already appear in the classical theory of Chapter 2, Chapter 3 uses a lot of ∞-category machinery, and we did not want to scare the potential unfamiliar reader. Chapter 3 is however essential to describe syntomic cohomology, and forms with Chapters 4 and 5 the main three faces of syntomic cohomology.

Chapters 4 and 5 are some quicks reviews of prismatic and motivic cohomology, respectively. Prismatic cohomology is a p-adic cohomology theory developed recently by Bhatt and Scholze, and an analogue in mixed characteristic of crystalline cohomology. To explain it, we will need to present the notions of δ-rings and prisms; we also introduce the (prismatic) Nygaard filtration, which is necessary to formulate the prismatic definition of syntomic cohomology (see Section 6.2). Motivic cohomology is a long-standing conjectural theory of "universal cohomology" for algebraic varieties. We review some of its history, and some ideas of how we can adapt it to the p-adic context.

Chapter 6 is the meeting point of the three tributaries Chapters 3, 4 and 5. As they meet, syntomic cohomology emerges. It can be defined in three distinct and independent ways: one via the topogical theory of Chapter 3, another via the prismatic theory of Chapter 4, and finally, as a form[3] of motivic cohomology theory as defined in Chapter 5. As a meeting point of three waterways, syntomic cohomology naturally bears deep consequences in p-adic arithmetic geometry. We present some of them after proving some fundational recent results on syntomic

[2]Syntomic cohomology existed since its introduction by Fontaine and Messing in the 1980's. However it gave the right object only rationally, and was defined in a much smaller generality.

[3]More precisely: as p-adic étale motivic cohomology.

cohomology. We end this chapter by a calm opening into the ocean, without wondering anymore about open questions or proofs; just sunbathing and ruffling the waves.

The Appendix is made of technicalities, which are either complements to the main text, or parts of it which would have made it too heavy to read. In general, we assume familiarity with derived categories, the cotangent complex, and more generally with homological algebra. For readability, we try to give an idea, at the beginning of each chapter or section, of what one can find in it.

Conventions

We denote by $\cong$ the notion of (strict) isomorphism (for instance an isomorphism of groups, or algebras), and by $\simeq$ the notion of equivalence (for instance in some ∞-derived category).

Degrees of graded objects are denoted by $*$ (*e.g.* HH_*); for complexes and simplicial objects we use $\bullet$ (*e.g.* $\mathrm{Bar}_\bullet$); finally for filtrations we use $\star$ (*e.g.* $\mathcal{N}^{\geqslant\star}$). When we regard an actual complex $K^\bullet$ as an object of the derived category, we simply write K.

Different kinds of Frobenius morphisms appear in the text. All of them will be denoted by the letter φ. For a ring (or a δ-ring) A, we write it φ_A, or simply φ if the context is clear. For cyclotomic spectra (such as topological Hochschild homology), we sometimes specify the underlying prime number p and write φ_p for the Frobenius morphism. This topological Frobenius morphism φ_p, when defined on the Nygaard filtration level $\mathcal{N}^{\geqslant i}\widehat{\Delta}$ ($i \in \mathbb{N}$), is not an endormorphism (it has target the whole object); moreover when $\widehat{\Delta}$ is interpreted as completed prismatic cohomology, the topological Frobenius φ_p gets identified with a divided Frobenius morphism, often denoted φ_i, and equal to φ/d^i, for d a distinguished element.

Every (commutative) ring A can be regarded as a ($\mathbb{E}_\infty$-) ring spectrum via the Eilenberg-MacLane functor: $A \mapsto HA$ (see A.2). For simplicity, we will denote A for both the ring and the corresponding ring spectrum. Moreover, schemes will be denoted with capital letters (*e.g.* X), while letters such as $\mathfrak{X}$ will denote formal schemes.

A (co)simplicial complex is called *discrete* if its homotopy groups are zero, outside of degree 0. Similarly, A (co)chain complex is called *discrete* if its (co)homology groups are all zero, except the one in degree 0.

If not said otherwise, K will denote a discretely valued extension of $\mathbb{Q}_p$ with perfect residue field k, and with ring of integers $\mathcal{O}_K$. C will denote a p-completed algebraic closure of K, with ring of integers $\mathcal{O}_C$ (*e.g.*, if K is equal to $\mathbb{Q}_p$, then $C \cong \mathbb{C}_p$). Finally we will denote $W(k)$ for the ring of Witt vectors over k.

The letter n will denote (co)homological degrees, while i will be used for indexing (typically, but not only, Nygaard) filtrations, (Breuil-Kisin) twists and (motivic) weights. We remark however that syntomic cohomology can be defined independently using a motivic filtration on topological cyclic homology, which is defined locally by truncation of homological degrees (and is the first historical definition for syntomic cohomology), or as Frobenius fixed points on the Nygaard filtration of (completed, absolute) prismatic cohomology. Morever it has an expected motivic nature, for which the indexing integer would correspond to some motivic weight. Hence the classical choice of conventions completely collapses in this situation. We choose here to respect this coincidence, and we will use both conventions for syntomic cohomology (that is, $\mathbb{Z}_p(i)$ or $\mathbb{Z}_p(n)$), in a locally coherent way.

Chapter 2

Hochschild and cyclic homology

In this chapter we present the classical theory of Hochschild and cyclic homology. This is mainly an excuse to introduce the topological theory of the next chapter in a more elementay way.

2.1 Classical theory

Classical Hochschild homology is defined by a simple chain complex, which surprisingly bears interesting informations about both algebraic and arithmetic objects. In this section we review its definition and the relation with its three variants: cyclic homology, negative cyclic homology, and periodic cyclic homology.

Definition 2.1.1. *Let k be a commutative ring, and A a flat algebra over k. The Hochschild complex* $\mathrm{HH}_\bullet(A/k)$ *is defined as*

$$\mathrm{HH}_\bullet(A/k) := A \xleftarrow{b} A \otimes_k A \xleftarrow{b} A \otimes_k A \otimes_k A \xleftarrow{b} \dots$$

with $b : \begin{cases} A^{\otimes_k (n+1)} \to A^{\otimes_k n} \\ a_0 \otimes \cdots \otimes a_n \mapsto a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n + \cdots + \\ \quad (-1)^n a_0 \otimes \cdots \otimes a_{n-1} a_n + (-1)^{n+1} a_n a_0 \otimes \cdots \otimes a_{n-1}. \end{cases}$

From now on, and if not said otherwise, k will denote a commutative ring and A a flat k-algebra. We need to check the object we just defined is indeed a chain complex:

Lemma 2.1.2. *The Hochschild complex* $\mathrm{HH}_\bullet(A/k)$ *is a chain complex of k-modules.*

Proof. We need to prove that for any integer $n > 1$ and $a_0, \dots a_n$ some elements of A, then $(b \circ b)(a_0 \otimes \cdots \otimes a_n) = 0$. And this is a direct calculation. ◆

Remark 2.1.3. Remark that A is not necessarily commutative in Definition 2.1.1. In the noncommutative case, Hochschild homology was used by Connes in non-commutative geometry as a replacement for the de Rham complex.

Even though they are quite similar, one should not mix up the definition of the Hochschild complex with the one of the Bar complex.

Definition 2.1.4. *Let k be commutative ring, and A a flat algebra over k. The Bar complex* $\mathrm{Bar}_\bullet(A/k)$ *is defined as*

$$\mathrm{Bar}_\bullet(A/k) := A \otimes_k A \xleftarrow{b'} A \otimes_k A \otimes_k A \xleftarrow{b'} \dots$$

$$\textit{with } b' : \begin{cases} A^{\otimes_k (n+1)} \to A^{\otimes_k n} \\ a_0 \otimes \cdots \otimes a_n \mapsto a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n + \cdots + \\ \quad (-1)^n a_0 \otimes \cdots \otimes a_{n-1} a_n. \end{cases}$$

We leave to the reader the proof this is indeed a chain complex. One can prove there is a quasi-isomorphism $\mathrm{Bar}(A/k) \xrightarrow{\simeq} A$, where A is considered as a discrete complex. This justifies the difference of flavours – and of use – between the two complexes: while the Bar complex $\mathrm{Bar}(A/k)$ is a canonical resolution of A, and in particular has trivial higher homology groups, the Hochschild complex contains a lot of relevant informations on A.

Example 2.1.5. For any $a, b, c \in A$,

$$b(a \otimes b) = ab - ba,$$
$$b(a \otimes b \otimes c) = ab \otimes c - a \otimes bc + ca \otimes b.$$

As usual, a canonical way to study a (co)chain complex (a usual one, or considered as an element of a derived category) is to identify its (co)homology groups.

Definition 2.1.6. *Hochschild homology is defined as the homology of the Hochschild complex* $\mathrm{HH}_\bullet(A/k)$ *(Definition 2.1.1). We denote these homology groups* $\mathrm{HH}_n(A/k)$ *(:= "*$\mathrm{H}_n(\mathrm{HH}(A/k))$*") for $n \in \mathbb{Z}$. In particular* $\mathrm{HH}_n(A/k) = 0$ *for any $n < 0$.*

Example 2.1.7. If A is commutative, then $\mathrm{HH}_0(A/k) = A$.

Example 2.1.8. If A is commutative, then $\mathrm{HH}_1(A/k) \xrightarrow{\simeq} \Omega^1_{A/k}$. This is because $\mathrm{HH}_1(A/k) = A \otimes_k A/(ab \otimes c - a \otimes bc + ca \otimes b; a, b, c \in A)$, where we recognize the quotient relation as the Leibniz rule via: $a \otimes b \mapsto a db$.

Example 2.1.9. If $A = k$ in Definition 2.1.1, then $\mathrm{HH}_n(k/k) = \begin{cases} k & \text{if } n = 0; \\ 0 & \text{if } n > 0. \end{cases}$

Definition 2.1.1 is probably the most elementary way to define Hochschild homology. The following two remarks deal with two other ways to consider Hochschild homology. These can be usefulon their own, either in adapting Definition 2.1.1 to more general contexts, or to make proofs easier.

Remark 2.1.10. (Shukla homology) The flatness condition in Definition 2.1.1 can be removed by taking derived versions of tensor products, and homology groups in the derived category $D(k)$. We usually impose the flatness condition to make tensor products well-behaved. In the derived context, the right (that is, well-behaved) analogue of Hochschild homology is called *Shukla homology.*

Remark 2.1.11. We can see both the Hochschild complex $\mathrm{HH}_\bullet(A/k)$ and the Bar complex $\mathrm{Bar}_\bullet(A/k)$ as simplicial k-modules (via Dold-Kan). If A is commutative, these are even simplicial A-algebras. In particular Hochschild homology groups $\mathrm{HH}_n(A/k)$, for $n \geqslant 0$, are A-modules in the commutative case.

Now we arrive to our first general result on Hochschild homology. This identifies the Hochschild homology groups to the de Rham cohomology groups in the smooth commutative case. But first, remark that for any flat commutative k-algebra A, there is a natural map of graded A-algebras: $\Omega^*_{A/k} \to \mathrm{HH}_*(A/k)$. Indeed, it can be defined by sending $adb_1 \wedge \cdots \wedge db_n$ to $\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\varepsilon(\sigma)} a \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(n)}$, and is called the antisymmetrisation map.

Theorem 2.1.12. (Hochschild-Kostant-Rosenberg, 1962) *If A is a smooth (commutative) algebra over a commutative ring k, then the antisymmetrisation map is an isomorphism*

$$\Omega^n_{A/k} \xrightarrow{\simeq} \mathrm{HH}_n(A/k),$$

for any $n \geqslant 0$.

Proof. We use a base change formula for Hochschild homology of étale extensions, and the formula: $\mathrm{HH}_*(A/k) \cong \mathrm{Tor}_*^{A \otimes_k A^{\mathrm{op}}}(A, A)$. Then one restricts to proving the result for polynomial rings over k and $n = 1$, which can be done explicitly. ◆

We now turn to the definition (which is more a construction than just a notation) of cyclic homology. The starting point is the following: for k a commutative ring and A a flat k-algebra, each $A^{\otimes_k (n+1)}$ in $\mathrm{HH}_\bullet(A/k)$ has a k-linear action of $\mathbb{Z}/(n+1)$ by permutation. Intuitively, this justifies the adjective "cyclic" in the name cyclic homology. For all $n \geqslant 0$, let $t_n = t := 1 \in \mathbb{Z}/(n+1)$ be a generator of the cyclic group, such that $t_n : a_0 \otimes \cdots \otimes a_n \mapsto a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$. The idea is then to use this operator, together with the ones we will now define, to construct cyclic homology from the Hochschild complex.

Definition 2.1.13. *Let $n \geqslant 0$ be an integer. The norm map $N : A^{\otimes_k (n+1)} \to A^{\otimes_k (n+1)}$ is defined by $N := \sum_{i=0}^n ((-1)^n t_n)^i = \sum_{i=0}^n (\pm t)^i$. The extra degeneracy map $s : A^{\otimes_k n} \to A^{\otimes_k (n+1)}$ sends $a_0 \otimes \cdots \otimes a_{n-1}$ to $1 \otimes a_0 \otimes \cdots \otimes a_{n-1}$. Finally, let $B := (1 - (-1)^n t_n) s N : A^{\otimes_k n} \to A^{\otimes_k (n+1)}$ be the Connes' boundary operator.*

Proposition 2.1.14. *The maps t, N, s, B of Definition 2.1.13, and the maps b, b' defining the Hochschild and Bar complexes, satisfy the following identities: $(1 - \pm t)b' = n(1 - \pm t)$; $b'N = Nb$; $sb' + b's = \mathrm{id}$; $B^2 = 0$; $Bb = -bB$.*

Sketch of proof. To prove that $B^2 = 0$, we use the fact that, for any $n \geqslant 0$, $\sum_{i=0}^n (\pm t)^i (1 - \pm t) = 1 - (\pm)^{n+1} = 0$ (where $\pm$ denotes $(-1)^n$). The other identities are easier, and we leave them to the reader. ◆

Before arriving to the construction of cyclic homology, let us refine Theorem 2.1.12 by identifying, this time in the general commutative case, the Connes' boundary operator to the de Rham differential.

Remark 2.1.15. The Connes operator B induces k-linear maps: $B : \mathrm{HH}_n(A/k) \to \mathrm{HH}_{n+1}(A/k)$ (this is because $Bb = -bB$). If A is commutative, this makes HH_* into a commutative differential graded algebra. One can prove the following diagram commutes (see Remark 2.1.3)

$$\begin{array}{ccc} \mathrm{HH}_n(A/k) & \xrightarrow{\mathrm{B}} & \mathrm{HH}_{n+1}(A/k) \\ \uparrow{\scriptstyle \varepsilon_n} & & \uparrow{\scriptstyle \varepsilon_{n+1}} \\ \Omega^n_{A/k} & \xrightarrow{\mathrm{d}} & \Omega^{n+1}_{A/k} \end{array}$$

Construction 2.1.16. Using the properties 2.1.14, we define the anticommuting bicomplex of k-modules:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\cdots & \xleftarrow[(1-\pm t)]{} & A^{\otimes 3} & \xleftarrow[N]{} & A^{\otimes 3} & \xleftarrow[(1-\pm t)]{} & A^{\otimes 3} & \xleftarrow[N]{} & \cdots \\
 & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\cdots & \xleftarrow[(1-\pm t)]{} & A^{\otimes 2} & \xleftarrow[N]{} & A^{\otimes 2} & \xleftarrow[(1-\pm t)]{} & A^{\otimes 2} & \xleftarrow[N]{} & \cdots \\
 & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\cdots & \xleftarrow[(1-\pm t)]{} & A & \xleftarrow[N]{} & A & \xleftarrow[(1-\pm t)]{} & A & \xleftarrow[N]{} & \cdots
\end{array}$$

We fix the convention that some "A" which is a target of $(1 - \pm t)$, is in the position $(0, 0)$ of the bicomplex.

Remark 2.1.17. Even columns of Construction 2.1.16 are the Hochschild complex $\mathrm{HH}_\bullet(A/k)$, and odd columns are the augmented Bar complex $\mathrm{Bar}_\bullet(A/k) \to A$, which are hence acyclic.

We are now equipped to state the definition of cyclic homology. We also give the definitions of the variants negative cyclic homology and periodic cyclic homology, since they play a part in the theory.

Definition 2.1.18. *Using this bicomplex, we define the following:*

- *Cyclic homology* $\mathrm{HC}_\bullet(A/k)$ *is the complex of k-modules defined as the totalization of the half plane* $x \geqslant 0$ *of Construction 2.1.16;*
- *Negative cyclic homology* $\mathrm{HC}^-_\bullet(A/k)$ *is the complex of k-modules defined as the product totalization of the half plane* $x \leqslant 0$ *of Construction 2.1.16;*
- *Periodic cyclic homology* $\mathrm{HP}_\bullet(A/k)$ *is the complex of k-modules defined as the product totalization of the (whole) bicomplex in Construction 2.1.16.*

A typical example of how these are related is given by the following:

Proposition 2.1.19. *There is a short exact sequence of complexes, called the Norm sequence*

$$0 \to \mathrm{HC}^-_\bullet(A/k) \to \mathrm{HP}_\bullet(A/k) \to \mathrm{HC}_\bullet(A/k)[2] \to 0.$$

Proof. The first arrow $\mathrm{HC}^-_\bullet(A/k) \to \mathrm{HP}_\bullet(A/k)$ is induced by the inclusion map on the bicomplex of Construction 2.1.16, and thus is injective levelwise. Then note that by horizontal 2-periodicity of the bicomplex of Construction 2.1.16 that the map $\mathrm{HP}_\bullet(A/k) \to \mathrm{HC}_\bullet(A/k)[2]$ is just some restriction at the level of the bicomplex, and is thus surjective. Finally, this is indeed an exact sequence by acyclicity of the odd columns of the defining bicomplex. ◆

A natural question at this point could be to wonder why we introduce such objects. According to Theorem 2.1.12, we saw that Hochschild homology is related (at least in the smooth commutative case) to the de Rham complex. When we study the de Rham complex (and de Rham cohomology), we are often interested in the Hodge filtration.

More precisely, for a commutative algebra A over a commutative ring k, one can define the de Rham complex $\Omega^\bullet_{A/k}$. The naive truncation yields the following short exact sequence of complexes

$$0 \to \Omega^{\geqslant i}_{A/k} \to \Omega^\bullet_{A/k} \to \Omega^{\leqslant i-1}_{A/k} \to 0.$$

Considering cyclic homology (and its variants) then allows to reconstruct the Hodge filtrations:

Theorem 2.1.20. (Connes, Loday-Quillen, Feigin-Tygan, 1980's) *Let k be a commutative base ring containing $\mathbb{Q}$, and A be a smooth k-algebra. There are natural equivalences:*

$$\mathrm{HC}^-(A/k) \simeq \prod_{i\in\mathbb{Z}} \Omega^{\geqslant i}_{A/k}[2i],$$
$$\mathrm{HP}(A/k) \simeq \prod_{i\in\mathbb{Z}} \Omega_{A/k}[2i],$$
$$\mathrm{HC}(A/k) \simeq \prod_{i\geqslant 0} \Omega^{\leqslant i}_{A/k}[2i].$$

Remark in this result that the equivalences are given in the derived category, as opposed to Theorem 2.1.12 for instance, which was stated only at the level of (co)homology groups.

Sketch of proof. We use the formalism of mixed complexes (we do not explain here; see for instance [Mor19], Section 2.2) to deal with Construction 2.1.16 in a more functorial way. In particular, this reduces to proof to a statement about $\mathrm{HH}(A/k)$ and the maps b and B. We then use Theorem 2.1.12 and some quasi-isomorphism constructed from a quasi-inverse of the antisymmetrisation map from $\Omega_{A/k}$ to $\mathrm{HH}(A/k)$. Remark that this quasi-inverse is the reason why we need the characteristic 0 hypothesis, since it is constructed by inverting some "$n!$". ◆

By definition of $\mathrm{HC}(A/k)$, one can prove easily the following result relating Hochschild and cyclic homology. In fact, this result is the first time we introduce a filtration on cyclic homology, whose graded pieces we identify to something of interest (here shifted Hochschild homology); this will become a central idea in the next chapter.

Proposition 2.1.21. *Let k be a commutative ring, and A a flat algebra over k. There is natural exhaustive increasing $\mathbb{N}$-indexed filtration on $\mathrm{HC}_\bullet(A/k)$, with graded pieces $\mathrm{gr}_i\mathrm{HC}_\bullet(A/k) = \mathrm{HH}_\bullet(A/k)[2i]$.*

Proof. By construction of $\mathrm{HC}_\bullet(A/k)$, there is short exact sequence of complexes $0 \to \mathrm{HH}(A/k) \to \mathrm{HC}(A/k) \xrightarrow{S} \mathrm{HC}(A/k)[2] \to 0$, where the first map is the inclusion of the 0-th column at the level of the bicomplex, and the second map is constructed by projection after removing the first two columns in the definition of $\mathrm{HC}_\bullet(A/k)$. The filtration is then defined by $\mathrm{Fil}^i\mathrm{HC}_\bullet(A/k) := \ker(S^i : \mathrm{HC}_\bullet(A/k) \to \mathrm{HC}_\bullet(A/k)[2i])$, for $i \geqslant 0$. It follows by definition it is exhaustive, increasing and $\mathbb{N}$-indexed, and its i-th graded piece is $\mathrm{HH}_\bullet(A/k)[2i]$ for each $i \geqslant 0$ thanks to our short exact sequence of complexes. ◆

Remark 2.1.22. Connes (1983) developed a notion of *cyclic objects* in any category $\mathcal{C}$; this is formalized by a simplicial object in the category $\mathcal{C}$ satisfying some axioms, which are the ones satisfied by the Hochschild complex (in the category of k-algebras), and necessary to make the construction of cyclic homology Construction 2.1.16.

A similar construction of Hochschild homology (and its three cyclic variants) for non-necessarily flat algebras exists, as we explain now.

Construction 2.1.23. Let k be commutative ring, and A a (non-necessarily flat) commutative k-algebra. In this case, one can define the complex $\mathrm{HH}_\bullet(A/k)$ as the diagonal of $\mathrm{HH}_\bullet(P_\bullet/k)$ - which is a bisimplicial object-, for $P_\bullet \to A$ a simplicial resolution of A by polynomial k-algebras.

Remark 2.1.24. The construction 2.1.23 is similar to the definition of the cotangent complex with respect to the de Rham complex. More generally, these are two illustrations of the notion of *left Kan extensions*: we extend a functor defined on polynomial k-algebras $\mathcal{G} : k\text{-alg}_\Sigma \to \mathcal{D}(k)$ (*e.g.* $\Omega^1_{-/k}$, $\mathrm{HH}(-/k)$) to a functor $\mathbb{L}\mathcal{G} : k\text{-alg} \to \mathcal{D}(k)$ by sifted colimits.

Similarly (Construction 2.1.23), we define $\mathrm{HC}(A/k)$ for non-flat k-algebras A by left Kan extension from polynomial k-algebras. This is possible since HC is a colimit of HH, and not a limit (such as HP and HC^-, which involve taking infinite products of HH). Then $\mathrm{HC}(A/k)$ is equipped by left Kan extension with an operator $S : \mathrm{HC}(A/k) \to \mathrm{HC}(A/k)[2]$ (see the proof of Proposition 2.1.21). We can then define $\mathrm{HP}(A/k)$ and $\mathrm{HC}^-(A/k)$ for general k-algebras A as $\mathrm{HP}(A/k) := \varprojlim \mathrm{HC}(A/k)[2n]$ where the transition maps are shifts of the periodicity operator S, and $\mathrm{HC}^-(A/k) := \mathrm{hofib}(\mathrm{HP}(A/k) \to \mathrm{HC}(A/k)[2])$. Remark that these definitions make sense only in the derived category, as opposed to our first definition of Hochschild homology (Definition 2.1.1). Hence from now on, we usually work only with complexes in derived categories.

We end this section by the following classical result that relates Hochschild homology to the cotangent complex.

Theorem 2.1.25 (Hochschild–Kostant–Rosenberg filtration)**.** *Let A be a commutative algebra over a commutative base ring k.*

(1) There is a natural complete descending $\mathbb{N}$-indexed filtration on $\mathrm{HH}(A/k)$ such that $\mathrm{gr}_i\mathrm{HH}(A/k) \simeq \mathbb{L}^i_{A/k}[i]$, for all $i \geqslant 0$.

(2) There is a natural complete descending $\mathbb{N}$-indexed filtration on $\mathrm{HC}(A/k)$ such that $\mathrm{gr}_i\mathrm{HC}(A/k) \simeq \bigoplus_{n\geqslant 0} \mathbb{L}^i_{A/k}[i+2n]$, for all $i \geqslant 0$.

Sketch of proof. For (1), recall that $\mathrm{HH}(A/k)$ is defined by totalising the bisimplicial object $\mathrm{HH}(P_\bullet/k)$, where $P_\bullet \to A$ is a simplicial resolution of A by free (that is, polynomial) k-algebras. The Postnikov filtration (that is, truncating the degrees) on each $\mathrm{HH}(P_n/k)$ formally induces a natural complete descending $\mathbb{N}$-indexed filtration on $\mathrm{HH}(A/k)$ whose i-th graded piece is $\mathrm{HH}_i(P_\bullet/k)[i]$, for any $i \geqslant 0$. The statement then follows from Theorem 2.1.12 applied to the polynomial k-algebras P_n, and the definition of the cotangent complex $\mathbb{L}^i_{A/k}$.

For (2) we use similar arguments to restrict to polynomial k-algebras A. We then use again Theorem 2.1.12, and some explicit filtration on the bicomplex (Construction 2.1.16) defining HC. ◆

Remark this result is valid over any commutative base ring (*e.g.* over $k = \mathbb{Z}$); compare to Theorem 2.1.20.

2.2 Hochschild homology in characteristic p

We focus in this section on Hochschild homology (and its variants) in the characteristic p situation. In a way, this is the first time we glance at the arithmetic relevance of Hochschild homology. We begin with a direct consequence of the Hochschild-Kostant-Rosenberg filtration (Theorem 2.1.25).

Corollary 2.2.1. *We have*

$$\mathrm{HH}_n(\mathbb{F}_p\,/\,\mathbb{Z}) \cong \begin{cases} \mathbb{F}_p & \text{if } n \geqslant 0 \text{ is even ;} \\ 0 & \text{otherwise.} \end{cases}$$

In fact, considering multiplicative structure : $\mathrm{HH}_*(\mathbb{F}_p\,/\,\mathbb{Z}) \cong \mathbb{F}_p\langle u\rangle$*, with* $u \in \mathrm{HH}_2(\mathbb{F}_p\,/\,\mathbb{Z})$ *any basis element, and* $\mathbb{F}_p\langle u\rangle$ *is the free divided power algebra over* $\mathbb{F}_p$ *on the element* u*.*

Remark 2.2.2. In particular, $u^p = 0 \in \mathrm{HH}_{2p}(\mathbb{F}_p\,/\,\mathbb{Z})$ (by definition of the divided power structure). This is usually considered as a flaw, which is corrected when considering the "topological" version of Hochschild homology $\mathrm{THH}(\mathbb{F}_p)$.

Proof. By Theorem 2.1.25, there is a (complete descending $\mathbb{Z}$-indexed) filtration of $\mathrm{HH}(\mathbb{F}_p\,/\,\mathbb{Z})$ with graded pieces $\mathrm{gr}^i\mathrm{HH}(\mathbb{F}_p\,/\,\mathbb{Z}) \simeq \mathbb{L}^i_{\mathbb{F}_p\,/\,\mathbb{Z}}[i]$. Now, $\mathbb{L}_{\mathbb{F}_p\,/\,\mathbb{Z}}$ is given by a (flat) $\mathbb{F}_p$-module N of rank 1, supported in homological degree 1. Moreover $\mathbb{L}^i_{\mathbb{F}_p\,/\,\mathbb{Z}}[-i] = \Gamma^i_{\mathbb{F}_p}(N)$ is the i-th divided power of N (this is a classical result of Quillen, valid for any complex given by a flat module N in homological degree 1, and replacing $\mathbb{L}^i_{\mathbb{F}_p\,/\,\mathbb{Z}}$ by the i^{th} derived wedge power of N), and in particular is supported in degree 0; that is, $\mathbb{L}^i_{\mathbb{F}_p\,/\,\mathbb{Z}}$ is supported in degree i. Each $\mathbb{F}_p$-module $\Gamma^i_{\mathbb{F}_p}(N)$ is isomorphic to $\mathrm{Sym}^i_{\mathbb{F}_p}(N) \cong \mathbb{F}_p$, but the algebra structure on $\Gamma^*_{\mathbb{F}_p}(N) := \bigoplus_{i\geqslant 0}\Gamma^i_{\mathbb{F}_p\,/\,\mathbb{Z}}(N)$ is given by

$$a.b = \frac{(i+j)!}{i!j!}ab \qquad a \in \Gamma^i_{\mathbb{F}_p}(N), b \in \Gamma^j_{\mathbb{F}_p}(N),$$

where $a.b$ denotes the multiplication in $\Gamma^*_{\mathbb{F}_p}(N)$ and ab denotes the multiplication in $\mathrm{Sym}^*_{\mathbb{F}_p}(N)$. The graded pieces of $\mathrm{HH}(\mathbb{F}_p\,/\,\mathbb{Z})$ are supported in degrees $2i$, $i \geqslant 0$, which implies the desired result by Section A.3. ◆

In fact, the previous result is true for any perfect field of characteristic p instead of $\mathbb{F}_p$, with the proof remaining unchanged.

We can also try to compute the Hochschild homology and its variants over $\mathbb{F}_p$; that is, $\mathrm{HH}(-\,/\,\mathbb{F}_p)$ instead of $\mathrm{HH}(-\,/\,\mathbb{Z})$. These are related by a canonical base change isomorphism $\mathrm{HH}(-\,/\,\mathbb{Z}) \otimes^{\mathbb{L}}_{\mathrm{HH}(\mathbb{F}_p\,/\,\mathbb{Z})} \mathbb{F}_p \xrightarrow{\sim} \mathrm{HH}(-\,/\,\mathbb{F}_p)$. It appears it does not give any interesting information for *perfect* $\mathbb{F}_p$-algebras (see Corollary 2.2.4). However, we can identify $\mathrm{HH}(-\,/\,\mathbb{F}_p)$ on a large class of *semiperfect* algebras (which will be of interest to define syntomic cohomology later). Recall semiperfect algebras are the $\mathbb{F}_p$-algebras whose Frobenius endomorphism is surjective.

Lemma 2.2.3. ([BMS19]) *Let* S *be a semiperfect* $\mathbb{F}_p$*-algebra. Assume that* S *is quasiregular, i.e. the cotangent complex* $\mathbb{L}_{A/\,\mathbb{F}_p}$ *is a flat* S*-module supported in homological degree* 1 *(see also Definition 3.2.7). Then* $\mathrm{HH}_*(S/\,\mathbb{F}_p)$*,* $\mathrm{HC}^-_*(S/\,\mathbb{F}_p)$*,* $\mathrm{HP}_*(S/\,\mathbb{F}_p)$ *and* $\mathrm{HC}_*(S/\,\mathbb{F}_p)$ *are all supported in even degrees. Moreover, the* $\mathbb{F}_p$*-algebra* $\mathrm{HP}_0(S/\,\mathbb{F}_p) \cong \mathrm{HC}^-_0(S/\,\mathbb{F}_p)$ *has a complete descending* $\mathbb{Z}$*-indexed filtration by ideals such that, for all* $i \geqslant 0$*,*

$$\mathrm{Fil}^i\mathrm{HP}_0(S/\,\mathbb{F}_p) \cong \mathrm{HC}^-_{2i}(S/\,\mathbb{F}_p), \quad \mathrm{HP}_0(S/\,\mathbb{F}_p)/\mathrm{Fil}^i\mathrm{HP}_0(S/\,\mathbb{F}_p) \cong \mathrm{HC}_{2i}(S/\,\mathbb{F}_p),$$

$$\mathrm{gr}^i\mathrm{HP}_0(S/\,\mathbb{F}_p) \cong \mathrm{HH}_{2i}(S/\,\mathbb{F}_p) \cong \pi_i(\mathbb{L}^i_{S/\,\mathbb{F}_p}).$$

Proof. First, the quasiregular hypothesis on S together with standard facts about divided wedge powers (recall that $\mathbb{L}^i := \wedge^i\mathbb{L}^1$) imply that $\mathbb{L}^i_{A/\,\mathbb{F}_p}$ is a flat module supported in degree i, for all $i \geqslant 0$. This implies that the graded pieces $\mathbb{L}^i_{S/\,\mathbb{F}_p}[i]$ of the HKR filtration of Theorem 2.1.12.(1)

are all in even degrees $2i$. Thus the associated spectral sequence degenerates, and we get the ending result about $\mathrm{HH}(S/\mathbb{F}_p)$.

The rest of the proof is a consequence of the fact that $\mathrm{HH}_*(S/\mathbb{F}_p)$ is concentrated in even degrees and of the Construction 2.1.16. More precisely, we first deduce that $\mathrm{HP}_*(S/\mathbb{F}_p)$, $\mathrm{HC}^-_*(S/\mathbb{F}_p)$ and $\mathrm{HC}_*(S/\mathbb{F}_p)$ are all supported in even degrees using some short exact sequences coming from Construction 2.1.16. We then use that $\mathrm{HP}_{2i}(S/\mathbb{F}_p) \cong \mathrm{HP}_{2i-2}(S/\mathbb{F}_p)$ for all $i \in \mathbb{Z}$ and some "periodicity" operator relating $\mathrm{HP}(S/\mathbb{F}_p)$ and $\mathrm{HC}^-(S/\mathbb{F}_p)$ to define the filtration on $\mathrm{HP}_0(S/\mathbb{F}_p)$, where $\mathrm{Fil}^i\mathrm{HP}_0(S/\mathbb{F}_p)$ is given by the image of $\mathrm{HC}^-_{2i}(A/\mathbb{F}_p)$. The rest is a formal consequence of Proposition 2.1.19. ◆

Thanks to Lemma 2.2.3, it suffices now, for A a quasiregular semiperfect $\mathbb{F}_p$-algebra, to understand $\mathrm{HP}_0(A/\mathbb{F}_p)$ and its filtration to get all the data given by $\mathrm{HH}_*(A/\mathbb{F}_p)$, $\mathrm{HC}^-_*(A/\mathbb{F}_p)$, $\mathrm{HP}_*(A/\mathbb{F}_p)$ and $\mathrm{HC}_*(A/\mathbb{F}_p)$. This filtration is contructed in [BMS19] via some degenerate spectral sequence.

Corollary 2.2.4. *Let A be a perfect $\mathbb{F}_p$-algebra. Then $\mathbb{L}_{A/\mathbb{F}_p} \simeq 0$, and hence $\mathrm{HH}_*(A/\mathbb{F}_p)$ is isomorphic to the $\mathbb{F}_p$-module A supported in degree 0 (compare to Example 2.1.9). Similarly, $\mathrm{HC}_*(A/\mathbb{F}_p)$ is supported in nonnegative even degrees, where it is isomorphic to A; $\mathrm{HP}_*(A/\mathbb{F}_p)$ is supported in even degrees, where it is isomorphic to A; and $\mathrm{HC}^-_*(A/\mathbb{F}_p)$ is supported in nonpositive even degrees, where it is isomorphic to A.*

Proof. The computation of the cotangent complex in this case comes from the fact the Frobenius morphism on A induces an endomorphism on $\mathbb{L}_{A/\mathbb{F}_p}$, which is both an isomorphism by "transport de structure" since A is perfect, and zero since the differential of the Frobenius is zero in characteristic p. Remark in particular that $\mathbb{L}_{S'/S} \simeq 0$ is valid more generally for any morphism $S \to S'$ of perfect $\mathbb{F}_p$-algebras.

The description of $\mathrm{HH}_*(A/\mathbb{F}_p)$ is then a consequence of Lemma 2.2.3 for $A = S$. The description of $\mathrm{HC}_*(A/\mathbb{F}_p)$ comes from Proposition 2.1.21. Finally, we get that $\mathrm{HP}_*(A/\mathbb{F}_p)$ and $\mathrm{HC}^-_*(A/\mathbb{F}_p)$ are concentrated in even degrees, where we compute the desired result via Lemma 2.2.3. ◆

There is also an analogue of Theorem 2.1.20 in characteristic p, proved in [BMS19].

Theorem 2.2.5. ([BMS19]) *Let k be a perfect field of characteristic p, and R a smooth algebra over k. There are complete descending $\mathbb{Z}$-indexed filtrations on $\mathrm{HC}^-(R/k)$, $\mathrm{HP}(R/k)$ and $\mathrm{HC}(R/k)$, with graded pieces gr_i given respectively by*

$$\Omega^{\geqslant i}_{R/k}[2i], \;\; \Omega_{R/k}[2i], \;\; \Omega^{\leqslant i}_{R/k}[2i].$$

This result is proved using descent on the so-called *quasisyntomic site*. This will be detailed in the next chapter.

Theorem 2.2.5 says basically that we have the same filtration as in characteristic 0, except it is not necessarily split anymore.

Remark 2.2.6. Theorem 2.2.5 is formulated only over a perfect field k (compare to the characteristic 0 case, Theorem 2.1.20). Antieau ([Ant18]) proved a generalization of this result to smooth algebras over any base commutative ring (and in fact to any − possibly non-smooth − algebra by replacing de Rham cohomology by its derived version).

We note that all the results here have their analogues in mixed-characteristic. The formulation relies on perfectoid rings instead of perfect $\mathbb{F}_p$-algebras, and is then a bit less elementary. This will be the point of view adopted in most of the next chapter, since it includes the characteristic p case.

Chapter 3

THH and TC

We now present topological Hochschild homology (denoted THH). It is equipped with an action of the circle group $\mathbb{T} = \mathrm{S}^1$ which is fundamental in defining topological cyclic homology (denoted TC). The main goal of this chapter is to introduce the "motivic" filtrations on topological Hochschild homology THH and its variants TC^-, TP, whose graded pieces bear lots of arithmetic significance. We define those by descent on the quasisyntomic site – which is defined in Section 3.2. Details about the "motivic" nature of this filtration can be found in Chapter 5 and Chapter 6.

3.1 THH and the $\mathbb{T}$-action

Here we give an idea for the construction of topological Hochschild homology, and we state without proof a fundamental result of Bökstedt computing $\mathrm{THH}_*(\mathbb{F}_p)$.

The definition of topological Hochschild homology relies on some ∞-categorical and spectra formalism, which is quite heavy. Hence we only try to give an idea of the construction here, and refer to Appendix A.4. or [BMS19] for some more details.

Construction 3.1.1. The somehow cyclic flavour in Definition 2.1.1 can be formalized as an "algebraic" action of the circle group $\mathbb{T} = \mathrm{S}^1$. The idea is to encode algebraically how the circle $\mathbb{T}$ can act on algebraic objects such as chain complexes. More precisely, the $\mathbb{T}$-action can be formalized with the notion of cyclic objects (see [Lod92]), which are simplicial objects in a category, with some algebraic action. The geometric realization of a cyclic object is then equipped with an action of the topological space $\mathbb{T}$ [1]. By doing so in the so-called category of spectra one can define *topological Hochschild homology* $\mathrm{THH}(A)$, for any ring A (in fact for any $\mathbb{E}_\infty$-ring spectrum). Roughly, one replaces k-linear tensor products in $\mathrm{HH}(A/k)$ (see Definition 2.1.1) with smash products, and takes geometric realization. One can keep as an intuition that smash product behaves just as k-linear tensor products.

Recall that (usual) Hochschild homology of $\mathbb{F}_p$ can be computed as some divided power polynomial algebra (Corollary 2.2.1). The situation for the topological theory is a bit nicer, thanks to the following hard result.

[1] As a remark, the classifying space $\mathrm{B}\mathbb{T}$ of the circle group $\mathbb{T}$ can be represented by the infinite dimensional projective space over $\mathbb{C}$: $\mathrm{B}\mathbb{T} \simeq \mathbb{C}P^\infty$.

Theorem 3.1.2 (Bökstedt)**.** *The homotopy groups of* $\mathrm{THH}(\mathbb{F}_p)$ *are*

$$\mathrm{THH}_n(\mathbb{F}_p) := \pi_n \mathrm{THH}(\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & \text{if } n \geqslant 0 \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

Regarding multiplicative structure, $\mathrm{THH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[u]$, *with* $u \in \mathrm{THH}_2(\mathbb{F}_p) = \mathrm{HH}_2(\mathbb{F}_p \,/\, \mathbb{Z})$ *any basis element.*

One can define topological analogs of negative cyclic homology and periodic cyclic homology using similar techniques. We denote these by TC^- and TP. Just like THH, these are spectra when applied to a given $\mathbb{E}_\infty$-ring; in particular homotopy groups, indexed by the integers, are associated to those. A satisfying definition of topological cyclic homology TC is somehow harder to give; this was done first by Bökstedt-Hsiang-Madsen in [BHM93], and this is reviewed in Section 3.5.

Although topological Hochschild homology is better-behaved than classical Hochschild homology, the latter is often easier to compute explicitly, and one uses the following comparison lemma to relate the two.

Lemma 3.1.3. ([BMS19], Lemma 2.5) *For any commutative ring A, there is a natural* $\mathbb{T}$*-equivariant isomorphism of* $\mathbb{E}_\infty$*-ring spectra*

$$\mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z} \simeq \mathrm{HH}(A).$$

Moreover, this induces an isomorphism of p-complete $\mathbb{E}_\infty$*-ring spectra*

$$\mathrm{THH}(A; \mathbb{Z}_p) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z} \simeq \mathrm{HH}(A; \mathbb{Z}_p).$$

The homotopy groups $\pi_i \mathrm{THH}(\mathbb{Z})$ *are finite for* $i > 0$.

Proof. This is classical fact that Hochschild and topological Hochschild homology satisfy a universal property, and, if we accept that the smash product behaves like a usual tensor product, then these imply the first statement. More precisely, $\mathrm{THH}(A)$ (respectively $\mathrm{HH}(A)$) is the universal $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-ring spectrum (respectively $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra) equipped with a non-equivariant map from A. The final statement follows from the description of $\mathrm{THH}(\mathbb{Z})$ as the colimit of the simplicial spectrum with terms $\mathbb{Z} \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} \mathbb{Z}$ and the finiteness of the stable homotopy groups of spheres. The statement about p-completions follows as soon as one checks that $\mathrm{THH}(A; \mathbb{Z}_p) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z}$ is still p-complete, which follows from finiteness of $\pi_i \mathrm{THH}(\mathbb{Z})$ for $i > 0$. ◆

3.2 The quasisyntomic site

He now present, with some motivations, the definition of the quasisyntomic site (Definition 3.2.12), which is fundamental in defining the motivic filtrations on THH, TC^-, TP and TC. For the sake of clarity, we state some definitions only in characteristic p (see Section A.1 or [BMS19] for the general definitions in mixed characteristic).

The starting point in defining the quasisyntomic site is the following theorem, due to Hesselholt [Hes06] if $R = \mathcal{O}_{\mathbb{C}_p}$.

Theorem 3.2.1. ([BMS19]) *Let R be a perfectoid ring. There is a canonical (in R) isomorphism*

$$\pi_0 \mathrm{TC}^-(R; \mathbb{Z}_p) \cong A_{\mathrm{inf}}(R),$$

where $\mathrm{TC}^-(R; \mathbb{Z}_p)$ *denotes the p-completion of the spectrum* $\mathrm{TC}^-(R)$.

Remark 3.2.2. There is a "Frobenius" endomorphism $\varphi : \pi_0\mathrm{TC}^-(A;\mathbb{Z}_p) \to \pi_0\mathrm{TP}(R;\mathbb{Z}_p) \cong \pi_0\mathrm{TC}^-(A;\mathbb{Z}_p)$, defined for any $\mathbb{E}_\infty$-ring spectrum A. One can prove the isomorphism 3.2.1, for $A = R$, is in fact φ-equivariant, where $A_{\mathrm{inf}}(R) := W(R^\flat)$ is naturally equipped with its Frobenius endormorphism φ.

The idea to define the so-called quasisyntomic site, starting from Theorem 3.2.1, is somehow a bit indirect. Let us make a small detour to be able to explain how it shows up.

In 2016, Bhatt, Morrow and Scholze defined, in [BMS18], the so-called A_{inf}-cohomology theory. This is an integral p-adic cohomology theory over $\mathcal{O}_C$ that unifies and strenghtens a lot of known results in (integral) p-adic Hodge theory. This A_{inf}-cohomology theory relies on the A_{inf} period ring, which is defined for any perfectoid ring R as $A_{\mathrm{inf}} = A_{\mathrm{inf}}(R) := W(R^\flat)$, and which also appears in Theorem 3.2.1. Hence, Theorem 3.2.1 suggests there could exist a more general result –or theory– which would compare the A_{inf}-cohomology theory of [BMS18] with topological negative cyclic cohomology (or, more generally, with all the variants of topological Hochschild homology).

This idea is realized in [BMS19]: if A is the p-adic completion of a smooth $\mathcal{O}_C$-algebra, the complex $A\Omega_A$[2] from A_{inf}-cohomology is recovered via flat descent from $\pi_0\mathrm{TC}^-(-;\mathbb{Z}_p)$ by passage to a perfectoid cover $A \to R$. We aim at explaining this now.

To simplify the notations and to make the intuition a bit more concrete, we state the following results only in characteristic p (we refer to [BMS19], or Section A.1for the mixed-characteristic version). First, let us say what it means for an ∞-sheaf to satisfy flat descent.

Definition 3.2.3. *Let k be a commutative base ring, and $\mathcal{F} : k\text{-alg} \to \mathcal{D} = \mathcal{D}(\mathbb{Z}), \mathcal{D}(k), \mathrm{Sp}$ (or any ∞-category with sifted colimits) a functor. We say $\mathcal{F}$ satisfies flat descent (or is an fpqc ∞-sheaf) if, for any faithfully flat map $A \to B$ of k-algebras, the induced*

$$\mathcal{F}(A) \xrightarrow{\sim} \lim\left(\mathcal{F}(B) \rightrightarrows \mathcal{F}(B \otimes_A B) \rightrightarrows\!\!\!\!\to \mathcal{F}(B \otimes_A B \otimes_A B) \rightrightarrows\!\!\!\!\rightrightarrows \cdots\right)$$

is an equivalence in $\mathcal{D}$.

The cosimplicial object $\mathcal{F}(B) \rightrightarrows \mathcal{F}(B \otimes_A B) \rightrightarrows\!\!\!\!\to \cdots$ in Definiton 3.2.3 is the image under the functor $\mathcal{F}$ of the Čech nerve Čech(B/A). Remark that in the 1-categorical definition, a functor satisfies descent for some topology if it satisfies, for any cover in this topoogy, the same condition as in Definition 3.2.3, only with a truncated Čech complex. Moreover, the (homotopy) limit can be interpreted here as a totalisation functor if $\mathcal{D} = \mathcal{D}(\mathbb{Z})$ or $\mathcal{D}(k)$ (where totalisation makes sense).

Example 3.2.4. The functor $\mathrm{TC}^-(-;\mathbb{Z}_p) : k\text{-alg} \to \mathcal{D}(k)$ satisfies flat descent. This is not obvious at all, and was proved in [BMS19] (see Theorem 3.4.4).

Now we give an example of flat cover which is exactly the kind of covers we will use in order to perform descent on the quasisyntomic site:

Construction 3.2.5. Let R be a smooth algebra over a perfect ring k of characteristic p. Then the Frobenius $\varphi : R \to R$ is flat, and the colimit perfection $R_{\mathrm{perf}} := \lim\limits_{\varphi} R$ is a flat R-algebra. Moreover the Frobenius φ is an homeomorphism on $\mathrm{Spec}(R)$, and in particular is surjective. Hence the map $R \to R_{\mathrm{perf}}$ is faithfully flat. So, if the functor $\mathcal{F} : k - \mathrm{alg} \to \mathcal{D}$ satisfies flat descent, then the following natural morphism is an equivalence

$$\mathcal{F}(R) \xrightarrow{\sim} \lim\left(\mathcal{F}(R_{\mathrm{perf}}) \rightrightarrows \mathcal{F}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \rightrightarrows\!\!\!\!\to \mathcal{F}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \rightrightarrows\!\!\!\!\rightrightarrows \cdots\right).$$

[2]Recall first (see [BMS18]) that A_{inf}-cohomology is defined by the complex denoted $A\Omega$. More precisely, given the p-completion of a smooth $\mathcal{O}_C$-algebra A, we have $R\Gamma_{A_{\mathrm{inf}}}(A) := R\Gamma(A_{\mathrm{Zar}}, A\Omega_A)$.

Construction 3.2.5 says roughly that the value of $\mathcal{F}$ on R can be recovered given the values of $\mathcal{F}$ on the $R_{\text{perf}} \otimes_R \cdots \otimes_R R_{\text{perf}}$. Hence, the idea is now to compute the value of our functor of interest (that is, $\mathrm{TC}^-(-;\mathbb{Z}_p)$) on these k-algebras "$R_{\text{perf}} \otimes_R \cdots \otimes_R R_{\text{perf}}$". But what are these algebras ? To compute anything about them, we first need to learn some of their properties. Recall that, for now, we just computed $\pi_0\mathrm{TC}^-(-;\mathbb{Z}_p)$ on perfect(oid) rings, in Theorem 3.2.1.

So, first, one note that the k-algebras $R_{\text{perf}}\otimes_R\cdots\otimes_R R_{\text{perf}}$ (with R still a smooth k-algebra) are not perfect. Instead, they are locally quotients of perfect k-algebras by regular sequences. Indeed, locally $R_{\text{perf}}\otimes_R\cdots\otimes_R R_{\text{perf}} = (k[\mathbf{t}]_{\text{perf}}\otimes_k\cdots\otimes_k k[\mathbf{t}]_{\text{perf}})\otimes_{k[\mathbf{t}]} R$, with $k[\mathbf{t}]_{\text{perf}} = k[t_1^{1/p^\infty}, \ldots, t_d^{1/p^\infty}]$. These are perfect k-algebras modulo some regular sequence (essentially because every smooth scheme is regular).

Example 3.2.6. If $R = k[t]$, then $R_{\text{perf}} = k[t^{1/p^\infty}]$, and $R_{\text{perf}}\otimes_R R_{\text{perf}} = k[t^{1/p^\infty}]\otimes_{k[t]} k[t^{1/p^\infty}] \cong k[x^{1/p^\infty}, y^{1/p^\infty}]/(x-y)$, where $t^{1/p^j}\otimes 1 \mapsto x^{1/p^j}$, $1\otimes t^{1/p^j} \mapsto y^{1/p^j}$, and $x-y$ is a regular sequence (with only one element).

However, locally a quotient of a perfect algebra by a regular sequence is not a good notion when dealing with non-noetherian rings (such as perfect or perfectoid rings). This is mainly because for non-noetherian rings, several (potentially good) definitions of a regular sequence –which were equivalent for noetherian rings– do not coincide anymore.

This motivates the following wider definition, which is well-defined for arbitrary non-noetherian rings:

Definition 3.2.7. *A $\mathbb{F}_p$-algebra A is quasiregular semiperfect (qrsp) if it is semiperfect (i.e. its Frobenius is surjective), and the cotangent complex $\mathbb{L}_{A/\mathbb{F}_p}$ is a flat A-module supported in homological degree* 1.

Example 3.2.8. An easy example of quasiregular semiperfect $\mathbb{F}_p$-algebra is given by perfect $\mathbb{F}_p$-algebras. Indeed, these are of course semiperfect, and their cotangent complex over $\mathbb{F}_p$ is trivial, hence a flat module supported in degree 1.

The name "quasiregular" historically comes from the following definition of Quillen.

Definition 3.2.9. *An ideal I of a ring A is quasiregular if I/I^2 is a flat A/I-module, and $\pi_n(\mathbb{L}_{(A/I)/A}) = 0$ for all $n > 1$. For instance, if I is an ideal I of a noetherian ring A and is locally generated by a regular sequence, then I is quasiregular.*

The following lemma relates the previous two definitions.

Lemma 3.2.10. *Let A be a $\mathbb{F}_p$-algebra. Then A is a quasiregular semiperfect algebra if and only if there exists a perfect $\mathbb{F}_p$-algebra S and a quasiregular ideal I of S such that $A \cong S/I$.*

Proof. In both senses of implications, A is supposed to be semiperfect. So let us fix a perfect $\mathbb{F}_p$-algebra S with a surjective morphism $S \to A$, and let I be the kernel of this morphism. We need to prove that A is quasiregular semiperfect if and only if I is a quasiregular ideal, that is, $\mathbb{L}_{A/\mathbb{F}_p}$ is a flat A-module supported in homological degree 1 if and only if $\mathbb{L}_{A/S}$ is. Recall that for a morphism $T \to S$ of perfect algebras, the cotangent complex $\mathbb{L}_{S/T} \simeq 0$ is trivial. This can be proved by taking a simplical resolution $P_\bullet$ of S by polynomial T-algebras, and then noting that the Frobenius morphism on $\Omega^1_{P_\bullet/S}$ is at the same time zero and an equivalence. In particular, $\mathbb{L}_{S/\mathbb{F}_p} \simeq 0$. So, by the transitivity sequence for the cotangent complex, we conclude that $\mathbb{L}_{A/\mathbb{F}_p} \simeq \mathbb{L}_{A/S}$, hence the result. ◆

We still need to check now that the algebras $R_{\text{perf}} \otimes_R \cdots \otimes_R R_{\text{perf}}$, for R a smooth k-algebra, are indeed quasiregular semiperfect algebras (Definition 3.2.7). To do so, one can either argue directly, or use the quasisyntomic topology (see Example 3.2.16 and Lemma 3.2.15).

Remark 3.2.11. The analog notion in mixed-characteristic for quasiregular semiperfect algebras is called *quasiregular semiperfectoid* algebras (see Section A.1 or [BMS19]). For most purposes, they behave as their characteristic p siblings, and in particular quasiregular semiperfectoid $\mathbb{F}_p$-algebras are exactly the quasiregular semiperfect algebras of Definition 3.2.7.

We finally arrive at the definition of the quasisyntomic site. Roughly, it formalises the idea that there exist other rings than just smooth ones that admit nice flat covers such as the $R \to R_{\mathrm{perf}}$ of Construction 3.2.5.

More precisely, we would like to perform some descent from smooth algebras over $\mathcal{O}_K$ or $\mathcal{O}_C$ to quasiregular semiperfectoid algebras. To do so, one needs to have a well-behaved site-theoretic construction which would include both types of algebras: on the first hand "nice" finite type algebras over the integers of a p-adic number field, and on the other hand highly non-noetherian -but whose cotangent complex is not so badly behaved, see Definition 3.2.7- algebras. The "well-behaved" notion naturally boils down to controlling the cotangent complex of a given morphism. A reason why the cotangent complex is a natural object to consider here is because it controls topological Hochschild homology (see Chapter 2, Theorem 2.1.25 for a version in the classical theory, and Theorem 3.4.15).

Definition 3.2.12. *A morphism $A \to B$ of $\mathbb{F}_p$-algebras is quasisyntomic if it is flat, and $\mathbb{L}_{B/A}$ has Tor-amplitude in $[-1;0]$ (i.e. for each B-module M, $\mathbb{L}_{B/A}\otimes^{\mathbb{L}}_B M$ is supported in cohomological degrees $[-1;0]$). It is a quasisyntomic cover if it is faitfully flat.*

A $\mathbb{F}_p$-algebra A is quasisyntomic if the morphism $\mathbb{F}_p \to A$ is. Let QSyn *(resp.* qSyn_A*, for a quasisyntomic $\mathbb{F}_p$-algebra A) be the category of quasisyntomic $\mathbb{F}_p$-algebras (resp. quasisyntomic A-algebras), with all maps (and not only quasisyntomic ones).*

The category QSyn defines naturally a site, called the *quasisyntomic site*:

Lemma 3.2.13. *The category* $\mathrm{QSyn}^{\mathrm{op}}$ *has the structure of a site, with covers given by the quasisyntomic covers. The same holds true for the (opposite) category* $\mathrm{QRSP}^{\mathrm{op}}$ *of quasiregular semiperfect $\mathbb{F}_p$-algebras.*

For any abelian presheaf $\mathcal{F}$ on the site qSyn_A (Definition 3.2.12), we write $R\Gamma_{\mathrm{syn}}(A, \mathcal{F}) := R\Gamma(\mathrm{qSyn}_A, \mathcal{F})$ for the cohomology of its sheafification.

Proof. We need to check the three axioms for a covering family. Isomorphisms are indeed quasisyntomic covers. Quasisyntomic covers are stable under compositions. So it remains to prove that the pushout of a cover along an arbitrary map exists, and is a cover. Let $C \leftarrow A \to B$ a diagram in QSyn, with $A \to B$ a quasisyntomic cover. Let $D := B \otimes_A C$ be the pushout in $\mathbb{F}_p$-algebras. By transitivity of the cotangent complex, one proves that $C \to D$ is a quasisyntomic cover. Because C is quasisyntomic, this implies that D is also quasisyntomic. Hence $C \to D$ provides a pushout of $A \to B$ in QSyn. The statement for quasiregular semiperfect $\mathbb{F}_p$-algebras is a consequence of the first part, and Example 3.2.14. ◆

The following example confirms that quasiregular semiperfect algebras are quasisyntomic, and gives a criterion on how to recognize them among all the quasisyntomic algebras.

Example 3.2.14. A $\mathbb{F}_p$-algebra A is quasiregular semiperfect if and only if it is quasisyntomic and semiperfect. The first implication is a consequence of Lemma 3.2.15. To prove the converse, we need to prove that $\pi_0(\mathbb{L}_{A/\mathbb{F}_p}) \simeq 0$ if A is semiperfect and quasisyntomic. And this is a consequence of the transitivity property for the cotangent complex, applied to the morphisms $\mathbb{F}_p \to A_{\mathrm{perf}} \to A$, together with the fact that $\pi_0(\mathbb{L}_{A/S}) = 0$ for any surjective morphism $S \to A$.

We are now able to confirm that the quasisyntomic site, equipped with its quasiregular semiperfect(oid) objects, is indeed well-suited to perform the type of descent announced at the beginning of this section.

Lemma 3.2.15. ([BMS19], Lemma 4.28) *A p-complete ring A lies in* QSyn *exactly when there exists a quasisyntomic cover $A \to S$ with $S \in$ QRSPerd. When this holds, all terms $S \otimes_A \cdots \otimes_A S$ of the Čech nerve are quasiregular semiperfectoid. In particular, quasiregular semiperfectoid rings form a basis for the quasisyntomic site.*

Proof. For simplicity, we do the proof only in characteristic p. First, note that if $A \to B$ is a quasisyntomic cover of p-complete rings, then A is quasisyntomic if and only if B is quasisyntomic. Hence if there exists a quasisyntomic cover $A \to S$ with S quasiregular semiperfectoid, then S is quasisyntomic and so is A. For the converse, suppose A is quasisyntomic, take a surjective morphism $\mathbb{F}_p[X_i, i \in I] \twoheadrightarrow A$ from a polynomial algebra over $\mathbb{F}_p$, and define $S := A \otimes_{\mathbb{F}_p[X_i, i \in I]} \mathbb{F}_p[X_i^{1/p^\infty}, i \in I]$. Then, by stability of quasisyntomic covers under pushout (see the proof of Lemma 3.2.13), the morphism $A \to S$ is a quasisyntomic cover. Finally, S is a quotient of $\mathbb{F}_p[X_i^{1/p^\infty}, i \in I]$ and is then semiperfect, so Example 3.2.14 (the reverse implication) implies that S is quasiregular semiperfect.

To prove the Čech statement, one proves by taking repeated pushouts and compositions that $A \to S^{\otimes_A n}$ is quasisyntomic. So $S^{\otimes_A n}$ is quasisyntomic. But it is also semiperfect, since it is a quotient of $\mathbb{F}_p[X_i^{1/p^\infty}, i \in I]^{\otimes_{\mathbb{F}_p[X_i, i \in I]} n}$ which is semiperfect. Hence the result by Example 3.2.14. ◆

Example 3.2.16. In characteristic p, one can construct the cover $A \to S$ of Lemma 3.2.15 in the following way: take a surjective morphism $\mathbb{F}_p[X_i, i \in I] \twoheadrightarrow A$ from a polynomial algebra over $\mathbb{F}_p$, and define $S := A \otimes_{\mathbb{F}_p[X_i, i \in I]} \mathbb{F}_p[X_i^{1/p^\infty}, i \in I]$.

In particular, if $A = R$ is a smooth k-algebra, we recover from Lemma 3.2.15 that $R_{\text{perf}} \otimes_R \cdots \otimes_R R_{\text{perf}}$ is quasiregular semiperfect. Indeed, the Frobenius on R is surjective, so the previous construction of S coincides with the perfection R_{perf} of R.

We conclude with an application of this formalism. As we said before, one can compare A_{inf}-cohomology to the functor $\pi_0 \text{TC}^-(-; \mathbb{Z}_p)$. This is done as follows.

For any quasisyntomic base ring A, $\text{TC}^-(-; \mathbb{Z}_p)$ is a presheaf on the site qSyn_A (in fact even a sheaf by Theorem 3.4.4), and so the association $B \mapsto \pi_0 \text{TC}^-(B; \mathbb{Z}_p)$ defines a presheaf of rings on the site qSyn_A.

Taking cohomology of its quasisyntomic sheafification then gives the following comparison result:

Theorem 3.2.17. ([BMS19]) *Let A be an $\mathcal{O}_C$-algebra that can be written as the p-adic completion of a smooth $\mathcal{O}_C$-algebra. There is a functorial (in A) isomorphism of $\mathbb{E}_\infty$-A_{inf}-algebras*

$$A\Omega_A \simeq R\Gamma_{\text{syn}}(A, \pi_0 \text{TC}^-(-; \mathbb{Z}_p)).$$

We will not prove this result, since it has no purpose in the rest of the text. But we will encounter one of the main tools in proving it, that is the filtration on $R\Gamma_{syn}(A, \pi_0\, TC^-(-; \mathbb{Z}_p))$ (see Section 3.4).

3.3 A Breuil-Kisin cohomology theory over $\mathcal{O}_K$

As it was a major motivation in defining the quasisyntomic site (see [BMS19]), we review the application to Breuil-Kisin cohomology theories. Roughly, the quasisyntomic site is used to define a well-behaved p-adic cohomology theory that refines étale, crystalline and de Rham cohomology.

More precisely, we use a relative version $\pi_0\mathrm{TC}^-(-/\mathbb{S}[z];\mathbb{Z}_p)$ of the functor $\pi_0\mathrm{TC}^-(-;\mathbb{Z}_p)$. Again, this is a presheaf for the quasisyntomic topology. And we use the cohomology of its sheafification to define a Breuil-Kisin cohomology. Remark that this idea is quite natural, since A_{inf}-cohomology already compared to the three main p-adic cohomology theories (over $\mathcal{O}_C$).

Theorem 3.3.1. ([BMS19], Theorem 1.2) *Let $\mathfrak{X}$ be proper smooth formal scheme over the ring $\mathcal{O}_K$. There is a $\mathfrak{S}$-linear cohomology theory $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ equipped with a φ-linear Frobenius map $\varphi : R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \to R\Gamma_{\mathfrak{S}}(\mathfrak{X})$, with the following properties:*

(1) After the extension to A_{inf}, it recovers the A_{inf}-cohomology theory

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \simeq R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}_{\mathcal{O}_C}).$$

In particular, $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ is a perfect complex of $\mathfrak{S}$-modules, and φ induces an isomorphism

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S},\varphi} \mathfrak{S}\left[\frac{1}{E}\right] \simeq R\Gamma_{\mathfrak{S}}(\mathfrak{X})\left[\frac{1}{E}\right],$$

and so all $\mathrm{H}^i_{\mathfrak{S}}(\mathfrak{X}) := \mathrm{H}^i(R\Gamma_{\mathfrak{S}}(\mathfrak{X}))$ are Breuil-Kisin modules. Moreover, after scalar extension to $A_{\mathrm{inf}}\left[\frac{1}{\mu}\right]$, one recovers étale cohomology.

(2) After scalar extension along some canonical map $\theta : \mathfrak{S} \twoheadrightarrow \mathcal{O}_K$ (see page 2 of [BMS19]), one recovers de Rham cohomology

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S}} \mathcal{O}_K \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_K).$$

(3) After scalar extension along the map $\mathfrak{S} = W(k)[\![u]\!] \twoheadrightarrow W(k)$ which is the Frobenius on $W(k)$ and sends u to 0, one recovers crystalline cohomology of the special fibre

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S}} W(k) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k)).$$

Proof. We give only an idea on how the proof goes on. The proof in itself would take us too far from the path of this text.

The Breuil-Kisin cohomology $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ in this result is constructed via topological Hochschild homology. More precisely, one uses a relative variant of topological Hochschild homology, that is, defined over the $\mathbb{E}_\infty$-ring spectrum $\mathbb{S}[z]$ instead of the sphere spectrum $\mathbb{S}$. An intuition one can keep about why we use this relative version instead of usual topological Hochschild homology is that we want to mimic the morphisms between the base rings of our p-adic cohomology (*e.g.* $\mathfrak{S} \to \mathcal{O}_K$, which sends the formal variable u to a uniformiser in $\mathcal{O}_K$), and to so we use the "formal variable" z in $\mathbb{S}[z]$.

A relative version of topological negative cyclic homology $\mathrm{TC}^-(-/\mathbb{S}[z];\mathbb{Z}_p)$ is also defined, and the Frobenius pullback of $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ is defined as

$$\varphi^* R\Gamma_{\mathfrak{S}}(\mathfrak{X}) := R\Gamma_{\mathrm{syn}}(\mathfrak{X}, \pi_0\mathrm{TC}^-(-/\mathbb{S}[z];\mathbb{Z}_p)).$$

The Frobenius descended object $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ of Theorem 3.3.1 is then constructed in a somewhat indirect way, using direct calculations on $\mathrm{TC}^-(-/\mathbb{S}[z];\mathbb{Z}_p)$ and $\mathrm{TP}(-/\mathbb{S}[z];\mathbb{Z}_p)$, and a version of the so-called Segal conjecture.

◆

Remark 3.3.2. As a comparison, the Frobenius descended object $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ in Theorem 3.3.1 can also be defined in a site-theoretic way and more directly, using the prismatic theory of [BS19] (see Section 4.3).

The proof of Theorem 3.3.1 relies on the construction of a filtration on $\pi_0\mathrm{TC}^-(-;\mathbb{Z}_p)$, where graded pieces can be identified on a large class of p-adic rings : that is, on quasisyntomic rings. Similar filtrations, we will describe in the next section, are defined on the spectra $\mathrm{THH}(A;\mathbb{Z}_p)$, $\mathrm{TC}^-(A;\mathbb{Z}_p)$, $\mathrm{TP}(A;\mathbb{Z}_p)$ and $\mathrm{TC}(A;\mathbb{Z}_p)$, for any quasisyntomic ring A.

3.4 Filtration on THH, TC$^-$ and TP

We arrive now at the core of the chapter, that is introducing the "motivic" filtrations on THH, TC$^-$ and TP. First we define them; then we describe them locally (on perfectoid rings, and then more generally on quasiregular semiperfectoid rings). Finally, we identify the graded pieces of these filtrations in general (that is, on any quasisyntomic ring). This is done via descent on the quasisyntomic site to quasiregular semiperfectoid rings, which form a basis for the quasisyntomic topology.

3.4.1 Definition of the filtration

We first define the object $\widehat{\mathbb{\Delta}}_-$, which is central to express the filtrations on THH, TC$^-$ and TP, and which already made short appearances in the previous sections. Then we shall define our filtrations on $\mathrm{THH}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\mathrm{TP}(-;\mathbb{Z}_p)$.

Definition 3.4.1. *For any quasisyntomic ring A, define the $\mathbb{E}_\infty$-$\mathbb{Z}_p$-algebra $\widehat{\mathbb{\Delta}}_A$ as follows*

$$\widehat{\mathbb{\Delta}}_A := R\Gamma_{\mathrm{syn}}(A, \pi_0\mathrm{TC}^-(-;\mathbb{Z}_p)).$$

For instance, if $A = S$ is a quasiregular semiperfectoid ring, then $\widehat{\mathbb{\Delta}}_S = R\Gamma_{\mathrm{syn}}(S, \pi_0\mathrm{TC}^-(-;\mathbb{Z}_p))$, as a sheaf of complexes evaluated on S, is concentrated in degree 0; this is a direct consequence of Theorem 3.4.16.(2). Hence we can write

$$\widehat{\mathbb{\Delta}}_S = \pi_0\mathrm{TC}^-(S;\mathbb{Z}_p).$$

Said in a more fancy way, the presheaf $\pi_0\mathrm{TC}^-(-;\mathbb{Z}_p)$, on a base of the site qSyn_A (given by quasiregular semiperfectoid rings S), is already a sheaf with vanishing higher cohomology.

More generally, $\widehat{\mathbb{\Delta}}_-$ will be a central object when studying the functors $\pi_*\mathrm{THH}(-;\mathbb{Z}_p)$, $\pi_*\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\pi_*\mathrm{TP}(-;\mathbb{Z}_p)$ in the p-adic context. Indeed, together with its Nygaard filtration (Theorem 3.4.16.(1)), it will encode all the informations given by the graded pieces of the "motivic" filtrations Definition 3.4.5 (see Theorem 3.4.20). One should be careful though that Nygaard filtration on $\widehat{\mathbb{\Delta}}_A = R\Gamma_{\mathrm{syn}}(A, \pi_0\, TC^-(-;\mathbb{Z}_p))$ is not, even intuitively, the restriction of the filtration on $\mathrm{TC}^-(A;\mathbb{Z}_p)$ of Definition 3.4.5 at the "π_0-level": the graded pieces of this filtration will correspond to the Nygaard filtration levels (and not the graded pieces) on $\widehat{\mathbb{\Delta}}_A$.

Remark 3.4.2. (Why a "$\mathbb{\Delta}$" symbol ?) The $\mathbb{E}_\infty$-$\mathbb{Z}_p$-algebra $\widehat{\mathbb{\Delta}}_A$ of Definition 3.4.1 is also the completion of prismatic cohomology $\mathbb{\Delta}_A$ with respect to its Nygaard filtration. The filtration of [BMS19] on $\widehat{\mathbb{\Delta}}_A$ –we aim at defining here, see Theorem 3.4.16–, also called Nygaard filtration, coincides with the filtration coming from the prismatic theory; see Theorem 4.4.4.

The idea to define our filtrations on $\mathrm{THH}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\mathrm{TP}(-;\mathbb{Z}_p)$ is similar to what we did in Section 3.2: we descend on the quasisyntomic site to quasiregular semiperfectoid algebras. And locally, we want to define the filtrations as simple Postnikov filtrations:

Definition 3.4.3. *For a spectrum X (e.g.* $\mathrm{THH}(-;\mathbb{Z}_p)$*,* $\mathrm{TC}^-(-;\mathbb{Z}_p)$ *or* $\mathrm{TP}(-;\mathbb{Z}_p)$ *for A a quasisyntomic ring), the Postnikov filtration on X is defined by:* $\mathrm{Fil}^n X := \tau_{\geqslant n} X$*, where $\tau_{\geqslant n} X$, $n \in \mathbb{Z}$, is a spectrum with homotopy groups:* $\pi_k(\tau_{\geqslant n} X) \cong \begin{cases} \pi_k(X) & \text{if } k \geqslant n \\ 0 & \text{if } k < n. \end{cases}$ *We remark that this construction is functorial in X.*

But first, we need to check that $\mathrm{THH}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\mathrm{TP}(-;\mathbb{Z}_p)$ are well-suited for such a descent:

Theorem 3.4.4. ([BMS19], Theorem 3.1, Corollary 3.4) *The functors:* $\mathrm{THH}(-)$*,* $\mathrm{TC}^-(-)$*,* $\mathrm{TP}(-)$ *on the category of commutative rings satisfy flat descent. In particular,* $\mathrm{THH}(-)$*,* $\mathrm{TC}^-(-)$*,* $\mathrm{TP}(-)$ *and their p-completed analogues* $\mathrm{THH}(-;\mathbb{Z}_p)$*,* $\mathrm{TC}^-(-;\mathbb{Z}_p)$*,* $\mathrm{TP}(-;\mathbb{Z}_p)$ *satisfy descent on the quasisyntomic site.*

Proof. This relies on the fact that, over a fixed base ring R, the cotangent complex $\mathbb{L}_{-/R} : A \mapsto \mathbb{L}_{A/R}$ satisfies flat descent. This is a profound result, proved by Bhatt in [Bha12]. Concretely, we first reduce to the case of $\mathrm{THH}(-)$ by using short exact sequences and limits over the HKR filtration. And then we use the notion of "weak Postnikov tower" and Lemma 3.1.3 to reduce to the case of $\mathrm{HH}(-)$. And we prove this functor satisfies flat descent by taking limit and by induction –the initialisation being with the cotangent complex– over the HKR filtration. The last assertion follows from the fact that the flat topology is finer than the quasisyntomic topology. ◆

Now these three functors also define presheaves for the quasisyntomic topology , and then we can define the filtrations on $\mathrm{THH}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\mathrm{TP}(-;\mathbb{Z}_p)$ as a Postnikov filtration, taken locally on the quasisyntomic site. We remark that these presheaves will vanish on any quasiregular semiperfect algebra for i odd (this is a part of Theorem 3.4.16), and so we will discard the odd terms and reindex the filtration[3]:

Definition 3.4.5. *Let A be quasisyntomic ring, and define*

$$\mathrm{Fil}^n \mathrm{THH}(A;\mathbb{Z}_p) = R\Gamma_{\mathrm{syn}}(A, \tau_{\geqslant 2n}\mathrm{THH}(-;\mathbb{Z}_p)),$$

$$\mathrm{Fil}^n \mathrm{TC}^-(A;\mathbb{Z}_p) = R\Gamma_{\mathrm{syn}}(A, \tau_{\geqslant 2n}\mathrm{TC}^-(-;\mathbb{Z}_p)),$$

$$\mathrm{Fil}^n \mathrm{TP}(A;\mathbb{Z}_p) = R\Gamma_{\mathrm{syn}}(A, \tau_{\geqslant 2n}\mathrm{TP}(-;\mathbb{Z}_p)).$$

These are complete exhaustive decreasing multiplicative $\mathbb{Z}$-indexed filtrations.

In fact, one should remark that, given the flat cover $R \to R_{\mathrm{perf}}$ from Construction 3.2.5, we can construct more explicitly the filtrations of Definition 3.4.5 using the following general consequence of Definition 3.2.3:

Corollary 3.4.6. *Let k be a perfect field of characteristic p and* $\mathcal{F} : k\text{-alg} \to \mathcal{D} = \mathcal{D}(\mathbb{Z}), \mathcal{D}(k), \mathrm{Sp}$ *a functor satisfying flat descent. Let R be a smooth k-algebra, and denote $\mathcal{F}_i(-) := \pi_i \mathcal{F}(-)$. Then $\mathcal{F}(R) \in \mathcal{D}$ has a natural complete descending $\mathbb{Z}$-indexed filtration with i^{th}-graded piece given by the $[i]$-shift of the cochain complex associated (via Dold-Kan) to the totalisation of the cosimplicial abelian group*

$$\mathcal{F}_i(R_{\mathrm{perf}}) \rightrightarrows \mathcal{F}_i(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \mathcal{F}_i(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow \\[-1em] \rightarrow} \cdots$$

[3]The fact that our functors are supported in even degrees is not anodyne, and is related to the notion of even cohomology theory. See Remark 6.4.3.

Proof. Each $\mathcal{F}(R_{perf} \otimes_R \cdots \otimes_R R_{perf})$ is an object of the category $\mathcal{D} = \mathcal{D}(\mathbb{Z}), \mathcal{D}(k), Sp$, and hence can be equipped with its natural complete, descending $\mathbb{Z}$-indexed Postnikov filtration $\tau_{\geqslant i}\mathcal{F}(R_{perf} \otimes_R \cdots \otimes_R R_{perf})$. By flat descent and functoriality of the Postnikov filtration, this formally induces a filtration on $\mathcal{F}(R)$. This filtration is still complete since the limits $\mathcal{F}(R_{perf} \otimes_R \cdots \otimes_R R_{perf}) \xrightarrow{\sim} \lim_i \tau_{<i}\mathcal{F}(R_{perf} \otimes_R \cdots \otimes_R R_{perf})$ commute with the totalisation. The graded pieces of this filtration on $\mathcal{F}(R)$ are then precisely the desired objects, that is

$$\mathrm{gr}^i \mathcal{F}(R) \simeq \lim \left(\mathcal{F}_i(R_{perf}) \rightrightarrows \mathcal{F}_i(R_{perf} \otimes_R R_{perf}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathcal{F}_i(R_{perf} \otimes_R R_{perf} \otimes_R R_{perf}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots\right)[i].$$

◆

Here is a first relation between these filtrations and $\widehat{\mathbb{\Delta}}_{-}$:

Example 3.4.7. The 0-th graded piece of the filtration of Definition 3.4.5 on TC^- is given by: $\mathrm{gr}^0\mathrm{TC}^-(A; \mathbb{Z}_p) \simeq \widehat{\mathbb{\Delta}}_A$. Compare with Theorem 3.4.20.

Remark 3.4.8. For A be a quasisyntomic ring, one should not confuse the objects $\pi_0\mathrm{TC}^-(A; \mathbb{Z}_p)$ and $\widehat{\mathbb{\Delta}}_A := R\Gamma_{\mathrm{syn}}(A; \pi_0\mathrm{TC}^-(-; \mathbb{Z}_p))$: the second one is the quasisyntomic sheafified version of the first. In particular, $\pi_0\mathrm{TC}^-(-; \mathbb{Z}_p)$ –which is quasisyntomic presheaf of rings– is not well-suited to perform descent to quasiregular semiperfectoid algebras, for which we use $\widehat{\mathbb{\Delta}}_{-} := R\Gamma_{\mathrm{syn}}(-; \pi_0\mathrm{TC}^-(-; \mathbb{Z}_p))$ –which is really a quasisyntomic sheaf, with values that are complexes in the derived category. As a comparison, the graded pieces of the "naive" Postnikov filtration (that is, the one on $\mathrm{TC}^-(A; \mathbb{Z}_p)$) are just the homotopy groups of $\mathrm{TC}^-(A; \mathbb{Z}_p)$, and hence bear less informations on the ring A.

Remark 3.4.9. The idea of defining a filtration locally for some topology goes back to the definition of the motivic filtration on complex K-theory (see Section 5.1). One takes a cover of a given topological space by open contractible subsets, defines locally the filtration on these subsets –which K-theory is easier to understand because they are homotopically equivalent to a point–, and then globalises the construction. In this type of constructions, the choice of the topology and the covering families is crucial. In our case, the key point to globalise our constructions from quasiregular semiperfectoids to any quasisyntomic algebra is the equivalence (given by the restriction functor) between the categories of abelian sheaves on $\mathrm{QRSPerd}^{\mathrm{op}}$ and $\mathrm{QSyn}^{\mathrm{op}}$ ([BMS19] Proposition 4.31, or page 25 of [Mor19]).

3.4.2 THH, TC$^-$ and TP on perfectoid rings

The reason why we like to perform descent to quasiregular semiperfectoid rings is that $\mathrm{THH}(-; \mathbb{Z}_p)$ and its variants are well-behaved on these. Perfectoid rings R are an important example of quasiregular semiperfectoid rings, and one can compute (the homotopy groups of) $\mathrm{THH}(R; \mathbb{Z}_p)$, $\mathrm{TC}^-(R; \mathbb{Z}_p)$ and $\mathrm{TP}(R; \mathbb{Z}_p)$ explicitly.

Proposition 3.4.10. *Let R be a perfectoid ring. There are natural isomorphisms*

$$\pi_*\mathrm{THH}(R; \mathbb{Z}_p) \cong R[u],$$
$$\pi_*\mathrm{TC}^-(R; \mathbb{Z}_p) \cong A_{\mathrm{inf}}[u, v]/(uv - \xi),$$
$$\pi_*\mathrm{TP}(R; \mathbb{Z}_p) \cong A_{\mathrm{inf}}[\sigma, \sigma^{-1}],$$

where u, σ are (formal variables) in degree 2, v is in degree -2, and $\xi \in A_{\mathrm{inf}}(R)$ is the usual generator of the kernel of $\theta : A_{\mathrm{inf}}(R) \twoheadrightarrow R$. Moreover, the canonical map from $\pi_*TP(R; \mathbb{Z}_p)$ to $\pi_*TC^-(R; \mathbb{Z}_p)$ maps $\sigma \mapsto \xi u$.

Proof. We sketch the proof for THH. First, we claim that $\mathrm{THH}(-;\mathbb{Z}_p)$ satisfies base change for perfectoid ring: that is, for any map $R \to R'$ of perfectoid rings, the induced map $\mathrm{THH}(R;\mathbb{Z}_p)\otimes^{\mathbb{L}}_R R' \to \mathrm{THH}(R';\mathbb{Z}_p)$ is an equivalence. This can be proved by reduction to $\mathrm{HH}(-;\mathbb{Z}_p)$ via Lemma 3.1.3, and then by reduction to the cotangent complex via the HKR filtration.

Now, the result is true for $R = \mathbb{F}_p$ thanks to Theorem 3.1.2. Hence the base change property implies the similar result for any perfectoid ring in characteristic p. Then the general case follows from a more careful use of the base change property.

The other calculations also reduce first to the characteristic p case, and then to the case of $R = \mathbb{F}_p$, even though the proof is more intricate. ◆

Note that this generalises Theorem 3.2.1. We now state a direct consequence of this structure result.

Corollary 3.4.11. *Let R be a perfectoid ring. For any algebra A over the perfectoid ring R, we can consider $\pi_*\mathrm{THH}(A;\mathbb{Z}_p)$ (resp. $\pi_*\mathrm{TC}^-(A;\mathbb{Z}_p)$, $\pi_*\mathrm{TP}(A;\mathbb{Z}_p)$) as a graded algebra over the graded ring $\pi_*\mathrm{THH}(R;\mathbb{Z}_p)$ (resp. $\pi_*\mathrm{TC}^-(R;\mathbb{Z}_p)$, $\pi_*\mathrm{TP}(R;\mathbb{Z}_p)$). Hence $\pi_*\mathrm{TP}(A;\mathbb{Z}_p)$ is 2-periodic (that is, $\pi_{*+2}\mathrm{TP}(A;\mathbb{Z}_p) \cong \pi_*\mathrm{TP}(A;\mathbb{Z}_p)$).*

In particular, if S is a quasiregular semiperfectoid ring, with a map $R \to S$ from a perfectoid ring R, then $\widehat{\mathbb{\Delta}}_S$ has a canonical structure of $A_{\mathrm{inf}}(R)$-module.

Proof. This is a consequence of the universal property for $\mathrm{THH}(R;\mathbb{Z}_p)$ (resp.$\mathrm{TC}^-(R;\mathbb{Z}_p)$, $\mathrm{TP}(R;\mathbb{Z}_p)$), that is there is a structural ($\mathbb{T}$-equivariant) morphism of $\mathbb{E}_\infty$-ring spectra $\mathrm{THH}(R;\mathbb{Z}_p) \to \mathrm{THH}(A;\mathbb{Z}_p)$, which induces a morphism of graded rings $\pi_*\mathrm{THH}(R;\mathbb{Z}_p) \to \pi_*\mathrm{THH}(A;\mathbb{Z}_p)$, making $\pi_*\mathrm{THH}(A;\mathbb{Z}_p)$ into a $\pi_*\mathrm{THH}(R;\mathbb{Z}_p)$-graded algebra. The last assertion is a consequence of Proposition 3.4.10 : $\pi_*\mathrm{THH}(R;\mathbb{Z}_p)$ is 2-periodic, and then any graded algebra over $\pi_*\mathrm{THH}(R;\mathbb{Z}_p)$ is 2-periodic. ◆

Remark 3.4.12. Despite the name, periodic topological cyclic homology is not always periodic. For instance, $\pi_*\mathrm{TP}(\mathbb{Z})$ is not.

3.4.3 THH, TC$^-$ and TP on quasiregular semiperfectoid rings

Despite the description of $\mathrm{THH}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\mathrm{TP}(-;\mathbb{Z}_p)$ is not as direct as for perfectoid rings, one can still compute them explicitly on any quasiregular semiperfectoid ring (Theorem 3.4.16 and Corollary 3.4.19).

Recall that $\mathrm{THH}(-)$ is a functor that takes an $\mathbb{E}_\infty$-ring spectrum A (*e.g.* some usual commutative ring) and builds an $\mathbb{E}_\infty$-ring spectrum $\mathrm{THH}(A)$ equipped with an action of the circle group $\mathbb{T} = S^1$.

One can form the homotopy fixed points for this action: $\mathrm{TC}^-(A) := \mathrm{THH}(A)^{h\mathbb{T}}$, and the Tate construction: $\mathrm{TP}(A) := \mathrm{THH}(A)^{t\mathbb{T}}$, *i.e.* the cone of the norm map $\mathrm{THH}(A)_{h\mathbb{T}} \to \mathrm{THH}(A)^{h\mathbb{T}}$ from homotopy orbits to homotopy fixed points. In particular, there is a canonical map

$$\mathrm{can} : \mathrm{TC}^-(A) \to \mathrm{TP}(A).$$

Remark 3.4.13. One should be careful we do not define topological cyclic homology TC naively as the homotopy orbits $\mathrm{THH}(-)_{h\mathbb{T}}$ –which could seem natural considering the classical definition of cyclic homology HC (see Proposition 2.1.19). Instead, the definition of TC relies on the “Frobenius” map $\varphi = \varphi_p^{h\mathbb{T}} : \mathrm{TC}^-(A;\mathbb{Z}_p) \to (\mathrm{THH}(A)^{t\mathbb{T}})^{h\mathbb{T}} \simeq \mathrm{TP}(A;\mathbb{Z}_p)$ (well-defined for any connective $\mathbb{E}_\infty$-ring spectrum A) introduced by Nikolaus-Scholze in [NS18] (see Section 3.5).

These structures allow us to define spectral sequences converging to $\pi_*\mathrm{TC}^-$ and $\pi_*\mathrm{TP}$:

Lemma 3.4.14. *Let A be an $\mathbb{E}_\infty$-ring spectrum. Then there is a homotopy fixed point spectral sequence*

$$E_2^{i,j} = \mathrm{H}^{-i+j}(\mathbb{T}, \pi_j \mathrm{THH}(A)) \implies \pi_{i+j}\mathrm{TC}^-(A),$$

and a similar spectral sequence for the Tate construction, converging to $\pi_*\mathrm{TP}(A)$.

Proof. This is an example of the Bousfield-Kan spectral sequence for homotopy limits. More informally, one can think of it as a "Grothendieck spectral sequence", associated to the composition of taking fixed points (for the $\mathbb{T}$-action) and taking π_0 (and similarly for the Tate construction). ◆

Theorem 3.4.15. ([BMS19], Theorem 7.1) *Let R be a perfectoid ring and $S \in \mathrm{QRSPerd}_R$. Then $\pi_*\mathrm{THH}(S;\mathbb{Z}_p)$ is concentrated in even degrees, and each $\pi_{2i}\mathrm{THH}(S;\mathbb{Z}_p)$, for $i \geqslant 0$, admits a finite decreasing filtration with graded pieces given in ascending order by $\Gamma_S^j(M)_p^\wedge$, for $0 \leqslant j \leqslant i$, with $M := \pi_1(\mathbb{L}_{R/S})_p^\wedge$. Here $\Gamma_S^j(M)$ denotes the j-th divided product of the S-module M over S, and $\Gamma_S^j(M)_p^\wedge$ is its p-completion.*

More informally, the last part of Theorem 3.4.15 says that $\mathrm{THH}(S;\mathbb{Z}_p)$ (for S a quasiregular semiperfectoid algebra over R) is controlled by the cotangent complex $\mathbb{L}_{S/R}$.

Proof. One can prove that $\mathrm{THH}(-;\mathbb{Z}_p)$ admits a complete descending multiplicative $\mathbb{N}$-indexed filtration on the categoryof p-complete R-algebras, with graded pieces

$$\mathrm{gr}^n\mathrm{THH} \simeq \bigoplus_{\substack{0\leqslant i\leqslant n \\ i-n \text{ even}}} (\wedge^i \mathbb{L}_{-/R})_p^\wedge[n].$$

This can be proved first on (quasi)smooth R-algebras, and then on any p-complete R-algebras by left Kan extension. Now if S is quasiregular semiperfectoid, then $(\wedge^i \mathbb{L}_{-/R})_p^\wedge$ has p-complete Tor amplitude concentrated in homological degree i, and in particular lives in degree i. Hence each term of the graded pieces lives in degree $i+n$, which is even. The completeness of the filtration implies that $\pi_*\mathrm{THH}(S;\mathbb{Z}_p)$ is concentrated in even degrees. The previous filtration also induces a filtration on each $\pi_{2i}\mathrm{THH}(S;\mathbb{Z}_p)$, which graded pieces are the terms of the previous graded pieces concentrated in degree $2i$, hence the result. ◆

Let R be a perfectoid ring, and $S \in \mathrm{QRSPerd}_R$. The canonical map $\pi_*\mathrm{TC}^-(S;\mathbb{Z}_p) \xrightarrow{can} \pi_*\mathrm{TP}(S;\mathbb{Z}_p)$ is injective in all degrees, and an isomorphism in degrees $\leqslant 0$. This is a formal consequence of the spectral sequences computing $\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\mathrm{TP}(-;\mathbb{Z}_p)$ in terms of $\mathrm{THH}(-;\mathbb{Z}_p)$, together with the fact that $\pi_*\mathrm{THH}(S;\mathbb{Z}_p)$ is supported in even degrees. In particular, $\pi_0\mathrm{TC}^-(S;\mathbb{Z}_p) \cong \pi_0\mathrm{TP}(S;\mathbb{Z}_p)$.

We now state the following structural result, relating $\widehat{\Delta}_S$ with $\mathrm{THH}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\mathrm{TP}(-;\mathbb{Z}_p)$. A consequence of this result is the expression, for any quasiregular semiperfectoid ring S, of all the homotopy groups $\pi_*\mathrm{THH}(S;\mathbb{Z}_p)$, $\pi_*\mathrm{TC}^-(S;\mathbb{Z}_p)$ and $\pi_*\mathrm{TP}(S;\mathbb{Z}_p)$ in terms of $\widehat{\Delta}_S$ and its Nygaard filtration.

Theorem 3.4.16. ([BMS19], Theorem 7.2) *Let R be a perfectoid ring, and $S \in \mathrm{QRSPerd}_R$.*

(1) The homotopy fixed point spectral sequence calculating $\mathrm{TC}^-(S;\mathbb{Z}_p)$ and the Tate spectral sequence calculating $\mathrm{TP}(S;\mathbb{Z}_p)$ degenerate. Each of these two (degenerate) spectral sequences endows $\widehat{\Delta}_S := \pi_0\mathrm{TC}^-(S;\mathbb{Z}_p) \cong \pi_0\mathrm{TP}(S;\mathbb{Z}_p)$ with the same complete descending $\mathbb{Z}$-indexed filtration $\mathcal{N}^{\geqslant\star}\widehat{\Delta}_S$, called the Nygaard filtration, for which it is complete (see Remark 3.4.2)[4].

[4]This Nygaard filtration satisfies $\mathcal{N}^{\geqslant i}\widehat{\Delta}_S = \widehat{\Delta}_S$ for all $i \leqslant 0$, and hence can also be considered as an $\mathbb{N}$-indexed filtration.

(2) Both $\pi_*\mathrm{TC}^-(S;\mathbb{Z}_p)$ and $\pi_*\mathrm{TP}(S;\mathbb{Z}_p)$ live only in even degrees.

(3) For each $i \geqslant 0$, there is a natural identification of the Nygaard filtration level

$$\mathcal{N}^{\geqslant i}\widehat{\Delta}_S \cong \pi_{2i}\mathrm{TC}^-(S;\mathbb{Z}_p).$$

More precisely, the isomorphism is given by the multiplication by $v^i \in \pi_{-2i}\mathrm{TC}^-(R;\mathbb{Z}_p)$: $\pi_{2i}\mathrm{TC}^-(S;\mathbb{Z}_p) \xrightarrow{v^i} \pi_0\mathrm{TC}^-(S;\mathbb{Z}_p)$ (see Proposition 3.4.10 and Corollary 3.4.11 for the definition of this map).

(4) For all $i \geqslant 0$, there are natural identifications of the Nygaard graded pieces $\mathcal{N}^i\widehat{\Delta}_S := \mathcal{N}^{\geqslant i}\widehat{\Delta}_S/\mathcal{N}^{\geqslant i+1}\widehat{\Delta}_S$

$$\mathcal{N}^i\widehat{\Delta}_S \cong \pi_{2i}\mathrm{THH}(S;\mathbb{Z}_p).$$

Proof. The first two points are a consequence of the fact that $\pi_*\mathrm{THH}(S;\mathbb{Z}_p)$ lives in even degrees. More precisely, the differential maps in the second page of the spectral sequences are all zero, so the spectral sequences degenerate and induce the desired filtration on $\pi_0\, TC^-(S;\mathbb{Z}_p) \cong \pi_0\, TP(S;\mathbb{Z}_p)$. A similar argument proves (2).

For (3), we use the homotopy fixed point spectral sequence and restrict to proving a similar statement about $\mathrm{THH}(S;\mathbb{Z}_p)$. This is then a consequence of the $\mathbb{T}$-equivariant $\mathrm{THH}(R;\mathbb{Z}_p)$-module spectrum structure on $\mathrm{THH}(S;\mathbb{Z}_p)$. Remark that the "$\mathbb{T}$-equivariant" is necessary here, since we take homotopy fixed points for the $\mathbb{T}$-action in the spectral sequence.

(4) is a consequence of the definition of the Nygaard filtration on $\widehat{\Delta}_S$ from the spectral sequences of (1), and again the fact that $\pi_*\mathrm{THH}(S;\mathbb{Z}_p)$ is concentrated in even degrees. It can also be proved as a consequence of (3), if we identify the morphism $\mathcal{N}^{\geqslant i}\widehat{\Delta}_S \leftarrow \mathcal{N}^{\geqslant i+1}\widehat{\Delta}_S$ with the multiplication by v on $\pi_{2i+2}\mathrm{TC}^-(S;\mathbb{Z}_p)$. Remark here that the Nygaard filtration is defined, as in Definition 3.4.5, with a double-speed Postnikov flavour. ◆

3.4.4 Breuil-Kisin twists

We define here the notion of Breuil-Kisin twist. This aims at avoiding some choices made in the previous results, and was introduced in [BMS19], §6.2.

The identifications in Theorem 3.4.16 depend on some choices: that is, the choice of coordinates u and v, and the choice of ξ (as a generator of the kernel of the map $\theta : A_{\mathrm{inf}} \twoheadrightarrow R$) in the isomorphism $\pi_*\mathrm{TC}^-(R;\mathbb{Z}_p) \cong A_{\mathrm{inf}}[u,v]/(uv-\xi)$ (Proposition 3.4.10). For convenience, we would like to make these identifications more canonical; and to do so, we choose an A_{inf}-module of rank 1, which we will call a (Breuil-Kisin) "twist" $A_{\mathrm{inf}}\{1\}$ of A_{inf}, to play the role of the degree 2 part in the graded ring structures $\pi_*\mathrm{TC}^-(R;\mathbb{Z}_p)$ and $\pi_*\mathrm{TP}(R;\mathbb{Z}_p)$. We do not consider specific elements of this ring $A_{\mathrm{inf}}\{1\}$. Given Proposition 3.4.10, a good candidate for this A_{inf}-module is $\pi_2\mathrm{TP}(R;\mathbb{Z}_p)$[5]: the other homotopy groups of interest can be expressed in term of this one. More generally, we define the *Breuil-Kisin* twist of any A_{inf}-module and any R-module as follows.

Definition 3.4.17. *Let R be a perfectoid ring, and $A_{\mathrm{inf}} := A_{\mathrm{inf}}(R)$ the associated period ring. For an A_{inf}-module M and $i \in \mathbb{Z}$, the Breuil-Kisin twist $M\{i\}$ is: $M\{i\} := M \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}\{1\}^{\otimes i}$, where $A_{\mathrm{inf}}\{1\} := \pi_2\mathrm{TP}(R;\mathbb{Z}_p)$ (see Proposition 3.4.10). If M is an R-module, then $M\{i\}$ denotes the corresponding twist when M is considered an A_{inf}-module via the map $\tilde{\theta} := \theta\circ\varphi^{-1} : A_{\mathrm{inf}} \twoheadrightarrow R$.*

One could wonder why we use the map $\tilde{\theta}$, instead of the usual map θ from A_{inf} to R to define the Breuil-Kisin twist of an R-module. The reason is that, for a perfectoid ring R, we

[5]Remind that a perfectoid ring R is hidden in the shortened notation A_{inf} of $A_{\mathrm{inf}}(R)$.

would like this Breuil-Kisin twist $R\{i\}$ to express canonically the isomorphism of graded rings $\pi_*\mathrm{THH}(R;\mathbb{Z}_p) \cong R[u]$ of Proposition 3.4.10 (that is, without a choice of coordinate u). More precisely, we aim at a result of the form

$$\pi_*\mathrm{THH}(R;\mathbb{Z}_p) = \bigoplus_{i\geqslant 0} R\{i\}.$$

Now, $\pi_2\mathrm{THH}(R;\mathbb{Z}_p)$ is canonically isomorphic to $\ker(\theta)/\ker(\theta)^2$ –which, as a ring, is isomorphic to R, but also bears a non-trivial graded ring structure from $\pi_*\mathrm{THH}(R;\mathbb{Z}_p)$. Indeed, $\pi_2\mathrm{THH}(R;\mathbb{Z}_p) \cong \pi_2\mathrm{HH}(R/\mathbb{Z}_p)$ (this is true for any commutative ring) and $\pi_*\mathrm{HH}(R/\mathbb{Z}_p)$ can be computed with the HKR filtration, which has graded pieces $(\wedge^i_R \mathbb{L}_{R/\mathbb{Z}_p})^\wedge_p[i] \simeq R[2i]$ (cf [BMS19], Proposition 4.19 for the last identification). Moreover, the base change $A_{\mathrm{inf}}\{1\} \otimes_{A_{\mathrm{inf}},\theta} R \cong R$ is canonically trivial, while the base change $A_{\mathrm{inf}}\{1\} \otimes_{A_{\mathrm{inf}},\tilde{\theta}} R \cong \ker(\theta)/\ker(\theta)^2$. This explains the choice of using $\tilde{\theta}$ in Definition 3.4.17.

Using these Breuil-Kisin twists, the identifications of Proposition 3.4.10 can be rewritten as follows.

Proposition 3.4.18. ([BMS19], Proposition 6.5) *Let R be a perfectoid ring. Defining the Nygaard filtration on A_{inf} as $\mathcal{N}^{\geqslant i}A_{\mathrm{inf}} = \xi^i A_{\mathrm{inf}}$ for $i \geqslant 0$ and $\mathcal{N}^{\geqslant i}A_{\mathrm{inf}} = A_{\mathrm{inf}}$ for $i \leqslant 0$, and the graded pieces $\mathcal{N}^i A_{\mathrm{inf}} := \mathcal{N}^{\geqslant i}A_{\mathrm{inf}}/\mathcal{N}^{\geqslant i+1}A_{\mathrm{inf}}$, there are natural isomorphisms:*

$$\pi_*\mathrm{THH}(R;\mathbb{Z}_p) \cong \bigoplus_{i\geqslant 0} R\{i\} = \bigoplus_{i\geqslant 0} \mathcal{N}^i A_{\mathrm{inf}},$$
$$\pi_*\mathrm{TC}^-(R;\mathbb{Z}_p) \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{N}^{\geqslant i}A_{\mathrm{inf}}\{i\},$$
$$\pi_*\mathrm{TP}(R;\mathbb{Z}_p) \cong \bigoplus_{i\in\mathbb{Z}} A_{\mathrm{inf}}\{i\},$$

under which the canonical map $TC^- \to TP$ corresponds to the inclusion $\mathcal{N}^{\geqslant i}A_{inf} \to A_{inf}$, and the Frobenius map $TC^- \to TP$ corresponds to the Frobenius $A_{inf}[\frac{1}{\xi}] \to A_{inf}[\frac{1}{\tilde{\xi}}]$ which sends $\mathcal{N}^{\geqslant i}A_{inf}\{i\}$ into $A_{inf}\{i\}$.

Proof. The first isomorphism follows from the multiplicative structure of Proposition 3.4.10, and the fact that $A_{\mathrm{inf}} \otimes_{A_{\mathrm{inf}},\tilde{\theta}} R \cong (\ker\theta)/(\ker\theta)^2 = R\{1\}$. The third isomorphism is also a consequence of the multiplicative structure described in Proposition 3.4.10, and the definition of $A_{\mathrm{inf}}\{1\}$. The second isomorphism follows from unwiding the definitions of the Frobenius maps on TC^- and $\mathcal{N}^{\geqslant\star}A_{\mathrm{inf}}$. ◆

We can also express the values of $\pi_*\mathrm{THH}(-;\mathbb{Z}_p)$, $\pi_*\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\pi_*\mathrm{TP}(-;\mathbb{Z}_p)$ on quasiregular semiperfectoid rings in a simpler way by using Breuil-Kisin twists. Remark we find back Proposition 3.4.18 when applied to perfectoid rings.

Corollary 3.4.19. *Let R be a perfectoid ring, and $S \in \mathrm{QRSPerd}_R$ be a quasiregular semiperfectoid algebra over R.*

(1) The homotopy groups $\pi_*\mathrm{TP}(S;\mathbb{Z}_p)$ are 2-periodic: $\pi_{*+2}\mathrm{TP}(S;\mathbb{Z}_p) \cong \pi_*\mathrm{TP}(S;\mathbb{Z}_p)$. More precisely, there is a canonical isomorphism of graded rings

$$\pi_*\mathrm{TP}(S;\mathbb{Z}_p) \cong \bigoplus_{i\in\mathbb{Z}} \widehat{\mathbb{\Delta}}_S\{i\} = \bigoplus_{i\in\mathbb{Z}} \pi_0\mathrm{TP}(S;\mathbb{Z}_p)\{i\}.$$

(2) *There is a canonical isomorphism of graded rings*

$$\pi_*\mathrm{TC}^-(S;\mathbb{Z}_p) \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{N}^{\geqslant i}\widehat{\mathbb{\Delta}}_S\{i\} = \bigoplus_{i\in\mathbb{Z}} \mathcal{N}^{\geqslant i}\pi_0\mathrm{TP}(S;\mathbb{Z}_p)\{i\}.$$

In particular, the canonical map $\pi_*\mathrm{TC}^-(S;\mathbb{Z}_p) \to \pi_*\mathrm{TP}(S;\mathbb{Z}_p)$ *is an isomorphism in non-positive degrees.*

(3) *There is a canonical isomorphism of graded rings*

$$\pi_*\mathrm{THH}(S;\mathbb{Z}_p) \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{N}^{i}\widehat{\mathbb{\Delta}}_S\{i\} = \bigoplus_{i\in\mathbb{Z}} \mathcal{N}^{i}\pi_0\mathrm{TP}(S;\mathbb{Z}_p)\{i\}.$$

Proof. The first part of (1) is a part of Corollary 3.4.11. The rest of (1) follows from Definition 3.4.1 of $\widehat{\mathbb{\Delta}}_S$ and Definition 3.4.17 of the Breuil-Kisin twists.

For (2), we use the multiplicative structure described in Theorem 3.4.16.(3) and again Definition 3.4.17 of the Breuil-Kisin twists.

Finally, forgetting the multiplicative structure, (3) is just a part of Theorem 3.4.16. The multiplicative structure is then ensured by applying (2) to the short exact sequence $0 \to \mathrm{TC}^-(S;\mathbb{Z}_p)[-2] \to \mathrm{TC}^-(S;\mathbb{Z}_p) \to \mathrm{THH}(S;\mathbb{Z}_p) \to 0$.

◆

3.4.5 Graded pieces of THH, TC$^-$ and TP

We conclude the section by the following result, which differs from Corollary 3.4.19 by its generality: it describes the graded pieces of the motivic filtrations for any quasisyntomic ring A, instead of just quasiregular semiperfectoid ones. Note that, in this generality, the filtrations need not to be split.

Theorem 3.4.20. ([BMS19], Theorem 1.12) *Let A be a quasisyntomic ring. There are natural isomorphisms:*

$$\begin{aligned} \mathrm{gr}^n\mathrm{THH}(A;\mathbb{Z}_p) &\simeq \mathcal{N}^n\widehat{\mathbb{\Delta}}_A\{n\}[2n] \simeq \mathcal{N}^n\widehat{\mathbb{\Delta}}_A[2n],\\ \mathrm{gr}^n\mathrm{TC}^-(A;\mathbb{Z}_p) &\simeq \mathcal{N}^{\geqslant n}\widehat{\mathbb{\Delta}}_A\{n\}[2n],\\ \mathrm{gr}^n\mathrm{TC}^-(A;\mathbb{Z}_p) &\simeq \widehat{\mathbb{\Delta}}_A\{n\}[2n]. \end{aligned}$$

These induce multiplicative spectral sequences:

$$\begin{aligned} \mathrm{E}_2^{i,j} &= \mathrm{H}^{i-j}(\mathcal{N}^{-j}\widehat{\mathbb{\Delta}}_A) \Rightarrow \pi_{-i-j}\mathrm{THH}(A;\mathbb{Z}_p),\\ \mathrm{E}_2^{i,j} &= \mathrm{H}^{i-j}(\mathcal{N}^{\geqslant -j}\widehat{\mathbb{\Delta}}_A\{-j\}) \Rightarrow \pi_{-i-j}\mathrm{TC}^-(A;\mathbb{Z}_p),\\ \mathrm{E}_2^{i,j} &= \mathrm{H}^{i-j}(\mathcal{N}^{-j}\widehat{\mathbb{\Delta}}_A\{-j\}) \Rightarrow \pi_{-i-j}\mathrm{TP}(A;\mathbb{Z}_p). \end{aligned}$$

Sketch of proof. The idea to prove Theorem 3.4.20 is to perform descent on the quasisyntomic site -which is allowed by Theorem 3.4.4- to quasiregular semiperfectoid algebras, where we already identified the graded pieces of our filtrations on $\mathrm{THH}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\mathrm{TP}(-;\mathbb{Z}_p)$ (Theorem 3.4.16).

More precisely, one proves this result first on R-algebras A, with R a fixed perfectoid ring. The reason why we do so is to make sense of the Breuil-Kisin twist (the objects $\mathrm{THH}(A;\mathbb{Z}_p)$, $\mathrm{TC}^-(A;\mathbb{Z}_p)$ and $\mathrm{TP}(A;\mathbb{Z}_p)$ are independant of the choice of R). In this context, we unfold the filtration by descent on the quasisyntomic site, that is we globalise the construction of the filtration, already constructed locally (namely on quasiregular semiperfectoids over R), to define it on any quasisyntomic ring A over R. To perform the unfolding, and because we are dealing with filtrations, we use the notion of filtered derived category in order to make the constructions canonical.

In general the complex $\widehat{\Delta}_A\{1\}$ is defined as the first graded piece of the filtration Definition 3.4.5: $\widehat{\Delta}_A\{1\} := \mathrm{gr}^1\mathrm{TP}(A;\mathbb{Z}_p)[-2]$. It is again equipped with a Nygaard filtration $\mathcal{N}^{\geqslant \star}\widehat{\Delta}_A\{1\}$ coming via quasisyntomic descent of the abutment filtration of Theorem 3.4.19. The proof then uses properties of the filtered derived category and some base change results to a perfectoid base ring. ◆

3.5 TC, after Nikolaus-Scholze

Before regarding the motivic filtration on topological cyclic homology TC in Section 6.1 –like we just did for THH, TC^- and TP– we still need to give a construction for TC, which is somehow more subtle than the definitions of TC^- and TP.

As we saw earlier, one could think of defining TC in the same way we define its classical version cyclic homology HC. However, it appeared to be a bit too naive way to define TC as the homotopy orbits $\mathrm{THH}(-)_{h\mathbb{T}}$ of THH under its $\mathbb{T}$-action. The main historical reason is that it does not satisfy McCarthy's theorem relating relative K-theory and relative TC of a nilpotent ideal. Here we explain how to define TC (as in [NS18]), using some arithmetic flavoured fiber sequence[6]: see Definition 3.5.5.

Definition 3.5.1 ([NS19], Definition II.1.1). *(1) A cyclotomic spectrum is a spectrum X with a $\mathbb{T}$-action together with $\mathbb{T}$-equivariant maps $\varphi_p : X \to X^{tC_p}$ for every prime number p. Here $C_p \subset \mathbb{T}$ denotes the cyclic subgroup of order p of $\mathbb{T}$.*

(2) For a fixed prime number p, a p-cyclotomic spectrum is a spectrum X with C_{p^∞}-action and a C_{p^∞}-equivariant map $\varphi_p : X \to X^{tC_p}$. Here $C_{p^\infty} \subset \mathbb{T}$ denotes the subgroup of $\mathbb{T}$ of p-power torsion elements, and C_p its cyclic subgroup of order p.

Remark that for arithmetic geometry applications, all the ring spectra are commutative ring spectra (that is, $\mathbb{E}_\infty$-ring spectra). However, most of the definitions and results concerning topological cyclic homology (*e.g.* the definition of TC Definition 3.5.3) can be given for general associative and unital ring spectra (that is, $\mathbb{E}_1$-ring spectra).

Example 3.5.2 ([NS19], Example II.1.2). (1) For every associative ring spectrum $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$, the topological Hochschild homology $\mathrm{THH}(R)$ is a cyclotomic spectrum . More precisely, given a prime number p, the Tate diagonal of R is, by definition, the map $\Delta_p : R \to (R^{\otimes p})^{tC_p}$. It extends to a $\mathbb{T}$-equivariant map $\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ defined levelwise by the Tate diagonal Δ_p, and hence defines the cyclotomic spectrum structure on $\mathrm{THH}(R)$. See [NS18, III.2.] for details.

(2) The sphere spectrum $\mathbb{S}$, with the trivial $\mathbb{T}$-action, has the structure of a cyclotomic spectrum called the *cyclotomic sphere*. One can prove it is equivalent to $\mathrm{THH}(\mathbb{S})$ as a cyclotomic spectrum, and that the maps φ_p are p-completions.

(3) For every cyclotomic spectrum we get a p-cyclotomic spectrym by restriction. In particular we can consider $\mathrm{THH}(R)$ and $\mathbb{S}$ as p-cyclotomic spectra (for any prime number p), and we do not distinguish these notationally.

[6] A (homotopy) fiber sequence is a $(\infty,1)$-categorical construction that formalises the notion of homotopy kernel (or homotopy fiber) of a morphism.

Definition 3.5.3 ([NS19], Definition II.1.8). *(1) Let $(X, (\varphi_p)_{p \in \mathcal{P}})$ be a cyclotomic spectrum. The integral topological cyclic homology* $\mathrm{TC}(X)$ *of* X *is the mapping spectrum* $\mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}, X) \in \mathrm{Sp}$.

(2) Let (X, φ_p) be a p-cyclotomic spectrum. The p-typical topological cyclic homology $\mathrm{TC}(X; \mathbb{Z}_p)$ *of X is the mapping spectrum* $\mathrm{map}_{\mathrm{CycSp}_p}(\mathbb{S}, X) \in \mathrm{Sp}$.

(3) Let $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$ be an associative ring spectrum. Then $\mathrm{TC}(R) := \mathrm{TC}(\mathrm{THH}(R))$ *and* $\mathrm{TC}(R; \mathbb{Z}_p) := \mathrm{TC}(\mathrm{THH}(R); \mathbb{Z}_p)$.

From the previous general definition, we will use mainly the following particular case:

Definition 3.5.4 (The Frobenius morphism on TC^-). *Let A be a connective $\mathbb{E}_\infty$-ring spectrum. The "Frobenius" $\varphi_p : \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ (from Definition 3.5.1) induces a map $\varphi_p := \varphi_p^{h\mathbb{T}}$:* $\mathrm{TC}^-(A; \mathbb{Z}_p) \to (\mathrm{THH}(A)^{tC_p})^{h\mathbb{T}} \simeq \mathrm{TP}(A; \mathbb{Z}_p)$.

Although the definition of topological cyclic homology in [NS18] (that is, Definition 3.5.3) is more natural and uses the notion of cyclotomic spectra, we give the following characterization of TC.

Definition 3.5.5 ([NS19], Corollary 1.5). *For any connective $\mathbb{E}_\infty$-ring spectrum, there is a natural fiber sequence*

$$\mathrm{TC}(A) \to \mathrm{THH}(A)^{h\mathbb{T}} \xrightarrow{(\varphi_p - \mathrm{can})} \prod_{p \in \mathbb{P}} (\mathrm{THH}(A)^{tC_p})^{h\mathbb{T}}$$

where $\mathrm{can} : \mathrm{THH}(A)^{h\mathbb{T}} \simeq (\mathrm{THH}(A)^{hC_p})^{h(\mathbb{T}/C_p)} = (\mathrm{THH}(A)^{hC_p})^{h\mathbb{T}} \to (\mathrm{THH}(A)^{tC_p})^{h\mathbb{T}}$ *denotes the canonical projection, with the isomorphism $\mathbb{T}/C_p \cong \mathbb{T}$ in the middle identification.*

Let A be a connective $\mathbb{E}_\infty$-ring spectrum. One could rewrite the fiber sequence of Definition 3.5.5 as an equalizer between the maps can, $(\varphi_p)_{p \in \mathbb{P}} : \mathrm{THH}(A)^{h\mathbb{T}} = \mathrm{TC}^-(A) \to \prod_{p \in \mathbb{P}} (\mathrm{THH}(A)^{tC_p})^{h\mathbb{T}} \simeq \prod_{p \in \mathbb{P}} \mathrm{TP}(A; \mathbb{Z}_p)$. Restricting to the p-completion for a given prime p, this leads to the following characterization of $\mathrm{TC}(-; \mathbb{Z}_p)$:

Corollary 3.5.6. *Let A be a connective $\mathbb{E}_\infty$-ring spectrum. The p-completion* $\mathrm{TC}(A; \mathbb{Z}_p)$ *of the spectrum* $\mathrm{TC}(A)$ *satisfies the following natural fiber sequence*

$$\mathrm{TC}(A; \mathbb{Z}_p) \to \mathrm{TC}^-(A; \mathbb{Z}_p) \xrightarrow{\varphi_p - \mathrm{can}} \mathrm{TP}(A; \mathbb{Z}_p).$$

Chapter 4

Prisms and prismatic cohomology

Prismatic cohomology finds its motivation in the Breuil-Kisin cohomology defined in [BMS19] (see Section 3.3). More precisely –and we will keep this analogy as a motivating thought in this chapter–, prismatic cohomology is a site-theoretic cohomology theory defined in mixed characteristic, which is analogous in its construction to crystalline cohomology in characteristic p. It is constructed in [BS19] from the notion of prisms, which was defined for the occasion.

The relevance of prismatic cohomology for us is that is brings a completely independant construction not only of the Breuil-Kisin cohomology $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ of Section 3.3, but also of the Nygaard filtration of Theorem 3.4.16. Forseing a bit the future, this will lead to another independant construction of syntomic cohomology (see Section 6.2).

This chapter first reviews the notion of δ-ring, which is a way to encode lifts of Frobenius on a large class of rings. We then define prisms (which are in particular δ-rings if we forget some additional structure) and prismatic cohomology, while giving some examples and applications. The Nygaard filtration is defined on (absolute) prismatic cohomology in Section 4.4 by descent on the quasisyntomic site.

4.1 δ-rings

Frobenius endomorphism is a central object in arithmetic geometry. When dealing with mixed-characteristic objects (for instance, finite extensions of $\mathbb{Q}_p$, or schemes over finite extensions of $\mathbb{Q}_p$), one uses the similar notion of lift of Frobenius, that is, an endomorphism which coincides modulo p with the usual Frobenius endomorphism.

Example 4.1.1. The ring of integers $\mathbb{Z}$ has reduction $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ modulo p, whose Frobenius endormorphism is the identity. In particular, the only possible lift of Frobenius on $\mathbb{Z}$ is the identity. Similarly, there is exactly one Frobenius lift on the p-adic integers $\mathbb{Z}_p$, given by the identity morphism.

More generally, if a ring R is p-torsionfree, then the Frobenius lift $\varphi : R \to R$ can be written as $\varphi : x \mapsto x^p + p\delta(x)$, for some map $\delta : R \to R$. This is by definition of being a Frobenius lift. Although the map δ is not a morphism of rings (because φ is, and the two are not compatible), this is not a random map from R to itself. Indeed, φ **is** a morphism of rings, and hence satisfies the usual multiplication and addition formulas: $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x + y) = \varphi(x) + \varphi(y)$. And since R is assumed to be p-torsionfree, one can write, for any $x \in R$: $\delta(x) = \frac{\varphi(x)-x^p}{p}$, which implies the map δ satisfies the two identities:

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y),$$
$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}.$$

Conversely, given a map δ satisfying these two properties, the map $\varphi : x \mapsto x^p + p\delta(x)$ is a lift of Frobenius.

When the ring R is not torsionfree, the notion of lift of Frobenius is a bit more subtle. For instance, keeping the previous notations of a map δ satisfying the two previous properties and the associated lift of Frobenius φ, it is a stronger condition for an ideal $I \subseteq R$ to be stable under δ than stable under φ.

Definition 4.1.2. *A δ-ring is a pair (R, δ) where R is a commutative ring and $\delta : R \to R$ is a map of sets with $\delta(0) = \delta(1) = 0$, satisfying the following two identities:*

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y),$$
$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}.$$

The category of δ-rings is the category with objects the δ-rings, and with morphisms the morphisms of rings which respect the δ-structure.

Remark 4.1.3. The expression $\frac{x^p+y^p-(x+y)^p}{p}$ in Definition 4.1.2 is a polynomial in $\mathbb{Z}[x, y]$. Thus it is well-defined, independently of any p-torsion issue.

A δ-ring (R, δ) (or simply R) is automatically equipped with a lift of Frobenius, defined by $\varphi(x) := x^p + p\delta(x)$. The defining conditions on δ are made so that this map φ is a ring endomorphism, and hence a lift of Frobenius. However, remark that not every lift of Frobenius comes from a δ-structure. For instance, there is no δ-structure on $\mathbb{F}_p$: $\delta(0) = \delta(p) = \delta(1 + \cdots + 1) = \delta(1) + \cdots + \delta(1) + \frac{1^p+\dots 1^p-(1+\cdots+1)^p}{p} = \frac{1^p+\cdots+1^p-(1+\cdots+1)^p}{p} = 1 - p^{p-1} = 1$, which is a contradiction. Similarly, several δ-structures on a given ring can lead to the same lift of Frobenius.

Example 4.1.4. The Frobenius lift on the ring of integers $\mathbb{Z}$ comes from a unique δ-structure, given by $\delta(n) := \frac{n-n^p}{p}$. This is well defined by the little Fermat's theorem.

Example 4.1.5. The ring of p-adic integers $\mathbb{Z}_p$ has a unique δ-ring structure, given by $\delta(n) := \frac{n-n^p}{p}$. The expression $\frac{n-n^p}{p}$ is well-defined in $\mathbb{Z}/p^k\mathbb{Z}$ for any $k \geqslant 1$ (see Example 4.1.4), and hence in $\mathbb{Z}_p$.

Definition 4.1.6. *An element d of a δ-ring R is called* distinguished *if $\delta(d)$ is a unit in R.*

Any morphism of δ-rings preserves distinguished elements.

Example 4.1.7. Let $\mathbb{Z}_p$ be the δ-ring of p-adic integers. The element $d := p$ is distinguished. Indeed, $\delta(p) = 1 - p^{p-1}$ is a unit in $\mathbb{Z}_p$. Remark that p is not distinguished in the ring of integers $\mathbb{Z}$.

Remark 4.1.8. The right definition of δ-rings (as given in [BS19]) restricts to $\mathbb{Z}_{(p)}$-algebras (instead of general rings). In particular $\mathbb{Z}$ is not strictly speaking a δ-ring, and $\mathbb{Z}_{(p)}$ is the initial object in the category of δ-rings.

4.2 Prisms

One of the motivations to introduce prismatic cohomology was to define a mixed-characteristic variant for crystalline cohomology. In this perspective, and in the same way we define divided power rings in the context of crystalline cohomology, we first introduce the following notion of prism.

Definition 4.2.1. *A prism is a pair (A, I) where A is a δ-ring (which induces a lift of the Frobenius on A/p, denoted φ_A), and $I \subseteq A$ is an ideal defining a Cartier divisor in* Spec(A)*, satisfying the following two conditions:*

- *The ring A is derived (p, I)-adically complete*[1].
- *The ideal $I + \varphi_A(I)A$ contains p.*

We say a ring A is J-adically complete if it is a-complete for each element $a \in J$. In particular, if the ideal I of Definition 4.2.1 is generated by an element d, the ring A is (p, I)-adically complete if and only if it is p-complete and d-complete.

Remark 4.2.2. Unwinding the definitions, a special case of *prism* is the data of a p-torsionfree ring A, a lift of Frobenius $\varphi_A : A \to A$ of the Frobenius on A/p, and a nonzerodivisor $d \in A$ such that A is (p, d)-complete, and $p \in \langle d, \varphi_A(d) \rangle$ (the ideal of A spanned by d and $\varphi_A(d)$).

In what follows, the ideal I is actually principal; we call any generator d of I a *distinguished element* of A.

Examples 4.2.3.

- (Crystalline Prisms) We call the prism $(A, (p))$ crystalline whenever A is a p-torsionfree and p-complete ring, with a Frobenius lift $\varphi : A \to A$. For example, the usual Frobenius lift $\varphi : W(k) \to W(k)$ on the Witt vectors $W(k)$ of a perfect field k of characteristic p defines a (crystalline) prism $(W(k), (p))$.
- (Breuil-Kisin-type Prisms) Let $\pi \in K$ be a uniformizer of K. We have a prism (A, I), defined as follows: the ring $A := W(k)[\![u]\!]$ has a δ-structure induced by the usual Frobenius lift on $W(k)$[2], and by sending u to u^p; we then define $I \subseteq A$ to be the kernel of the projection $A = W(k)[\![u]\!] \twoheadrightarrow \mathcal{O}_K$ sending u to π.
- (q-crystalline Prism) Let $(A, I) := (\mathbb{Z}_{\mathrm{p}}[\![q-1]\!], ([p]_q))$ be the prism such that: $\mathbb{Z}_{\mathrm{p}}[\![q-1]\!]$, the $(p, q-1)$-adic completion of $\mathbb{Z}[q]$, has a δ-structure via the Frobenius lift sending q to q^p; and $[p]_q := 1 + q + \cdots + q^{p-1} = \frac{q^p-1}{q-1}$ is the q analog of p.

Definition 4.2.4. *A prism (A, I) is bounded if there is some integer n such that $A/I[p^\infty] = A/I[p^n]$.*

When defining the prismatic site and prismatic cohomology, we will use in fact essentially only bounded prisms. Finally, we state without proof the following nice result, relating perfect prisms (that is, prisms (A, I) such that the Frobenius φ on A, coming from its δ-structure, is an isomorphism) to perfectoid rings:

Theorem 4.2.5 ([BS22], Theorem 3.10)**.** *The categories of perfectoid rings R and of perfect prisms (A, I) are equivalent. The functors between these two are $R \mapsto A_{\mathrm{inf}}(R), \ker(\theta))$ and $(A, I) \mapsto A/I$.*

[1] Remark that the notion of derived completeness coincide with the usual completeness for *bounded* prisms.

[2] $W(k)$ is a p-torsionfree ring, since k is a perfect field of characteristic p.

4.3 Prismatic cohomology

The notion of prisms leads to the definition of a site, called the prismatic site. Prismatic cohomology is then defined as the cohomology of some structural sheaf on this site. In particular, this recovers the construction of a Breuil-Kisin cohomology given in Chapter 3 (following [BMS19]), via a completely site-theoretic construction.

Fix a bounded prism (A, I), and a p-adic formal scheme X over A/I. The p-adic formal scheme X is the object of interest here. The idea is to consider specific prisms over (A, I) in order to get informations on X. One can see a similarity with the previously existing crystalline theory, where one studies objects of mixed-characteristic (*e.g.* over the ring of Witt vectors) to study characteristic p objects. See also Example 4.2.3.

Proposition 4.3.1 ([BS22], Proposition 1.5)**.** *If $(A, I) \to (B, J)$ is a map of prisms, then $J = IB$.*

Sketch of proof. One can prove that any prism (A, I) is locally for the flat topology a prism (that is, a δ-pair satisfying the properties of Definition 4.2.1) of the form $(A', (d))$, with d a distinguished element of A' ([BS19, Lemma 3.1.(3)]). We can thus restrict, by faithfully flat descent, to the case where (A, I) and (B, J) are of this form. Let d and e be distinguished elements of A and B such that $I = (d)$ and $J = (e)$. Now by definition of a map of δ-pairs, there is an element $f \in B$ such that $d = ef$, and unwinding the definitions shows that f is a unit in B. This proves the ideals (d) and (e) coincide in B, which is what we wanted. ◆

In other words, when working over a fixed base prism, the ideal I is not varying anymore.

Definition 4.3.2 (The prismatic site)**.** *Let (A, I) be a bounded prism and X a smooth p-adic formal scheme over the ring A/I. Let $(X/A)_{\Delta}$ be the category of maps $(A, I) \to (B, IB)$ of bounded prisms together with a map $\mathrm{Spf}(B/IB) \to X$ over A/I, and with the natural notion of morphism. We shall denote such an objects by*

$$(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X) \in (X/A)_{\Delta}$$

when the context is clear. A map $(\mathrm{Spf}(C) \leftarrow \mathrm{Spf}(C/IC) \to X) \to (\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$ in $(X/A)_{\Delta}$ is a flat cover if $(B, IB) \to (C, IC)$ is a faithfully flat map of prisms, that is, C is (p, IB)-completely flat over B (see Section A.1). The category $(X/A)_{\Delta}$ with the topology defined by flat covers is called the prismatic site of X/A.

Remark 4.3.3. (Relation between prismatic and quasisyntomic topologies) There is no direct relation between the prismatic site and the quasisyntomic site from Chapter 3. For a prism (A, I) and a formal scheme $X = \mathrm{Spf}(R)$ over A/I, one can consider the quasisyntomic site over R (of quasisyntomic R-algebras S) and the prismatic site $(X/A)_{\Delta}$, as described above. Then a sheaf on the prismatic site does not induce a sheaf on the quasisyntomic site. However, there exists a relation between the cohomology of the prismatic structure sheaf (Definition 4.3.4) and a certain quasisyntomic sheaf: see Theorem 4.4.5. This last quasisyntomic sheaf is constructed by unfolding from quasiregular semiperfectoid algebras, and via the deep comparison result Theorem 4.4.4 between the prismatic site and the theory of [BMS19] for quasiregular semiperfectoid algebras.

Definition 4.3.4 (Prismatic structure sheaf)**.** *Let (A, I) be a bounded prism and X a p-adic formal scheme over the ring A/I. The assignment $(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X) \mapsto B$ defines a presheaf $\mathcal{O}_{\Delta}$ of commutative A-algebras on the site $(X/A)_{\Delta}$. It can be proved that $\mathcal{O}_{\Delta}$ is a sheaf, and we call it the structure sheaf on $(X/A)_{\Delta}$.*

If we keep in mind (as a motivation) the analogy between crystalline and prismatic cohomology, we would like to have good properties for prismatic cohomology that we know to be true for crystalline cohomology (comparison to the de Rham complex, Cartier isomorphism, ...). This is what ensures (some parts of) the following result.

Theorem 4.3.5. *Fix a bounded prism (A, I) (which plays the role of coefficients of prismatic cohomology), and let X be a p-adic formal smooth scheme over A/I. Prismatic cohomology is defined as $R\Gamma_{\Delta}(X/A) := R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta})$, and satisfies:*

- *$R\Gamma_{\Delta}(X/A) \in D(A)$ is equipped with a φ_A-linear map φ.*
- *(Crystalline comparison) If $I = (p)$, then there is a canonical φ-equivariant isomorphism*

$$R\Gamma_{\mathrm{crys}}(X/A) \cong \varphi_A^* R\Gamma_{\Delta}(X/A)$$

  of commutative algebras in the category $D(A)$.
- *(Etale comparison) Assume A is perfect. Then, for any $n \geqslant 0$ we can compute the étale cohomology of the generic fiber X_η of X (ie over $\mathbb{Q}_{\mathrm{p}}$) with the following canonical isomorphism in $D(\mathbb{Z}/p^n)$*

$$R\Gamma_{\acute{e}t}(X_\eta, \mathbb{Z}/p^n\mathbb{Z}) \cong \left(R\Gamma_{\Delta}(X/A)/p^n\left[\frac{1}{I}\right]\right)^{\varphi=\mathrm{id}}.$$

- *(de Rham comparison) There is a canonical isomorphism*

$$R\Gamma_{\mathrm{dR}}(X/(A/I)) \cong R\Gamma_{\Delta}(X/A) \otimes^{\wedge,\mathbb{L}}_{A,\varphi_A} A/I$$

  of commutative algebras in the category $D(A)$, which extends naturally to an isomorphism of commutative differential graded algebras.
- *(Base change) Let $(A, I) \to (B, J)$ be a map of bounded prisms, and let Y be the base change of X along B/J. Then the natural map induces an isomorphism*

$$R\Gamma_{\Delta}(X/A) \otimes^{\wedge,\mathbb{L}}_{A} B \cong R\Gamma_{\Delta}(Y/B),$$

  where the completion on the left is the derived (p, J)-adic completion.

The following is a consequence of the previous big result of the prismatic theory. The aim is to see more clearly how prismatic cohomology really interpolates étale, crystalline and de Rham cohomologies via the so-called Breuil-Kisin cohomology.

Corollary 4.3.6. *We consider the Breuil-Kisin prism $(\mathfrak{S} = W(k)[\![u]\!], I)$ as defined in Example 4.2.3, and X a proper smooth scheme over $\mathcal{O}_K$. Let $E \in \mathfrak{S}$ be an element generating I (and which is then a polynomial in u). Then the Breuil-Kisin cohomology $R\Gamma_{\mathfrak{S}}(X) := R\Gamma_{\Delta}(X/\mathfrak{S})$ interpolates crystalline, étale and de Rham cohomologies in the following sense, where the quotients and fixed points are some examples of (homotopy) limits in the derived category:*

(1) $\left(R\Gamma_{\mathfrak{S}}(X) \otimes_{\mathfrak{S},\varphi} W(C^\flat)\right)^{\varphi=\mathrm{id}} \simeq R\Gamma_{\acute{e}t}(X_{\overline{K}}, \mathbb{Z}_{\mathrm{p}})$.

(2) $\left(R\Gamma_{\mathfrak{S}}(X) \otimes_{\mathfrak{S},\varphi} \mathfrak{S}\right)/u \simeq R\Gamma_{\mathrm{crys}}(X_k/W(k))$.

(3) $\left(R\Gamma_{\mathfrak{S}}(X) \otimes_{\mathfrak{S},\varphi} \mathfrak{S}\right)/E \simeq R\Gamma_{\mathrm{dR}}(X/\mathcal{O}_K)$.

Proof. We remark that the Frobenius fixed points of $W(C^\flat) = A_{\mathrm{inf}}(C)$ correspond to the ring $\mathbb{Z}_p$, hence (1) is reformulation of the étale comparison theorem for prismatic cohomology, stated in the case of $A = \mathfrak{S}$ and taking the (derived) inverse limit over n.

For (2), we use the crystalline comparison to the smooth scheme X_k and the prism $(W(k), (p))$, and the base change property for prismatic cohomology: $(R\Gamma_{\mathfrak{S}}(X) \otimes_{\mathfrak{S},\varphi} \mathfrak{S})/u = (R\Gamma_{\mathfrak{S}}(X) \otimes_{\mathfrak{S},\varphi} \mathfrak{S}) \otimes_{\mathfrak{S}} W(k) \simeq R\Gamma_{\Delta}(X_k/W(k)) \otimes_{W(k),\varphi} W(k)$.

The proof of (3) is similar to the proof of (2), replacing the crystalline comparison for prismatic cohomology with the de Rham comparison. ◆

The Breuil-Kisin cohomology $R\Gamma_{\mathfrak{S}}(X)$ coincides with the one defined in [BMS19] using topological cyclic homology. The existence of this Breuil-Kisin cohomology (constructed in either way) has the following consequences.

Example 4.3.7.

- Assume that $I = (p)$, and A is a (p-completely) smooth lift of a smooth k-algebra equipped with a Frobenius lift, for some perfect field k. Then the length of the torsion in crystalline cohomology, when finite, is a multiple of p.
- Let K be the discrete valuation field with ring of integers $\mathcal{O}_K = \mathbb{Z}_p[p^{1/p}]$. Then the torsion of $R\Gamma_{\mathrm{dR}}(X/\mathcal{O}_K)$ does not have a factor of the form $\mathcal{O}_K/p^{1/p}$.
- Let K be any discrete valuation field of mixed characteristic, and with perfect residue field k of characteristic p. The length of the torsion of $\mathrm{H}^n_{\mathrm{crys}}(X_k/W(k))$ is greater than or equal to the length of the torsion of $\mathrm{H}^n_{\acute{e}t}(X_{\overline{K}}, \mathbb{Z}_{\mathrm{p}})$.

Remark 4.3.8. Because the base prism A has to be perfect in the étale comparison case, there is no analog of Example 4.3.7.(1) about the torsion of étale cohomology.

The last example in particular (via the classical Grothendieck comparison result between Betti and étale cohomologies over $\mathbb{C}$) implies that if M is a projective, generically smooth $\mathbb{Z}$-scheme, with good reduction at p, then $\mathrm{H}^n_{\mathrm{dR}}(M/\mathbb{Z})$ has more p-primary torsion than $\mathrm{H}^n_{\mathrm{Betti}}(M(\mathbb{C}), \mathbb{Z})$.

4.4 The Nygaard filtration

The goal of this section is to review the Nygaard filtration on prismatic cohomology $\Delta_{-/A}$, defined for any bounded prism A. More precisely, it is defined on the Frobenius twist $\Delta^{(1)}_{-/A}$ of the quasisyntomic sheaf $\Delta_{-/A}$, which is itself constructed by descent to quasiregular semiperfectoid algebras.

First let us give some intuition of when one uses Nygaard-type filtrations. For p-adic rings, we suppose equipped with a Frobenius endormorphism φ, the *Nygaard filtration* can be defined as follows:

Definition 4.4.1. *Let A be a $\mathbb{Z}_{(p)}$-algebra, and $\varphi : A \to A$ a Frobenius map (that is a lift of Frobenius if A is not an $\mathbb{F}_p$-algebra). The Nygaard filtration $\mathcal{N}^{\geqslant i}A$ is defined by*

$$\mathcal{N}^{\geqslant i}A := \{x \in A \mid \varphi(x) \in p^i A\}.$$

We denote by $\mathcal{N}^i A := \mathcal{N}^{\geqslant i}A/\mathcal{N}^{\geqslant i-1}A$ for the corresponding graded pieces.

Note that the Nygaard filtration is trivial over a ring A in which p is invertible (the graded pieces are either equal to A or 0). Hence the Nygaard filtration is an interesting object only in the integral context. It was defined in the context of crystalline cohomology, and in particular when studying characteristic p objects. We will now present an analogue of this filtration (we also call

Nygaard filtration) in the prismatic context; as usual, the prime number p will be replaced, for general prisms (A, I) –that is, not necessarily satisfying $I = (p)$, which would correspond to the crystalline context– by the ideal I (see Theorem 4.4.3).

First, one can define the notion of *absolute prismatic site* $(X)_{\Delta}$ for any p-adic formal scheme X by disregarding the base (A, I) in Definition 4.3.2. More precisely, this is the category of prisms (B, J) together with a map $\mathrm{Spf}(B/J) \to X$, with (again) the flat topology. If $X = \mathrm{Spf}(S)$ is a p-adic affine formal scheme, we also denote by $(S)_{\Delta}$ the absolute prismatic site over X. If this ring S is moreover semiperfectoid, we have a better understanding of the absolute prismatic site $(S)_{\Delta}$.

Definition 4.4.2. *A ring S is semiperfectoid if it is a derived p-adically complete quotient of a perfectoid ring. If S is a $\mathbb{F}_p$-algebra, then S is semiperfectoid if and only it is semiperfect, that is if its Frobenius morphism is surjective.*

Theorem 4.4.3 ([BS22], Theorem 1.11 (1))**.** *Let S be a semiperfectoid ring. Then the category $(S)_{\Delta}$ (underlying the absolute prismatic site $(S)_{\Delta}$) has an initial object (Δ, I), and $I = (d)$ is principal.*

Moreover, using the notion of perfection for prisms and Theorem 4.4.3, one can define the perfectoidization S_{perfd} of any semiperfectoid ring S.

We define now a filtration (called the *Nygaard filtration*) on this initial prism Δ_S^{init}, when S is a semiperfectoid ring which is moreover quasiregular (Definition 3.2.7 and Definition A.1.3). Recall quasiregular semiperfectoid rings are of importance in the sense they form a basis for the quasisyntomic topology.

Theorem 4.4.4. *[BS22], Theorem 1.13] Let S be a quasiregular semiperfectoid ring (Definition 3.2.7). Write $\Delta_S = \Delta_S^{\mathrm{init}}$ for the initial prism $(\Delta_S^{\mathrm{init}}, I)$, with the understanding that Δ_S is better behaved when S is quasiregular. Then the ring Δ_S admits a natural $\mathbb{Z}$-indexed decreasing ("Nygaard") filtration, described for nonnegative degrees i by*

$$\mathcal{N}^{\geqslant i}\Delta_S = \{x \in \Delta_S \mid \varphi(x) \in I^i \Delta_S\}.$$

The ring $\widehat{\Delta}_S = \pi_0 \mathrm{TC}^-(S; \mathbb{Z}_p)$ (Definition 3.4.1) is φ-equivariantly isomorphic to the completion of Δ_S with respect to its Nygaard filtration, and in particular admits a functorial δ-structure.

Let (A, I) be a bounded prism. The construction of the Nygaard filtration of Theorem 4.4.4 now globalizes (that is, sheafifies) to smooth p-adic formal schemes over A/I. By comparing the (quasi)syntomic cohomology of this sheaf to prismatic cohomology, this induces a filtration on prismatic cohomology.

Theorem 4.4.5 ([BS22], Theorem 1.15)**.** *Let (A, I) be a bounded prism and let $X = \mathrm{Spf}(R)$ be an affine smooth p-adic formal scheme over A/I.*

(1) On the quasisyntomic site qSyn_X, one can define a sheaf of (p, I)-completely flat δ-A-algebras $\Delta_{-/A}$ equipped with a Nygaard filtration on its Frobenius twist

$$\Delta_{-/A}^{(1)} := \Delta_{-/A} \widehat{\otimes}^{\mathbb{L}}_{A,\varphi} A$$

given by

$$\mathcal{N}^{\geqslant i}\Delta_{-/A}^{(1)} = \{x \in \Delta_{-/A}^{(1)} \mid \varphi(x) \in I^i \Delta_{-/A}.\}$$

(2) There is a canonical isomorphism

$$R\Gamma_{\Delta}(X/A) \simeq R\Gamma(X_{\mathrm{qsyn}}, \Delta_{-/A})$$

and we endow prismatic cohomology with the Nygaard filtration

$$\mathcal{N}^{\geqslant i} R\Gamma_{\Delta}(X/A)^{(1)} = R\Gamma(X_{\mathrm{qsyn}}, \mathcal{N}^{\geqslant i} \Delta^{(1)}_{-/A}).$$

This filtration on prismatic cohomology will be of importance to us, since it will become a tool in defining syntomic cohomology in Chapter 6.

Chapter 5

Motivic cohomology

This chapter is a (small) review of the wide subject named motivic cohomology. The even wider subject it came from, that is the theory of motives, is also way more conjectural than its derived counterpart at this day. The goal here is to give some historical insights, as well as a few more recent results, to understand what we mean by the "motivic nature" of both the filtrations defined in Chapter 3, and syntomic cohomology (which is, according to titles, the subject of the next chapter and of this whole text).

5.1 A bit of history

In the end of the XIX$^{\text{th}}$ century, Poincaré defined singular cohomology as an good invariant for topological varieties. Several other invariants (*e.g.* simplicial cohomology, de Rham cohomology) for these varieties were defined by Poincaré and others, and were called cohomology theories because their similar nature. The question then arised of defining what **is** a cohomology theory. A satisfying set of axioms (including a Mayer-Vietoris property, compatibility with homotopy, and additivity) was proposed, and we know call (in this context of algebraic topology) a cohomology theory any theory satisfying these Eilenberg-Steenrod axioms.

The starting point for motivic cohomology is the following spectral sequence.

Theorem 5.1.1 (Atiyah-Hirzebruch spectral sequence in algebraic topology)**.** *Let* E *be a multiplicative cohomology theory and* X *a topological space. Then there is a spectral sequence converging to the cohomology groups* $\mathrm{E}^*(X)$

$$E_2^{p,q} = \mathrm{H}^p(X, \mathrm{E}^q(\mathrm{pt})) \Rightarrow \mathrm{E}^{p+q}(X),$$

where pt *is the single point topological space, and* H^* *is the singular cohomology theory. In particular, when the differentials in the spectral sequence turn out to be understandable, one can compute the cohomology groups* $\mathrm{E}^*(X)$ *given only the cohomology groups* $\mathrm{E}^*(\mathrm{pt})$ *and singular cohomology.*

This spectral sequence says that singular cohomology is "universal" in this context. In algebraic geometry, one uses also a lot of cohomology theories, some of which being algebraic analogues of the topological ones; *e.g.* (algebraic) de Rham cohomology is the algebraic analogue of de Rham cohomology. However, this is not clear - at first - how to adapt this idea of "universal cohomology theory" in algebraic geometry. To avoid confusion, such a universal theory in algebraic geometry will be called *motivic cohomology*.

In the 1980's, it was predicted (by Beilinson, Deligne, ...) that a theory of motivic cohomology $\mathrm{H}^n(X, \mathbb{Z}(i))$ for schemes X should exist, where $n \in \mathbb{Z}$ is the cohomological degree, and $i \geqslant 0$ is the motivic weight indexing the so-called Tate twist $\mathbb{Z}(i)$. One of the main motivations to define motivic cohomology is the (conjectural) theory of *motives* of Grothendieck. Motives are supposed to explain some similarities one observes in studying arithmetic and geometric aspects of algebraic varieties. In this fashion, motivic cohomology should be some form of derived version of the theory of motives:

Remark 5.1.2. (Motivic cohomology from the point of view of motives) These (motivic) cohomology groups $\mathrm{H}^n(X, \mathbb{Z}(i))$ should be, from the point of view of motives, the Ext groups between X and the i-th Tate twist in the abelian category of motives. More precisely, one can associate (conjecturally) a motive $\mathrm{M}(X)$ to any scheme X (possibly with some hypothesis on X). The i-th Tate twist $\mathbb{Z}(i)$ is constructed as an object in the abelian category of motives. The associated derived category allows one to form the Ext groups between these two objects $\mathrm{M}(X)$ and $\mathbb{Z}(i)$. Remark that, contrary to the abelian structure on the category of motives, the associated derived category of motives exists (at least in some cases).

The derived category associated to theFrom this abelian structure (which one of the major issues in defining the category of motives) allows one to form the Ext groups between these two objects $\mathrm{M}(X)$ and $\mathbb{Z}(i)$.

Remark 5.1.3. When the derived category of motives is defined, the analogues $\mathbb{F}_p(i)$ (for any prime number p), $\mathbb{Z}/p^r\mathbb{Z}(i)$ for $r \geqslant 1$, $\mathbb{Q}(i)$, $\mathbb{Z}_p(i)$, $\mathbb{Q}_p(i)$ (for $i \geqslant 0$) of the Tate twists $\mathbb{Z}(i)$ can also be defined, in the same we define $\mathbb{F}_p$, $\mathbb{Z}/p^r\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$ or $\mathbb{Q}_p$ from $\mathbb{Z}$ in the category of rings.

An important feature of motivic cohomology is its relation to algebraic K-theory. Algebraic K-theory a multiplicative cohomology theory defined for (as general as you want) algebraic varieties. It captures enough information on arithmetic, algebraic geometry and topology to formulate some of the main conjectures in arithmetic geometry (*e.g.* the standard conjectures or the Beilinson conjectures). There is a conjectural[1] "motivic" filtration on (connective) algebraic K-theory, coming from the so-called Atiyah-Hirzebruch spectral sequence (and related to the skeleton of a CW-complex in the topological setting), whose graded pieces are motivic cohomology $\mathrm{H}^n(X, \mathbb{Z}(i))$.

In the topological setting (that is, in algebraic topology), motivic cohomology is given by singular cohomology. The direct analogue of singular cohomology in algebraic geometry is étale cohomology (for instance, because its construction mimics the local analytic structure of algebraic topology). However the étale theory is, by nature, a geometric theory. Indeed, it is defined only after base change to some algebraic closure of the base field. The Zariski motivic cohomology is in this sense more universal - and arithmetic - than étale motivic cohomology. In particular, one expects Zariski motivic cohomology to detects lots of subtle arithmetic phenomena (such as relations between periods of integrals, or special values of L-functions), while étale motivic cohomology should be related to more geometric aspects of algebraic varieties (such as Galois representations, or ramification theory).

Although the abelian category of motives does not yet exist, motivic cohomology was contructed by Bloch and Voevodsky for smooth varieties over a field. In particular, we can talk about the Tate twist $\mathbb{Z}(i)_{X_{\mathrm{Zar}}}$ and the étale Tate twist $\mathbb{Z}(i)_{X_{\text{ét}}}$ (for $i \geqslant 0$) in this context. The variants of Remark 5.1.3 are also well-defined.

Example 5.1.4. For a field F of characteristic p, the complex $(\mathbb{Z}(i)/p^r)_{\mathrm{Spec}(F)_{\mathrm{Zar}}}$, for any $i \geqslant 0$ and $r \geqslant 1$, of motivic cohomology modulo p^r, is supported in degree i. This will be a consequence of the result Theorem 5.3.4 of Geisser-Levine.

[1]It currently exists for smooth varieties, and smooth schemes over a Dedekind ring.

As it is common in arithmetic and arithmetic geometry, one could try to understand a problem given over the integers $\mathbb{Z}$ (here, the construction of motivic cohomology) by decomposing this problem into “smaller” ones over on one hand the rationals $\mathbb{Q}$, and on the other hand over each prime number p. Respectively, this gives rise in our context to the theory of rational motivic cohomology, and that of “p-adic” motivic cohomology for each prime number p. More precisely, we distinguish two kinds of “p-adic” motivic cohomologies[2]: first ℓ-adic motivic cohomology cohomology, when the given prime ℓ is different from the characteristic of our base scheme X; and p-adic motivic cohomology otherwise. As usual, the behaviour in the ℓ-adic context is somehow simpler than in the p-adic context. Moreover, it is expected that one can reconstruct (global) motivic cohomology from its (local) rational, ℓ-adic and p-adic versions, which are still conjectural, via the so-called *arithmetic fracture square* (see [nLa]).

5.2 The Beilinson-Lichtenbaum conjecture and ℓ-adic étale motivic cohomology

The Beilinson-Lichtenbaum conjecture, which is stated in Theorem 5.2.1, relates usual (that is, Zariski) motivic cohomology to étale motivic cohomology. In the ℓ-adic context, it says in particular that ℓ-adic étale motivic cohomology –that is, étale sheafification of ℓ-adic motivic cohomology $\mathbb{Z}_l(i)_{X_{\mathrm{Zar}}}$– is simply ℓ-adic étale cohomology, and gives an explicit procedure to construct ℓ-adic motivic cohomology from ℓ-adic étale cohomology. A standard formulation of the Beilinson-Lichtenbaum conjecture is as follows.

Theorem 5.2.1. *(Beilinson-Lichtenbaum conjecture) Let X be a smooth scheme over a field k in which the prime ℓ is invertible. Then for any $i \geqslant 0$, there is an isomorphism of sheaves of complexes in the Zariski site of X*

$$\mathbb{Z}/\ell\,\mathbb{Z}(i)_{X_{\mathrm{Zar}}} \simeq \tau^{\leqslant i}\mathrm{R}\varepsilon_*(\mu_\ell^{\otimes i})$$

where $\varepsilon : X_{\text{ét}} \to X_{\mathrm{Zar}}$ is the restriction from the étale site to the Zariski site.

One should be careful that the functor $\tau^{\geqslant i}$ is here the Zariski sheafification of the usual truncation of complexes. In particular, both sides of the equivalence are concentrated in degrees $\leqslant i$ only Zariski locally. The Beilinson-Lichtenbaum conjecture is essentially equivalent to the Bloch-Kato conjecture in the ℓ-adic context, and was proved by Voevodsky.

Corollary 5.2.2. *Let X be a smooth scheme over a field k in which the prime ℓ is invertible. Then the mod ℓ Zariski motivic cohomology $\mathbb{Z}/\ell\,\mathbb{Z}(i)_{X_{\mathrm{Zar}}}$ is Zariski locally concentrated in degrees $[0;i]$, and thus in general concentrated in degrees $[0; i+\dim(X)]$. In degrees $n \leqslant i$, it is computed in terms of the étale cohomology of $\mu_\ell^{\otimes i}$:*

$$\mathrm{H}^n(X_{Zar}, \mathbb{Z}/\ell\,\mathbb{Z}(i)_{X_{\mathrm{Zar}}}) = \mathrm{H}^n_{\text{ét}}(X, \mu_\ell^{\otimes i}).$$

Proof. The last part is just a reformulation of Theorem 5.2.1 when taking cohomology groups on the two sides, and using that the Zariski cohomology of $\mathrm{R}\varepsilon_*(\mu_\ell^{\otimes i})$ is the étale cohomology of $\mu_\ell^{\otimes i}$. The global assertion about $\mathbb{Z}/\ell\,\mathbb{Z}(i)_{X_{\mathrm{Zar}}}$ is a consequence of the local one, via the local-global spectral sequence: $\mathrm{H}^n(X, \mathrm{H}^j(\mathbb{Z}/\ell\,\mathbb{Z}(i)_{X_{\mathrm{Zar}}})) \Rightarrow \mathrm{H}^{n+j}(X, \mathbb{Z}/\ell\,\mathbb{Z}(i)_{X_{\mathrm{Zar}}})$[3], and the definition of the cohomological dimension $\dim(X)$. ◆

[2]Here we restrict our attention to schemes defined over a given field. Hence the characteristic of our schemes is well-defined.

[3]Remark that in the current ∞-category context this local-global sequence requires some technical assumptions to be satisfied. One can read Lurie for more details.

One should not mistake motivic cohomology $\mathbb{Z}/\ell\,\mathbb{Z}(i)_{X_{Zar}}$ and étale cohomology $\mu_\ell^{\otimes i}$ (which are respectively sheaves of complexes on the Zariski and the étale site): their cohomology groups coincide only in degrees $\leqslant i$. In higher degrees, one can deduce from the Gersten conjecture that $H^n(X_{\mathrm{Zar}}, \mathbb{Z}/\ell\,\mathbb{Z}(i)_{X_{Zar}}) = 0$ for any $n > 2i$.

5.3 p-adic (étale) motivic cohomology

In the p-adic setting, Milne and Illusie understood étale cohomology (with constant coefficients $\mu_\ell^{\otimes j}$) was not anymore étale motivic cohomology theory, as in the ℓ-adic context. Instead, one should replace the sheaf "$\mu_p^{\otimes j}$" by the log de Rham-Witt sheaf: $W_r\Omega^i_{X,\log}[-i]$. Remark this sheaf is, by definition, concentrated in degree 0 (we also say it is *discrete*). It was proved by Geisser-Levine, in the case of a p-adic smooth scheme X, this indeed corresponds to Zariski motivic cohomology (as defined by Voevodsky).

Definition 5.3.1. *Let A be commutative ring. The logarithmic de Rham group $\Omega^j_{A,\log}$, for $j \geqslant 0$, is the subgroup of the de Rham group Ω^j_A generated Zariski locally*[4] *by the logarithmic forms $\frac{\mathrm{d}x}{x}$, for $x \in A^\times$. This definition globalizes to general schemes X, and defines the logarithmic de Rham groups $\Omega^\bullet_{X,\log}$. The de Rham-Witt analogue $W\Omega^\bullet_{X,\log}$ (or $W_r\Omega^\bullet_{X,\log} := W\Omega^\bullet_{X,\log}/p^r$, for $r \geqslant 1$) is defined analogously.*

Remark 5.3.2. The de Rham-Witt complex $W\Omega$ is a mixed-characteristic analogue of the (algebraic) de Rham complex. As the usual Witt vectors construction for rings, it has a Frobenius and a Verschiebung endomorphism. For proper smooth schemes over a perfect field of characteristic p, it computes crystalline cohomology.

Example 5.3.3. Let X be a smooth scheme over a field of characteristic p. There is a natural quasi-isomorphism of complexes: $W_0\Omega_X = W\Omega_X/p \simeq \Omega_X$. The same result holds for the logarithmic analogue.

The following deep result, we will not prove here, compares (Zariski) motivic cohomology as constructed by Voevodsky and the logarithmic de Rham-Witt sheaf.

Theorem 5.3.4. (Geisser-Levine, [GL00]) *Let X be a smooth scheme over a field k of characteristic p. Then for any $r \geqslant 1$ and $i \geqslant 0$, there is a natural isomorphism of sheaves of complexes on the Zariski site of X:*

$$W_r\Omega^i_{X,\log}[-i] \simeq \mathbb{Z}(i)_{X_{\mathrm{Zar}}}/p^r.$$

Remark 5.3.5. ($\mathbb{Z}(i)/p^r$ is supported in degree i) In this remark, we try to give some insights to explain the shift "$[-i]$" appearing in Theorem 5.3.4. Concretely, it says that $\mathbb{Z}(i)/p^r$ is a complex of sheaves supported in degree i; and we claim this is not a coincidence: it means that it is purely symbolic, generated by multiplication of terms coming from $\mathbb{Z}(1)[1] = \mathbb{G}_m$.

More precisely, motivic cohomology is equipped with a bigraded multiplicative structure. That is, for any A a ring whose motivic cohomology is well-defined, there is a map of complexes $\mathbb{Z}(i)(A)\otimes\mathbb{Z}(j)(A) \to \mathbb{Z}(i+j)(A)$ for each i and j nonnegative integers, which are compatible with each other. Taking cohomology induces maps $\mathrm{H}^n(A,\mathbb{Z}(i))\otimes\mathrm{H}^m(A,\mathbb{Z}(j)) \to \mathrm{H}^{n+m}(A,\mathbb{Z}(i+j))$. In general $\mathbb{Z}(1)(A)$ is $A^\times[-1] = \mathbb{G}_m(A)[-1]$, whence $\mathrm{H}^1(A,\mathbb{Z}(1)) = A^\times$. So, given units $f_1,\dots,f_j \in A$, we can then use the product structure to build an element $\{f_1,\dots,f_j\} \in \mathrm{H}^j(A,\mathbb{Z}(j))$: such elements are called *symbols*. The previous construction usually refines to a map from Milnor

[4] One can alternatively define Ω^j_A as the forms generated étale locally by the logarithmic forms, and then restrict to the Zariski site. The definitions coincide, thanks to [Mor15], Corollary 4.2(i).

K-theory $\mathrm{K}_j^{\mathrm{M}}(A) \to \mathrm{H}^j(A, \mathbb{Z}(j))$, which is usually an isomorphism (called the Nesterenko-Suslin isomorphism).

In practice, this means that $\mathbb{Z}(j)(A)$ nearly always has classes in cohomological degree j. Saying that it is "purely symbolic" means that it only has these "obvious" classes, *i.e.*, that it is supported in degree j. For instance, the previous deep result of Geisser-Levine shows that $\mathbb{Z}(i)(A)/p^r$, and in particular $\mathbb{Z}(i)(A)/p$, is purely symbolic; hence for local rings A (*i.e.* rings such that there is no higher Zariski cohomology), the Atiyah-Hirzebruch spectral sequence from motivic cohomology to K-theory completely degenerates and we see that $\mathrm{K}_j(A; \mathbb{Z}/p) = \mathrm{H}^j(A, \mathbb{Z}(j)/p) = \mathrm{K}_j^{\mathrm{M}}(A)/p$. Overall, this shows that every element in $\mathrm{K}_j(A; \mathbb{Z}/p)$ is generated by products of j elements from $\mathrm{K}_1(A; \mathbb{Z}/p) = A^\times/p$, *i.e.*, the K-group mod p is purely symbolic, in a similar sense.

Now let us go back to logarithmic forms. A priori, this is not clear from Theorem 5.3.4 why these objects should be related to motivic cohomology. And in fact, the main reason why logarithmic forms are of any importance is because of this same motivic nature we can see in Theorem 5.3.4 (this really comes, historically, from observations of Milne and Illusie). Intuitively, one could say that (usual) algebraic differential forms, given by the de Rham or the de Rham-Witt complex, have too much structure to be motivic. That is, motivic cohomology needs to compare to all other cohomology theories (by definition of being motivic), and then can not bear too much structural informations (such as a Frobenius endomorphism, or a Verschiebung operator). This structure on the de Rham-Witt complex is killed when restricting to logarithmic forms. In fact, this is intuitively explained by the following, saying that logarithmic forms are the Frobenius fixed points of these usual algebraic differential forms:

Proposition 5.3.6. ([BMS19], Proposition 8.4) *Let X be a smooth scheme over a perfect field k of characteristic p. Then for all $i \geqslant 0$, there is a short exact (i.e. exact in each degree) sequence of sheaves of complexes on the pro-étale site of X*

$$0 \to W\Omega^i_{X,\log}[-i] \to \mathcal{N}^{\geqslant i} W\Omega^\bullet_X \xrightarrow{\varphi_i - 1} W\Omega^\bullet_X \to 0$$

where $\varphi_i : \mathcal{N}^{\geqslant i} W\Omega^\bullet_X \to W\Omega^\bullet_X$ is the Frobenius on the Nygaard filtration (see [BMS19] for its construction in this case) of the de Rham-Witt sheaf $W\Omega^\bullet_X$.

We note that the last result is formulated in the (pro-)étale topology, instead of the Zariski topology. This means that the de Rham-Witt complex sheaf corresponds to these Frobenius fixed-points only in the étale context. In the Zariski topology, there is a similar fiber sequence, but the map $\varphi_i - 1$ is not surjective, and hence this does not defines a distinguished triangle as in the previous result. In fact, Proposition 5.3.6 is the first time we see a relation between (the Frobenius fixed points of) the Nygaard filtration and p-adic étale motivic cohomology – where the étale part is related to what we just wrote. See Section 6.2 and Section 6.6.

Chapter 6

Syntomic cohomology

We now arrive to the core of the text: that is, syntomic cohomology. The last three chapters covered three different topics in arithmetic geometry, which correspond to the three flavours one can taste when looking –with his or her mind– at syntomic cohomology. The first corresponds historically to the first definition of syntomic cohomology as we know it, and is related to Chapter 3 (Section 6.1). The prismatic theory of Chapter 4 can be applied to produce an independent equivalent definition of syntomic cohomology (Section 6.2). These two definitions can be used to prove some applications to algebraic K-theory and p-adic Hodge theory (Section 6.4), and can be compared to the previous definition[1] of syntomic cohomology given by Fontaine and Messing (Section 6.5). Finally, we prove that syntomic cohomology (as defined in Section 6.1) is p-adic étale motivic cohomology, hence ending the triple-flavoured description of syntomic cohomology.

6.1 As graded pieces of TC

We begin with the first definitionof syntomic cohomology. Following the methods of Chapter 3, it is defined as the graded pieces of $\mathrm{TC}(-;\mathbb{Z}_p)$ with respect to its "motivic" filtration.

Let A be a quasisyntomic ring. Recall that the $\mathbb{E}_\infty$-$\mathbb{Z}_p$-algebra $\widehat{\Delta}_A$ is defined by $\widehat{\Delta}_A := R\Gamma_{\mathrm{syn}}(A;\pi_0\mathrm{TC}^-(-;\mathbb{Z}_p))$, and is equipped with its Nygaard filtration $\mathcal{N}^{\geqslant\star}\widehat{\Delta}_A$. Moreover, recall there are (complete exhaustive decreasing multiplicative) indexed respectively by $\mathbb{Z}_{\geqslant 0}$, $\mathbb{Z}$ and $\mathbb{Z}$ filtrations on $\mathrm{THH}(A;\mathbb{Z}_p)$, $\mathrm{TC}^-(A;\mathbb{Z}_p)$ and $\mathrm{TP}(A;\mathbb{Z}_p)$, given by the Postnikov filtration locally on the quasisyntomic site

$$\begin{aligned}\mathrm{Fil}^n\mathrm{THH}(A;\mathbb{Z}_p) &:= R\Gamma_{\mathrm{syn}}(A;\tau_{\geqslant 2n}\mathrm{THH}(-;\mathbb{Z}_p)),\\ \mathrm{Fil}^n\mathrm{TC}^-(A;\mathbb{Z}_p) &:= R\Gamma_{\mathrm{syn}}(A;\tau_{\geqslant 2n}\mathrm{TC}^-(-;\mathbb{Z}_p)),\\ \mathrm{Fil}^n\mathrm{TP}(A;\mathbb{Z}_p) &:= R\Gamma_{\mathrm{syn}}(A;\tau_{\geqslant 2n}\mathrm{TP}(-;\mathbb{Z}_p)),\end{aligned}$$

with graded pieces:

$$\begin{aligned}\mathrm{gr}^n\mathrm{THH}(A;\mathbb{Z}_p) &:= \mathcal{N}^n\widehat{\Delta}_A\{n\}[2n],\\ \mathrm{gr}^n\mathrm{TC}^-(A;\mathbb{Z}_p) &:= \mathcal{N}^{\geqslant n}\widehat{\Delta}_A\{n\}[2n],\\ \mathrm{gr}^n\mathrm{TP}(A;\mathbb{Z}_p) &:= \widehat{\Delta}_A\{n\}[2n].\end{aligned}$$

[1]This definition was not as general, and not as well-behaved as the recent one. However it gave, apart from its name, some ideas of results which have now been proved also in the bigger generality allowed by the recent definition.

Roughly, this says that the graded pieces of the filtrations on $\mathrm{THH}(A;\mathbb{Z}_p)$, $\mathrm{TC}^-(A;\mathbb{Z}_p)$ and $\mathrm{TP}(A;\mathbb{Z}_p)$ are understood by the object $\widehat{\Delta}_A$, together with its Nygaard filtration $\mathcal{N}^{\geqslant\star}\widehat{\Delta}_A$.

Moreover, the sheaves $\widehat{\Delta}_-$, $\mathcal{N}^{\geqslant\star}\widehat{\Delta}_-$, and $\mathcal{N}^{\star}\widehat{\Delta}_-$ take discrete values, defined via topological Hochschild homology, on quasiregular semiperfectoid algebras.

Question 6.1.1. How about a similar filtration on $\mathrm{TC}(A;\mathbb{Z}_p)$?

This question is answered by the following theorem: that is, the filtrations on $\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\mathrm{TP}(-;\mathbb{Z}_p)$ induce a natural filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$.

Theorem 6.1.2. ([BMS19], Theorem 1.12.(5)) *Let A be a quasisyntomic ring. The map φ_p : $\mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{TP}(A;\mathbb{Z}_p)$ (Corollary 3.5.6) induces natural maps φ_p : $\mathrm{Fil}^n\mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{Fil}^n\mathrm{TP}(A;\mathbb{Z}_p)$, thereby giving a natural filtration*

$$\mathrm{Fil}^n\mathrm{TC}(A;\mathbb{Z}_p) := \mathrm{hofib}(\varphi_p - \mathrm{can} : \mathrm{Fil}^n\mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{Fil}^n\mathrm{TP}(A;\mathbb{Z}_p))$$

on topological cyclic homology $\mathrm{TC}(A;\mathbb{Z}_p) = \mathrm{hofib}(\varphi_p - \mathrm{can} : \mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{TP}(A;\mathbb{Z}_p))$.

We can now state the definition of syntomic cohomology.

Definition 6.1.3. *Let A be a quasisyntomic ring. Denote by $\mathbb{Z}_p(n)(A)$ the graded pieces of the filtration on* $\mathrm{TC}(A;\mathbb{Z}_p)$

$$\mathbb{Z}_p(n)(A) := \mathrm{gr}^n\mathrm{TC}(A;\mathbb{Z}_p)[-2n] = \mathrm{hofib}(\varphi - \mathrm{can} : \mathcal{N}^{\geqslant n}\widehat{\Delta}_A\{n\} \to \widehat{\Delta}_A\{n\}),$$

where $\varphi : \mathcal{N}^{\geqslant n}\widehat{\Delta}_A\{n\} \to \widehat{\Delta}_A\{n\}$ is a natural Frobenius endomorphism of the Breuil-Kisin twist $\widehat{\Delta}_A\{n\}$. The complexes $\mathbb{Z}_p(n)(A)$, indexed by integers n, are called syntomic cohomology *of A.*

Fontaine and Messing, in [FM87], defined a version of syntomic cohomology for noetherian schemes. This is related, in characteristic p, to crystalline cohomology. In both cases, the most interesting examples of coverings for an affine scheme $\mathrm{Spec}(A)$ are the extensions of A generated by the p^n-th roots of elements of A. This difference is that in the version of [BMS19], one can add all of these roots (for all n and all elements of A) at once. The ring we obtain is a quasiregular semiperfectoid ring (which is indeed a quasisyntomic ring, hence considered by the theory of [BMS19]). In particular, this makes most of the constructions and the proofs more natural.

Remark 6.1.4. Despite the name, the so-called "Frobenius" map φ_p defined on $\pi_0\mathrm{TC}^-(-;\mathbb{Z}_p)$ (or similar topological objects) is not obviously, nor always, related to the Frobenius of some characteristic p object. However, when applied to mixed characteristic or characteristic p objects, one can show in most cases (*e.g.* Definition 6.1.3) that this "Frobenius" morphism φ_p corresponds to (some lift of) the Frobenius φ on the associated object.

Example 6.1.5. For instance, if R is a perfectoid ring, then $\widehat{\Delta}_R = \pi_0\mathrm{TC}^-(R;\mathbb{Z}_p) \cong A_{\mathrm{inf}}(R)$ (see Section 3.4), and the topological "Frobenius" map φ_p on $\pi_0\mathrm{TC}^-(R;\mathbb{Z}_p)$ (coming the p-cyclotomic spectrum structure of $\mathrm{TC}^-(-;\mathbb{Z}_p)$) corresponds to the classical lift of Frobenius on the ring $A_{\mathrm{inf}}(R) := W(R^\flat)$.

Remark 6.1.6. Let S be a quasiregular semiperfectoid ring. A consequence of the prismatic theory [BS19, Remark 1.14.] is that the (topological) "Frobenius" map φ_p on the commutative ring $\pi_0\mathrm{TC}^-(S;\mathbb{Z}_p) = \pi_0\mathrm{TP}(S;\mathbb{Z}_p)$ lifts the Frobenius on the (characteristic p) ring $\pi_0\mathrm{TC}^-(S;\mathbb{Z}_p)/p$.

The syntomic cohomology complexes $\mathbb{Z}_p(n)$ (or $\mathbb{Z}_p(n)(-)$) are sheaves of complexes on the quasisyntomic site $\mathrm{QSyn}^{\mathrm{op}}$. For any integer n, it follows from the Definition 6.1.3 (and from the local structure of TC^- and $\widehat{\Delta}$) that these complexes $\mathbb{Z}_p(n)$ are locally on qSyn_A concentrated in (cohomological) degrees 0 and 1. More precisely, in the case of a quasiregular semiperfectoid ring S, the filtration on $\mathrm{TC}^-(-;\mathbb{Z}_p)$ and $\mathrm{TP}(-;\mathbb{Z}_p)$ is just the usual double-speed Postnikov filtration (thanks to Theorem 3.4.16.(2)), and the syntomic sheaf $\mathbb{Z}_p(n)$ is thus given by the two term complex

$$\mathbb{Z}_p(n)(S) = (\tau_{[2n-1;2n]}\mathrm{TC}(S;\mathbb{Z}_p)[-2n]) = \mathrm{hofib}(\varphi - \mathrm{can} : \pi_{2n}\mathrm{TC}^-(S;\mathbb{Z}_p) \to \pi_{2n}\mathrm{TP}(S;\mathbb{Z}_p)),$$

with cohomology[2]

$$\mathrm{H}^0(\mathbb{Z}_p(n)(S)) = \mathrm{TC}_{2n}(S;\mathbb{Z}_p), \qquad \mathrm{H}^1(\mathbb{Z}_p(n)(S)) = \mathrm{TC}_{2n-1}(S;\mathbb{Z}_p).$$

In fact, one can prove that locally, the sheaves of complexes $\mathbb{Z}_p(n)$ are even discrete (see Theorem 6.3.1). In particular, $\mathbb{Z}_p(n)$ locally identifies with its global section functor. And because the derived global section functor applied to $\mathbb{Z}_p(n)$, and $\mathbb{Z}_p(n)$ itself are quasisyntomic sheaves, one can write for any quasisyntomic ring A

$$\mathbb{Z}_p(n)(A) = R\Gamma_{\mathrm{syn}}(A, \mathbb{Z}_p(n))$$

as the cohomology of a sheaf on the quasisyntomic site of A –justifying the name *syntomic cohomology*.

Now we give some examples of computations for syntomic cohomology (Definition 6.1.3).

Lemma 6.1.7 ($\mathbb{Z}_p(n)$ for $n < 0$)**.** *In negative degree $n < 0$, the complexes $\mathbb{Z}_p(n) = 0$ vanish.*

Proof. Intuitively, we could prove this by analogy to the classical theory of Chapter 2: this would give us an exact sequence (which is just an analogy, since the Definition 3.5.3 of TC does not have such a property) $0 \to \mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{TP}(A;\mathbb{Z}_p) \to \mathrm{TC}(A;\mathbb{Z}_p) \to 0$, we can apply to any quasisyntomic ring A. And then by taking homotopy groups and using that the canonical map $\pi_*\mathrm{TC}^-(A;\mathbb{Z}_p) \to \pi_*\mathrm{TP}(A;\mathbb{Z}_p)$ is an isomorphism in degrees $\leqslant 0$ (Theorem 3.4.20), we could conclude that $\mathbb{Z}_p(n)$ vanishes for $n < 0$ (and is concentrated in degree 0 for $n = 0$). However the only concrete relation we have at disposal instead of this exact sequence, is the fiber sequence from Corollary 3.5.6, and this is not sufficient to make this proof work.

The actual proof ([BMS19], Proposition 7.16) that $\mathbb{Z}_p(n) = 0$ for $n < 0$ is in fact not that far from the previous intuition: we use a comparison to the classical definition of TC, given in [NS18, Theorem II.4.10], which implies that for any connective ring spectrum (and then any quasisyntomic ring) A, we have $\pi_i\mathrm{TC}(A;\mathbb{Z}_p) = 0$ for any $i < -1$. In particular, $\mathbb{Z}_p(n)$ vanishes locally (according to the previous discussion), and hence $\mathbb{Z}_p(n) = 0$ for $n < 0$. ◆

Corollary 6.1.8. *Let A be a quasisyntomic ring. Then there is a spectral sequence*

$$\mathrm{E}_2^{i,j} = \mathrm{H}^{i-j}(\mathbb{Z}_p(-j)(A)) \Rightarrow \pi_{-i-j}\mathrm{TC}(A;\mathbb{Z}_p).$$

This spectral sequence degenerates locally on the quasisyntomic site. For a quasisyntomic ring S on which it degenerates, we have the following identification

$$\pi_n\mathrm{TC}(S;\mathbb{Z}_p) = \begin{cases} \mathrm{H}^0(\mathbb{Z}_p(n/2)(S)) & \text{if } n \geqslant 0 \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}.$$

[2]Remark that the conventions for TC are those of homology, while they are cohomological for the complexes $\mathbb{Z}_p(n)$.

Proof. The first assertion is a direct application of the formalism of spectral sequences induced by a filtered complex, applied to Theorem 6.1.2. In particular, the graded pieces of $\pi_n \mathrm{TC}(A; \mathbb{Z}_p)$ are indeed given by: $\mathrm{gr}^i \pi_n \mathrm{TC}(A; \mathbb{Z}_p) = \mathrm{H}^{n+2i}(\mathbb{Z}_p(n+i)(A))$.

For the second assertion, we use the deep result (Theorem 6.3.1) of the previous discussion, which asserts that locally on the quasisyntomic site, $\mathbb{Z}_p(n)$ takes discrete values. Since it is defined via a double-speed Postnikov filtration, this implies that all the differentials on the E_2-page of our spectral sequence are zero (by an argument of parity). The final assertion is just reindexing the indices, and Lemma 6.1.7. ◆

Proposition 6.1.9 ($\mathbb{Z}_p(n)$ for $n = 0$)**.** *In weight* 0*, syntomic cohomology* $\mathbb{Z}_p(0)$ *coincide quasisyntomic locally with the constant sheaf* $\mathbb{Z}_p := \lim\limits_{\leftarrow} \mathbb{Z}/p^r\,\mathbb{Z}$*.*

Sketch of proof. This is proved in [BMS19], Proposition 7.16, by showing first that $\pi_{-1}\mathrm{TC}(-; \mathbb{Z}_p)$ vanishes locally on $\mathrm{QRSPerd}^{\mathrm{op}}$, hence $\mathbb{Z}_p(0)$ is locally concentratred in degree 0. This proves that, as an ∞-sheaf, $\mathbb{Z}_p(0)$ is discrete (that is, it takes discrete values locally on the quasisyntomic site)[3]. Then, by looking at pro-étale covers on $\mathrm{QRSPerd}^{\mathrm{op}}$, a result of [CMM18] comparing algebraic K-theory and $\tau_{\geqslant 0}\mathrm{TC}$ proves the result. ◆

Proposition 6.1.10 ($\mathbb{Z}_p(n)$ for $n = 1$)**.** *In weight* 1*, syntomic cohomology* $\mathbb{Z}_p(1)$ *is quasisyntomic locally concentrated in degree* 0*, where it is given by* $T_p\mathbb{G}_m$*.*

Note that here $T_p(-)$ denotes the p-adic Tate module, defined on abelian groups by $A \mapsto T_p(A) := \lim\limits_{\leftarrow} A[p^n] = \lim\limits_{\leftarrow}\{a \in A \mid p^n.a = 0\}$.

Sketch of proof. This is proved in [BMS19], Proposition 7.17 in the following way: one uses the result from [CMM18] comparing TC and algebraic K-theory to reduce to the same computation in K-theory; then one proves the result for rings which are both quasiregular semiperfectoid and w-local (that is, local for the pro-étale topology, as defined in [BS15]); finally, one proves that such rings form a basis for QRSPerd in the quasisyntomic topology (this suffices to prove the result on QSyn). ◆

Remark 6.1.11. A general motto in this story is that the objects $\mathrm{THH}(A; \mathbb{Z}_p)$, $\mathrm{TC}^-(A; \mathbb{Z}_p)$, $\mathrm{TP}(A; \mathbb{Z}_p)$, $\mathrm{TC}(A; \mathbb{Z}_p)$ admits filtrations for A any quasisyntomic ring, and that the graded pieces of these filtrations are locally discrete (that is, concentrated in only one degree) in the quasisyntomic topology.

6.2 As Frobenius fixed points on the Nygaard filtration of prismatic cohomology

We now give the prismatic definition for syntomic cohomology. More precisely, syntomic cohomology corresponds to the Frobenius fixed points on the Nygaard filtration on prismatic cohomology. Here the Frobenius comes directly from the prismatic theory (and the δ-ring structures), and has not the same nature than the topological Frobenius endormorphism of Section 6.1.

[3](Classical and ∞-sheaves) Remark that $\mathbb{Z}_p$ is here a constant classical sheaf (that is, it takes values in abelian groups), considered, via composition by $\mathrm{Ab} \to \mathcal{D}(\mathbb{Z})$, as an ∞-presheaf with values in the derived (∞-)category $\mathcal{D}(\mathbb{Z})$. In particular the ∞-sheafification of this ∞-presheaf $\mathbb{Z}_p$, which is also the cohomology $R\Gamma(-, \mathbb{Z}_p)$ of the classical sheaf $\mathbb{Z}_p$, is equal to $\mathbb{Z}_p(0)$ on any object of the quasisyntomic site. Thus one can not hope $\mathbb{Z}_p$ is already an ∞-sheaf, since it has in general higher cohomology groups.

The object $\widehat{\mathbb{\Delta}}$ (and its Nygaard filtration $\mathcal{N}^{\geqslant\star}\widehat{\mathbb{\Delta}}$) from the topological theory, have a prismatic interpretation (Theorem 4.4.4): it is the (quasisyntomic sheafified) Nygaard completion of the prism $\mathbb{\Delta}$. Recall that syntomic cohomology is defined as sheaves $\mathbb{Z}_p(n)$ for $n \geqslant 0$ on the quasisyntomic site, via the Nygaard filtration on $\widehat{\mathbb{\Delta}}$:

$$\mathbb{Z}_p(i)(-) := \mathrm{hofib}(\varphi - \mathrm{can} : \mathcal{N}^{\geqslant i}\widehat{\mathbb{\Delta}}\{i\} \to \widehat{\mathbb{\Delta}}\{i\}).$$

The construction of the non-completed object $\mathbb{\Delta}$ is completely independant of the topological definition of $\widehat{\mathbb{\Delta}}$. This suggests an alternative definition of syntomic cohomology, given in Definition 6.2.2.

Definition 6.2.1. (Divided Frobenius maps) *Let S be a quasiregular semiperfectoid ring with associated prism $(\mathbb{\Delta}_S, (d))$. For $i \geqslant 0$, the divided Frobenius map φ_i on the (Nygaard-completed) prism $\widehat{\mathbb{\Delta}}_S$ (Definition 3.4.1 or Definition 4.4.4) is*

$$\varphi_i := \frac{\varphi}{d^i} : \mathcal{N}^{\geqslant i}\widehat{\mathbb{\Delta}}_S\{i\} \to \widehat{\mathbb{\Delta}}_S\{i\},$$

where φ is the Frobenius map coming from the δ-ring structure of $\mathbb{\Delta}$, and d is a nonzero divisor. One can prove that φ_i corresponds to the topological Frobenius φ.

This map is well-defined because the image of φ in $\widehat{\mathbb{\Delta}}_S$ is contained in $d^i\widehat{\mathbb{\Delta}}_S$.

In particular the (topological) Frobenius φ identifies with the divided (arithmetic) Frobenius $\frac{\varphi}{p^i}$, and one can define syntomic cohomology $\mathbb{Z}_p(i)$ in the following way.

Definition 6.2.2. *Let S be a quasiregular semiperfectoid ring. Then*

$$\mathbb{Z}_p(i)(S) \simeq \mathrm{hofib}(\varphi_i - 1 : \mathcal{N}^{\geqslant i}\widehat{\mathbb{\Delta}}_S \to \widehat{\mathbb{\Delta}}_S).$$

This construction forms a sheaf on the site of quasiregular semiperfectoid algebras. One can then globalize to get a sheaf $\mathbb{Z}_p(i)$ on the whole quasisyntomic site (which is called "unfolding" in the litterature).

Note that syntomic cohomology appears not in the same fashion here, compared to the first definition via the filtration on TC. Indeed, syntomic cohomology $\mathbb{Z}_p(n)$ is on the one hand the graded pieces of the filtration on TC (Definition 6.1.3); and on the other hand, it is the Frobenius fixed points on the Nygaard filtration levels $(\mathcal{N}^{\geqslant n}\mathbb{\Delta}_-)^{\varphi_i = 1} = (\mathcal{N}^{\geqslant i}\mathbb{\Delta}_-)^{\varphi = p^i}$ of prismatic cohomology (Definition 6.2.2).

6.3 Foundational results on syntomic cohomology

After defining syntomic cohomology in the two last sections, the goal in this section is to present some structural results about it. For instance, we identify syntomic cohomology in some cases, and present some general properties about it (torsionfreeness, connectivity, ...).

The syntomic cohomology sheaves $\mathbb{Z}_p(i)$ are concentrated in positive (cohomological) degrees. In fact, we can prove that locally (in the quasisyntomic topology), these are even discrete complexes[4]. More precisely, we have the following result which is a consequence of the prismatic theory (and whose proof correspond to [BS19, Section 7], which we will not reproduce here).

[4]We call such sheaves simply *discrete* sheaves. Similarly, we say in general that a sheaf which takes values in $\mathcal{D}(\mathbb{Z}_p)^{\geqslant 0}$ is *p-torsion free* if its cohomology groups are all p-torsion free on some basis of the corresponding topology.

Theorem 6.3.1 ([BS22], Theorem 1.17)**.** *The quasisyntomic sheaves of complexes* $\mathbb{Z}_p(n)$ *are concentrated in degree* 0 *and p-torsion free*[5]*. More precisely, given* $R \in \mathrm{QSyn}$*, there exists a cover* $R \to R'$ *in* QSyn *such that* $\mathbb{Z}_p(n)(R')$ *is discrete and p-torsion free.*

Remark 6.3.2. One should not confuse what we call "the syntomic cohomology sheaf $\mathbb{Z}_p(n)$" and "a quasisyntomic sheaf", which is only a sheaf on the quasisyntomic site. In particular, quasisyntomic cohomology $\mathbb{Z}_p(n)$ is a sheaf on the quasisyntomic site for each integer n.

In characteristic p, one can even identify syntomic cohomology locally in the quasisyntomic site. But before stating the corresponding precise result, let us recall a few facts about the object $\mathbb{A}_{\mathrm{crys}}(-)$, which is a generalisation of the crystalline Fontaine's period ring A_{crys} (see Construction 6.4.8).

$\mathbb{A}_{\mathrm{crys}}(-)$ is defined, for a semiperfect $\mathbb{F}_p$-algebra S, as the p-completion of the divided power envelope of $W(S^\flat) \to S$ (see [BMS19], Definition 8.9). In particular we have $\mathbb{A}_{\mathrm{crys}}(S)/p = D_{S^\flat}(I)$, where $D_{S^\flat}(I)$ is the divided power envelope of $S^\flat$ along the ideal $I := \ker(S^\flat \twoheadrightarrow S) \subseteq S^\flat$. The Frobenius morphism on S induces by functoriality a Frobenius endormorphism $\varphi : \mathbb{A}_{\mathrm{crys}}(S) \to \mathbb{A}_{\mathrm{crys}}(S)$, so that we can define a Nygaard filtration on $\mathbb{A}_{\mathrm{crys}}(S)$ (Definition 4.4.1): $\mathcal{N}^{\geqslant i}\mathbb{A}_{\mathrm{crys}}(S) := \{x \in \mathbb{A}_{\mathrm{crys}}(S) \mid \varphi(x) \in p^i\mathbb{A}_{\mathrm{crys}}(S)\}$, for $i \geqslant 0$. The object $\widehat{\mathbb{A}}_{\mathrm{crys}}(S)$ denotes the completion of $\mathbb{A}_{\mathrm{crys}}(S)$ with respect to this Nygaard filtration, and thus is also equipped with its (completed) Nygaard filtration $\mathcal{N}^{\geqslant i}\widehat{\mathbb{A}}_{\mathrm{crys}}(S)$.

On quasiregular semiperfect $\mathbb{F}_p$-algebras S, one can prove that $\mathbb{A}_{\mathrm{crys}}(S)$ coincides with the derived de Rham-Witt complex $\mathbb{L}W\Omega_S$ ([BMS19], Theorem 8.14). One can then deduce that $\mathbb{A}_{\mathrm{crys}}(S)$ is p-torsionfree on these algebras, that it defines a sheaf on QRSP, which unfolding to $\mathrm{QSyn}_{\mathbb{F}_p}$ computes crystalline cohomology $R\Gamma_{\mathrm{crys}}(A/\mathbb{Z}_p)$, for regular $\mathbb{F}_p$-algebras A.

Now, $\widehat{\Delta}_S$ compares to $\widehat{\mathbb{A}}_{\mathrm{crys}}(S)$ for quasiregular semiperfect algebras:

Theorem 6.3.3. ([BMS19], Theorem 8.17) *Let S be a quasiregular semiperfect $\mathbb{F}_p$-algebra. There is a functorial φ-equivariant isomorphism of rings $\widehat{\Delta}_S \cong \widehat{\mathbb{A}}_{crys}(S)$, with an identification of the two Nygaard filtrations.*

In fact, this is how we prove that $\widehat{\Delta}_A$ computes crystalline cohomology of A, for A a smooth algebra over a perfect field of characteristic p; this is an analogue of Theorem 3.2.17 in characteristic p. Instead of proving this result (whose proof goes through several successive reductions, up to the case $S = \mathbb{F}_p[T^{\pm 1/p^\infty}]/(T-1)$), we present in detail the following one, more directly related to syntomic cohomology, which gives an explicit version of Theorem 6.3.1 in characteristic p:

Proposition 6.3.4. ([BMS19], Proposition 8.20) *Let S be a quasiregular semiperfect $\mathbb{F}_p$-algebra, and $i > 0$. Then the complex $\mathbb{Z}_p(i)(S)$ is concentrated locally in degree* 0 *and given by the p-torsionfree group $\mathbb{A}_{\mathrm{crys}}(S)^{\varphi=p^i}$. For $i = 0$, the same result holds for the sheaf $\mathbb{Z}_p(0)$ only locally on the site $\mathrm{QRSP}^{\mathrm{op}}$: $\mathbb{Z}_p(0)$ takes discrete values locally on this site, given by the p-torsionfree group sheaf*[6] $\mathbb{A}_{\mathrm{crys}}(-)^{\varphi=p^i}$.

Proof. Recall that for a quasiregular semiperfect $\mathbb{F}_p$-algebra S, we have the following description of syntomic cohomology: $\mathbb{Z}_p(i)(S) = \mathrm{hofib}(\mathcal{N}^{\geqslant i}\widehat{\Delta}_S \xrightarrow{\varphi_i - 1} \widehat{\Delta}_S)$ (see Definition 6.2.2), with $\varphi_i := \frac{\varphi}{p^i}$ the divided Frobenius map.

[5]The definitions for discrete and p-torsion free sheaves here is the one for ∞-sheaves, that is sheaves taking values in the derived ∞-category $\mathcal{D}(\mathbb{Z})$ which are locally discrete / p-torsion free. Roughly, there is an equivalence between discrete sheaves A of abelian groups (that is, sheaves that take discrete values on any object of the defining site), and discrete ∞-sheaves with values in $\mathcal{D}(\mathbb{Z})$, via the functor $A \mapsto R\Gamma(-, A)$.

[6]Remark that here we mean ∞-sheaf: being p-torsionfree then means it takes locally (discrete) p-torsionfree values.

Then, one can prove ([BMS19], Lemma 8.19) that the operator $\varphi_i - 1 : \mathcal{N}^{\geqslant i}\widehat{\mathbb{A}}_{\mathrm{crys}}(-) \to \widehat{\mathbb{A}}_{\mathrm{crys}}(-)$ is surjective (directly for any quasiregular semiperfect algebra S if $i > 0$, and only as a map of sheaves, that is locally, if $i = 0$). Hence, according to the previous result, we can identify syntomic cohomology $\mathbb{Z}_p(i)$ on any quasiregular semiperfect $\mathbb{F}_p$-algebra with

$$\mathbb{Z}_p(i)(S) = \ker(\mathcal{N}^{\geqslant i}\widehat{\mathbb{A}}_{\mathrm{crys}}(S) \xrightarrow{\varphi_i - 1} \widehat{\mathbb{A}}_{\mathrm{crys}}(S)).$$

It remains to prove that the natural map

$$\alpha : \ker(\mathcal{N}^{\geqslant i}\mathbb{A}_{\mathrm{crys}}(S) \xrightarrow{\varphi_i - 1} \mathbb{A}_{\mathrm{crys}}(S)) \to \ker(\mathcal{N}^{\geqslant i}\widehat{\mathbb{A}}_{\mathrm{crys}}(S) \xrightarrow{\varphi_i - 1} \widehat{\mathbb{A}}_{\mathrm{crys}}(S))$$

is an isomorphism. By definition of the Nygaard filtration[7], the divided Frobenius map $\varphi_i : \mathcal{N}^{\geqslant i}\mathbb{A}_{\mathrm{crys}}(S) \to \mathbb{A}_{\mathrm{crys}}(S)$ factors through the completion map $\alpha : \mathcal{N}^{\geqslant i}\mathbb{A}_{\mathrm{crys}}(S) \to \mathcal{N}^{\geqslant i}\widehat{\mathbb{A}}_{\mathrm{crys}}(S)$; we call β the corresponding quotient map. This ensures the injectivity of α : if $x \in \mathcal{N}^{\geqslant i}\mathbb{A}_{\mathrm{crys}}(S)$ satisfies $\varphi_i(x) = x$, and $\alpha(x) = 0$, then $\varphi_i(x) = \beta(\alpha(x)) = 0$, and thus $x = \varphi_i(x) = 0$. Moreover the quotient map $\beta : \mathcal{N}^{\geqslant i}\widehat{\mathbb{A}}_{\mathrm{crys}}(S) \to \mathbb{A}_{\mathrm{crys}}(S)$ satisfies the identity: $\varphi_i(y) = \alpha(\beta(y))$ for any $y \in \mathcal{N}^{\geqslant i}\widehat{\mathbb{A}}_{\mathrm{crys}}(S)$, and thus a similar argument implies the surjectivity of α. ◆

Again in characteristic p, and for smooth varieties over a perfect field (and in particular not only locally in the quasisyntomic site), one can compute explicitly syntomic cohomology $\mathbb{Z}_p(n)$.

Theorem 6.3.5. ([BMS19], Theorem 1.15.(1)) *Let A be a smooth k-algebra, where k is a perfect field of characteristic p. Then there is an isomorphism of sheaves of complexes on the pro-étale site of $X = \mathrm{Spec}(A)$*

$$\mathbb{Z}_p(i) \simeq W\Omega^i_{X,\log}[-i].$$

Remark 6.3.6. We use the pro-étale site in the formulation of Theorem 6.3.5 so that the logarithmic de Rham-Witt sheaf is well-defined (the "pro-" part being because we work with p-adic coefficients, see [BS15]). $\mathbb{Z}_p(i)$ is indeed a sheaf of complexes on the pro-étale site by restriction from the quasisyntomic site: étale morphisms are quasisyntomic, and qSyn_A is stable under filtered colimits.

In mixed-characteristic, we can also compute an analogue of Theorem 6.3.5:

Theorem 6.3.7. ([BMS19], Theorem 1.15.(2)) *Let A be the p-adic completion of a smooth $\mathcal{O}_C$ algebra, where C is an algebraically closed complete extension of $\mathbb{Q}_p$. Then there is an isomorphism of sheaves of complexes on the pro-étale site of $\mathfrak{X} = \mathrm{Spf}(A)$*

$$\mathbb{Z}_p(i) \simeq \tau^{\leqslant i} R\psi\, \mathbb{Z}_p(i),$$

where on the right-hand side, $\mathbb{Z}_p(i)$ denotes the usual (pro-)étale sheaf on the generic fiber X of $\mathfrak{X}$, and $R\psi$ denotes the nearby cycles functor.

As we will see in Section 6.6, syntomic cohomology is a form of motivic cohomology. There is a more general relation between motivic cohomology and the nearby cycles. Indeed, the localisation sequence for motivic cohomology relates motivic cohomology of a scheme (with some assumptions, *e.g.* smooth over the ring of integers $\mathcal{O}_K$ of a p-adic field K) to its generic and special fibers; then the Beilinson Lichtenbaum conjecture (Theorem 5.2.1) expresses the motivic cohomology of the generic fiber as étale cohomology; remark finally that a connectivity result

[7]More precisely, we use the fact that $\mathbb{A}_{\mathrm{crys}}(S)$ is p-complete, and so if for an element x the Frobenius $\varphi(x)$ is in $p^i\mathbb{A}_{\mathrm{crys}}(S)$ for any $i \geqslant 0$, then $\varphi(x) = 0$.

on the motivic cohomology of the special fiber explains the $\tau^{\leqslant i}$ in the previous result, which is necessary to make nearby cycles appear. In higher cohomological degrees, one thus expects motivic cohomology of our scheme to be the same as the motivic cohomology of its generic fiber[8].

In low degrees, one can also compute syntomic cohomology $\mathbb{Z}_p(i)$ only in terms derived de Rham theory, instead of TC or the prismatic theory. This is a consequence of a comparison result to the theory of Fontaine-Messing, given in [AMMN20].

Notation 6.3.8. Let R be a commutative ring. There is an analogue of de Rham complex, which is the right well-behaved version for non-smooth commutative rings R. It is defined in [Bha12], and called the *derived de Rham cohomology* $L\Omega_R$ of R. It is equipped with a derived Hodge filtration $L\Omega_R^{\geqslant\star}$, a crystalline Frobenius $\varphi : L\Omega_R \to L\Omega_R$, and, for $i < p$, a "divided" Frobenius $\varphi/p^i : L\Omega_R^{\geqslant i} \to L\Omega_R$.

Remark 6.3.9. The derived de Rham cohomology is a complex defined only in the derived category. The intuition is similar to that of the cotangent complex with respect to Ω^1.

Theorem 6.3.10. ([AMMN20], Theorem F) *Let A be a quasisyntomic ring.*

(1) For each $i \leqslant p-2$, there is a natural identification

$$\mathbb{Z}_p(i)(A) \simeq \mathrm{hofib}(\varphi/p^i - \mathrm{id} : L\Omega_A^{\geqslant i} \to L\Omega_A).$$

(2) For each $i \geqslant 0$, there is a natural identification

$$\mathbb{Q}_p(i)(A) \simeq \mathrm{hofib}(\varphi - p^i : L\Omega_A^{\geqslant i} \to L\Omega_A)_{\mathbb{Q}_p},$$

where $(-)_{\mathbb{Q}_p}$ on the right-hand side means we take rationalisation.

The proof of Theorem 6.3.10 relies on the following connectivity result on syntomic cohomology.

Theorem 6.3.11. ([AMMN20], Theorem G) *Let A be a quasisyntomic ring. Then the syntomic cohomology complex $\mathbb{Z}_p(i)(A)$ is concentrated in cohomological degrees $\leqslant i+1$. Moreover, if A is a strictly henselian local ring, then the complex $\mathbb{Z}_p(i)(A)$ is concentrated in cohomological degrees $\leqslant i$.*

This connectivity result finds its meaning when we interpret syntomic cohomology $\mathbb{Z}_p(i)$ as a form of motivic cohomology (see Section 6.6). Indeed, it is expected that a fixed (homology) group of algebraic K-theory $\pi_n K(-)$ does not receive, via the Atiyah-Hirzebruch spectral sequence, a contribution for all weights i of motivic cohomology $\mathbb{Z}(i)(-)$. Here the (p-adic étale) analogue for syntomic cohomology $\mathbb{Z}_p(i)$ is that $\pi_n \mathrm{TC}(-;\mathbb{Z}_p)$ receives a contribution only from $\mathrm{H}^{-n}(\mathbb{Z}_p(0)(-))$, $\mathrm{H}^{2-n}(\mathbb{Z}_p(1)(-))$, $\mathrm{H}^{4-n}(\mathbb{Z}_p(2)(-))$, ... Hence it receives non-trivial data, thanks to the previous connectivity result, only from $\mathbb{Z}_p(i)$ for $i \leqslant n+1$.

6.4 Applications to K-theory and p-adic Hodge theory

We now give some applications of our fundational results of Section 6.3. One should not that Proposition 6.4.4 is only a consequence of the Beilinson fiber square, and Theorem 6.4.6 is a consequence of Theorem 6.5.9 and the Beilinson fiber square. As we do not discuss here the Beilinson fiber square (which is fully developped in [AMMN20]), these are not strictly applications of Section 6.3. The following is a consequence of the discreteness of syntomic cohomology (see [BS19], Section 14).

[8] A version of such a result is given in [BS22], Theorem 9.4 for perfectoid rings. A more general statement for smooth schemes over a perfectoid ring will be published soon by other coauthors.

Corollary 6.4.1 ([BS22], Corollary 14.2)**.** *Locally on the quasisyntomic site, the functor $K(-;\mathbb{Z}_p)$*[9] *is concentrated in even degrees, i.e. $\pi_n K(-;\mathbb{Z}_p)$ for n odd vanishes after quasisyntomic sheafification.*

Proof. The paper [CMM18] identifies K-theory with topological cyclic homology on a large class of rings. In particular, if shows that for S a ring which is Henselian along pS and such that S/pS is semiperfect (*e.g.* if S is quasiregular semiperfectoid), the trace map $K(S;\mathbb{Z}_p) \to \tau_{\geqslant 0}\mathrm{TC}(S;\mathbb{Z}_p)$ is an equivalence.

Now, for a quasiregular semiperfectoid ring S, $\mathrm{TC}(S;\mathbb{Z}_p)$ (and hence $\tau_{\geqslant 0}\mathrm{TC}(S;\mathbb{Z}_p)$) admits a $\mathbb{N}$-indexed filtration given by the doube-speed Postnikov filtration, and with graded pieces $\mathrm{gr}^i\mathrm{TC}(S;\mathbb{Z}_p) = \mathbb{Z}_p(i)(S)$. And by Theorem 6.3.1, these are discrete quasisyntomic sheaves. So $\pi_n K(-;\mathbb{Z}_p)$ is locally zero for n odd on the quasisyntomic site, and hence its quasisyntomic sheafification is also zero. ◆

Another consequence of the proof of Corollary 6.4.1 is that the trace map from algebraic K-theory induces an identification $K_{2n}(-;\mathbb{Z}_p)[0] \simeq \mathbb{Z}_p(n)$, where the left-hand side is the quasisyntomic sheafified K-group. Hence (Proposition 6.3.4), one deduces in particular the following structure result for K-theory on quasiregular semiperfect $\mathbb{F}_p$-algebras.

Corollary 6.4.2. ([BMS19], Corollary 8.23) *For any quasiregular semiperfect $\mathbb{F}_p$-algebra S, the algebraic K-theory $K_*(S;\mathbb{Z}_p)$ vanishes in odd degrees, and we have an isomorphism of graded rings*

$$K_{\mathrm{even}}(S;\mathbb{Z}_p) \cong \bigoplus_{i\geqslant 0} \mathbb{A}_{\mathrm{crys}}(S)^{\varphi=p^i}.$$

Remark 6.4.3. Being concentrated in even degrees is not a condition without meaning. A cohomology theory is called an *even cohomology theory* when it is concentrated in even degrees (for instance, algebraic K-theory on the site of quasiregular semiperfectoid algebras). The fundamental structure behind even cohomology theory is that of complex vector bundles, which are central objects in K-theory, and in geometry in general: these are represented by the infinite-dimensional complex projective space $\mathbb{C}P^\infty$, which has (singular, and hence, by the Atiyah-Hirzebruch spectral sequence, any) cohomology concentrated in even degrees.

Moreover, remark the local situation of Corollary 6.4.2 is drastically different from the one in Remark 5.3.5, which is about syntomic cohomology in the global situation.

Using the fundational results on TC and syntomic cohomology (which are related by their "motivic" filtration, via the definition given in Definition 6.1.3), one can reprove some known calculations on the K-theory of p-adic fields (see [AMMN20], section 7). Here is an example.

Proposition 6.4.4. ([AMMN20], Example 7.2) *If the field F is a finite extension of $\mathbb{Q}_p$ of degree d, then for any integer $s \geqslant 0$*

$$\dim_{\mathbb{Q}_p}\mathrm{K}_s(F;\mathbb{Q}_p) = \begin{cases} 0 & \text{if } s<0 \text{ or } s \text{ even,}\\ d+1 & \text{if } s=1,\\ d & \text{otherwise.}\end{cases}$$

Sketch of proof. For $s<0$, this is by definition of (connective) K-theory. Let k be the residue field of F. Since k is perfect, we have $K_s(k;\mathbb{Z}_p) = \mathbb{Z}_p$ for $s=0$ and 0 otherwise (this is classical result in K-theory, see [AMMN, Theorem 7.1]). Then the dévissage cofiber sequence in K-theory with $\mathbb{Z}_p$-coefficients $K(k;\mathbb{Z}_p) \to K(\mathcal{O}_F;\mathbb{Z}_p) \to K(F;\mathbb{Z}_p)$ proves that $K_s(\mathcal{O}_F;\mathbb{Z}_p) \cong K_s(F;\mathbb{Z}_p)$

[9]Here $K(-)$ denotes the connective algebraic K-theory of a ring, and $K(-;\mathbb{Z}_p)$ its p-completion.

for $s \neq 1$ and that there is an exact sequence $0 \to K_1(\mathcal{O}_F;\mathbb{Z}_p) \to K_1(F;\mathbb{Z}_p) \to \mathbb{Z}_p \to 0$, where the map $K_1(F;\mathbb{Z}_p) \cong F^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathbb{Z}_p$ is induced by the p-adic valuation.

Then, one can prove using the Beilinson fiber square and a computation on cyclic homology of $\mathcal{O}_F$ that $K_{2s-1}(\mathcal{O}_F;\mathbb{Q}_p) = F$ for $s = 1$ and 0 otherwise, and $K_{2s}(\mathcal{O}_F;\mathbb{Q}_p) = 0$ for $s > 0$.

Finally, we get the desired result by taking dimension of the $\mathbb{Q}_p$-vector spaces in the previous identifications. ◆

Remark 6.4.5. This dimension calculation is in accordance to the Beilinson-Lichtenbaum conjecture, which predicts in this case that $\mathrm{K}_{2s-1}(F;\mathbb{Q}_p) \simeq \mathrm{H}^1_{\text{ét}}(F,\mathbb{Q}_p(s))$ and $\mathrm{K}_{2s-2}(F;\mathbb{Q}_p) \simeq \mathrm{H}^2_{\text{ét}}(F,\mathbb{Q}_p(s))$ for any $s > 0$.

We can also perform the same kind of calculations directly for syntomic cohomology:

Theorem 6.4.6 ([AMMN20], Theorem 7.5)**.** *Let $\mathcal{O}_F$ be a complete discrete valuation ring of mixed characteristic with perfect residue field k of characteristic p (e.g. the ring of integers of a finite extension of $\mathbb{Q}_p$). Then there is a natural identification*

$$\mathbb{Q}_p(i)(\mathcal{O}_F) \simeq \begin{cases} R\Gamma_{\text{proét}}(\mathrm{Spec}(k),\mathbb{Q}_p) & \text{if } i = 0; \\ F[-1] & \text{if } i > 0. \end{cases}$$

And, if $\mathcal{O}_F$ is unramified (e.g. if $\mathcal{O}_F$ is the ring of integers of a finite unramified extension F of $\mathbb{Q}_p$), the previous identification has an integral version, though only in low degrees

$$\mathbb{Z}_p(i)(W(k)) \simeq \begin{cases} R\Gamma_{\text{proét}}(\mathrm{Spec}(k),\mathbb{Z}_p) & \text{if } i = 0; \\ W(k)[-1] & \text{if } 0 < i \leqslant p-2. \end{cases}$$

Sketch of proof. The first part comes from the fiber square ([AMMN, Theorem 6.17]) comparing $\mathbb{Q}_p(i)(\mathcal{O}_F)$ and $\mathbb{Q}_p(i)(\mathcal{O}_F/p)$, and some comparison result between $\mathbb{Q}_p(i)$ and $\mathbb{Q}_p(i)^{\mathrm{FM}}$.

The integral result follows from the same comparison result Theorem 6.5.9 between $\mathbb{Z}_p(i)$ and $\mathbb{Z}_p(i)^{\mathrm{FM}}$, and the fact that the map $\varphi : W(k) \to W(k)$ is the identity. ◆

The filtration on étale K-theory (that is, on TC) is somewhat harder to compute in mixed-characteristic. However, we can still compute things on some examples. We give now an application when applying our results to perfectoid rings. More precisely, we use the calculation of (rational) syntomic cohomology Theorem 6.3.10 on perfectoid rings, and deduce a new proof of the so-called "fundamental exact sequence of p-adic Hodge theory".

Theorem 6.4.7 (The fundamental exact sequence for $\mathcal{O}_C$)**.** *Let $\mathcal{O}_C$ be the ring of integers of a complete algebraically closed nonarchimedean field C (for instance $C = \mathbb{C}_p$). Then there is the following short exact sequence of abelian groups, for any $i > 0$:*

$$0 \to \mathbb{Q}_p(i)(\mathcal{O}_C) \to \mathrm{B}^+_{\mathrm{crys}}(\mathcal{O}_C)^{\varphi=p^i} \to \mathrm{B}^+_{\mathrm{dR}}(\mathcal{O}_C)/\mathrm{Fil}^{\geqslant i}\mathrm{B}^+_{\mathrm{dR}}(\mathcal{O}_C) \to 0.$$

Remark the first historical version of this exact sequence was given in the case of $i = 0$, where the middle term is then called $\mathrm{B}_e := \mathrm{B}^+_{\mathrm{crys}}(\mathcal{O}_C)^{\varphi=1}$, and the last term is replaced as follows

$$0 \to \mathbb{Q}_p \to \mathrm{B}_e \to \mathrm{B}_{\mathrm{dR}}/\mathrm{B}^+_{\mathrm{dR}} \to 0.$$

We recall for convenience the definitions of the period rings (that is, A_{inf}, A_{crys}, $\mathrm{B}^+_{\mathrm{dR}}$) appearing in Theorem 6.4.7. These are stated in the widest generality, that is for a general perfectoid ring R, but one recovers the previous rings by taking $R = \mathcal{O}_C$.

Construction 6.4.8 (Period rings, see [Bei12], [Bha12])**.** Let R be a perfectoid ring.

(1) The Fontaine's ring $A_{\text{inf}}(R) := W(R^\flat)$ is equipped with the canonical map $\theta : A_{\text{inf}}(R) \twoheadrightarrow R$ with kernel generated by some element ξ. Here $A_{\text{inf}}(R)$ is also isomorphic to the completed prismatic cohomology $\widehat{\Delta}_R$, where the corresponding Nygaard filtration is given by the ξ-adic filtration on $A_{\text{inf}}(R)$.

(2) The ring $A_{\text{crys}}(R) = A_{\text{crys}}(R/p)$ is the p-adic completion of the divided power envelope of $(\xi) \subset A_{\text{inf}}(R)$, that is the divided power envelope along the morphism $A_{\text{inf}}(R) \twoheadrightarrow R$. We have $A_{\text{crys}}(R) \simeq \mathbb{L}\Omega_R$, where the right-hand side is the derived de Rham cohomology of R; the Hodge filtration is identified with the divided power filtration via this equivalence. We also define $\mathrm{B}^+_{\text{crys}}(R) := A_{\text{crys}}(R)[1/p] = (\mathbb{L}\Omega_R)_{\mathbb{Q}_p}$, which inherits a Frobenius operator φ from $A_{\text{crys}}(R)$.

(3) Finally, let $\mathrm{B}^+_{\text{dR}}(R)$ be the ring $\mathrm{B}^+_{\text{dR}}(R) := \lim\limits_{\leftarrow}(A_{\text{inf}}(R)/\xi^n[1/p])$, which is also the Hodge completion $(\widehat{\mathbb{L}\Omega}_R)_{\mathbb{Q}_p}$ of $(L\Omega_R)_{\mathbb{Q}_p}$. Remark that, via this identification, the ξ-adic filtration on $\mathrm{B}^+_{\text{dR}}(R)$ (for which it is complete) corresponds to the Hodge filtration on $(\widehat{\mathbb{L}\Omega}_R)_{\mathbb{Q}_p}$.

As a lot of constructions or results concerning the ring $\mathcal{O}_C$ of Theorem 6.4.7, there is a more general version of the fundamental exact sequence of p-adic Hodge theory for general perfectoid rings R. This general version is recovered by the fundational results obtained on syntomic cohomology, as we explain now.

Theorem 6.4.9 (The fundamental exact sequence, [AMMN20], Theorem 7.7)**.** *Let R be a perfectoid ring and $i > 0$. Then there is a natural pullback square in $\mathcal{D}(\mathbb{Q}_p)$*

$$\begin{array}{ccc} R\Gamma_{\text{proét}}(\mathrm{Spec}(R[1/p]), \mathbb{Q}_p(i)) & \longrightarrow & \mathrm{B}^+_{\text{crys}}(R)^{\varphi = p^i} \\ \downarrow & & \downarrow \\ \mathrm{Fil}^{\geqslant i}\mathrm{B}^+_{\text{dR}}(R) & \longrightarrow & \mathrm{B}^+_{\text{dR}}(R). \end{array}$$

Proof. We use the so-called Beilinson fiber square, studied in depth in the article [AMMN20]. Roughly, this is a functorial pullback square relating $\mathrm{TC}(R)$, $\mathrm{TC}(R/p)$, $\mathrm{HC}^-(R)$ and $\mathrm{HP}(R)$ for any commutative ring henselian along (p). Combined with the following pullback square –generalisation of McCarthy's theorem– from [CMM21]

$$\begin{array}{ccc} K(R) & \longrightarrow & K(R/I) \\ {\scriptstyle \mathrm{tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}} \\ \mathrm{TC}(R) & \longrightarrow & \mathrm{TC}(R/I) \end{array}$$

applied to the henselian pair $(R, I) = (R, (p))$, we deduce that for any perfectoid ring R there is a fiber square:

$$\begin{array}{ccc} \mathbb{Q}_p(i)(R) & \longrightarrow & \mathbb{Q}_p(i)(R/p) \\ \downarrow & & \downarrow \\ (\widehat{\mathbb{L}\Omega}^{\geqslant i}_R)_{\mathbb{Q}_p} & \longrightarrow & (\widehat{\mathbb{L}\Omega}_R)_{\mathbb{Q}_p}. \end{array}$$

More precisely, this follows from [AMMN20], Theorem 6.17. Now the first term is identified with $R\Gamma_{\text{proét}}(\text{Spec}(R[1/p]), \mathbb{Q}_p(i))$ by [BS19, Theorem 9.4]; the proof of this identifications relies on successive reductions, and uses descent on the arc-topology, as developed in [BM18][10].

Moreover, the ring R/p is quasiregular semiperfect, hence by Proposition 6.3.4, $\mathbb{Q}_p(i)(R/p) \simeq \mathbb{A}_{\text{crys}}(R/p)[1/p] = \text{A}_{\text{crys}}(R/p)[1/p] = \text{A}_{\text{crys}}(R)[1/p] =: \text{B}^+_{\text{crys}}(R)^{\varphi = p^i}$.

And finally, the bottom arrow identifies with the map $\text{Fil}^i\text{B}^+_{\text{dR}}(R) \to \text{B}^+_{\text{dR}}(R)$ by construction of the period ring $\text{B}^+_{\text{dR}}(R)$ (Construction 6.4.8). ◆

6.5 Comparison with the theory of Fontaine-Messing

We present here the comparison result given in [AMMN20] between syntomic cohomology (as defined in [BMS19]), and the historically first notion of syntomic cohomology, as defined by Fontaine and Messing (in [FM87]). First, let us review some of the classical definitions of the theory of Fontaine-Messing, such as the *syntomic site* (which gave the name to the more recent quasisyntomic site).

Definition 6.5.1 (Mazur)**.** *A finite type map $f : Y \to X$ of schemes is syntomic if it is both a local complete intersection and flat.*

The corresponding *syntomic topology* is aimed to be richer than the flat topology, without being too complicated to handle. See [FM87], [Bau92] or [KT03] for some applications.

Remark 6.5.2. The notion of "local complete intersection" relies on regular closed embeddings, which are non-ambiguously defined only for noetherian schemes. In Fontaine-Messing, all schemes are then implicitly noetherian when dealing with syntomic topology.

In the noetherian context, we have the following:

Theorem 6.5.3 (Quillen)**.** *Let A and B be noetherian rings, and $A \to B$ a map of finite type. Then this is a local complete intersection if and only if the cotangent complex $\mathbb{L}_{B/A}$ has Tor amplitude in $[-1, 0]$ (indexing conventions for the derived category are cohomological).*

Remark 6.5.4. The finite type hypothesis in the previous result can be removed, by replacing "local complete intersection" by a more general notion valid for non-finite type morphisms of noetherian rings, as in [Avr99]. Following [FM87], we will not use these conventions here.

In this regard, the idea of the quasisyntomic topology developed in [BMS19] is to discard the noetherian and finite type hypotheses. As a comparison, we recall here the definition of quasisyntomic morphism. As usual in this context, all the constructions satisfy descent for the Zariski topology, so we may restrict to the affine context:

Definition 6.5.5. *A map of rings $R \to S$ is quasisyntomic if it is flat, and the cotangent complex $\mathbb{L}_{S/R}$ has Tor amplitude in $[-1, 0]$.*

Remark that, when considering p-complete rings, one may prefer to replace the "flat" and "Tor amplitude" conditions by their p-complete analogues (as we do in Definition A.1.3). To avoid this type of complications, let us work over $\mathbb{F}_p$.

[10] The local parts on the arc-topology are given by (ultra)products of p-complete valuation rings of rank $\leqslant 1$ with algebraically closed fraction field. The main deep property used in [BS19, Theorem 9.4] about the arc-topology is that both $\mathbb{Z}_p(n)$ and $R\Gamma(\text{Spec}(R[1/p]), \mathbb{Z}_p(n))$ are p-complete arc-sheaves when restricted to the category of perfectoid rings, and thus one can perform descent on the arc-topology.

Definition 6.5.6 ([FM86], II.1.1.). *Let X be a scheme. The syntomic site* Syn_X *consists of the category of X-schemes, endowed with the topology generated by the surjective syntomic X-morphisms of affine schemes. When $X = \mathrm{Spec}(A)$ is affine, we also denote* Syn_A *the site consisting of the category of affine schemes over* $\mathrm{Spec}(A)$*, with the same topology.*

Let $\mathrm{QSyn}_{\mathbb{F}_p}$ be the category of quasisyntomic $\mathbb{F}_p$-algebras (and all maps between them), and $\mathrm{QSyn}^{\mathrm{ft}}_{\mathbb{F}_p} \subseteq \mathrm{QSyn}_{\mathbb{F}_p}$ the full subcategory of quasisyntomic finite type $\mathbb{F}_p$-algebras. We turn both categories (or more precisely their opposites) into sites by declaring covers to be quasisyntomic faithfully flat maps.

Lemma 6.5.7. *There is a (trivial) canonical equivalence of sites* $\mathrm{Syn}_{\mathbb{F}_p} \cong \mathrm{QSyn}^{\mathrm{ft}\ \mathrm{op}}_{\mathbb{F}_p}$.

Proof. By definition, an object of $\mathrm{Syn}_{\mathbb{F}_p}$ is a finite type $\mathbb{F}_p$-algebra which is flat and a local complete intersection. By Theorem 6.5.3, this corresponds to quasisyntomic $\mathbb{F}_p$-algebras of finite type, *ie* to objects of $\mathrm{QSyn}^{\mathrm{ft}\ \mathrm{op}}_{\mathbb{F}_p}$. The maps in both categories are all morphisms (with the usual change of direction between rings and schemes). Finally, a map of rings is faithfully flat if and only if it is flat, and the induced map on spectra is surjective, so the claim follows (again using Theorem 6.5.3). ◆

Even if Fontaine-Messing defined syntomic cohomology $\mathbb{Z}_p(i)$ only in the noetherian context, their definition can be adapted to the more recent context of quasisyntomic rings.

Definition 6.5.8 (Syntomic cohomology of Fontaine–Messing, [AMMN, Definition 6.9.). *] We define sheaves $\mathbb{Z}_p(i)^{\mathrm{FM}}$ for $0 \leqslant i \leqslant p-2$, and $\mathbb{Q}_p(i)^{\mathrm{FM}}$ for $i \geqslant 0$, on the quasiregular semiperfectoid site* QRSPerd*, via*

$$\mathbb{Z}_p(i)^{\mathrm{FM}}(S) = \mathrm{fib}(\varphi/p^i - \mathrm{id} : L\Omega_S^{\geqslant i} \to L\Omega_S),$$
$$\mathbb{Q}_p(i)^{\mathrm{FM}}(S) = \mathrm{fib}(\varphi - p^i : L\Omega_S^{\geqslant i} \to L\Omega_S)_{\mathbb{Q}_p}.$$

These are sheaves on QRSPerd *because $S \mapsto L\Omega_S^{\geqslant i}$ is a sheaf. Unfolding (that is globalising to the whole quasisyntomic site), this defines similar sheaves on* QSyn*, for which we give the same notations.*

The following important result implies in particular the calculation of syntomic cohomology (of [BMS19]) in terms of derived de Rham cohomology Theorem 6.3.10.

Theorem 6.5.9 ([AMMN20], Theorem 6.22). *Let A be a quasisyntomic ring. Then there are natural, multiplicative equivalences of sheaves of complexes:*

$$\mathbb{Z}_p(i)^{\mathrm{FM}}(A) \simeq \mathbb{Z}_p(i)(A) \quad \text{for } i \leqslant p-2$$
$$\mathbb{Q}_p(i)^{\mathrm{FM}}(A) \simeq \mathbb{Q}_p(i)(A) \quad \text{for all } i \geqslant 0.$$

Sketch of proof. First we can reduce to proving the result for $A = S$ a quasiregular semiperfectoid $\mathbb{Z}_p$-algebra. If we fix some $i \geqslant 0$, we can also suppose that $\mathbb{Z}_p(i)(S)$ is concentrated in degree 0 thanks to Theorem 6.3.1. We then prove, using a version of the Beilinson fiber square ([AMMN, Theorem 6.17]) that $\mathbb{Q}_p(i)(S/p)$ is concentrated in degree 0. This implies that the map $\varphi - p^i$: $(\mathbb{L}\Omega_S)_{\mathbb{Q}_p} \to (\mathbb{L}\Omega_S)_{\mathbb{Q}_p}$ is surjective, thus giving us a cartesian and cocartesian square of abelian groups:

$$\begin{array}{ccc} \mathbb{Q}_p(i)(R) & \longrightarrow & \mathbb{Q}_p(i)(R/p) \\ \downarrow & & \downarrow \\ (\mathbb{L}\Omega_R^{\geqslant i})_{\mathbb{Q}_p} & \longrightarrow & (\mathbb{L}\Omega_R)_{\mathbb{Q}_p}. \end{array}$$

We then prove by a structure result on the right map that $\varphi - p^i : (\mathbb{L}\Omega_S^{\geqslant i})_{\mathbb{Q}_p} \to (\mathbb{L}\Omega_S)_{\mathbb{Q}_p}$ is surjective. Putting these observations together, this shows a natural identification

$$\mathbb{Q}_p(i)(S) = (\mathbb{L}\Omega_S)_{\mathbb{Q}_p}^{\varphi=p^i} \cap (\mathbb{L}\Omega_S^{\geqslant i})_{\mathbb{Q}_p} \simeq \mathrm{hofib}(\varphi - p^i : (\mathbb{L}\Omega_S^{\geqslant i})_{\mathbb{Q}_p} \to (\mathbb{L}\Omega_S)_{\mathbb{Q}_p}),$$

which is exactly the second desired identification.

The first assertion, about $\mathbb{Z}_p(i)$, is more intricate, and follows from an analogue of Theorem 6.3.1 proving that $\mathbb{Z}_p(i)^{\mathrm{FM}}(-)$, for $i \leqslant p-2$, is a discrete sheaf on the quasisyntomic site (see [AMMN, Section 6.4]). ◆

Remark 6.5.10. The previous equivalences are given only in nonnegative (cohomological) degrees. This is because syntomic cohomology $\mathbb{Z}_p(i)$ and $\mathbb{Q}_p(i)$ is zero in negative degrees (see Example 6.1.7).

It was established (in [Gei04, Theorem 1.3.]) that, for $i \leqslant p-2$ and formally smooth schemes over DVRs, syntomic cohomology is p-adic étale motivic cohomology. In fact, one expects this is the case in general; and the new definition of syntomic cohomology being valid in a much more general context then the one of Fontaine-Messing (that is, on the quasisyntomic site), this suggests a far wider definition for p-adic étale motivic cohomology (see the next section for details).

6.6 Syntomic cohomology is p-adic étale motivic cohomology

The aim of this section is to present the motivic flavour of syntomic cohomology. In particular we sketch the proof that syntomic cohomology is p-adic étale motivic cohomology.

On the one hand, algebraic K-theory has a filtration with graded pieces motivic cohomology (see Section 5.1). On the other hand, topological cyclic homology TC has a filtration –in the p-adic context– with graded pieces syntomic cohomology (see Section 6.1). Moreover, there is a map, called the *cyclotomic trace map*, which is an equivalence in some contexts of interest, that one can use to compare algebraic K-theory and TC. An adventurous mind would like to try to compare these two filtrations. This (arguably good) way of thinking leads to the intuition that syntomic cohomology has, in some sense, some motivic nature. This is the idea we develop now.

Algebraic K-theory is related to topological cyclic homology TC in the following way. Let R be a commutative ring. The *cyclotomic trace* is a map from algebraic K-theory to topological cyclic homology

$$\mathrm{K}(R) \to \mathrm{TC}(R).$$

This map can be shown to be universal in some sense ([BGT]), and is an extremely useful tool in studying algebraic K-theory. First, because TC is often easier to calculate directly than K-theory - for instance in the p-adic setting by using results of [BMS19]. And the cyclotomic trace is also an effective approximation to algebraic K-theory in a lot of contexts. In this way, for p-adic rings, TC and p-adic étale K-theory agree in nonnegative degrees, via the cyclotomic trace map. We formulate this last result for strictly henselian local rings R, since they are the local rings in the étale topology.

Theorem 6.6.1 ([CMM21], Theorem C)**.** *Let R be a strictly henselian local ring with residue field of characteristic p. Then the map $K(R) \to \mathrm{TC}(R)$ is a p-adic equivalence, that is, it is an equivalence after p-completion.*

One may then expect the filtration of Theorem 6.1.2 to be the étale sheafification of the filtration on algebraic K-theory with associated graded motivic cohomology (see Section 5.1 on motivic coho). In particular one expects the $\mathbb{Z}_p(i)$ to be the p-adic étale motivic cohomology (or, more precisely, of p-adic étale Tate twist). More concretely, it should be the complex obtained by étale sheafifying and p-adically completing the (Tate twist) complex $\mathbb{Z}(i)$.

This intuition is supported by all the computations surrounding syntomic cohomology. For instance, the results of [BMS19] together with Theorem 6.6.1 provide us a p-adic étale Atiyah-Hirzebruch type spectral sequence. Another example is the following result in equal characteristic p.

Theorem 6.6.2. *Let X be a smooth scheme over a perfect field k of characteristic p. Then there is an isomorphism of sheaves of complexes in the étale site of X, between syntomic cohomology $\mathbb{Z}_p(n)$ and p-adic étale sheafified motivic cohomology $\mathbb{Z}_p(n)_{\mathrm{Mot}}$ (which is constructed in this case by Voevodsky).*

Proof. This is a consequence of the computation Theorem 6.3.5 of syntomic cohomology in terms of the logarithmic de Rham-Witt sheaf, and the classical result of Geisser-Levine Theorem 5.3.4 on p-adic motivic cohomology. ◆

In the p-adic context, one hopes to define p-adic motivic cohomology on a large class of objects using syntomic cohomology (that is, p-adic étale motivic cohomology) and a similar Beilinson-Lichtenbaum type construction.

Remark 6.6.3. While the ℓ-adic and p-adic motivic cohomology are only "local" parts of motivic cohomology, one can expect to construct a "global" theory of motivic cohomology from these, when combined with the (also conjectural, and largely independant) rational motivic cohomology (see also the end of Section 5.1).

Example 6.6.4. As an example, the situation in equal characteristic p is well-understood. Indeed, for smooth algebras over a perfect field of characteristic p, p-adic étale motivic cohomology is known to be given by the logarithmic de Rham-Witt sheaves; see Theorem 6.3.5.

6.7 An hedonist picture of syntomic cohomology

It is often nice to get a look from above once we learned several details of a theory. Here we try to summarize some of the main aspects we encountered about syntomic cohomology.

Remind that syntomic cohomology can be defined via topological Hochschild homology (using techniques of Chapter 3), or independently via prismatic cohomology (reviewed in Chapter 4), and that it corresponds to p-adic étale motivic cohomology (reviewed in Chapter 5).

Syntomic cohomology is a collection of sheaves $\mathbb{Z}_p(i) = \mathbb{Z}_p(i)(-)$, for $i \in \mathbb{N}$, on the *quasisyntomic site* (as defined in Section 3.2 in characteristic p, and in Section A.1 in mixed characteristic). The *quasiregular semiperfectoid* $\mathbb{Z}_p$-algebras form a basis for the quasisyntomic site, on which it is easier to compute syntomic cohomology $\mathbb{Z}_p(i)$; similarly when restricting to characteristic p, quasiregular semiperfect $\mathbb{F}_p$-algebras form a basis for the quasisyntomic site (restricted to $\mathbb{F}_p$-algebras), on which we can express more explicitly the values of the sheaves $\mathbb{Z}_p(i)$. A major technique when proving or defining anything about syntomic cohomology is thus to perform quasisyntomic descent to quasiregular semiperfectoid rings. There exist some rational and mod p variants of syntomic cohomology as well, we denote by $\mathbb{Q}_p(i)$ and $\mathbb{F}_p(i)$ respectively, and for which the previous remarks also apply.

By restricting syntomic cohomology to the étale site (or the pro-étale site), we can identify syntomic cohomology with *p-adic étale motivic cohomology.* Since p-adic étale motivic cohomology (in fact any form of motivic cohomology) is still a conjectural object in general, this identification makes sense only when the motivic side makes sense (for instance, for a smooth scheme over a field). However, some important conjectural properties of p-adic étale motivic cohomology (such as an Atiyah-Hirzebruch type spectral sequence) are satisfied by syntomic cohomology, and in a greater generality than where motivic cohomology is already well-defined. This suggests some new possible developments in motivic cohomology theory, arising from syntomic cohomology. Remark that, comparing syntomic cohomology $\mathbb{Z}_p(i)$ with motivic cohomology, the integer $i \geqslant 0$ is interpreted as a *motivic weight* (see Section 5.1).

The prismatic theory is not conjectural at all in its construction, and provides a construction of the sheaves $\mathbb{Z}_p(i)$ which is in a way more arithmetical (in particular, it does not use the topological theory surrounding TC). It is based on the so-called prismatic site, and gives an analogue in mixed characteristic of some construction which previously existed only equal characteristic p. For instance, think about crystalline cohomology, which takes a (smooth proper) variety over a perfect field k of characteristic p to a cohomology theory defined over the mixed characteristic ring $W(k)$. It is represented by an explicit complex $W\Omega^\bullet$ (or $\mathbb{L}W\Omega^\bullet$ in its derived version), also of mixed characteristic. These objects are extremely useful to study questions in p-adic Hodge theory, and the prismatic cohomology then gives a similar construction for varieties of mixed characteristic. *Prismatic cohomology*, denoted Δ_-, is again a sheaf on the quasisyntomic site, and is equipped with a natural filtration $\mathcal{N}^{\geqslant i}\Delta_-$ ($i \geqslant 0$), called the *Nygaard filtration.* In this picture, syntomic cohomology appears as the divided Frobenius fixed points on this Nygaard filtration: $\mathbb{Z}_p(i)(-) = (\mathcal{N}^{\geqslant i}\Delta_-)^{\varphi_i=1}$, where $\varphi_i := \frac{\varphi}{p^i}$, and is thus concretely related to most of the prismatic cohomology theory. Finally, the relation between the prismatic theory (Chapter 4) and the topological theory (Chapter 3) is in the fact that the Nygaard completion $\widehat{\Delta}_-$ of prismatic cohomology corresponds (φ-equivariantly) to the quasisyntomic sheaf $\widehat{\Delta}_- := R\Gamma_{\mathrm{syn}}(-, \pi_0\mathrm{TC}^-(-;\mathbb{Z}_p))$.

We now summarize these ideas in the following, where we compare the informations in mixed characteristic (that is, the general situation) to their characteristic p analogues, where we have sometimes more precise informations.

General picture of syntomic cohomology

For any $i \geqslant 0$, syntomic cohomology $\mathbb{Z}_p(i)$ is a sheaf of complexes on the quasisyntomic site QSyn, and its values are concentrated in cohomological degrees $[0, i]$. Moreover, for any quasisyntomic ring A,

$$\begin{aligned}
\mathbb{Z}_p(i)(A) &= \mathrm{gr}^i \mathrm{TC}(A; \mathbb{Z}_p)[-2i] && \text{(Chapter 3, motivic filtration on TC)} \\
&= (\mathcal{N}^{\geqslant i} \widehat{\Delta}_A)^{\varphi_i = 1} && \text{(Chapter 4, Frobenius fixed points on prismatic cohomology)} \\
&= \mathbb{Z}_p(i)_{\mathrm{Spec}(A)_{\text{ét}}} && \text{(Chapter 5, } p\text{-adic étale motivic cohomology)}
\end{aligned}$$

In mixed characteristic

On quasiregular semiperfectoid algebras S, the complexes $\mathbb{Z}_p(i)(S)$ are concentrated in degrees $[0, 1]$; this comes from the fact that the complex $\widehat{\Delta}_S$ is discrete (that is, concentrated in degree 0).

Even more locally in the quasisyntomic topology, $\mathbb{Z}_p(i)$ is in fact discrete for any $i \geqslant 0$. This implies that p-completed algebraic K-theory $K(-; \mathbb{Z}_p)$ is, locally in the quasisyntomic topology, concentrated in even degrees.

Examples:

- $\widehat{\Delta}_R \simeq A_{\mathrm{inf}}(R)$ for any perfectoid ring R. In particular $\widehat{\Delta}_R$ is concentrated in degree 0.
- $\mathbb{Z}_p(i)(R) \simeq \left(\varphi^{-1}(d)^i A \xrightarrow{\frac{\varphi}{d^i} - \mathrm{id}} A\right)$ for R any perfectoid ring, $(A, (d))$ an associated perfect prism, and $i > 0$.
- The Nygaard filtration on the ring $\mathbb{Z}$ is simply the p-adic filtration: $\mathcal{N}^{\geqslant i} \mathbb{Z} = p^i \mathbb{Z}$. Its Nygaard completion is thus its p-adic completion $\mathbb{Z}_p$.

In characteristic p

On quasiregular semiperfect $\mathbb{F}_p$-algebras S, the complexes $\mathbb{Z}_p(i)(S)$ are given by the p-torsion free ring $\mathbb{A}_{\mathrm{crys}}(S)^{\varphi = p^i}$, concentrated in degree 0. This implies that the p-completed algebraic K-theory on S is concentrated in even degrees, and given by $K_{2i}(S; \mathbb{Z}_p) \cong \mathbb{A}_{\mathrm{crys}}(S)^{\varphi = p^i}$.

Examples:

- For k a perfect $\mathbb{F}_p$-algebra (*e.g.* $k = \mathbb{F}_p$), $\widehat{\Delta}_k \cong A_{\mathrm{inf}}(k) = W(k)$.
- For k a perfect $\mathbb{F}_p$-algebra, $\mathbb{Z}_p(i)(k) \simeq W(k)^{\varphi = p^i}[0] \simeq \left(\varphi^{-1}(p)^i W(k) \xrightarrow{\frac{\varphi}{p^i} - \mathrm{id}} W(k)\right)$ is zero whenever $i > 0$. For instance if $k = \mathbb{F}_p$, then $\mathbb{Z}_p(i)(\mathbb{F}_p) \simeq \mathbb{Z}_p[0]$ if $i = 0$, and is zero otherwise.
- For any quasisyntomic $\mathbb{F}_p$-algebra A, the Frobenius endormorphism on A acts by multiplication by p^i on $\mathbb{Z}_p(i)(A)$.

Appendix A

Technicalities

Here are some more technical tools and definitions that were used throughout the text.

A.1 Some useful definitions in mixed characteristic

We present here some definitions of objects we encountered in the main text, and for which we only gave the definition in characteristic p. We thus state the definitions of perfectoid rings, of quasiregular semiperfectoid rings, and finally the definition of the quasisyntomic site in mixed characteristic, as given in [BMS18], [BMS19] or [AMMN20].

We begin with the definition of perfectoid rings, as stated in [BMS18][1]. Let R be a commutative ring which is π-adically complete and separated for some element $\pi \in R$ dividing p. Note that this condition implies that R is also p-adically complete. Let $\varphi : R/pR \to R/pR$ be the (absolute) Frobenius on R/pR (in fact, all the induced Frobenius morphisms will also be denoted by φ). We denote by $R^\flat$ the *tilt* of R

$$R^\flat := \varprojlim_{\varphi} R/pR,$$

which is a perfect $\mathbb{F}_p$-algebra. Remark that for such a ring, one can define its Fontaine's ring $A_{\mathrm{inf}}(R)$: $A_{\mathrm{inf}}(R) := W(R^\flat)$, which is still equipped with a Frobenius automorphism φ ($W(-)$ denotes the ring of Witt vectors, hence the Frobenius is a lift of Frobenius here).

Definition A.1.1 (Perfectoid ring, [BMS18], Definition 3.5). *A ring S is* perfectoid *if and only if it satisfies the following conditions:*

(1) S is π-adically complete for some element $\pi \in S$ such that π^p divides p;

(2) the Frobenius map $\varphi : S/pS \to S/pS$ is surjective;

(3) the kernel of the map $\theta : A_{inf}(S) \to S$ is principal.

Remark that a ring of characteristic p is perfectoid if and only if it is perfect (take $\pi = 0$). We will not try to really motivate this definition, but we shall just say that perfectoid rings have played an important role in p-adic arithmetic geometry, by providing some tool to study highly non-Notherian objects. For instance, perfectoid rings arise naturally when considering local questions on the pro-étale topology. In fact, the situation is similar to what we encountered

[1] There exist a lot of different definitions for perfectoid rings in the litterature, which often apply to different contexts. The definition given in [BMS18] unifies most of them.

in Chapter 3, where the quasiregular semiperfect(oid) rings form a basis for the quasisyntomic topology. We now define these (quasiregular semiperfectoid and quasisyntomic rings) in mixed characteristic.

First, we need the following definition:

Definition A.1.2 (p-complete (faithful) flatness and Tor amplitude, [BMS19], Definition 4.1)**.** *Let R be a commutative ring. An R-module M is called p-completely flat (resp. p-completely faithfully flat) if $M \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ is a flat (resp. faithfully flat) R/p-module concentrated in degree zero. Similarly, an object $N \in \mathcal{D}(R)$ has p-complete Tor amplitude in $[a, b]$ if $N \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ has Tor amplitude in $[a, b]$.*

Remark that, if R is a $\mathbb{F}_p$-algebra, then p-complete flatness (resp. p-complete Tor amplitude in $[a, b]$) is the same as flatness (resp. Tor amplitude in $[a, b]$).

Definition A.1.3 (The quasisyntomic site))**.** *(1) A commutative ring R is called* quasisyntomic *if it is p-complete, has bounded p-power torsion (that is, $R[p^\infty] = R[p^n]$ for some integer $n \geqslant 0$), and $\mathbb{L}_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1, 0]$. We let* QSyn *be the category of quasisyntomic rings, with all ring homomorphisms (and not just the quasisyntomic ones).*

(2) We call a map $A \to B$ in QSyn *a quasisyntomic map (resp. quasisyntomic cover) if $A \to B$ is p-completely flat (resp. p-completely faithfully flat), and if $\mathbb{L}_{B/A} \in \mathcal{D}(B)$ has p-complete Tor-amplitude in $[-1, 0]$. The opposite category of QSyn acquires the structure of a site by declaring covers to be quasisyntomic covers.*

(3) An object $S \in$ QSyn *is* quasiregular semiperfectoid *if S admits a map from a perfectoid ring, and if the Frobenius morphism on S/p is surjective. We denote by* QRSPerd $\subset$ QSyn *the full subcategory spanned by quasiregular semiperfectoid rings.*

Example A.1.4. (1) An $\mathbb{F}_p$-algebra is quasiregular semiperfectoid (resp. quasisyntomic) if and only if it is quasiregular semiperfect (resp. quasisyntomic as defined in Chapter 3) (see Definitions 3.2.7 and 3.2.12 respectively).

(2) Any perfectoid ring R is a quasiregular semiperfectoid ring. Indeed, the cotangent complex $\mathbb{L}_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude concentrated in degree -1, and has bounded p-torsion (in fact, $R[p^\infty] = R[p]$).

The proofs of the foundational results about the quasisyntomic site (for instance, quasiregular semiperfectoid rings form a basis for the quasisyntomic site) correspond to their characteristic p analogues (see [BMS19] for more details).

A.2 ∞-categories and spectra

The definition of an ∞-category has motivation in both (usual) category formalism and homotopy theory. The basic idea is that we would like to keep trace of the choices we make when we say "these two objects are equal": what we usually mean by this (at least in algebraic contexts) is that there is an isomorphism from the first to the second object. But then there may exist other such isomorphisms, from the first and to the second same object. But usually, one can relate these different isomorphisms by some "2-morphisms", which can also be isomorphisms. And so on, we define some "higher" morphisms, starting from our two concrete objects. To formalise this notion (and to be able to use such ideas concretely, without specifying all the higher data we are dealing with), we use simplicial objects. More precisely, the definition of an ∞-category comes as follows:

Definition A.2.1 (∞-category)**.** *We say a simplicial set $S_\bullet$ is an ∞-category if it satisfies the following condition*

For $0 < i < n$, every map $\sigma_0 : \Lambda_i^n \to S_\bullet$ from the i-th simplicial horn admits an extension to the n-simplex: $\sigma : \Delta^n \to S_\bullet$.

This condition is called the weak Kan extension condition (and ∞-categories are also called weak Kan complexes).

In practice, the notion of ∞-category is often used in algebraic geometry in the case where $S_\bullet$ is the derived category $\mathcal{D}(k)$ over some ring k: the "elements" (we usually call without quotation marks) of the ∞-derived category $\mathcal{D}(k)$ –which correspond to the 0-simplexes of $S_\bullet$– are the usual complexes of k-modules, seen as elements in the derived category, "1-morphisms" are usual morphisms, "2-morphisms" are homotopies between morphisms, "3-morphisms" are homotopies between homotopies between morphisms, and so on. This example of ∞-category leads us directly[2] to the notion of *spectrum*.

The category of spectra is, in a way, a well-behaved category to deal with algebraic objects (such as rings, or modules) in the ∞-categorical setting. A good approximation for the category of spectra is the derived ∞-category $\mathcal{D}(\mathbb{Z})$ over the integers $\mathbb{Z}$, which embeds into the category of spectra. This theory was developed progressively from the 1980's. It started from algebraic topology issues, and was formalized in its modern version in the setting of ∞-categories of Lurie. This is only recently that the theory of spectra has had arithmetic applications (typically via topological cyclic homology, as in [BMS19]). Instead of giving a precise definition, we present here only some intuition for the definition of spectra.

Definition A.2.2. *Intuitively, a* spectrum *is a sequence of pointed spaces $\{X_n\}_{n\geqslant 0}$ and maps $\mathrm{S}^1 \wedge X_n \to X_{n+1}$ (where $\mathrm{S}^1 \wedge X_n = \mathrm{S}^1 \times X_n/((\mathrm{S}^1 \times \star) \cup (\star \times X_n))$). Given a spectrum X, one can extract homotopy groups $\pi_i(X)$, $i \in \mathbb{Z}$, which are all abelian (one can think of these π_* as an analogue of H_*).*

The category of spectra Sp is a symmetric monoidal ∞-category, with a symmetric monoidal structure called "smash product". Notions of rings and modules still make sense in spectra, and the ∞-derived category $\mathcal{D}(k)$ over a fixed commutative ring k embeds into the category Sp.

Definition A.2.3. *Ring objects in spectra are called $\mathbb{E}_\infty$-rings. The initial object in $\mathbb{E}_\infty$-rings is the so-called sphere spectrum $\mathbb{S}$.*

There is a functor, called the *Eilenberg-Maclane functor*, which takes any complex $C^\bullet$ (and in particular, any usual commutative ring when considered as a discrete complex) to a $\mathbb{E}_\infty$-ring spectrum HC. This defines a fully faithful map to the symmetric monoidal category of spectra.

As an example, one can speak about sheaves taking values not only in the derived category $\mathcal{D}(\mathbb{Z})$, but in $\mathbb{E}_\infty$-ring spectra. This is the case for $\mathrm{THH}(-)$ for instance.

A.3 Of filtrations and spectral sequences

The notion of filtration is used for lots of different types of objects. We define it here only for chain complexes.

[2]The history of the two notions is a bit more complicate.

Definition A.3.1 (Filtration on a chain complex)**.** *Let $C^\bullet$ a complex of modules over a commutative ring. A descending, $\mathbb{N}$-indexed filtration on $C^\bullet$ is the data of a descending chain of subcomplexes: $C^\bullet = \mathrm{Fil}^0 C^\bullet \supseteq \mathrm{Fil}^1 C^\bullet \supseteq \mathrm{Fil}^2 C^\bullet \supseteq \dots$. The graded pieces of this filtration are the complexes $\mathrm{gr}^i C := \mathrm{Fil}^i C/\mathrm{Fil}^{i+1} C$, for $i \geqslant 0$; one checks easily these are indeed complexes. Such a filtration is said to be complete if the canonical map $C \to \lim_i C/\mathrm{Fil}^i C$ is an isomorphism; that is, if it is an isomorphism in each degree.*

The similar notions of $\mathbb{Z}$-indexed, or ascending filtrations are defined in the same way. We say the filtration is then *exhaustive* if the colimit up to the filtration is the complex itself. Remark this property is trivial for $\mathbb{N}$-indexed filtrations, and non trivial for $\mathbb{Z}$-indexed ones.

Now, one can associate to a ($\mathbb{Z}$- or $\mathbb{N}$-indexed) filtered chain complex C a spectral sequence. This spectral sequence is a tool for computing the (co)homology of the complex C from the (co)homology of the associated graded objects –which is in general simpler. Recall a spectral sequence is given in general by some bicomplexes, indexed by some pages $E_r^{p,q}$, among which we usually specify only a first page (for instance the E_2-page). We do not recall the theory of spectral sequences here, but only what is necessary for us.

One says that a spectral sequence *converges* to H with an increasing filtration F if $E_\infty^{p,q} = \mathrm{Fil}^p \mathrm{H}^{p+q}/\mathrm{Fil}^{p+1}\mathrm{H}^{p+q}$. Moreover, a spectral sequence *degenerates* (at the second page) if $E_\infty = E_2$. This happens in particular when all the differentials on the second page are zero.

Going the other way, we can define a spectral sequence converging to a given complex equipped with a filtration. More precisely, let us fix a cochain complex $C^\bullet$ together with subcomplexes $\mathrm{Fil}^p C^\bullet$, forming a complete exhaustive decreasing filtration, where p ranges across the integers. We require the boundary map to be compatible with the filtration. Then we define the first page of the spectral sequence as: $E_1^{p,q} := \mathrm{Fil}^p C^{p+q}/\mathrm{Fil}^{p+1} C^{p+q}$. That is, E_1 corresponds to the graded pieces of the filtration. By restricting the differential of $C^\bullet$ on this first page, we can define recursively the higher pages of the spectral sequence. And, thanks to the completeness and exhaustiveness properties of our given filtration, this spectral sequence converges to $E_\infty^{p,q} = \mathrm{Fil}^p \mathrm{H}^{p+q}(C^\bullet)$. This gives us an equivalence between spectral sequences and filtrations on a complex of modules.

Moreover, when the starting complex $C^\bullet$ admits a ring structure (and hence its cohomology has the structure of a graded ring $\mathrm{H}(C^\bullet)$), then it induces a *multiplicative* structure on the induced spectral sequence: that is, each page E_r is differential graded algebra (instead of just a geaded bicomplex), and these multiplicative structures on each page are compatible via the boundary map.

Now, we sometimes consider complexes C that are defined directly in a derived category, without a canonical complex to represent it (this is the case for crystalline cohomology, who needed to wait for the de Rham-Witt complex before being represented by a canonical complex in some cases, or even topological Hochschild homology, which is by nature only an object in the derived sense). In this case, our previous discussion needs to be adapted: for instance, even the notion of "subcomplex" is not well-defined. This is what we do now.

Definition A.3.2 (Filtration in the derived category)**.** *We say that a descending $\mathbb{N}$-indexed filtration on the complex C is then a collection of complexes $\mathrm{Fil}^i C$, $i \geqslant 0$, and maps between them: $C = \mathrm{Fil}^0 C \leftarrow \mathrm{Fil}^1 C \leftarrow \mathrm{Fil}^2 C \leftarrow \dots$. The graded pieces of the filtration are computed as the cofiber of the maps $\mathrm{Fil}^{\star+1} C \to \mathrm{Fil}^\star C$.*

Such a (descending, $\mathbb{N}$-indexed) filtration is said to be complete if the canonical map $C \to \mathrm{Rlim}_i C/\mathrm{Fil}^i C$ is an equivalence, where Rlim *means we take the (inverse) limit in the derived sense, and $C/\mathrm{Fil}^i C$ denotes the cofiber of the map $\mathrm{Fil}^i C \to C$. This condition is equivalent to asking that $\mathrm{Rlim}_i \mathrm{Fil}^i C \simeq 0$.*

The previous discussion can now be translated directly in the language of derived categories.

Remark A.3.3 (Filtration on a sheaf)**.** We can also speak about filtrations on a given sheaf. If this sheaf takes values in the category of rings, or k-modules for a given ring k, then we are in the first situation. If this sheaf takes values in the derived category over some ring (for instance $\mathcal{D}^{\geqslant 0}(\mathbb{Z}_p)$), then we use the second definition. In both cases, we define the filtration directly on the values of our sheaf.

In fact, the notion of filtered objects in derived category can be formalized in a more functorial way when using sheaves taking values in this category. More precisely, there is a notion of *filtered derived category*, which formalises the fact for a sheaf to be equipped with a natural filtration. In particular, this allows some good formalism when dealing with compatibility between tensor products and filtered objects. See [BMS19], Section 5.1 for some details on the filtered derived category.

Bibliography

[AMMN20] Benjamin Antieau et al. "On the Beilinson fiber square". In: *arXiv:2003.12541 [math]* (Mar. 2020). arXiv: 2003.12541. URL: http://arxiv.org/abs/2003.12541 (visited on 08/12/2020).

[Ant19] Benjamin Antieau. "Periodic cyclic homology and derived de Rham cohomology". en. In: *Annals of K-Theory* 4.3 (2019), pp. 505–519.

[Avr99] Luchezar L. Avramov. "Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology". In: *Ann. of Math.* 150.2 (1999), pp. 455–487.

[Bei12] A. Beilinson. "p-adic periods and derived de Rham cohomology". en. In: *Journal of the American Mathematical Society* 25.3 (2012), pp. 715–738. (Visited on 08/13/2020).

[BGT13] Andrew J Blumberg, David Gepner, and Gonçalo Tabuada. "A universal characterization of higher algebraic K-theory". In: *Geometry & Topology* 17.2 (Apr. 2013), pp. 733–838.

[Bha12] Bhargav Bhatt. "p-adic derived de Rham cohomology". In: *arXiv:1204.6560* (2012).

[BHM93] M. Bökstedt, W. C. Hsiang, and I. Madsen. "The cyclotomic trace and algebraic K-theory of spaces". In: *Inventiones Mathematicae* 111.1 (1993), pp. 465–539.

[BLM21] Bhargav Bhatt, Jacob Lurie, and Akhil Mathew. "Revisiting the de Rham–Witt complex". In: *Astérisque* 424 (2021), pp. viii+165.

[BM21] Bhargav Bhatt and Akhil Mathew. "The arc topology". In: *Duke Math. J.* 170.9 (2021). , pp. 1899–1988.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 128 (2018), pp. 219–397.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Topological Hochschild homology and integral p-adic Hodge theory". In: *arXiv:1802.03261 [math]* (Apr. 2019). arXiv: 1802.03261. URL: http://arxiv.org/abs/1802.03261 (visited on 08/12/2020).

[BS15] Bhargav Bhatt and Peter Scholze. "The pro-étale topology for schemes". In: *Astérisque* 369 (2015), pp. 99–201.

[BS22] Bhargav Bhatt and Peter Scholze. "Prisms and prismatic cohomology". In: *Ann. of Math. (2)* 196.3 (2022), pp. 1135–1275.

[CMM21] Dustin Clausen, Akhil Mathew, and Matthew Morrow. "K-theory and topological cyclic homology of henselian pairs". In: *J. Amer. Math. Soc.* 34.2 (2021), pp. 411–473.

[DF85] William G. Dwyer and Eric M. Friedlander. "Algebraic and étale K-theory". In: *Transactions of the American Mathematical Society* 292.1 (Jan. 1985), pp. 247–247. (Visited on 08/13/2020).

[FM86] J.-M Fontaine and W Messing, eds. *p-adic periods and p-adic etale cohomology.* English. OCLC: 123328869. 1986.

[Fon82] Jean-Marc Fontaine. "Sur Certains Types de Representations p-Adiques du Groupe de Galois d'un Corps Local; Construction d'un Anneau de Barsotti-Tate". In: *The Annals of Mathematics* 115.3 (1982), p. 529.

[Fri05] Eric M. Friedlander. "Motivic complexes of Suslin and Voevodsky". In: *Handbook of K-theory. Vol. 1, 2.* Springer, Berlin, 2005, pp. 1081–1104.

[Fu11] Lei Fu. *Etale cohomology theory.* Nankai tracts in mathematics v. 13. OCLC: ocn587124404. Hackensack, N.J: World Scientific, 2011. ISBN: 978-981-4307-72-7.

[GDK] Alexandre Grothendieck, Pierre Deligne, and Nick Katz. *Groupes de monodromie en géométrie algébrique : SGA 7.*

[GL00] Thomas Geisser and Marc Levine. "The K-theory of fields in characteristic p". en. In: *Inventiones mathematicae* 139.3 (2000), pp. 459–493.

[Hes06] Lars Hesselholt. "On the topological cyclic homology of the algebraic closure of a local field". In: *arXiv:math/0508309* 399 (2006). arXiv: math/0508309, pp. 133–162. DOI: 10.1090/conm/399/07517. URL: http://arxiv.org/abs/math/0508309 (visited on 08/13/2020).

[HN19] Lars Hesselholt and Thomas Nikolaus. "Topological cyclic homology". In: *arXiv:1905.08984 [math]* (May 2019). arXiv: 1905.08984. URL: http://arxiv.org/abs/1905.08984 (visited on 08/13/2020).

[KM21] Shane Kelly and Matthew Morrow. "K-theory of valuation rings". In: *Compos. Math.* 157.6 (2021), pp. 1121–1142.

[Lod92] Jean-Louis Loday. *Cyclic homology.* Vol. 301. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Appendix E by María O. Ronco. Springer-Verlag, Berlin, 1992, pp. xviii+454.

[Lur09] Jacob Lurie. *Higher topos theory.* Annals of mathematics studies no. 170. OCLC: ocn244702012. Princeton, N.J: Princeton University Press, 2009. ISBN: 978-0-691-14048-3 978-0-691-14049-0.

[Mat] Akhil Mathew. "Simplicial commutative rings, I". en. In: (), p. 9. URL: https://math.uchicago.edu/~amathew/SCR.pdf.

[Mor15] Matthew Morrow. "K-theory and logarithmic Hodge–Witt sheaves of formal schemes in characteristic p". In: *arXiv:1512.04703 [math]* (Dec. 2015). arXiv: 1512.04703. URL: http://arxiv.org/abs/1512.04703 (visited on 09/08/2020).

[Mor19] Matthew Morrow. "Topological Hochschild homology in arithmetic geometry". swc-math.github.io/aws/2019/2019MorrowNotes.pdf. 2019.

[nLa] nLab. *fracture theorem in nLab.* URL: https://ncatlab.org/nlab/show/fracture+theorem (visited on 08/13/2020).

[NS19] Thomas Nikolaus and Peter Scholze. "Correction to "On topological cyclic homology" [MR3904731]". In: *Acta Math.* 222.1 (2019), pp. 215–218.

[Qui70] Daniel Quillen. "On the (co-) homology of commutative rings". In: *Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968)*. Proc. Sympos. Pure Math., XVII. Amer. Math. Soc., Providence, RI, 1970, pp. 65–87.

[Tat67] J. T. Tate. "p-Divisible Groups". en. In: *Proceedings of a Conference on Local Fields*. Ed. by T. A. Springer. Berlin, Heidelberg: Springer, 1967, pp. 158–183. ISBN: 978-3-642-87942-5. DOI: 10.1007/978-3-642-87942-5_12.